Lindblad master equation for twisted photons in a turbulent atmosphere

Tim Ehret,* Vyacheslav Shatokhin, and Andreas Buchleitner
Albert-Ludwigs-Universität Freiburg, Hermann-Herder-Strasse 3, D-79104 Freiburg, Germany

(Received 3 July 2026; accepted 23 October 2026; published 10 November 2026)

We revisit a Lindblad master equation for the propagation of photons carrying orbital angular momentum through a turbulent atmosphere. This equation is here independently derived from the stochastic Hamiltonian formulation of paraxial optics, accounts for the Kolmogorov power spectrum of turbulence, and relies on an exact ensemble average over refractive index fluctuations. We show that, in contrast to previous representations, our present formulation admits efficient numerical solutions on suitably truncated Hilbert spaces, which we exploit in a study of the crosstalk of a single twisted photon propagating through the atmosphere.

DOI: 10.1103/m4rt-sss4

I. INTRODUCTION

The great potential of structured light modes in diverse applications ranging from microscopy [1] to high-dimensional quantum cryptography [2] primarily stems from the theoretically unbounded orthonormal basis furnished by the associated beam solutions of the paraxial wave equation in vacuum [3]. Within the zoology of structured light beams [4], Laguerre-Gaussian (LG) modes occupy a special role, since they carry an orbital angular momentum (OAM) $l\hbar$ ($l = 0, \pm 1, \pm 2, \ldots$) per photon imprinted in their (twisted) helical phase profiles $\exp(il\varphi)$, with φ the azimuthal angle in the plane transverse to the optical axis [3]. Unfortunately, in free space the phase and intensity profiles of twisted light modes—in fact, of any modes except singular modes [5] or eigenmodes [6,7] of turbulence—are severely distorted by fluctuations of the refractive index of air, resulting in turbulence-induced power loss, intermodal crosstalk, and the deterioration of the nonclassical properties of light [8–12], rendering the realization of high-dimensional optical communication protocols in free-space channels highly challenging [13–17].

The propagation of a scalar monochromatic optical wave with an arbitrary transverse spatial profile across a turbulent medium is governed by the stochastic parabolic equation (SPE) [18]. Despite substantial progress in understanding the SPE's properties under a broad range of turbulence conditions [19], its analytical solutions are known in only a few simple cases [18,20,21], while in general the SPE is solved numerically. For instance, the density matrix of the propagated twisted photon(s) can be obtained indirectly from the SPE via the split-step method based on discrete phase screens representing a three-dimensional random medium [22–24], including dynamically evolving media [5,25]. By properly specifying the initial field, one can describe the transmission of classical and quantum beams alike, be they, e.g., twisted [26], squeezed [27], or entangled [28]. However, obtaining ensemble averages relies on many random realizations of the phase screens, which is computationally expensive. In addition, the generation of phase screens that faithfully represent atmospheric turbulence is a nontrivial task which constitutes a research area of its own [23–25]. Finally, the distance between neighboring screens is not strictly determined, but is deduced from empirical conditions that intensity fluctuations be sufficiently small [28,29] or that positions of turbulence-induced vortices can be tracked [26]. These empirical conditions lead to distinct prescriptions for interscreen intervals for identical turbulence conditions.

The split-step method's trade-offs can be avoided by directly deriving the dynamics of the ensemble-averaged density matrix; in this vein, an infinitesimal propagation equation (IPE) describing the evolution of the quantum state of single or entangled twisted photons in atmospheric turbulence was derived from the SPE [30]. The IPE was obtained in the LG [30,31] and plane-wave [32] basis, but both resulting forms suffer from certain limitations. While, in the LG basis, the IPE can only be solved under severe basis truncation in Ref. [30], rendering in general unreliable solutions [33], in the plane-wave basis it can be solved under the quadratic approximation of the phase structure function [34] and for small numbers of populated LG modes, with $|l| \leqslant 5$ and radial index $p = 0$ determining the modal structure along the transverse radius. Within this framework, the authors of Ref. [31] studied a Lindbladian representation of the IPE in the LG basis, with a discrete set of Lindblad operators provided in terms of a generating function, and explored the dynamics of the twisted photon's density operator on a truncated space $|l|, p \leqslant 4$.

In the present contribution, we streamline the derivation of the Lindblad master equation, employing methods from the theory of open quantum systems, and cast the equation in a form that is numerically superior to the one presented in Refs. [31,32] and grants additional structural insight concealed in Refs. [31,32].

Note that the theory of wave propagation through disordered media is closely related to the theory of open quantum

*Contact author: tim.ehret@physik.uni-freiburg.de

Published by the American Physical Society under the terms of the Creative Commons Attribution 4.0 International license. Further distribution of this work must maintain attribution to the author(s) and the published article's title, journal citation, and DOI.

systems [35], where one is concerned with the effective dynamics of a subsystem embedded into an environment. An effective description is obtained by performing a partial trace over the environment, which can be understood as a statistical average over the environment state. In close analogy, in the study of disordered systems, one resorts to a statistical treatment and seeks to assess ensemble-averaged quantities with the average taken over different realizations of the disorder. This link has been explored in multiple works, demonstrating how exact [36,37] or perturbative [38] master equations governing the disorder-averaged time evolution of disordered systems can be derived. We exploit this link to derive a Lindblad master equation for the evolution of a photon carrying OAM in atmospheric turbulence. It should be further mentioned that the Hamiltonian for twisted photons in turbulence bears close similarity to the Hamiltonian for plane-wave photons [39,40], while our ensemble-averaged Lindblad master equation is related to the quantum Boltzmann equation for the photon distribution function (PDF) in the atmosphere [39,40].

Our presentation proceeds in the following section with a formulation of the SPE, followed by a derivation of the Hamiltonian describing an infinite set of quantized harmonic oscillators associated with LG modes in the atmosphere. The latter is approximated by a nonabsorbing turbulent medium that is characterized by the Kolmogorov power spectrum. In Sec. III, we provide a Lindblad master equation for the density operator of the oscillators by performing an exact average over the refractive index fluctuations, following the procedure outlined in Ref. [38]. We then study the evolution of turbulence-induced crosstalk among LG modes in Sec. IV, and conclude our work in Sec. V.

II. QUANTIZATION IN INHOMOGENEOUS DIELECTRICS AND THE HAMILTONIAN

A. Stochastic parabolic equation

We set out our derivation of the Hamiltonian with a formulation of the SPE [18], which governs the propagation, in the atmosphere, of the spatial envelope $\psi(\mathbf{r})$ of the monochromatic field with wave number k_0 and frequency ω, $\phi(\mathbf{r}) = \psi(\mathbf{r})e^{i(k_0 z-\omega t)}$,

$$-i\frac{\partial}{\partial z}\psi(\boldsymbol{\rho}, z) = \left(\frac{1}{2k_0}\Delta_\perp + k_0\delta n(\boldsymbol{\rho}, z)\right)\psi(\boldsymbol{\rho}, z), \quad (1)$$

with $\boldsymbol{\rho} = (x, y)$ a vector in the transverse plane, $\delta n(\boldsymbol{\rho}, z)$ the zero-mean refractive index fluctuations, and $\Delta_\perp = \partial^2/\partial x^2 + \partial^2/\partial y^2$ the transverse Laplace operator responsible for wave diffraction. Equation (1) governs a *scalar* field amplitude, reflecting negligibly small depolarization effects in the atmosphere [18], and is equivalent to the two-dimensional Schrödinger equation, with the spatial coordinate z playing the role of time. This promotion of z renders $\delta n(\boldsymbol{\rho}, z)$ a stochastic process, placing the problem into the realm of stochastic Hamiltonians. Note also that Eq. (1) ignores quadratic terms $\mathcal{O}(\delta n(\boldsymbol{\rho}, z)^2) \approx 10^{-6}$ [21] and assumes a flat gradient of the envelope $\phi(\boldsymbol{r})$, such that $|\partial_z^2\phi| \ll |k_0\partial_z\phi| \ll |k_0^2\phi|$, where henceforth we adopt the notation $(\partial_x, \partial_y, \partial_z) := (\partial/\partial x, \partial/\partial y, \partial/\partial z)$. In addition, fluctuations are assumed to be delta correlated along the propagation axis, which amounts to the Markov approximation [41],

$$\overline{\delta n(\boldsymbol{\rho}_1, z_1)\delta n(\boldsymbol{\rho}_2, z_2)} = \delta(z_1 - z_2)B_n^\perp(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2), \quad (2)$$

with the overbar indicating the ensemble average over the refractive index fluctuations, and

$$B_n^\perp(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2) = 2\pi \int d^2\boldsymbol{q}\ \Phi_K(|\boldsymbol{q}|)e^{i\boldsymbol{q}\cdot(\boldsymbol{\rho}_1-\boldsymbol{\rho}_2)}, \quad (3)$$

the transverse correlation function [18] given by the Fourier transform of the Kolmogorov power spectrum of turbulence [18],

$$\Phi_K(q) = 0.033\, C_n^2 q^{-11/3}, \quad (4)$$

with C_n^2 the turbulence structure constant and $\boldsymbol{q}$ the transverse momentum.

B. Hamiltonian

A Hamiltonian formulation of paraxial optics can be obtained by postulating a Lagrangian consistent with the paraxial wave equation [42]. The scalar character of Eq. (1) simplifies the derivation of the Hamiltonian within the framework of elementary QED, by fixing the photon's polarization and focusing on the scalar field components. Thus, we avoid the more involved quantization of vector fields (see Refs. [43,44]). It is easy to show that a theory consistent with the SPE requires the incorporation of refractive index fluctuations into the Lagrangian of Ref. [42], resulting in the following Lagrangian:

$$\mathcal{L} = i\psi^*\partial_z\psi + k_0\delta n(\boldsymbol{\rho}, z)\psi^*\psi - \frac{1}{2k_0}\delta_{ij}\partial_i\psi^*\partial_j\psi, \quad (5)$$

with the sum convention in force, $i, j = x, y$, and a single canonically conjugate pair of variables, $(\psi(\boldsymbol{\rho}, z), \Pi(\boldsymbol{\rho}, z))$, with canonical momentum

$$\Pi = \frac{\partial\mathcal{L}}{\partial(\partial_z\psi)} = i\psi^*, \quad \Pi^* = 0. \quad (6)$$

We recover Eq. (1) by applying the Euler-Lagrange equation to Eq. (5). The Hamiltonian density is straightforwardly obtained by Legendre transformation, to give

$$\mathcal{H}(\boldsymbol{\rho}, z) = \frac{1}{2k_0}(\boldsymbol{\nabla}_\perp\psi^*)(\boldsymbol{\nabla}_\perp\psi) - k_0\delta n(\boldsymbol{\rho}, z)\psi^*\psi, \quad (7)$$

which yields, upon transverse integration, the total Hamiltonian

$$H(z) = \int d^2\boldsymbol{\rho}\left[-\frac{1}{2k_0}\psi^*\Delta_\perp\psi - k_0\delta n(\boldsymbol{\rho}, z)\psi^*\psi\right], \quad (8)$$

where partial integration of the first term on the right-hand side of Eq. (7) was used to arrive at the equality, with surface terms dropped on account of the wave function's decay. Field quantization follows the standard procedure [45], with the promotion of scalar fields to operators, $\psi(\boldsymbol{\rho}, z) \to \hat{\psi}(\boldsymbol{\rho}, z)$, $\psi^*(\boldsymbol{\rho}, z) \to \hat{\psi}^\dagger(\boldsymbol{\rho}, z)$, postulation of the canonical commutation relations

$$[\hat{\psi}^\dagger(\boldsymbol{\rho}, z), \hat{\psi}^\dagger(\boldsymbol{\rho}', z)] = [\hat{\psi}(\boldsymbol{\rho}, z), \hat{\psi}(\boldsymbol{\rho}', z)] = 0,$$
$$[\hat{\psi}(\boldsymbol{\rho}, z), \hat{\psi}^\dagger(\boldsymbol{\rho}', z)] = \delta(\boldsymbol{\rho} - \boldsymbol{\rho}'),$$

and the Heisenberg equation of motion

$$i\partial_z\hat{\psi}(\boldsymbol{\rho}, z) = [\hat{\psi}(\boldsymbol{\rho}, z), \hat{H}(z)]. \tag{9}$$

The term $-\Delta_\perp/(2k_0)$ in Eq. (8) corresponds to the kinetic term $\boldsymbol{p}_\perp^2/(2k_0)$; we can readily solve for the evolution due to this term by using the transverse momentum basis,

$$\hat{\Psi}(\boldsymbol{q}, z) = \int d^2\boldsymbol{\rho}\ \hat{\psi}(\boldsymbol{\rho}, z)e^{-i\boldsymbol{q}\cdot\boldsymbol{\rho}}, \tag{10}$$

$$\hat{\psi}(\boldsymbol{\rho}, z) = \frac{1}{(2\pi)^2}\int d^2\boldsymbol{q}\ \hat{\Psi}(\boldsymbol{q}, z)e^{i\boldsymbol{q}\cdot\boldsymbol{\rho}}, \tag{11}$$

which diagonalizes $\hat{H}_0 := -(2k_0)^{-1}\int d^2\boldsymbol{\rho}\ \hat{\psi}^\dagger\Delta_\perp\hat{\psi}$, to find the solution to the equation $i\partial_z\hat{\Psi}(\boldsymbol{q}, z) = [\hat{\Psi}(\boldsymbol{q}, z), \hat{H}_0]$ given by

$$\hat{\Psi}(\boldsymbol{q}, z) = \hat{\Psi}(\boldsymbol{q}, 0)e^{-iz\boldsymbol{q}^2/(2k_0)}. \tag{12}$$

In the interaction picture with respect to $\hat{H}_0$, we thus find

$$\begin{aligned}\hat{H}^{\text{int}} &= -e^{-i\hat{H}_0 z}\left[\int d^2\boldsymbol{\rho}\ k_0\delta n(\boldsymbol{\rho}, z)\hat{\psi}^\dagger\hat{\psi}\right]e^{i\hat{H}_0 z}\\ &= -\frac{k_0}{(2\pi)^4}\int d^2\boldsymbol{\rho}\int d^2\boldsymbol{q}\int d^2\boldsymbol{q}'\ e^{-i\boldsymbol{\rho}\cdot\boldsymbol{q}+iz\boldsymbol{q}^2/(2k_0)}\\ &\quad\times\hat{\Psi}^\dagger(\boldsymbol{q}, 0)\delta n(\boldsymbol{\rho}, z)\hat{\Psi}(\boldsymbol{q}', 0)e^{i\boldsymbol{\rho}\cdot\boldsymbol{q}'-iz\boldsymbol{q}'^2/(2k_0)}.\end{aligned} \tag{13}$$

We deal with the propagation of light quanta populating LG modes. It is therefore convenient to introduce the photonic annihilation and creation operators of LG modes,

$$\hat{a}_{p,l} = \frac{1}{(2\pi)^2}\int d^2\boldsymbol{q}\ \tilde{u}^*_{p,l}(\boldsymbol{q}, 0)\hat{\Psi}(\boldsymbol{q}, 0), \tag{14}$$

where $\tilde{u}_{p,l}(\boldsymbol{q}, z)$ is the Fourier transform of the LG mode function $u_{p,l}(\boldsymbol{\rho}, z)$, as defined in the Supplemental Material [46], and the creation operators are given by the Hermitian conjugates of $\hat{a}_{p,l}$. Throughout, tildes denote Fourier-transformed quantities. Using the identity [54]

$$e^{i(\boldsymbol{\rho}\cdot\boldsymbol{q}-z\boldsymbol{q}^2/(2k_0))} = \sum_{p,l}\tilde{u}^*_{p,l}(\boldsymbol{q}, 0)u_{p,l}(\boldsymbol{\rho}, z), \tag{15}$$

we can rewrite Eq. (13) in a concise form,

$$\hat{H}(z) = \sum_{\alpha,\beta} g_{\alpha\beta}(z)\hat{a}^\dagger_\alpha\hat{a}_\beta, \tag{16}$$

where we dropped, for the remainder of this paper, the superscript "int" of the Hamiltonian, and the coupling constants read

$$g_{\alpha\beta}(z) = -k_0\int d^2\boldsymbol{\rho}\,\delta n(\boldsymbol{\rho}, z)u^*_\alpha(\boldsymbol{\rho}, z)u_\beta(\boldsymbol{\rho}, z), \tag{17}$$

with $\alpha = (p_\alpha, l_\alpha)$ a compound index. Equation (16) describes the linear coupling, due to refractive index fluctuations, between quantized harmonic oscillators associated with LG modes α and β.

Vacuum propagation effects, such as diffraction and the accumulation of a Gouy phase arising due to transverse spatial confinement [55], are accounted for by $\hat{H}_0$.

III. MASTER EQUATION

A. Ensemble average

The open quantum systems theory derivation of a master equation in the weak-coupling limit, which is certainly justified in our present scenario since $\delta n \ll 1$, typically begins with a separation of the total system into a system of interest S and a reservoir R. The second-order master equation for the system's reduced density operator ρ_S then takes the form [35]

$$\partial_z\rho_S(z) = -\int_{z_0}^{z} dz'\ \mathrm{Tr}_R[\hat{H}(z), [\hat{H}(z'), \rho_{SR}(z)]]. \tag{18}$$

However, a separation between system and reservoir is ill defined in the present case since all LG modes should be treated equally in a description of light propagating through the atmosphere. Indeed, one can show that the splitting of the quantized oscillators into system and reservoir followed by the standard Born approximation, where the reservoir modes are assumed to be in a vacuum steady state, amounts to complete neglect of intermodal crosstalk, or a coupling of the initially excited LG mode(s) to other LG modes—the very effect we intend to study. We therefore stress that, instead of pursuing a standard open quantum systems approach, we consider the system of LG modes to be *closed* and rely on the master equation derived in Ref. [37] to obtain an exact master equation for the ensemble-averaged density operator $\bar{\rho}$, in the spirit of Refs. [37,38,56]. Crucially, considering a closed rather than an open system introduces a conceptual subtlety: unlike open systems, where decoherence arises from the partial trace of an entangled system-environment state, decoherence here arises from the averaging of stochastically evolving coherences over a disordered potential landscape [36,38].

Consider the stochastic Hamiltonian (16), written as $\hat{H}(z) = \sum_{\alpha,\beta} g_{\alpha\beta}(z)\hat{X}_{\alpha\beta}$ with operators $\hat{X}_{\alpha\beta} := \hat{a}^\dagger_\alpha\hat{a}_\beta$ introduced for brevity, and its induced dynamics given by the von Neumann equation

$$\partial_z\rho(z) = -i[\hat{H}(z), \rho(z)]. \tag{19}$$

Using variational calculus, Budini derived an exact time-convolutionless master equation [37] for the ensemble-averaged density operator $\bar{\rho}$ associated with a broad class of stochastic Hamiltonians encompassing (16). For the Hamiltonian under consideration, the equation reads

$$\begin{aligned}\partial_z\bar{\rho}(z) = -\int_0^z dz'\ &\overline{g^*_{\alpha\beta}(z)g_{\alpha'\beta'}(z')}\\ &\times[\hat{X}^\dagger_{\alpha\beta}, \overline{[\mathcal{G}(z, z')\hat{X}_{\alpha'\beta'}\mathcal{G}(z, z')^\dagger, \rho(z)]}],\end{aligned} \tag{20}$$

with propagator

$$\mathcal{G}(z_1, z_2) = \mathcal{Z}\exp\left(-i\int_{z_2}^{z_1} dz'\hat{H}(z')\right) \tag{21}$$

and $\mathcal{Z}$ the spatial equivalent of the time-ordering operator. While the formal result Eq. (20) is generally hard to solve, the Markov approximation (2) implies that the correlator satisfies $\overline{g^*_{\alpha\beta}(z)g_{\alpha'\beta'}(z')} \propto \delta(z - z')$, thus collapsing the spatial integral and rendering the master equation tractable, as the propagator reduces to $\mathcal{G}(z, z) = \mathbf{1}$ and the average on the right-hand side of (20) is readily performed. Using $g^*_{\alpha\beta} = g_{\beta\alpha}$ and the

symmetry $\overline{g_{\alpha\beta}(z)g_{\alpha'\beta'}(z')} = \overline{g_{\alpha'\beta'}(z)g_{\alpha\beta}(z')}$, we can rearrange terms to arrive at

$$\partial_z\bar{\rho}(z) = \sum_{\alpha,\beta,\alpha',\beta'} \gamma_{\alpha\beta\beta'\alpha'}(z)\left[\hat{X}_{\alpha\beta}\bar{\rho}\hat{X}^{\dagger}_{\alpha'\beta'} - \frac{1}{2}\{\hat{X}^{\dagger}_{\alpha'\beta'}\hat{X}_{\alpha\beta}, \bar{\rho}\}\right], \tag{22}$$

with the intermodal coupling (crosstalk) tensor

$$\gamma_{\alpha\beta\alpha'\beta'}(z) = \overline{g_{\alpha\beta}(z)g_{\alpha'\beta'}(z)}. \tag{23}$$

Note that, so far, we have not imposed any restriction on the photonic initial state, nor on the power spectrum of the turbulent medium, such that Eq. (22) is fully general.

Equation (22) is one of the central results of this paper, whose consequences will be examined once we have presented an analytical expression for the relaxation tensor, in the following section.

Structurally, Eq. (22) exhibits the form of a Gorini-Kossakowski-Lindblad-Sudarshan (GKLS) generator [35], except for the z dependence of the relaxation tensor and a potential violation of positive semidefiniteness of the Kossakowski matrix corresponding to $\gamma_{\alpha\beta\alpha'\beta'}$. We will show in Sec. III B that the generator is approximately independent of z for practically relevant propagation distances $z \ll z_R$ that are much shorter than the Rayleigh length z_R, which corresponds to the distance at which the beam's area has doubled due to diffraction. Moreover, we comment on positive semidefiniteness for arbitrary z in Sec. III C.

B. Relaxation tensor

1. General formula

We now seek to deduce a closed-form expression for the relaxation tensor, formally defined in Eq. (23). The key difficulty of the master-equation-based approach lies in the analytical evaluation of the correlation function $\overline{g_{\alpha\beta}g_{\alpha'\beta'}}$. In Ref. [31], the latter average is inferred via a generating function. However, as discussed below, the closed-form expression for the tensor $\gamma_{\alpha\beta\alpha'\beta'}$, obtained in the present contribution, holds numerical advantages over Ref. [31].

Introducing $U_{\alpha\beta}(\boldsymbol{\rho}, z) = u^*_{\alpha}(\boldsymbol{\rho}, z)u_{\beta}(\boldsymbol{\rho}, z)$ and its Fourier-transformed counterpart $\tilde{U}_{\alpha\beta}(\boldsymbol{q}, z) = \mathcal{F}^{2\mathrm{D}}\{U_{\alpha\beta}(\boldsymbol{\rho}, z)\}$, we substitute Eqs. (17) and (2) into Eq. (23), to find

$$\begin{aligned}\gamma_{\alpha\beta\alpha'\beta'} &= k_0^2 \int d^2\boldsymbol{\rho}_1 \int d^2\boldsymbol{\rho}_2\, U_{\alpha\beta}(\boldsymbol{\rho}_1, z)U_{\alpha'\beta'}(\boldsymbol{\rho}_2, z) \\ &\quad \times B_n^{\perp}(\boldsymbol{\rho}_1 - \boldsymbol{\rho}_2) \\ &= 2\pi k_0^2 \int d^2\boldsymbol{q}\; \Phi_K(|\boldsymbol{q}|)\tilde{U}_{\alpha'\beta'}(\boldsymbol{q}, z)\tilde{U}_{\alpha\beta}(-\boldsymbol{q}, z), \end{aligned} \tag{24}$$

where the last equality follows from Eq. (3). The analytical evaluation of the integral (24) is cumbersome; hence, we defer it to the Supplemental Material [46], where the derivation and explicit expressions are provided.

2. Remarks on approximations

Before discussing the structure of the solution, we briefly comment on the employed and avoided approximations of our approach.

(i) We evaluate the correlation function using the Kolmogorov power spectrum [see Eq. (4)]. Therefrom, the phase structure function can be deduced as [18]

$$D_{\phi}(|\delta\boldsymbol{\rho}|) \simeq 6.88\left(\frac{|\delta\boldsymbol{\rho}|}{r_0}\right)^{5/3}, \tag{25}$$

with r_0 the Fried parameter that defines the transverse coherence length of turbulence. The characteristic 5/3-power-law scaling of $D_{\phi}(|\delta\boldsymbol{\rho}|)$ often complicates or even prevents analytical considerations. To overcome this difficulty, the quadratic approximation ($5/3 \to 2$) is commonly applied [9,34,57,58]. However, this approximation can produce artifacts [59], e.g., an underestimation of the entanglement decay in turbulence [60]. Our present approach does *not* rely on the quadratic approximation.

(ii) As discussed in the Supplemental Material [46], our adoption of the Kolmogorov power spectrum leads to diverging integrals in the special case $l_{\alpha} = l_{\beta}$ [$\Rightarrow l_{\beta'} = l_{\alpha'}$; see Eq. (30)]. However, we prove that this is not problematic because diverging contributions to the master equation cancel exactly—a feature encountered in earlier studies of dynamics under the exact Kolmogorov power spectrum [30,31]. The strategy to deal with diverging integrals is thus to regularize the singularity of the Kolmogorov spectrum by *ad interim* adopting the modified von Karman spectrum, i.e. in (4), $q^{-11/3} \to 1/(q^2 + q_0^2)^{11/6}$, ensuring convergence of Eq. (30) for the special case $l_{\alpha} = l_{\beta}$ and allowing us to solve the integral in Eq. (24). The solution thus obtained is expanded in a series in q_0, where singularities [which, as we show, mutually cancel on the right-hand side of Eq. (22)] can be dropped; finally, we let $q_0 \to 0$ to return to the Kolmogorov spectrum $\propto q^{-11/3}$, thus retaining only the zeroth-order term in the q_0 expansion of the solution to Eq. (24). It must be stressed that this procedure is fully rigorous, without any informal or heuristic arguments; the interim adoption of the modified von Karman power spectrum is merely a trick.

3. Analysis of $\gamma_{\alpha\beta\alpha'\beta'}(z)$ at different z

Let us now present a closed-form solution for the relaxation tensor defined in Eq. (24), which is another central result of this work [46]:

$$\gamma_{\alpha\beta\alpha'\beta'}(z) = C_n^2 k_0^2 \sigma^{-5/6}(z) e^{-i\varphi_{\alpha\beta\alpha'\beta'}(z)}\Gamma_{\alpha\beta\alpha'\beta'}, \tag{26}$$

where

$$\sigma(z) = 2/w^2(z), \tag{27}$$

is expressed through the beam radius,

$$w(z) = w_0\sqrt{1 + (z/z_R)^2}, \tag{28}$$

with w_0 the initial beam's radius and z_R the Rayleigh length; $\varphi_{\alpha\beta\alpha'\beta'}(z)$ is a phase given by differences of the Gouy phases $\xi(z)$ associated with LG modes α, β, α', and β',

$$\varphi_{\alpha\beta\alpha'\beta'}(z) := \xi_{\beta}(z) - \xi_{\alpha}(z) + \xi_{\beta'}(z) - \xi_{\alpha'}(z), \tag{29}$$

and $\Gamma_{\alpha\beta\alpha'\beta'}$ is a complicated tensor which, however, exhibits no dependence on turbulence and beam parameters [46].

We further find [46] that Eq. (26) obeys a strict selection rule: $\gamma_{\alpha\beta\alpha'\beta'}(z)$ vanishes unless

$$l_{\alpha} - l_{\beta} = l_{\beta'} - l_{\alpha'}, \tag{30}$$

i.e., the difference of involved OAM pairs is an additional invariant in turbulence, beyond the well-known invariance of OAM itself [26,61]. This becomes apparent by returning to the master equation (22) and noting that Eq. (30) leads to a pairing of operators creating and annihilating the same number of OAM quanta. The conservation of the OAM difference can be ascribed to the isotropy of the Gaussian random field $\delta n(\boldsymbol{\rho}, z)$ in the transverse plane. As we show in Sec. IV, the associated selection rule (30) substantially simplifies the numerical simulation of the master equation.

The structure of the relaxation tensor gives rise to another important insight: Note that Eq. (26) factorizes into a product of beam or turbulence parameters, with a simple z-dependent function $\sigma(z)^{-5/6}$, on the one side, and a tensor $\Gamma_{\alpha\beta\alpha'\beta'}$, on the other side. The benefit of this factorization is evident, as the evaluation of $\Gamma_{\alpha\beta\alpha'\beta'}$—even if computationally demanding—needs to be performed once, with the result stored and recycled for different choices of the beam and turbulence parameters. We stress that the importance of our work lies not with the master equation (22) which, as we show in Secs. III D and III E, is equivalent to the classical parabolic equation [21] and the IPE expressed in the LG basis [31], but with the convenient factorization and reusability of the relaxation tensor, which greatly accelerates the numerical solution of the master equation. In addition, our closed-form expression for the relaxation tensor allows us to deduce a number of its fundamental properties, as discussed below.

Considering the regimes distinguished by the ratio of the propagation distance z and the Rayleigh length z_R, we can further simplify the structure of Eq. (26). For $z \ll z_R$, we can set $w(z) \approx w_0$, $\varphi_{\alpha\beta\alpha'\beta'}(z) \approx 0$, $\sigma(z) \approx 2/w_0^2$, to find that the relaxation tensor, and hence also the generator of the evolution in Eq. (22), become independent of the propagation distance. Together with the semidefiniteness in the single-photon subspace (see Sec. III C), this establishes the GKLS form. Similarly, for $z \gg z_R$, we have $\xi_{p,l}(z) \approx \pi(2p + |l| + 1)/2$, such that the only z dependence resides with $\sigma(z) \propto (z_R/z)^2$, which is independent of indices $\alpha, \beta, \alpha', \beta'$. Then, it follows that the decay rates are increasing with the propagation distance as $\sigma(z)^{-5/6} \propto (z/z_R)^{5/3}$. To summarize, we deduce

$$\gamma_{\alpha\beta\alpha'\beta'}(z) \simeq 2^{-5/6} w_0^{5/3} C_n^2 k_0^2 \, \Gamma_{\alpha\beta\alpha'\beta'} \times \begin{cases} 1 \\ \text{for } z \ll z_R \\ (1 + (z/z_R)^2)^{5/6} e^{-i\varphi_{\alpha\beta\alpha'\beta'}(z)} \\ \text{for } z \sim z_R \\ (z/z_R)^{5/3} e^{-i\varphi_{\alpha\beta\alpha'\beta'}(\infty)} \\ \text{for } z \gg z_R. \end{cases} \tag{31}$$

The scaling $\propto (z/z_R)^{5/3}$ for $z \gg z_R$ can be intuitively understood as a result of the interplay between the size of the beam and the turbulent eddies. In fact, this power law can be rigorously deduced by examining Eq. (24), expressing the relaxation tensor $\gamma_{\alpha\beta\alpha'\beta'}$ in terms of the mode functions $u_\alpha(\boldsymbol{\rho}, z)$ and the Kolmogorov power spectrum.

4. Derivation of the power law $(z/z_R)^{5/3}$

To deduce this power law, we need to elucidate how the Fourier-transformed mode product, $\tilde{U}_{\alpha\beta}(\boldsymbol{q}, z)$, depends on z. We start with the case $\alpha = \beta$ and then generalize the result for $\alpha \neq \beta$.

Using the LG modes' definition [46], we can write $U_{\alpha\alpha} = u_\alpha^* u_\alpha = |u_\alpha|^2$ as

$$|u_\alpha(\boldsymbol{\rho}, z)|^2 = w^{-2}(z) \left| u_\alpha^{\mathrm{red}}[\boldsymbol{\rho}/w(z)] \right|^2, \tag{32}$$

with the reduced mode

$$\left| u_\alpha^{\mathrm{red}}(\boldsymbol{\rho}/w(z)) \right|^2 := (C_\alpha)^2 \left(\frac{2\rho^2}{w^2(z)} \right)^{|l_\alpha|} \left\{ L_p^{|l_\alpha|}[2\rho^2/w^2(z)] \right\}^2 \exp[-2\rho^2/w^2(z)], \tag{33}$$

where details can be found in Ref. [46]. Note that the z dependence of $|u_\alpha^{\mathrm{red}}|^2$ is entirely contained in its argument. Then it follows from Eq. (32) that

$$\int d^2\boldsymbol{\rho} |u_\alpha(\boldsymbol{\rho}, z)|^2 = \int \frac{d^2\boldsymbol{\rho}}{w^2(z)} \left| u_\alpha^{\mathrm{red}}[\boldsymbol{\rho}/w(z)] \right|^2 = \int \frac{d^2\boldsymbol{\rho}'}{w_0^2} \left| u_\alpha^{\mathrm{red}}(\boldsymbol{\rho}'/w_0) \right|^2, \tag{34}$$

where $\boldsymbol{\rho}' := \boldsymbol{\rho} w_0/w(z)$, and the last equality of Eq. (34) expresses the total power conservation law of the mode α in the transverse plane at any propagation distance z. Consequently, for the Fourier transform of $U_{\alpha\alpha}$ it holds that

$$\tilde{U}_{\alpha\alpha}(\boldsymbol{q}, z) = \int \frac{d^2\boldsymbol{\rho}}{w^2(z)} \left| u_\alpha^{\mathrm{red}}[\boldsymbol{\rho}/w(z)] \right|^2 e^{-i\boldsymbol{\rho}\cdot\boldsymbol{q}} = \int \frac{d^2\boldsymbol{\rho}'}{w_0^2} \left| u_\alpha^{\mathrm{red}}(\boldsymbol{\rho}'/w_0) \right|^2 e^{-i\boldsymbol{\rho}'\cdot\boldsymbol{q}'}, \tag{35}$$

where $\boldsymbol{q}' := \boldsymbol{q} w(z)/w_0$. The last equation means that $\tilde{U}_{\alpha\alpha}(\boldsymbol{q}, z)$ depends on z only through its momentum argument, $\tilde{U}_{\alpha\alpha}(\boldsymbol{q}, z) = \tilde{U}_{\alpha\alpha}[\boldsymbol{q} w(z)/w_0]$.

It is easy to show that for $z \gg z_R$, this is a property not only of $\tilde{U}_{\alpha\alpha}$, but more generally of $\tilde{U}_{\alpha\beta}$, such that $\tilde{U}_{\alpha'\beta'}(\boldsymbol{q}, z)\tilde{U}_{\alpha\beta}(-\boldsymbol{q}, z) = \tilde{U}_{\alpha'\beta'}[\boldsymbol{q} w(z)/w_0]\tilde{U}_{\alpha\beta}[-\boldsymbol{q} w(z)/w_0]$. Hence, Eq. (24) can be represented in the form

$$\gamma_{\alpha\beta\alpha'\beta'}(z) = 2\pi k_0^2 \int d^2\boldsymbol{q} \; \Phi_K(|\boldsymbol{q}|) \times \tilde{U}_{\alpha'\beta'}[\boldsymbol{q} w(z)/w_0]\tilde{U}_{\alpha\beta}[-\boldsymbol{q} w(z)/w_0]. \tag{36}$$

Finally, recalling from Eq. (4) that $\Phi_K(|\boldsymbol{q}|) \propto |\boldsymbol{q}|^{-11/3}$ and changing the integration variable $\boldsymbol{q} \to \boldsymbol{q}'$, we obtain

$$\begin{aligned} \gamma_{\alpha\beta\alpha'\beta'}(z) &= 2\pi k_0^2 \left(\frac{w_0}{w(z)} \right)^{-11/3+2} \int d^2\boldsymbol{q}' \; \Phi_K(|\boldsymbol{q}'|) \times \tilde{U}_{\alpha'\beta'}(\boldsymbol{q}')\tilde{U}_{\alpha\beta}(-\boldsymbol{q}') \\ &\simeq 2\pi k_0^2 \left(\frac{z}{z_R} \right)^{5/3} \int d^2\boldsymbol{q}' \; \Phi_K(|\boldsymbol{q}'|) \times \tilde{U}_{\alpha'\beta'}(\boldsymbol{q}')\tilde{U}_{\alpha\beta}(-\boldsymbol{q}') \\ &= \left(\frac{z}{z_R} \right)^{5/3} \gamma_{\alpha\beta\alpha'\beta'}(0) e^{-i\varphi_{\alpha\beta\alpha'\beta'}(\infty)}. \end{aligned} \tag{37}$$

We have thus recovered the scaling $(z/z_R)^{5/3}$ that we found previously by direct analysis of the solution (26), and showed

that the scaling 5/3 is due to an interplay of the characteristic scaling 11/3 of the Kolmogorov spectrum and the beam's (transverse) area increasing as $\propto (z/z_R)^2$.

5. *Incorporation of $C_n^2(z)$*

So far, the structure constant C_n^2 has been assumed constant along the channel [see Eq. (4)]. However, including a z-dependent structure constant, $C_n^2(z)$, only amounts to the replacement $C_n^2 \to C_n^2(z)$ in the expression (26) for the relaxation tensor, which does not significantly increase computational complexity. This feature of the relaxation tensor, which stems from its factorization into spatially and turbulence-dependent terms, on the one hand, and the tensor dependent on modal indices, on the other hand, highlights another advantage of our method over existing approaches [21], where a spatially varying structure constant typically introduces substantial additional complexity.

C. Complete positivity of the generator

Let us briefly comment upon the complete positivity of the generator of Eq. (22) when restricted to the single-photon sector in the representation given in Eq. (48) below: given the canonical form thereof in terms of continuously parametrized jump operators and the power spectrum assuming the role of (positive) decay rates, complete positivity is implied [62].

Together with the z independence of the tensor $\gamma_{\alpha\beta\alpha'\beta'}$ established in the previous section, we conclude that the generator is of GKLS form in the regime $z \ll z_R$.

In contrast, for $z \sim z_R$ and $z \gg z_R$, the z dependence of the decay rates [see Eq. (31)] immediately implies that the generator is *not* of GKLS form. Nevertheless, as the decay rates remain positive for all z, the resulting dynamical map is completely positive divisible [63] in the regimes $z \sim z_R$ and $z \gg z_R$.

D. Equivalence to Ishimaru's parabolic equation

With the density operator at hand, we can assess arbitrary correlation functions of the propagated field. Of particular interest is the mutual coherence function (MCF), defined as [21]

$$\Gamma(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2, z) = \overline{\psi(\boldsymbol{\rho}_1, z)\psi^*(\boldsymbol{\rho}_2, z)}. \tag{38}$$

In the single-photon sector, the MCF can be equivalently expressed as the quantum-mechanical expectation value: $\Gamma(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2, z) = \mathrm{Tr}[\bar{\rho}(z)\hat{\psi}^\dagger(\boldsymbol{\rho}_2, z)\hat{\psi}(\boldsymbol{\rho}_1, z)]$, with

$$\hat{\psi}(\boldsymbol{\rho}, z) = \sum_\alpha u_\alpha(\boldsymbol{\rho}, z)\hat{a}_\alpha. \tag{39}$$

It then follows that

$$\Gamma(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2, z) = \bar{\rho}(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2, z), \tag{40}$$

with $\bar{\rho}(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2, z) = \langle \boldsymbol{\rho}_1 | \, \bar{\rho}(z) \, | \boldsymbol{\rho}_2 \rangle$ the density operator expressed in the transverse position basis. Under the same assumptions that were used to derive the SPE, i.e., the refractive index fluctuations represent a Gaussian random field that is delta correlated along the propagation axis (see Sec. II A), the MCF obeys the equation [21,64]

$$\begin{aligned}\partial_z \Gamma(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2, z) = {} & \frac{i}{2k_0}(\Delta_{\perp,1} - \Delta_{\perp,2})\Gamma(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2, z) \\ & + k_0^2[B_n^\perp(\boldsymbol{\rho}_1 - \boldsymbol{\rho}_2) - B_n^\perp(0)]\Gamma(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2, z),\end{aligned} \tag{41}$$

with $\Delta_{\perp,1/2}$ the Laplace operator with respect to the transverse coordinate $\boldsymbol{\rho}_{1/2}$.

Equation (40) and the analogous use of the Markov approximation suggest a close relation between the master equation (22) and Eq. (41). Indeed, equivalence holds, as we now show: We first derive an equation of motion for $\bar{\rho}_{\mu\nu}$. Using the single-photon identities $\langle\mu|\hat{a}_\alpha^\dagger\hat{a}_\beta\bar{\rho}\hat{a}_{\alpha'}^\dagger\hat{a}_{\beta'}|\nu\rangle = \delta_{\mu\alpha}\delta_{\beta'\nu}\bar{\rho}_{\beta\alpha'}$, $\langle\mu|\hat{a}_\alpha^\dagger\hat{a}_\beta\hat{a}_{\alpha'}^\dagger\hat{a}_{\beta'}\bar{\rho}|\nu\rangle = \delta_{\mu\alpha}\delta_{\alpha'\beta}\bar{\rho}_{\beta'\nu}$, and $\langle\mu|\bar{\rho}\hat{a}_\alpha^\dagger\hat{a}_\beta\hat{a}_{\alpha'}^\dagger\hat{a}_{\beta'}|\nu\rangle = \delta_{\beta\alpha'}\delta_{\nu\beta'}\bar{\rho}_{\mu\alpha}$, with $|\mu\rangle$ and $|\nu\rangle$ the one-photon Fock states of modes μ and ν, respectively, we arrive at

$$\partial_z\bar{\rho}_{\mu\nu} = \gamma_{\mu\alpha\beta\nu}\bar{\rho}_{\alpha\beta} - \tfrac{1}{2}\gamma_{\mu\alpha\alpha\beta}\bar{\rho}_{\beta\nu} - \tfrac{1}{2}\bar{\rho}_{\mu\alpha}\gamma_{\alpha\beta\beta\nu}, \tag{42}$$

with sum convention in force. Returning to the Schrödinger picture, where the master equation exhibits a free propagation part $\hat{H}_0$ and explicitly writing out the definitions of $\gamma_{\alpha\beta\alpha'\beta'}$ (24), and the operators $\tilde{U}_{\alpha\beta}$, we find

$$\begin{aligned}\partial_z\bar{\rho}_{\mu\nu} = {} & -i\langle\mu|\,[\hat{H}_0, \bar{\rho}]\,|\nu\rangle + 2\pi k_0^2 \int d^2\boldsymbol{q} \int d^2\boldsymbol{\rho} \int d^2\boldsymbol{\rho}' \Phi_K(\boldsymbol{q}) e^{-i(\boldsymbol{\rho}'-\boldsymbol{\rho})\cdot\boldsymbol{q}} \Big[\bar{\rho}_{\alpha\beta}u_\mu^*(\boldsymbol{\rho}, 0)u_\alpha(\boldsymbol{\rho}, 0)u_\nu(\boldsymbol{\rho}', 0)u_\beta^*(\boldsymbol{\rho}', 0) \\ & - \frac{1}{2}u_\mu^*(\boldsymbol{\rho}, 0)u_\alpha(\boldsymbol{\rho}, 0)u_\alpha^*(\boldsymbol{\rho}', 0)u_\beta(\boldsymbol{\rho}', 0)\bar{\rho}_{\beta\nu} - \frac{1}{2}\bar{\rho}_{\mu\alpha}u_\alpha^*(\boldsymbol{\rho}, 0)u_\beta(\boldsymbol{\rho}, 0)u_\beta^*(\boldsymbol{\rho}', 0)u_\nu(\boldsymbol{\rho}', 0)\Big].\end{aligned} \tag{43}$$

We recognize that $u_\alpha(\boldsymbol{\rho}, 0)\bar{\rho}_{\alpha\beta}u_\beta^*(\boldsymbol{\rho}', 0) = \langle\boldsymbol{\rho}|\,\bar{\rho}\,|\boldsymbol{\rho}'\rangle = \bar{\rho}(\boldsymbol{\rho}, \boldsymbol{\rho}')$, and use the completeness relation of the LG basis to perform the sum over α (β, respectively) in the third and fourth term of Eq. (43), subsequently applying Eq. (3). Together with $u_\beta(\boldsymbol{\rho}, 0)\bar{\rho}_{\beta\nu} = \int d^2\boldsymbol{\rho}'\bar{\rho}(\boldsymbol{\rho}, \boldsymbol{\rho}')u_\nu(\boldsymbol{\rho}', 0)$ and $\bar{\rho}_{\mu\alpha}u_\alpha^*(\boldsymbol{\rho}', 0) = \int d^2\boldsymbol{\rho}\bar{\rho}(\boldsymbol{\rho}, \boldsymbol{\rho}')u_\mu^*(\boldsymbol{\rho}, 0)$, we arrive at [65]

$$\begin{aligned}\partial_z\langle\mu|\,\bar{\rho}\,|\nu\rangle = {} & -i\langle\mu|\,[\hat{H}_0, \bar{\rho}]\,|\nu\rangle \\ & + k_0^2\langle\mu|\,\bar{\rho}(\boldsymbol{\rho}, \boldsymbol{\rho}')[B_n^\perp(\boldsymbol{\rho} - \boldsymbol{\rho}') - B_n^\perp(0)]\,|\nu\rangle.\end{aligned} \tag{44}$$

Equivalence of Eq. (44) to Eq. (41) follows from

$$-i[\hat{H}_0, \bar{\rho}(\boldsymbol{\rho}, \boldsymbol{\rho}')] = \frac{i}{2k_0}(\Delta_{\perp,1} - \Delta_{\perp,2})\bar{\rho}(\boldsymbol{\rho}, \boldsymbol{\rho}'), \tag{45}$$

implying

$$\begin{aligned}\partial_z\bar{\rho}(\boldsymbol{\rho}, \boldsymbol{\rho}') = {} & \frac{i}{2k_0}(\Delta_{\perp,1} - \Delta_{\perp,2})\bar{\rho}(\boldsymbol{\rho}, \boldsymbol{\rho}') \\ & + k_0^2[B_n^\perp(\boldsymbol{\rho} - \boldsymbol{\rho}') - B_n^\perp(0)]\bar{\rho}(\boldsymbol{\rho}, \boldsymbol{\rho}').\end{aligned} \tag{46}$$

TABLE I. Relevant quantities appearing in the master equation of Ref. [32], and their definitions in the present contribution. Note that Ref. [32] adheres to a different convention with respect to the Fourier transform and to the definition of the power spectrum.

Variable or function	Definition in Ref. [32]	Our definition
$2\pi\boldsymbol{a}$		$\boldsymbol{q}$
$\Phi_1(\boldsymbol{a})$	$0.033\ (2\pi)^3 C_n^2\ \lvert 2\pi\boldsymbol{a}\rvert^{-11/3}$	$(2\pi)^3\Phi_K(\lvert\boldsymbol{q}\rvert)$
$G_\mu(\boldsymbol{a}, z)$	$\int d^2\boldsymbol{\rho}\ u_\mu(\boldsymbol{\rho}, z)e^{i2\pi\boldsymbol{a}\cdot\boldsymbol{\rho}}$	$\mathcal{F}^{2\mathrm{D}}\{u_\mu\}\lvert_{-\boldsymbol{q}} = \tilde{u}_\mu(-\boldsymbol{q}, z)$
$W_{\mu\nu}(\boldsymbol{a}, z)$	$\int d^2\boldsymbol{a}'\ G_\mu^*(\boldsymbol{a}'+\boldsymbol{a}, z)G_\nu(\boldsymbol{a}', z)$	$\int d^2\boldsymbol{a}'\ \tilde{u}_\mu^*(-(\boldsymbol{q}+\boldsymbol{q}'), z)\tilde{u}_\nu(-\boldsymbol{q}', z)$
$\mathcal{P}_{\mu\nu}$	$\frac{2\pi^2}{k_0}\int d^2\boldsymbol{a}\ \lvert\boldsymbol{a}\rvert^2 G_\mu^*(\boldsymbol{a}, z)G_\nu(\boldsymbol{a}, z)$	$\frac{1}{2k_0}\int d^2\boldsymbol{q}\ \lvert\boldsymbol{q}\rvert^2\tilde{u}_\mu^*(\boldsymbol{q}, z)\tilde{u}_\nu(\boldsymbol{q}, z) = \langle\mu\lvert\hat{H}_0\rvert\nu\rangle$

Consequently, in the single-photon sector, our master equation is no more "quantum" than the classical parabolic equation. Further, it follows that the numerical approach developed in this contribution is likewise applicable to find numerical solutions for the MCF.

E. Equivalence to Roux's Lindblad equation for a single photon

We have alluded to the equivalence of the IPE expressed in Lindblad form [31,32] with our master equation (22). Since we chose a different route to derive a master equation, relying on a stochastic Hamiltonian, this equivalence is not immediately evident from Eq. (22); to contrast the two approaches, we briefly sketch the derivation in Ref. [32].

Roux expands arbitrary basis states in the transverse momentum basis; for ease of comparison, we here choose the LG basis, such that mode $\alpha \equiv (p_\alpha, l_\alpha)$ can be written in notation of Ref. [32] as

$$|\alpha\rangle = \int d^2\boldsymbol{a}\ G_\alpha(\boldsymbol{a}, z)\,|\boldsymbol{a}\rangle\,, \tag{47}$$

with $\boldsymbol{a}$ related to the 2D momentum vector via $2\pi\boldsymbol{a} = \boldsymbol{q}$. The momentum space wave functions G_α obey the stochastic parabolic equation (in the Fourier domain) for infinitesimal propagation, allowing Roux to derive an equation for elements of the infinitesimal evolution operator dU with $\rho(z+dz) = dU\,\rho(z)dU^\dagger$, and, following this, an equation for $\rho(z+dz)$ that is turned into a first-order differential equation. Further steps in the derivation include the Markov approximation and an ensemble average over refractive index fluctuations. The final single-photon Lindblad equation reads [32]

$$\begin{aligned}\partial_z\bar{\rho}_{\mu\nu}(z) = i[\mathcal{P}, \bar{\rho}(z)]_{\mu\nu} + k_0^2\int d^2\boldsymbol{a}\ \Phi_1(\boldsymbol{a})\\ \times\Big[W_{\mu\alpha}(\boldsymbol{a}, z)\bar{\rho}_{\alpha\beta}(z)W^\dagger_{\beta\nu}(\boldsymbol{a}, z)\\ -\frac{1}{2}W^\dagger_{\mu\alpha}(\boldsymbol{a}, z)W_{\alpha\beta}(\boldsymbol{a}, z)\bar{\rho}_{\beta\nu}(z)\\ -\frac{1}{2}\bar{\rho}_{\mu\alpha}(z)W^\dagger_{\alpha\beta}(\boldsymbol{a}, z)W_{\beta\nu}(\boldsymbol{a}, z)\Big],\end{aligned} \tag{48}$$

where $\langle\mu|\bar{\rho}|\nu\rangle = \bar{\rho}_{\mu\nu}$, and $\mathcal{P}$ is a free kinetic term. The summation convention is in force in Eq. (48), and the definitions of $\Phi_1(\boldsymbol{a})$, $W_{\alpha\beta}(\boldsymbol{a}, z)$, and $\mathcal{P}$ are given in Table I.

Upon substitution of Eq. (24) in the single-photon master equation written for components of the density operator, Eq. (42), we already find some structural resemblance of the dissipators:

$$\begin{aligned}\partial_z\bar{\rho}_{\mu\nu} = 2\pi k_0^2\int d^2\boldsymbol{q}\ \Phi_K(|\boldsymbol{q}|)\Big[\tilde{U}_{\mu\alpha}(-\boldsymbol{q}, z)\bar{\rho}_{\alpha\beta}\tilde{U}_{\beta\nu}(\boldsymbol{q}, z)\\ -\frac{1}{2}\tilde{U}_{\mu\alpha}(-\boldsymbol{q}, z)\tilde{U}_{\alpha\beta}(\boldsymbol{q}, z)\bar{\rho}_{\beta\nu}\\ -\frac{1}{2}\bar{\rho}_{\mu\alpha}\tilde{U}_{\alpha\beta}(-\boldsymbol{q}, z)\tilde{U}_{\beta\nu}(\boldsymbol{q}, z)\Big].\end{aligned} \tag{49}$$

It remains to establish the relation between $\tilde{U}_{\mu\nu}$ and $W_{\mu\nu}$ in (48) and (49), respectively. Let us denote the transverse Fourier transform by $\mathcal{F}^{2\mathrm{D}}\{\cdot\}$, and the convolution by $\star$. From the convolution theorem for Fourier transforms, it follows that

$$\begin{aligned}\tilde{U}_{\mu\nu}(\boldsymbol{q}, z) &= \mathcal{F}^{2\mathrm{D}}\{u_\mu^*(\rho, z)u_\nu(\rho, z)\}\\ &= \mathcal{F}^{2\mathrm{D}}\{u_\mu(\rho, z)\}|^*_{-\boldsymbol{q}}\star\mathcal{F}^{2\mathrm{D}}\{u_\nu(\rho, z)\},\end{aligned} \tag{50}$$

with $\mathcal{F}^{2\mathrm{D}}\{\cdot\}|_{-\boldsymbol{q}}$ indicating the Fourier transform evaluated at $-\boldsymbol{q}$, and, therefore,

$$\begin{aligned}\tilde{U}_{\mu\nu}(\boldsymbol{q}, z) &= \int d^2\boldsymbol{q}'\ \tilde{u}_\mu^*(-(\boldsymbol{q}-\boldsymbol{q}'), z)\tilde{u}_\nu(\boldsymbol{q}', z)\\ &= \int d^2\boldsymbol{q}'\ \tilde{u}_\mu^*(-(\boldsymbol{q}+\boldsymbol{q}'), z)\tilde{u}_\nu(-\boldsymbol{q}', z)\\ &= W_{\mu\nu}(\boldsymbol{a}, z),\end{aligned} \tag{51}$$

where the last equality follows from comparison with Table I.

The Hermiticity $W^\dagger_{\mu\nu} = W^*_{\nu\mu}$ [32] further implies

$$\tilde{U}_{\mu\nu}(-\boldsymbol{q}, z) = W^\dagger_{\mu\nu}(\boldsymbol{a}, z). \tag{52}$$

Substituting Eqs. (50) and (52) and Roux's definition of the power spectrum (see Table I) in the single-photon master equation Eq. (49), we recover the dissipator of Eq. (48). This completes the proof of the equivalence between the two master equations. A few remarks are in order:

(i) The "kinetic term" $\mathcal{P}$ in Ref. [32] corresponds to our free Hamiltonian $\hat{H}_0$. Transforming from the interaction to the Schrödinger picture, we can recover the "kinetic term" in our master equation (49).

(ii) The equivalence of the master equations in the single excitation subspace is expected, given that we used a Hamiltonian which is *by construction* compatible with the stochastic parabolic equation and imposed the Markov approximation—in complete analogy to Ref. [32]—in order to arrive at the second-order master equation (22).

(iii) The equivalence proved in this section together with the result of Sec. III D implies that the single-photon IPE

is just a different representation of the classical parabolic equation [21].

(iv) Although formal equivalence holds between Eqs. (48) and (42), it must be stressed that the former equation and its generalizations [34] are not well adapted for numerical simulations. For instance, in the study of entanglement evolution in Ref. [34], the propagation of maximally entangled twisted biphotons was considered in the plane-wave basis, followed by a projection onto the LG basis using generating functions. Evaluation of projections amounts to multifold integrals that can be taken only under the quadratic approximation to the turbulence phase structure function. Furthermore, although this elaborate procedure does not include any basis truncation, performing projections onto LG modes with $p > 0$ and $|l| \geqslant 5$ becomes rather challenging.

The authors of Ref. [31] improved the numerical capacity of the Lindbladian representation of the IPE for the study of LG modes by deriving a generating function for a discrete set of Lindblad operators in the LG basis, though the Hilbert space accessible via this approach remains limited (p, $|l| \leqslant 4$), and truncation-based losses pose a non-negligible issue [31].

In contrast, below we report numerical solutions on a truncated Hilbert space but in Kolmogorov turbulence and for $p \leqslant 9$ and $|l| \leqslant 26$.

F. Relation to a quantum kinetic description of light propagation

To study turbulence-induced scintillation, the authors of Ref. [40] derived a Boltzmann-Langevin equation for the PDF, defined in close analogy to the Wigner function,

$$\hat{f}(\boldsymbol{r}, \boldsymbol{k}, t) = \frac{1}{V} \sum_{\boldsymbol{k}'} e^{-i\boldsymbol{k}\cdot\boldsymbol{r}} b^{\dagger}_{\boldsymbol{k}+\boldsymbol{k}'/2} b_{\boldsymbol{k}-\boldsymbol{k}'/2}, \tag{53}$$

with $b^{\dagger}_{\boldsymbol{k}}$ the bosonic creation operator of a photon with wave vector $\boldsymbol{k}$ and V the normalization volume. While the kinetic equation is more general than the master equation, including source fluctuations and going beyond the monochromatic limit, we briefly show that, under suitable approximations, a close structural similarity with Eq. (22) can be unveiled. The kinetic equation is derived from the following Hamiltonian:

$$\hat{H} = \sum_{\boldsymbol{k}} \hbar\omega_{\boldsymbol{k}} b^{\dagger}_{\boldsymbol{k}} b_{\boldsymbol{k}} - \sum_{\boldsymbol{k},\boldsymbol{k}'} \hbar\omega_{\boldsymbol{k}} n_{\boldsymbol{k}'} b^{\dagger}_{\boldsymbol{k}} b_{\boldsymbol{k}+\boldsymbol{k}'}, \tag{54}$$

with $n_{\boldsymbol{k}'} = \int dV \, \exp(i\boldsymbol{k}'\boldsymbol{r})\delta n(\boldsymbol{r})/V$ the Fourier transform of $\delta n(\boldsymbol{r})$ under the conventions of Ref. [40]. The first term of Eq. (54) is the free Hamiltonian, whereas the relation of the second term in (54) to the second term of our Hamiltonian (8) becomes apparent by imposing the paraxial approximation, $\omega_{\boldsymbol{k}} \simeq \omega_0 = ck_0$, and returning to real space, where the second term of (54) can be written as

$$\propto \hbar\omega_0 \int d^3\boldsymbol{r} \, \delta n(\boldsymbol{r}) b^{\dagger}_{\boldsymbol{r}} b_{\boldsymbol{r}}. \tag{55}$$

From Eq. (54), the kinetic equation for the PDF is derived,

$$(\partial_t + c_{\boldsymbol{k}} \cdot \boldsymbol{\nabla}_{\boldsymbol{r}})\hat{f} = \hat{K} - \hat{v}_{\boldsymbol{k}}\{\hat{f}\}, \tag{56}$$

with $c_{\boldsymbol{k}} = \boldsymbol{\nabla}_{\boldsymbol{k}}\omega_{\boldsymbol{k}}$, $\hat{K}$ the Langevin source describing fluctuations around the ensemble-averaged kinetic evolution ($\overline{\hat{K}} = 0$ in the ensemble-averaged picture), and

$$\hat{v}_{\boldsymbol{k}}\{\hat{f}\} \propto \int d^2\boldsymbol{q} \; \Phi(\boldsymbol{q})[\hat{f}(\boldsymbol{r}, \boldsymbol{k}, t) - \hat{f}(\boldsymbol{r}, \boldsymbol{k} + \boldsymbol{q}, t)], \tag{57}$$

where $\Phi(\boldsymbol{q})$ is the (e.g. Kolmogorov) power spectrum of refractive index fluctuations, and the addition of a three-dimensional wave vector $\boldsymbol{k}$ and a two-dimensional transverse wave vector $\boldsymbol{q}$ is to be understood as a zero padding of the latter. Upon promoting the propagation distance to the evolution variable, $z = ct$, and invoking the paraxial approximation to write $c_{\boldsymbol{k}} \simeq c\boldsymbol{e}_z$ in Eq. (56), where $\boldsymbol{e}_z$ is the unit vector in the z direction, one finds that the mean PDF, $\bar{f}$, obeys

$$D_z\bar{f}(\boldsymbol{r}, \boldsymbol{k}) = 2\pi k_0^2 \int d^2\boldsymbol{q} \; \Phi(\boldsymbol{q})[\bar{f}(\boldsymbol{r}, \boldsymbol{k} + \boldsymbol{q}) - \bar{f}(\boldsymbol{r}, \boldsymbol{k})], \tag{58}$$

with D_z a total derivative with respect to z. To illustrate structural similarities, consider the master equation in the plane-wave basis, Eq. (46). Using Eq. (3), we can write Eq. (46) as

$$\partial_z\bar{\rho} = -i[\hat{H}_0, \bar{\rho}] + 2\pi k_0^2 \int d^2\boldsymbol{q} \; \Phi_K(\boldsymbol{q}) \times [e^{i\boldsymbol{q}\cdot\boldsymbol{\rho}}\bar{\rho}(\boldsymbol{\rho}, \boldsymbol{\rho}')e^{-i\boldsymbol{q}\cdot\boldsymbol{\rho}'} - \bar{\rho}(\boldsymbol{\rho}, \boldsymbol{\rho}')], \tag{59}$$

which exhibits the form of a collisional decoherence master equation [66]. Moving to Fourier space, the dissipative part of Eq. (59) assumes the form

$$2\pi k_0^2 \int d^2\boldsymbol{q} \; \Phi_K(\boldsymbol{q})[\bar{\rho}(\boldsymbol{q}_1 - \boldsymbol{q}, \boldsymbol{q}_2 - \boldsymbol{q}) - \bar{\rho}(\boldsymbol{q}_1, \boldsymbol{q}_2)]. \tag{60}$$

Restricting $\bar{\rho}(\boldsymbol{q}_1, \boldsymbol{q}_2, z)$ to the diagonal, $\bar{n}(\boldsymbol{q}, z) = \bar{\rho}(\boldsymbol{q}, \boldsymbol{q}, z)$, and exploiting the isotropy $\Phi_K(\boldsymbol{q}) = \Phi_K(-\boldsymbol{q})$, we thus find, in close resemblance to Eq. (58), the collision integral

$$\partial_z\bar{n}(\boldsymbol{q}, z) = 2\pi k_0^2 \int d^2\boldsymbol{q}' \; \Phi_K(\boldsymbol{q}')[\bar{n}(\boldsymbol{q} + \boldsymbol{q}') - \bar{n}(\boldsymbol{q}, z)], \tag{61}$$

where the first and second terms on the right-hand side describe, respectively, gain and loss of the population of the spatial mode with transverse momentum $\boldsymbol{q}$ due to scattering on the refractive index inhomogeneities.

IV. APPLICATION: EVOLUTION OF CROSSTALK

A. Dimension of the truncated Hilbert space

To demonstrate the numerical feasibility of our single-photon master equation, we now determine the turbulence-induced crosstalk among LG modes. As we are mainly interested in the spatial evolution of the density operator's populations, we can take advantage of the selection rule stated by Eq. (30), implying that the Liouville subspace spanned by dyadics $|p_1, l\rangle\langle p_2, l|$, with suitably truncated radial and azimuthal index range $p_{1,2} = 0, ..., p_{\max}$ and $l = -l_{\max}, ..., l_{\max}$, is invariant under evolution due to Eq. (22). Consequently, we can focus on this subspace whose dimension scales as

$$\dim(\text{Liouville space}) = (p_{\max} + 1)^2(2l_{\max} + 1). \tag{62}$$

Note the linear scaling in $l_{\max}$ in (62), which is rather favorable for numerical simulations, since the primary effect of atmospheric turbulence is to induce errors in the beam's phase front, causing OAM crosstalk, or spreading of the OAM distribution. The quadratic scaling in $p_{\max}$ is less benign, but still tractable. In this contribution, we consider the regime of weak scintillations, allowing for numerically tractable choices of $p_{\max}$ and $l_{\max}$. However, it should be noted that our method is not fundamentally limited to this regime; under the caveat of a sufficiently large Hilbert space, it is also possible to explore the regime of pronounced scintillations.

For weak scintillation, it is reasonable to assume that crosstalk among the radial modes is dominated by that among modes with the same radial index but distinct azimuthal indices, justifying a basis choice with $p_{\max} < l_{\max}$. In the following, we report our results on crosstalk for $p_{\max} = 9$, $l_{\max} = 26$.

B. Comparison to single-phase-screen models

The regime of weak scintillations is characterized by Rytov variance $\sigma_R^2 < 1$ [67]. In this case the effect of turbulence is well captured by a single phase screen distorting the field's wavefront without altering its amplitude [8]. Based on this model, a number of maps were proposed to describe the effect of a screen on twisted photons [8,68,69]. Let us first compare the crosstalk predicted by our master equation to the crosstalk generated by the turbulence map Λ derived in Ref. [68], relating the input density matrix to the ensemble-averaged output density matrix summed over the radial index:

$$\bar{\sigma}_{l_\alpha,l_\beta}(z) := \sum_p \bar{\rho}_{p,l_\alpha,p,l_\beta}(z). \tag{63}$$

Specifically, for the initially populated LG_{0,l_0} mode, the output populations summed over the radial index are obtained according to

$$\bar{\sigma}_{l,l}(z) = \Lambda_{l,l}^{0,l_0,0,l_0}\bar{\rho}_{0,l_0,0,l_0}(0), \tag{64}$$

with elements of the map given by [60]

$$\Lambda_{l,l}^{0,l_0,0,l_0} = \frac{1}{2\pi}\int_0^\infty r dr \int_0^{2\pi} d\vartheta\; R^*_{0,l_0}(r)R_{0,l_0}(r) \times e^{-i\frac{\vartheta}{2}[2l-2l_0]}\exp\left(-\frac{1}{2}D_\phi[2r\sin(\vartheta/2)]\right), \tag{65}$$

where R_{0,l_0} denotes the radial part of the input LG mode, and D_ϕ the phase structure function (25). The integral in Eq. (65) can be solved numerically. Note that the phase structure function (25) depends on the Fried parameter, which in turn depends on the propagation distance. As examples, we consider the propagation of LG_{0,l_0} modes with $l_0 \in \{0, 3, 5\}$ through turbulence. Throughout, we set the beam waist to $w_0 = 0.07$ m and wavelength to $\lambda = 1064$ nm, which corresponds to the Rayleigh length $z_R = 14.5$ km. We further use a typical value $C_n^2 = 1 \times 10^{-14}$ m$^{-2/3}$ for the turbulence structure constant [21] and consider propagation distances up to $z = 700$ m, which corresponds to Rytov variance $\sigma_R^2 \leqslant 0.16$ and turbulence strength $w_0/r_0 \leqslant 1.13$. Since we are within the regime $z \ll z_R$, the relaxation tensor is z independent [see Eq. (31)], such that for all considered examples, the master equation can be efficiently solved numerically.

The resulting crosstalk histograms generated from solutions of our master equation for $z = 250$ m (this corresponds to $\sigma_R^2 = 0.024$ and $r_0 \approx 0.12$ m) and from Eq. (64), with the same r_0 for the map (65), are shown in Figs. 1(a)–1(c), displaying good quantitative agreement. We note that the bars derived from the master equation slightly deviate from those based on the map (64), which is an artifact of the basis truncation in the radial index p of the former method.

Both, the master equation (22) and the turbulence map (64), predict a symmetric distribution of crosstalk about the initially excited mode, LG_{0,l_0} [see Figs. 1(a)–1(c)]. Due to averaging over the radial index, the symmetric structures in Figs. 1(a)–1(c) conceal the crosstalk among the radial modes. The master equation (22) directly provides this information, which we present in Figs. 1(d)–1(f) for the propagation distances $z = 0, 200, 500$ m of the injected LG_{0,l_0} mode with $l_0 = 5$. The OAM crosstalk for distinct radial modes reveals asymmetric distributions for each value of p, with the sum over p resulting in the symmetric features as in Figs. 1(a)–1(c).

The general behavior of crosstalk for different radial modes can be understood by considering the case $p = 0$. At distances $z \gtrsim 200$ m, the histograms feature double peaks [see Figs. 1(e) and 1(f)]. Such crosstalk distributions were predicted previously using the multiple phase screens technique [28,70] and observed experimentally [71]. These structures can be explained as follows [28]: while the fragile phase profile of LG modes is severely distorted in turbulence, their intensity profiles are affected to a lesser degree in the considered propagation regime. Furthermore, for $|l| > 0$ the LG modes $(0, l)$ and $(0, -l)$ exhibit the same single-ring intensity profiles. Thus, their spatial overlap is significant, resulting in strong coupling by the Hamiltonian Eq. (16). In contrast, the fundamental mode $\mathrm{LG}_{0,0}$, characterized by a Gaussian intensity profile in the transverse plane, overlaps weakly with the $\mathrm{LG}_{0,5}$ mode, so that the corresponding crosstalk amplitude is suppressed. Overall, this leads to the double-peak distributions in Fig. 1.

To benchmark our results for OAM crosstalk at $p = 0$, we compare them with the crosstalk distribution from Ref. [69], expressed analytically in the quadratic approximation by the formula

$$\bar{\rho}_{0,l,0,l}(z) \simeq \frac{\Gamma^2(\kappa)}{|l_0|!|l|!|\Delta l|!}\frac{\epsilon^{2|\Delta l|}}{(1+\epsilon^2)^{2\kappa}} \times {}_2F_1\left[\kappa,\kappa;|\Delta l|+1;\frac{\epsilon^4}{(1+\epsilon^2)^2}\right], \tag{66}$$

with $\kappa = (|l| + |l_0| + |\Delta l|)/2 + 1$, $|\Delta l| = |l - l_0|$, $\epsilon = 0.89\, w_0(C_n^2 k_0^2 z)^{3/5}$, $\Gamma(x)$ the Gamma function, and ${}_2F_1(a, b; c; z)$ the hypergeometric function. The comparison is depicted in Figs. 1(g)–1(i) for propagation distances $z = 300, 500, 700$ m, displaying good quantitative agreement. At the same time, the master equation based on the Kolmogorov model, and the formula (66) based on the quadratic approximation, exhibit slight albeit systematic deviations, which were previously observed in benchmark simulations of Eq. (66) using the multiple phase screens method [69].

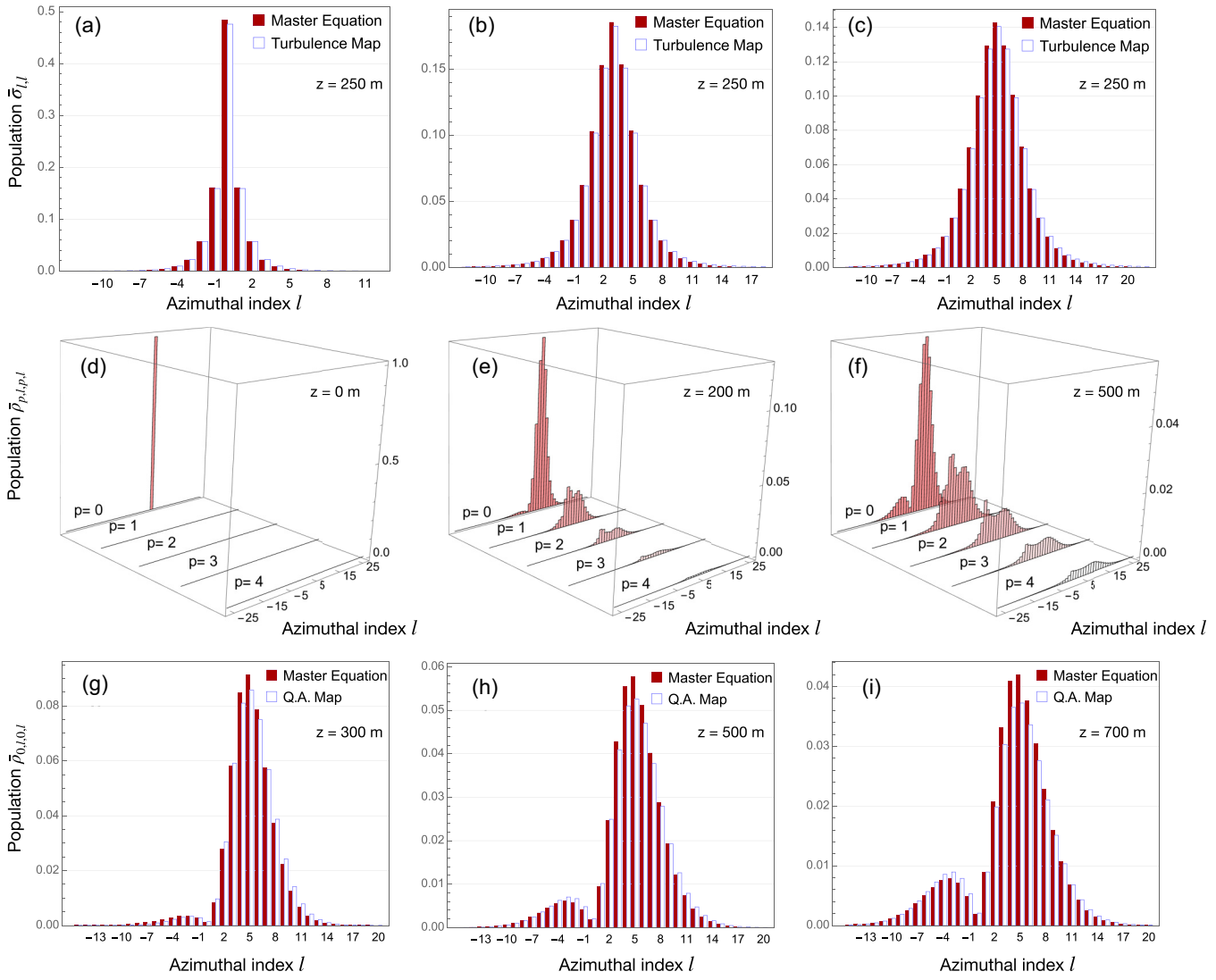


FIG. 1. Crosstalk histograms for fixed structure constant $C_n^2 = 1 \times 10^{-14}$ m$^{-2/3}$, beam waist $w_0 = 0.07$ m, and wavelength $\lambda = 1064$ nm. The propagation distance varies throughout (a)–(i) and is displayed in the upper-left corner of each subfigure; note that the y scale varies greatly through (a)–(i), reflecting the rapidly decaying survival probability of the initially excited mode. The initial state of each (single-photon) simulation is stated below. (a)–(c) Crosstalk histograms obtained using the turbulence map [68] and the master equation (22) for three different initial LG modes $(0, l_0)$, with $l_0 = 0$ in (a), $l_0 = 3$ in (b), and $l_0 = 5$ in (c). (d)–(f) Evolution of crosstalk for initial mode (0,5) at intermediate propagation distances; slices correspond to projections onto the $p = 0, ..., 4$ subspace. (g)–(i) Crosstalk histograms in $p = 0$ for initial mode (0,5) produced by the master equation and a map [69] derived under the quadratic approximation (abbreviated as "Q.A. Map").

V. CONCLUSION

We derived a Lindblad master equation for the twisted photon's averaged density operator, by performing an exact ensemble average over refractive index fluctuations described by the Kolmogorov power spectrum. This master equation can be efficiently solved numerically due to the closed-form expression of the relaxation tensor. We proved the equivalence of our master equation with the master equation of Ref. [32], and presented some simulation results for the evolution of crosstalk in atmospheric turbulence, which agree quantitatively with known results in the weak scintillation regime [68,69].

We expect that expanding the truncated Hilbert space, which will be the subject of future work, will enable the study of moderate to strong scintillation regimes and render our master equation a competitive alternative to the well-established multiple phase screens approach [22–24], thereby evading the need for numerically expensive ensemble average and sophisticated techniques of phase-screen generation. Although we have focused on the density operator's diagonal elements in our present contribution, it is evident that the true potential of a master equation based approach lies in the access to the full density operator, paving the way to the study of "classical" quantities such as the mutual coherence function, and "quantum" applications, such as the propagation of

an entangled photon pair through a turbulent channel. Future work will be dedicated to these applications.

ACKNOWLEDGMENTS

Fruitful discussions with Franklin L. S. Rodrigues, David Bachmann, and Robert Alicki are gratefully acknowledged. We thank the anonymous referees for constructive remarks and pointing out the relation to the quantum kinetic theory of light propagation. The authors acknowledge support by the state of Baden-Württemberg through bwHPC and support by the DFG-funded Research Training Group "Dynamics of Controlled Atomic and Molecular Systems" (RTG 2717).

DATA AVAILABILITY

There are no publicly available research data or software supporting this manuscript. Requests for further information or data should be sent to the authors.

[1] D. Cui, X. Yi, and L.-P. Yang, Back cover: Quantum imaging exploiting twisted photon pairs, Adv. Quantum Technol. **6**, 2370053 (2023).

[2] M. Mirhosseini, O. S. Magaña-Loaiza, M. N. O'Sullivan, B. Rodenburg, M. Malik, M. P. J. Lavery, M. J. Padgett, D. J. Gauthier, and R. W. Boyd, High-dimensional quantum cryptography with twisted light, New J. Phys. **17**, 033033 (2015).

[3] L. Allen, M. W. Beijersbergen, R. J. C. Spreeuw, and J. P. Woerdman, Orbital angular momentum of light and the transformation of Laguerre-Gaussian laser modes, Phys. Rev. A **45**, 8185 (1992).

[4] A. Forbes, M. de Oliveira, and M. R. Dennis, Structured light, Nat. Photon. **15**, 253 (2021).

[5] D. Bachmann, M. Isoard, V. Shatokhin, G. Sorelli, N. Treps, and A. Buchleitner, Highly transmitting modes of light in dynamic atmospheric turbulence, Phys. Rev. Lett. **130**, 073801 (2023).

[6] V. N. Shatokhin, D. Bachmann, G. Sorelli, N. Treps, and A. Buchleitner, Spatial eigenmodes of light in atmospheric turbulence, in *Environmental Effects on Light Propagation and Adaptive Systems III*, edited by K. Stein and S. Gladysz, Conference Proceedings (SPIE, Bellingham, WA, 2020), Vol. 11532, p. 115320G.

[7] A. Klug, C. Peters, and A. Forbes, Robust structured light in atmospheric turbulence, Adv. Photon. **5**, 016006 (2023).

[8] C. Paterson, Atmospheric turbulence and orbital angular momentum of single photons for optical communication, Phys. Rev. Lett. **94**, 153901 (2005).

[9] B. J. Smith and M. G. Raymer, Two-photon wave mechanics, Phys. Rev. A **74**, 062104 (2006).

[10] A. A. Semenov and W. Vogel, Quantum light in the turbulent atmosphere, Phys. Rev. A **80**, 021802 (2009).

[11] D. Y. Vasylyev, A. A. Semenov, and W. Vogel, Toward global quantum communication: Beam wandering preserves nonclassicality, Phys. Rev. Lett. **108**, 220501 (2012).

[12] D. Vasylyev, A. A. Semenov, and W. Vogel, Atmospheric quantum channels with weak and strong turbulence, Phys. Rev. Lett. **117**, 090501 (2016).

[13] G. Vallone, V. D'Ambrosio, A. Sponselli, S. Slussarenko, L. Marrucci, F. Sciarrino, and P. Villoresi, Free-space quantum key distribution by rotation-invariant twisted photons, Phys. Rev. Lett. **113**, 060503 (2014).

[14] A. Sit, F. Bouchard, R. Fickler, J. Gagnon-Bischoff, H. Larocque, K. Heshami, D. Elser, C. Peuntinger, K. Günthner, B. Heim, C. Marquardt, G. Leuchs, R. W. Boyd, and E. Karimi, High-dimensional intracity quantum cryptography with structured photons, Optica **4**, 1006 (2017).

[15] M. P. J. Lavery, C. Peuntinger, K. Günthner, P. Banzer, D. Elser, R. W. Boyd, M. J. Padgett, C. Marquardt, and G. Leuchs, Free-space propagation of high-dimensional structured optical fields in an urban environment, Sci. Adv. **3**, e1700552 (2017).

[16] M. A. Cox, N. Mphuthi, I. Nape, N. Mashaba, L. Cheng, and A. Forbes, Structured light in turbulence, IEEE J. Sel. Top. Quantum Electron. **27**, 1 (2021).

[17] A. E. Willner, Z. Zhao, C. Liu, R. Zhang, H. Song, K. Pang, K. Manukyan, H. Song, X. Su, G. Xie, Y. Ren, Y. Yan, M. Tur, A. F. Molisch, R. W. Boyd, H. Zhou, N. Hu, A. Minoofar, and H. Huang, Perspectives on advances in high-capacity, free-space communications using multiplexing of orbital-angular-momentum beams, APL Photon. **6**, 030901 (2021).

[18] V. I. Tatarskii, *Wave Propagation in a Turbulent Medium* (McGraw-Hill, New York, 1961).

[19] J.-P. Fouque, J. Garnier, G. Papanicolaou, and K. Solna, *Wave Propagation and Time Reversal in Randomly Layered Media*, Stochastic Modelling and Applied Probability Vol. 56 (Springer, New York, 2007).

[20] R. Dashen, Path integrals for waves in random media, J. Math. Phys. **20**, 894 (1979).

[21] A. Ishimaru, *Wave Propagation and Scattering in Random Media*, IEEE/OUP Series on Electromagnetic Wave Theory (IEEE, New York, 1997).

[22] J. M. Martin and S. M. Flatté, Intensity images and statistics from numerical simulation of wave propagation in 3-D random media, Appl. Opt. **27**, 2111 (1988).

[23] M. Charnotskii, Comparison of four techniques for turbulent phase screens simulation, J. Opt. Soc. Am. A **37**, 738 (2020).

[24] D. Bachmann, M. Isoard, V. N. Shatokhin, G. Sorelli, and A. Buchleitner, Accurate Zernike-corrected phase screens for arbitrary power spectra, Opt. Eng. **64**, 058102 (2025).

[25] E. Anzuola and S. Gladysz, Modeling dynamic atmospheric turbulence using temporal spectra and Karhunen-Loève decomposition, Opt. Eng. **56**, 071508 (2017).

[26] M. P. J. Lavery, Vortex instability in turbulent free-space propagation, New J. Phys. **20**, 043023 (2018).

[27] M. Klen and A. A. Semenov, Numerical simulations of atmospheric quantum channels, Phys. Rev. A **108**, 033718 (2023).

[28] G. Sorelli, N. Leonhard, V. N. Shatokhin, C. Reinlein, and A. Buchleitner, Entanglement protection of high-dimensional states by adaptive optics, New J. Phys. **21**, 023003 (2019).

[29] J. Schmidt, *Numerical Simulation of Optical Wave Propagation with Examples in MATLAB* (SPIE, Bellingham, WA, 2010).

[30] F. S. Roux, Infinitesimal-propagation equation for decoherence of an orbital-angular-momentum-entangled biphoton state in atmospheric turbulence, Phys. Rev. A **83**, 053822 (2011).

[31] J. R. González Alonso and T. A. Brun, Error correction with orbital angular momentum of multiple photons propagating in a turbulent atmosphere, Phys. Rev. A **95**, 032320 (2017).

[32] F. S. Roux, The Lindblad equation for the decay of entanglement due to atmospheric scintillation, J. Phys. A **47**, 195302 (2014).

[33] A. H. Ibrahim, F. S. Roux, M. McLaren, T. Konrad, and A. Forbes, Orbital-angular-momentum entanglement in turbulence, Phys. Rev. A **88**, 012312 (2013).

[34] F. S. Roux, T. Wellens, and V. N. Shatokhin, Entanglement evolution of twisted photons in strong atmospheric turbulence, Phys. Rev. A **92**, 012326 (2015).

[35] H.-P. Breuer and F. Petruccione, *The Theory of Open Quantum Systems* (Oxford University, New York, 2007).

[36] C. M. Kropf, C. Gneiting, and A. Buchleitner, Effective dynamics of disordered quantum systems, Phys. Rev. X **6**, 031023 (2016).

[37] A. A. Budini, Quantum systems subject to the action of classical stochastic fields, Phys. Rev. A **64**, 052110 (2001).

[38] L. Yu and D. F. V. James, Average quantum dynamics of closed systems over stochastic Hamiltonians, Sci. Rep. **15**, 30984 (2025).

[39] G. P. Berman and A. A. Chumak, Photon distribution function for long-distance propagation of partially coherent beams through the turbulent atmosphere, Phys. Rev. A **74**, 013805 (2006).

[40] R. A. Baskov and O. O. Chumak, Laser-beam scintillations for weak and moderate turbulence, Phys. Rev. A **97**, 043817 (2018).

[41] V. I. Klyatskin, Applicability of the approximation of a Markov random process in problems relating to the propagation of light in a medium with random inhomogeneities, Sov. Phys. JETP **30**, 520 (1970).

[42] A. Aiello, Field theory of monochromatic optical beams: II. Classical and quantum paraxial fields, J. Opt. **22**, 014002 (2020).

[43] R. J. Glauber and M. Lewenstein, Quantum optics of dielectric media, Phys. Rev. A **43**, 467 (1991).

[44] C. A. Dartora, G. G. Cabrera, K. Z. Nobrega, V. F. Montagner, M. H. K. Matielli, F. K. R. de Campos, and H. T. S. Filho, Lagrangian-Hamiltonian formulation of paraxial optics and applications: Study of gauge symmetries and the optical spin Hall effect, Phys. Rev. A **83**, 012110 (2011).

[45] C. Cohen-Tannoudji, J. Dupont-Roc, and G. Gilbert, *Photons and Atoms: Introduction to Quantum Electrodynamics* (Wiley, New York, 2004).

[46] See Supplemental Material at http://link.aps.org/supplemental/10.1103/m4rt-sss4 for an analytical evaluation of the relaxation tensor and a discussion of divergence due to the Kolmogorov power spectrum, which includes Refs. [47–53].

[47] P. Appell, Sur les fonctions hypergéométriques de deux variables, J. Math. Pures Appl. 3e Série **8**, 173 (1882).

[48] V. V. Kotlyar, E. G. Abramochkin, A. A. Kovalev, and A. A. Savelyeva, Product of two Laguerre-Gaussian beams, Photon. **9**, 496 (2022).

[49] L. Poh-aun, S.-H. Ong, and H. M. Srivastava, Some integrals of the products of Laguerre polynomials, Int. J. Comput. Math. **78**, 303 (2001).

[50] G. Sorelli, Quantum state transfer in diffractive and refractive media, Ph.D. dissertation, Albert-Ludwigs-Universität Freiburg, 2019.

[51] M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables* (Dover, New York, 1965).

[52] A. P. Prudnikov, Y. A. Brychkov, and O. I. Marichev, *Integrals and Series* (Gordon and Breach, New York, 1986), Vol. 2.

[53] S. Paycha, *Regularised Integrals, Sums, and Traces: An Analytic Point of View*, University Lecture Series (American Mathematical Society, Providence, 2012).

[54] G. F. Calvo, A. Picón, and E. Bagan, Quantum field theory of photons with orbital angular momentum, Phys. Rev. A **73**, 013805 (2006).

[55] S. Feng and H. G. Winful, Physical origin of the Gouy phase shift, Opt. Lett. **26**, 485 (2001).

[56] A. A. Budini, Non-Markovian Gaussian dissipative stochastic wave vector, Phys. Rev. A **63**, 012106 (2000).

[57] J. Carl Leader, Atmospheric propagation of partially coherent radiation, J. Opt. Soc. Am. **68**, 175 (1978).

[58] W. He, S. Guha, J. H. Shapiro, and B. A. Bash, Performance analysis of free-space quantum key distribution using multiple spatial modes, Opt. Express **29**, 19305 (2021).

[59] M. Charnotskii, Common omissions and misconceptions of wave propagation in turbulence: Discussion, J. Opt. Soc. Am. A **29**, 711 (2012).

[60] D. Bachmann, V. N. Shatokhin, and A. Buchleitner, Universal entanglement decay of photonic orbital angular momentum qubit states in atmospheric turbulence: An analytical treatment, J. Phys. A: Math. Theor. **52**, 405303 (2019).

[61] G. Gbur and R. K. Tyson, Vortex beam propagation through atmospheric turbulence and topological charge conservation, J. Opt. Soc. Am. A **25**, 225 (2008).

[62] R. Alicki and K. Lendi, *Quantum Dynamical Semigroups and Applications*, Lecture Notes in Physics (Springer-Verlag, Berlin, 2007).

[63] H.-P. Breuer, E.-M. Laine, J. Piilo, and B. Vacchini, *Colloquium*: Non-Markovian dynamics in open quantum systems, Rev. Mod. Phys. **88**, 021002 (2016).

[64] The parabolic equation (20–42) in Ref. [21] is given in terms of the correlation function of fluctuations of the relative dielectric constant, leading to an additional factor 1/4 in front of the structure function compared to our Eq. (41).

[65] The correlation function of the exact Kolmogorov power spectrum is ill defined; see the Supplemental Material [46] for a rigorous treatment.

[66] R. Alicki, Search for a border between classical and quantum worlds, Phys. Rev. A **65**, 034104 (2002).

[67] L. Andrews and R. Phillips, *Laser Beam Propagation Through Random Media* (SPIE, Bellingham, WA, 2005).

[68] N. D. Leonhard, V. N. Shatokhin, and A. Buchleitner, Universal entanglement decay of photonic-orbital-angular-momentum qubit states in atmospheric turbulence, Phys. Rev. A **91**, 012345 (2015).

[69] H. F. Elder and P. Sprangle, Mode power spectrum for Laguerre-Gauss beams in Kolmogorov turbulence, Opt. Lett. **47**, 3447 (2022).

[70] J. A. Anguita, M. A. Neifeld, and B. V. Vasic, Turbulence-induced channel crosstalk in an orbital angular momentum-multiplexed free-space optical link, Appl. Opt. **47**, 2414 (2008).

[71] D. Bachmann, A. Klug, M. Isoard, V. N. Shatokhin, G. Sorelli, A. Buchleitner, and A. Forbes, Universal crosstalk of twisted light in random media, Phys. Rev. Lett. **132**, 063801 (2024).