

Multivariate bicycle codes

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Quantum error correction suppresses noise in quantum systems to allow for high-precision computations. In this Letter, we introduce multivariate bicycle quantum low-density parity-check (QLDPC) codes, via an extension of the framework developed by Bravyi *et al.* [S. Bravyi, A. W. Cross, J. M. Gambetta, D. Maslov, P. Rall, and T. J. Yoder, *Nature (London)* **627**, 778 (2024)], and particularly focus on trivariate bicycle (TB) codes. Unlike the weight-6 codes proposed in their study, we offer concrete examples of weight-5 TB-QLDPC codes which promise to be more amenable to near-term experimental setups. We show that TB-QLDPC codes up to weight 6 have a biplanar structure and often possess a two-dimensional toric layout. Under circuit level noise, we find substantially better encoding rates than comparable surface codes while offering similar error suppression capabilities. For example, we can encode four logical qubits with distance 5 into 60 physical qubits using weight-5 check measurements of circuit depth 7, while a surface code with these parameters requires 200 physical qubits. The high encoding rate and compact layout make our codes highly suitable candidates for near-term hardware implementations, paving the way for a realizable quantum error correction protocol.

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Introduction. Quantum computing promises to solve problems intractable for classical computers [1–3]. Despite advancements in improving hardware quality, noise remains the most significant challenge for realizing practical implementations to ultimately achieve both reliable and scalable quantum computations. As a consequence, quantum error correction (QEC) is required to reduce noise of the physical hardware [4,5]. Among many QEC codes [6–8], the surface code has emerged which currently leads the forefront in experimental quantum computing due to its relatively high threshold error rates [8–13] and suitability for implementation with current hardware devices [13]. Despite its advantages, the surface code requires high qubit overhead leading to an asymptotically zero encoding rate while the distance scales as the square root of the number of physical qubits [14].

Recently, quantum low-density parity-check (QLDPC) codes have attracted attention for their potential to require fewer physical qubits than surface codes [15]. LDPC codes are well established in classical error correction [16–18] for their efficiency and decoding performance. In a recent breakthrough [19], Bravyi *et al.* presented a set of QLDPC codes that are significantly more qubit efficient than the surface code of the same number of logical qubits. These codes, called bivariate bicycle (BB) QLDPC codes, have origins in the work by Kovalev and Pryadko [20]. Reference [19] introduced codes with weight-6 stabilizer checks which can be encoded into a biplanar toric layout. This layout allows their code to accommodate four local stabilizer checks and two long-range checks. However, QLDPC codes present significant challenges for current experimental capabilities due to the required long-range connections. Considering the example of superconducting qubits, current experimental capabilities struggle to provide extensive long-range connections across the entire circuit board, although individual long-range connections have had considerable progress in their gate fidelities recently [21]. Therefore, an experimental proof of concept for the viability of QLDPC codes should be a minimalistic progression from fully local codes such as the surface code, e.g. by requiring only one long-range connection per qubit.

Here, we introduce multivariate bicycle (MB) QLDPC codes to provide several novel low-weight codes with better encoding rates than comparable surface codes. We define

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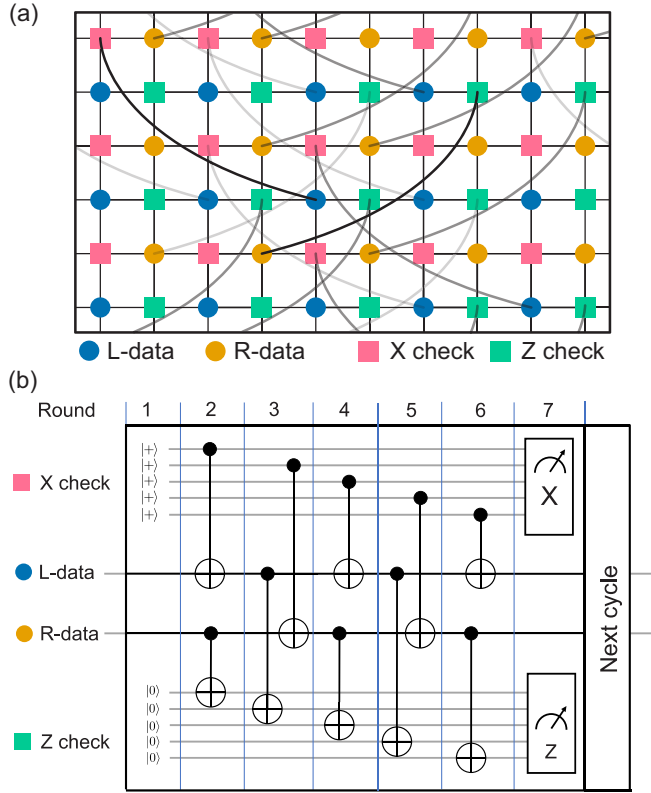


FIG. 1. (a) Our weight-5 $[[30, 4, 5]]$ TB-QLDPC code can be arranged in a biplanar grid with periodic boundary conditions using 30 data qubits (which belong either to the L or R sublattice) and 30 qubits for parity check measurements. Each parity check involves four neighboring data qubits, and one nonlocal interaction described by a translational invariant lattice vector (x, y) , where x is the horizontal and y the vertical axis of the grid. For X checks and the L sublattice, the interaction vector is $(-4, 3)$, while for Z checks and the R sublattice we have $(4, 3)$. For better visibility, we only plot representatives of the nonlocal interactions. (b) Measurement syndrome circuit for our weight-5 $[[30, 4, 5]]$ TB-QLDPC with only depth 7 and circuit level distance $d_{\text{circ}} \leq 5$, which our numerics suggest is tight.

MB-QLDPC codes in a general manner, and then focus on the concrete case of trivariate bicycle (TB) codes, which have three variable in the generating polynomials. As a highlight, we find a weight-5 $[[30, 4, 5]]$ TB code with a single long-range check which encodes four logical qubits using only 60 physical qubits in a biplanar toric architecture, while a comparable surface code would require 200 qubits (see Fig. 1). We design a depth-7 syndrome measurement circuit which preserves the distance even under circuit-level noise, in contrast to conventional depth-10 circuits. Under circuit level noise, our code has comparable noise suppression to surface codes and previously proposed weight-6 BB codes [19,22]. We also find for our codes several fault-tolerant logical circuits that can be implemented with transversal operations and SWAP gates via automorphism groups [23]. We also make progress on the properties of TB codes by showing that TB codes up to weight 6 possess a biplanar structure. Furthermore, we provide a weight-independent criterion to check whether a TB code has a toric layout structure. We also show that all stabilizer checks are translationally invariant when the MB-QLDPC codes have

a toric layout. Finally, we show that weight-4 TB codes can possess a tangled toric layout, where the Tanner graph corresponds to a deformed torus with some mismatched edges.

Framework. We give an introduction to quantum error correction in the Supplemental Material (SM) Sec. A [24] and now proceed to introduce the MB-QLDPC code and its group algebra with r variables $\mathbb{F}_2[G_r]$, where $G_r = \frac{\mathbb{Z}}{l_1} \times \frac{\mathbb{Z}}{l_2} \times \cdots \times \frac{\mathbb{Z}}{l_r}$. In the special case of $r = 2$, we recover the group algebra of the BB code. The group algebra $\mathbb{F}_2[G_r]$ is isomorphic to the multivariate polynomial quotient ring $R_r = \frac{\mathbb{F}_2[x_1, x_2, \dots, x_r]}{(x_1^{l_1} - 1, x_2^{l_2} - 1, \dots, x_r^{l_r} - 1)}$. Here, the quotient ring R_r has a basis consisting of the monomials $x_1^{i_1} x_2^{i_2} \cdots x_r^{i_r}$, where $0 \leq i_1 \leq l_1 - 1, 0 \leq i_2 \leq l_2 - 1, \dots, 0 \leq i_r \leq l_r - 1$. In this Letter, we define the variable x_i with $1 \leq i \leq r$, using the following encoding.

(1) Let $\text{enc}_{\text{bin}}(i) = b_q b_{q-1} \cdots b_2 b_1$ be the binary encoding of i using $q = \lfloor \log_2 r \rfloor + 1$ bits.

(2) Let $t_\alpha \equiv S_{l_i}^{b_\alpha}$ where S_{l_i} is a cyclic shift matrix of size $l_i \times l_i$.

(3) $x_i \equiv t_q \otimes t_{q-1} \otimes \cdots \otimes t_2 \otimes t_1$.

Note that the above encoding is not unique and has been chosen for its simplicity and one can trivially define other encodings. Moreover, the different variables are not independent. For example, in the trivariate case, the quotient ring is given by

$$R'_3 = \frac{\mathbb{F}_2[x_1, x_2, x_3]}{(x_1^{l_1} - 1, x_2^{l_2} - 1, x_3 - x_1 x_2)},$$

which is a subset of

$$R_3 = \frac{\mathbb{F}_2[x_1, x_2, x_3]}{(x_1^{l_1} - 1, x_2^{l_2} - 1, x_3^{l_3} - 1)}.$$

Thus, our choice of encoding leads to a reduction in the size of the set R_r . In general, however, one can trivially define variables such that they are all independent. One such candidate encoding could be one-hot encoding of i , defined as $\text{enc}_{\text{one-hot}}(i) = b_q b_{q-1} \cdots b_2 b_1$ with $q = r$. For example, $\text{enc}_{\text{one-hot}}(4) = 01000$, for $r = 5$. Steps 2 and 3 remain the same as the last encoding. Note that the BB code corresponds to the $r = 2$ case with either choice of encoding, one-hot or binary. This is because the aforementioned two encodings are the same for $r = 2$.

For concreteness, in this Letter, we will focus on $r = 3$ with binary encoding, which we call TB codes. We start with the identity matrix I_m and the cyclic shift matrix S_l of size $l \times l$ with $(S_l)_{kj} = \delta_{j, (k+1) \bmod l}$ for integer-valued l and m . Let now

$$x \equiv x_1 = S_l \otimes I_m, \quad y \equiv x_2 = I_l \otimes S_m, \quad z \equiv x_3 = S_l \otimes S_m$$

be matrices of dimension $(l \cdot m) \times (l \cdot m)$ that commute with each other $[x_i, x_j] = 0 \forall i, j \in \{1, 2, 3\}$. Now, we define the pair of matrices

$$A = \sum_{j=1}^{\mathcal{W}_A} A_j \quad \text{and} \quad B = \sum_{j=1}^{\mathcal{W}_B} B_j \quad (1)$$

generating a code of weight $\mathcal{W} = \mathcal{W}_A + \mathcal{W}_B$ with each matrix A_j and B_j being powers of x, y , or z . Working in the binary field $\mathbb{F}_2 = \{0, 1\}$, we ensure avoiding cancellation of terms

TABLE I. Best TB-QLDPC codes that we found ordered by weight $\mathcal{W}$ of the stabilizer checks. The code parameters $[[n, k, d]]$ correspond to k logical qubits, n physical qubits, and code distance d . The encoding rate $r = k/n$ is rounded to the next smallest integer value. In column $r_{\text{TB}}/r_{\text{SC}}$, we show the improvement factor in encoding rate compared to a surface code with the same code distance and number of logical qubits. Last columns indicate whether the code has toric structure, whether it can be encoded on a biplanar lattice, and the number N_L of logical circuits implementable by transversal operations and SWAP gates found via automorphism groups.

$[[n, k, d]]$	$\mathcal{W}$	k/n	$r_{\text{TB}}/r_{\text{SC}}$	Toric	Biplanar	N_L
$[[144, 2, 12]]$	4	1/72	2.0	×	✓	1
$[[30, 4, 5]]$	5	1/7	3.3	✓	✓	7
$[[30, 6, 4]]$	6	1/5	3.2	✓	✓	4
$[[30, 4, 5]]$	7	1/7	3.3	✓	?	4

by only allowing for unique terms in both A and B . In order to fully describe a TB code, we need to further define the stabilizer check matrices

$$H_X = [A|B] \quad \text{and} \quad H_Z = [B^T|A^T].$$

Following this definition, we can then define X -type and Z -type check operators being a row $v \in \mathbb{F}_2$ of H_X and H_Z , respectively. Since $[A, B] = 0$, these satisfy the stabilizer code condition requiring all X -type checks to commute with all Z -type checks, translating to $H_X H_Z^T = 0$. We emphasize that the A_j and B_j terms can be understood as introducing edges between checks and vertices on the Tanner graph of the TB code, one per term. This viewpoint is crucial in understanding physical implementations in the “Code layout” section. Further details are provided in Sec. F of the SM [24].

We then find the TB code with parameters $[[n, k, d]]$ in accordance with Lemma 1 of Ref. [19]:

$$\begin{aligned} n &= 2lm, \quad k = 2\dim[\ker(A) \cap \ker(B)], \\ d &= \min\{|v| : v \in \ker(H_X) \setminus \text{rs}(H_Z)\} \end{aligned} \quad (2)$$

with $|v|$ as the Hamming weight of vector v .

Results. In Table I, we provide an overview of our TB codes of weights 4–7. For a given distance d and number of logical qubits k , our codes require less physical qubits than surface codes which need $n = kd^2$ physical qubits [8,9]. For our codes, we also find N_L fault-tolerant logical circuits, which can be implemented via transversal operation and SWAP gates, via automorphism groups [23] (see SM [24] for explicit circuits). Note that we found even more TB codes of weight 4–7 with favorable properties, which are shown in Sec. C in the SM [24].

To characterize the error capabilities of our codes, we study the pseudthreshold p_0 . It is defined as the value satisfying the break-even relation $p_L(p) = p_{\text{unenc}}(p)$ with $p_{\text{unenc}}(p)$ being the probability that at least one unencoded qubit suffers an error. We also fit the logical error p_L with the heuristic formula [19,25]

$$p_L = p^{d_{\text{fit}}/2} \exp(c_0 + c_1 p + c_2 p^2) \quad (3)$$

where for surface codes and bicycle codes it has been observed that the fitted distance corresponds to the actual distance via $d_{\text{fit}}/2 = \lceil d/2 \rceil$ [25].

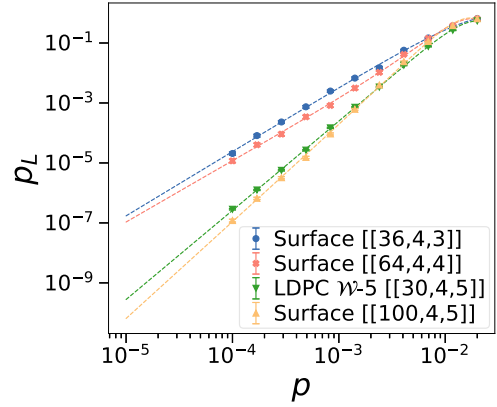


FIG. 2. Circuit-level noise for our weight-5 $[[30, 4, 5]]$ TB code and surface codes. We show logical error p_L against physical error rate p . Using Eq. (3), we find $d_{\text{fit}} = \{4.3, 4.1, 6.0, 6.4\}$, and pseudthreshold $p_0 = \{1.2 \times 10^{-3}, 2.2 \times 10^{-3}, 4.0 \times 10^{-3}, 3.6 \times 10^{-3}\}$.

We simulate our codes under circuit-level depolarizing noise, where we explicitly simulate the syndrome measurement circuit including one-qubit, two-qubit, measurement, state preparation and idling noise following Ref. [19] (see SM Sec. B [24]). To decode, a linearized noise model is constructed by simulating each error individually and constructing an effective parity check matrix, which neglects cancellations between errors [19]. To decode, we use the ordered statistics postprocessing step decoder of Refs. [26,27] where we find the most likely logical error given the measured syndrome. We study our weight-5 $[[30, 4, 5]]$ TB code under circuit-level depolarizing noise in Fig. 2. As comparison, we show surface codes with the same number of logical qubits. We find that our code has the same scaling with p as a surface code with more than three times the number of qubits.

In Fig. 3, we compare the noise suppression characteristics of our weight-5 $[[30, 4, 5]]$ TB code (Fig. 1) with the weight-6 $[[72, 12, 6]]$ code of Ref. [19] and the weight-6 $[[72, 8, 6]]$ code

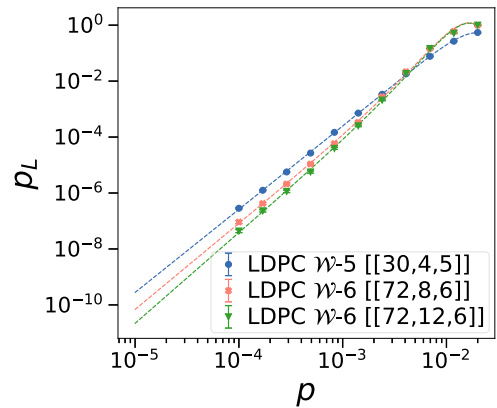


FIG. 3. Circuit-level noise for our weight-5 $[[30, 4, 5]]$ TB code (blue), the previously proposed weight-6 $[[72, 8, 6]]$ BB code of Ref. [22] (orange), and the weight-6 $[[72, 12, 6]]$ BB code of Ref. [19] (green). We show the dashed line fit with Eq. (3), where we find $d_{\text{fit}} = \{6.0, 6.1, 6.5\}$, and pseudthreshold $p_0 = \{4.0 \times 10^{-3}, 5.0 \times 10^{-3}, 6.0 \times 10^{-3}\}$.

of Ref. [22]. We find similar noise performance and scaling for all three codes, where notably our code requires less than half of the qubits to implement. We show additional results on the performance of our codes in Secs. D and E of the SM [24].

Code layout. We now discuss the physical layout needed to implement our TB-QLDPC codes where we consider biplanarity and toric layout. Biplanarity means that the Tanner graph has thickness 2, where thickness refers to the minimum number of separate two-dimensional planar layers necessary to embed the graph. Operationally, thickness tells us how many circuit boards are needed in a physical construction of the QLDPC code.

Proposition 1 (biplanar architecture). All TB-QLDPC codes of weight 4, where $A = A_1 + A_3$ and $B = B_1 + B_3$, all codes of weight 5 such that $A = A_1 + A_3$ and $B = B_1 + B_2 + B_3$, and all codes of weight 6 such that $A = A_1 + A_2 + A_3$ and $B = B_1 + B_2 + B_3$ (the case presented in Ref. [19]), or $A = A_1 + A_3$ and $B = B_1 + B_2 + B_3 + B_4$ allow for a biplanar architecture of thickness $\theta = 2$. The biplanar decomposition can be computed in time $O(n)$.

The proof is outlined in Proposition 6 in SM [24]. It implies that our weight-4 to weight-6 codes shown in Table I can be implemented in a biplanar lattice architecture (see Fig. 1), which makes them suitable candidates for near-term hardware implementations. It is currently unknown whether the weight-7 code is biplanar. Beyond biplanarity, we use the concept of a toric layout to simplify the design process in physical implementations as elaborated in Proposition 5 (SM [24]). We view the Tanner graph as embedded on a torus, allowing for edge crossings (see Fig. 1).

Definition 1 (toric layout [19]). A TB-QLDPC code has a toric layout $\iff \exists$ with positive integers μ and λ such that its Tanner graph has a spanning subgraph isomorphic to the Cayley graph of $\mathbb{Z}_{2\mu} \times \mathbb{Z}_{2\lambda}$. The parameters 2μ and 2λ determine the width and length of the toric layout for the MB code's Tanner graph.

Using the proposition below, we obtain an easy check for whether a TB code has a toric layout as further explained in SM [24]. We first define M to be the set of monomials $M = \{x^i y^j \mid i = 0, 1, \dots, l-1, j = 0, 1, \dots, m-1\}$.

Proposition 2 (TB-QLDPC toric layout criterion for arbitrary weight, generalized from Ref. [19]). A TB-QLDPC code $[\text{QC}(A, B)]$ of weight $\mathcal{W} = \mathcal{W}_A + \mathcal{W}_B$ has a toric layout $\iff \exists i, j \in \{1, \dots, \mathcal{W}_A\}, g, h \in \{1, \dots, \mathcal{W}_B\}$ such that (1) $\langle A_i A_j^T, B_g B_h^T \rangle = M$ and (2) $\text{ord}(A_i A_j^T) \text{ord}(B_g B_h^T) = lm$.

Although Ref. [19] proves Proposition 2 for the weight-6 case where $A = A_1 + A_2 + A_3$ and $B = B_1 + B_2 + B_3$, we emphasize that the proposition holds more generally because none of their argument requires that $i, j, g, h \in \{1, 2, 3\}$. In fact, we may carry out an identical argument for generic weights $A = \sum_{j=1}^{\mathcal{W}_A} A_j$ and $B = \sum_{j=1}^{\mathcal{W}_B} B_j$. As an example of the generality of Proposition 2, our weight-7 $[[30, 4, 5]]$ code was numerically confirmed to have a toric layout (see Sec. C in the SM [24]).

Our codes have $r = 2$ and thus according to our toric layout criterion can be realized on a two-dimensional Euclidean space, which is directly accessible for current experiments. In general, for multivariate group algebra of r independent variables, we obtain an analogous toric layout definition and

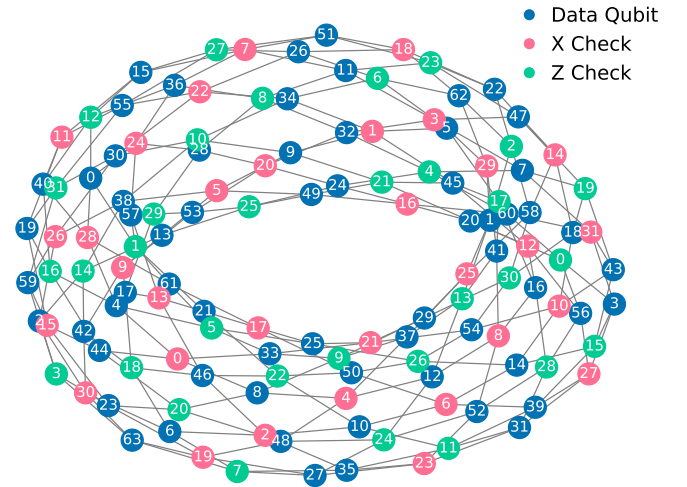


FIG. 4. Tanner graph of the weight-4 $[[64, 2, 8]]$ code, as a tangled torus with layout parameters $\mu = l = 8, \lambda = m = 4$. The usual torus topology is disrupted by a tangle on the right side of the figure.

criterion where the code can be realized on a r -dimensional torus for $\mathbb{Z}_{2\mu_1} \times \dots \times \mathbb{Z}_{2\mu_r}$. This higher-dimensional toric layout arises because each independent variable introduces an independent orthogonal direction in Euclidean space.

Proposition 3 (TB-QLDPC toric layout edge translational invariance). For any TB-QLDPC code (with Tanner graph G) of sparsity $\mathcal{W}$ with a toric layout, for any vertex v_T of a fixed type $T \in \{L, R, X, Z\}$, all of the $\mathcal{W}$ edges of v_T , including the long-range edges, have translationally invariant interaction vectors on the toric layout.

For a weight- $\mathcal{W}$ TB code, four interactions remain local, while $\mathcal{W} - 4$ interactions are nonlocal with translational symmetry. An example is shown in Fig. 1. This is facilitated by the cyclic property of H_X and H_Z , as elaborated in the SM [24]. An analogous proposition holds for MB codes with a toric layout as well, by repeating the arguments in higher spatial dimensions. Using the test in Proposition 2, we further conclude that all the weight-4 codes do not have a toric layout. However, they can possess a tangled toric layout, such as in Fig. 4 by using the tool from Ref. [28].

Intuitively, a tangled toric layout means that the code behaves as a surface code in the bulk, but with long-range edges produced by modifying the periodic boundary conditions along the horizontal and vertical edges of the square grid. We find that this gives a factor 2 improved encoding rate compared to regular toric codes. Yet, we believe the tangled layout can be easier to implement than biplanar layout for some experimental platforms, and thus could be an interesting experimental direction. In the Bravyi-Poulin-Terhal bound $kd^2 \leq cn$ [29], for the toric code the locality constant $c = 1$ saturates the bound, but in our weight-4 $[[64, 2, 8]]$ code we would need $c = 2$, verifying its enhanced performance. We provide an $O(n)$ algorithm to determine the tangling parameters (σ, τ) . This algorithm applies more generally to two-block codes, defined as CSS QLDPC codes such that $H_X = [A|B]$ and $H_Z = [B^T|A^T]$, where A and B commute and have the same size.

Proposition 4 [algorithm for weight-4 (σ, τ) -tangled toric layout parameters]. For any two-block CSS QLDPC code of sparsity 4 with equal X and Z degree per data qubit (two each), suppose it has a tangled toric layout known to have torus parameters μ and λ . Then there is an $O(n)$ time algorithm to find σ and τ .

The key idea of the algorithm is to embed a spanning subgraph onto the rectangular two-dimensional grid of size $2\mu \times 2\lambda$. The unassigned edges on the grid give us σ and τ respectively.

Discussion. In this Letter, we introduce MB-QLDPC codes as multivariable generalization of BB-QLDPC codes, which have only two variables. We study TB-QLDPC with three variables in detail and present compact codes with high encoding rates and strong noise suppression. Moreover, by optimized search we find codes with more resource-efficient parameters than corresponding surface codes. We obtain promising codes with only weights 4 and 5, which could offer a more hardware-friendly implementation compared to recently found weight-6 BB-QLDPC codes from Refs. [19,22,30]. For example, we find a highly compact weight-5 $[[30, 4, 5]]$ code which has a toric biplanar layout and a single long-range connection, which shows promise for practical implementations on current hardware. In particular, our code can be implemented with 60 physical qubits with comparable performance to surface codes of 200 qubits, as well as weight-6 codes of Refs. [19,22]. Thus, our code is especially promising when the qubit number is limited, as is the case with current quantum computers. We also found a highly compact depth-7 syndrome measurement circuit for weight-5 codes, which we find preserves the distance even under circuit-level noise and could be relevant for other weight-5 codes. We also find seven fault-tolerant logical operations via automorphism groups, which can be implemented transversally combined with SWAP gates [23]. We highlight that our code is easier to implement in practice as it has only weight-5

parity checks with only one long-range connection, compared to weight-6 checks with two long-range connections [19,22]. As long-range connection is a major bottleneck in superconducting qubit setup requiring ancillas or long-range couplers [22], our code essentially reduces the cost of this problem in half.

Further, we make progress on the properties of TB-QLDPC codes: We show that up to weight 6, all TB codes have a biplanar structure and further provide methods to check for a toric layout, which could be used to systematically search for codes with experimental-friendly layout, significantly reducing the search space of codes. Future research could combine our MB-QLDPC codes with subsystem codes [31,32] and Floquet codes [33–38] for subsystem MB bicycle codes and Floquet MB bicycle codes which could reduce the weight of check operators. A potential method to construct such codes is through Floquetification via ZX calculus [39,40]. In the future, automating the discovery of MB-QLDPC with favorable parameters, such as weight, Tanner graph thickness, encoding rate, and distance, could be achieved using machine learning techniques like reinforcement learning. Alternatively, one could study the structure of polynomials analytically to directly find MB codes with favorable parameters and layout. Finally, it would be interesting to develop adaptive MB-QLDPC codes using quantum combs [41]. Finally, it would be interesting to find efficient methods to check whether a code is biplanar, which could be done by leveraging the symmetries in the Tanner graph of QLDPC codes such as its toric layout or translation invariance.

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