

Topological quantum floating phase of dipolar bosons in an optical ladder

Henning Korbmaier^{1,*}, Gustavo A. Domínguez-Castro¹, Mateusz Łacki²,

Jakub Zakrzewski^{2,3,†} and Luis Santos^{1,‡}

¹*Institut für Theoretische Physik, Leibniz Universität Hannover, Germany*

²*Institute of Theoretical Physics, Jagiellonian University in Krakow, ul. Łojasiewicza 11, 30-348 Kraków, Poland*

³*Mark Kac Complex Systems Research Center, Jagiellonian University in Krakow, Łojasiewicza 11, 30-348 Kraków, Poland*



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Ultracold dipolar hard-core bosons in optical ladders provide exciting possibilities for the quantum simulation of anisotropic spin ladders. We show that introducing a tilt along the rungs results in a rich phase diagram at unit rung filling. In particular, for a sufficiently strong dipolar strength, the interplay between the long-range tail of the dipolar interactions and the tilting leads to the emergence of a quantum floating phase, a critical phase with incommensurate density-density correlations. Interestingly, the floating phase is topological, constituting an intermediate gapless stage in the melting of a crystal into a gapped topological Haldane phase. This novel scenario for topological floating phases in dipolar spin ladders can be investigated in ongoing experiments.

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Ultracold gases in optical lattices offer excellent possibilities for simulating spin models relevant to quantum magnetism [1]. For short-range interacting atoms, superexchange [2] results in weak nearest-neighbor spin-spin interactions, typically of Heisenberg type, which for fermions lead to long-range antiferromagnetic order [3]. Additionally, van der Waals interacting Rydberg gases naturally realize the quantum Ising model (QIM) [4]. The exquisite control available in these experiments enables the exploration of various lattice topologies and dimensionalities, including optical ladders.

Spin ladders play a prominent role in quantum magnetism, as an interesting intermediate case between one- and two-dimensions [5,6]. The interplay between leg and rung interactions results in rich physics for two-leg spin ladders, both in what concerns its highly nontrivial ground-state properties [7–9] and its dynamics [10–13]. These physics have been the focus of recent research in optical lattice experiments, including the realization of the Haldane phase using Fermi-Hubbard ladders [14], or the exploration of ballistic-to-diffusive dynamics in XX spin ladders using hard-core bosons [15].

Recent experiments with dipolar gases, including magnetic atoms [16–19], polar molecules [20–24], and Rydberg gases [25–29] have realized lattice models with strong intersite dipolar interactions, which may go well beyond nearest neighbors [30]. These experiments open exciting possibilities for

the controlled study and simulation of quantum magnetism. In particular, dipolar particles in optical ladders, formed by optical lattices or tweezers, provide a well-suited platform for the quantum simulation of fully anisotropic XXZ spin ladders.

This Letter illustrates some of these possibilities for the particular case of a square ladder at unit rung filling. The interplay between hopping, dipolar interactions, and a tilt of the ladder along the rung direction (readily possible in experiments) results in rich ground-state physics. Most interestingly, for finite tilting, the dipolar bosons develop a quantum floating phase, a critical phase that is intermediate between a crystal and a disordered system with incommensurate density-density (or equivalently spin-spin) correlations. The floating phase, originally introduced in the context of classical commensurate-incommensurate transitions in solids [31,32], has more recently been theoretically studied in various quantum systems [33–42]. Recent experiments on QIM realized with van der Waals interacting Rydberg atoms have observed the emergence of a quantum floating phase [43].

Our work shows that dipoles in ladders constitute a novel scenario in ultracold gases for the study of quantum floating phases in an anisotropic XXZ ladder. In contrast to the QIM, the floating phase in the XXZ ladder constitutes an intermediate stage in the melting of a crystalline structure into a disordered yet symmetry-protected gapped topological Haldane phase. Interestingly, the resulting floating phase, constitutes an intriguing example of a gapless yet topological phase [44–47].

(a) *Model.* We consider dipolar bosons in a two-leg ladder with L rungs on the xy plane with legs oriented along x [see Fig. 1(a)]. As in recent experiments on magnetic atoms [19], we assume strongly repulsive on-site interactions that result in the hard-core condition of maximally one particle per site. The dipole moments are oriented along a direction $\vec{e}_d$ on the yz plane forming an angle θ with the z -axis. For sufficiently deep lattices, the system is described by the extended

*Contact author: henning.korbmaier@itp.uni-hannover.de

†Contact author: jakub.zakrzewski@uj.edu.pl

‡Contact author: santos@itp.uni-hannover.de

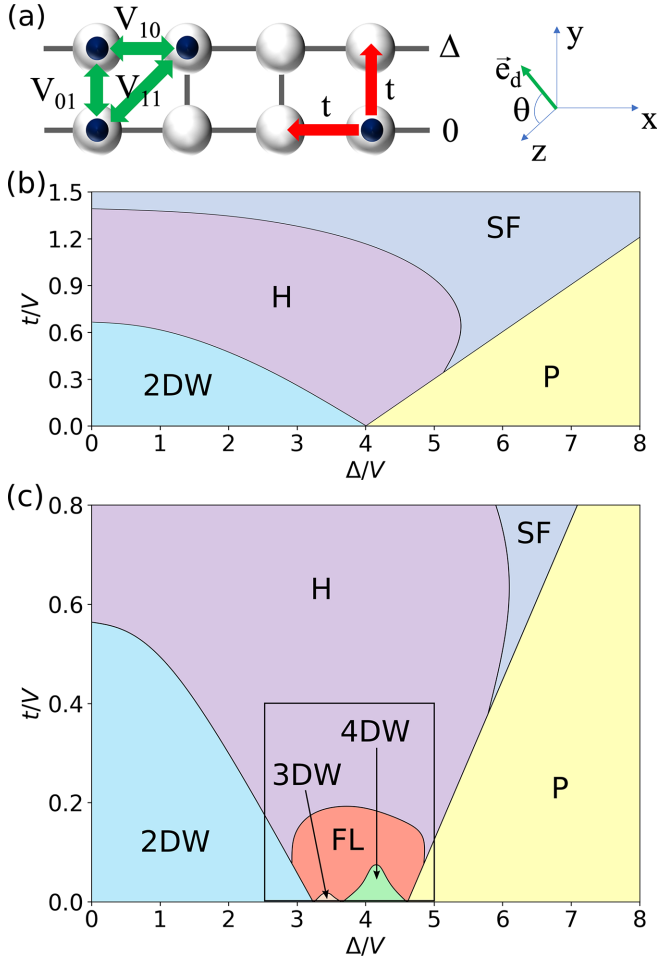


FIG. 1. (a) Scheme of the ladder system. (b) Phase diagram for the nearest-neighbor model showing the superfluid (SF) phase, the two-site period density wave (2DW), the polarized (P) phase, and the Haldane (H) phase. (c) Phase diagram including the dipolar tail (up to the sixth neighbor) showing additional density modulated phases (3DW and 4DW) and the floating (FL) phase. The rectangle shows the parameter region studied in Fig. 2. The phase boundaries are evaluated using DMRG calculations with $L = 96$ rungs, and monitoring the order parameters and correlations discussed in the text.

Bose-Hubbard Hamiltonian

$$H = -t \sum_j \left(\sum_{\alpha} \hat{b}_{j,\alpha}^{\dagger} \hat{b}_{j+1,\alpha} + \hat{b}_{j,1}^{\dagger} \hat{b}_{j,2} + \text{H.c.} \right) + \Delta \sum_j \hat{n}_{j,2} + \sum_{j,\alpha} \sum_{r>0} V_{r,0} \hat{n}_{j,\alpha} \hat{n}_{j+r,\alpha} + \sum_{j,\alpha \neq \beta} \sum_{r \geq 0} V_{r,1} \hat{n}_{j,\alpha} \hat{n}_{j+r,\beta}, \quad (1)$$

where $j \in \{1, \dots, L\}$ is the rung index, $\alpha, \beta \in \{1, 2\}$ are the two legs, $\hat{b}_{j,\alpha}$ ($\hat{b}_{j,\alpha}^{\dagger}$) is the bosonic annihilation (creation) operator at site j of leg α , and $\hat{n}_{j,\alpha} = \hat{b}_{j,\alpha}^{\dagger} \hat{b}_{j,\alpha}$ is the corresponding particle number operator. For simplicity, we assume that the hopping amplitude t is the same along the legs and the rungs (although they may be in principle different [48]). We consider leg 2 to have a different chemical potential $\Delta > 0$ compared to leg 1. This energy bias, which may be readily

realized by employing tilted ladders, plays a crucial role in the discussion below. The intersite dipolar couplings are of the form:

$$V_{i_x, i_y} = \frac{V}{(i_x^2 + i_y^2)^{3/2}} \left(1 - 3 \frac{i_y^2}{i_x^2 + i_y^2} \sin^2 \theta \right), \quad (2)$$

where $V \equiv V_{1,0} = \frac{C_{dd}}{4\pi\lambda^3}$ is the nearest-neighbor (NN) interaction along the leg, with λ the lattice spacing, $C_{dd} = \mu_0\mu$ for magnetic dipoles, with μ_0 the vacuum permeability and μ the magnetic dipole, and $C_{dd} = d^2/\epsilon_0$ for electric dipoles, with ϵ_0 the vacuum dielectric constant, and d the electric dipole moment. We are interested below in the ground-state phases at unit rung filling, i.e., when $\sum_{j,\alpha} \langle \hat{n}_{j,\alpha} \rangle = L$. Due to the hard-core constraint, Eq. (1) is equivalent to the spin-1/2 XXZ ladder:

$$H = t \sum_j \left[\sum_{\alpha} \hat{S}_{j,\alpha}^{+} \hat{S}_{j+1,\alpha}^{-} - \hat{S}_{j,1}^{+} \hat{S}_{j,2}^{-} + \text{H.c.} \right] - \Delta \sum_j \hat{S}_{j,2}^z + \sum_{j,\alpha} \sum_{r>0} V_{r,0} \hat{S}_{j,\alpha}^z \hat{S}_{j+r,\alpha}^z + \sum_{j,\alpha \neq \beta} \sum_{r \geq 0} V_{r,1} \hat{S}_{j,\alpha}^z \hat{S}_{j+r,\beta}^z, \quad (3)$$

where we introduce the spin-1/2 operators $\hat{S}_{j,\alpha}^{\pm} = (-1)^j \hat{b}_{j,\alpha}^{\dagger} \hat{b}_{j,\alpha}$ and $\hat{S}_{j,\alpha}^z = 1/2 - \hat{n}_{j,\alpha}$, and the tilting Δ acts as an effective magnetic field gradient between the two legs. The results below are obtained by combining DMRG, with up to bond dimension $\chi = 600$ and considering different L with periodic boundary conditions, and iDMRG, with up to $\chi = 300$ and considering a unit cell of 12 rungs. In both cases we employ the TeNPy library [49].

(b) *Nearest-neighbor model.* We consider first dipoles oriented along y ($\theta = \pi/2$). We neglect at this point the $1/r^3$ dipolar tail, considering an NN model, with only $V_{1,0} = V$ and $V_{0,1} = -2V$. Due to the ferro-(antiferro)-magnetic rung (leg) coupling, the XXZ ladder (3) presents for $\Delta = 0$ three phases [8,9]. Large Ising interactions (low-enough t/V) result in the stripe-Néel phase, with staggered magnetization along the legs, i.e., a density wave of period two (2DW) for the bosons, reflected in a peak at $k = \pi$ of the structure factor

$$S(k) = \frac{1}{L^2} \sum_{i,j} e^{ik(i-j)} \langle \hat{n}_i \hat{n}_j \rangle, \quad (4)$$

with $\hat{n}_j = \sum_{\alpha} \hat{n}_{j,\alpha}$. At $t = 0$, the 2DW displays a two-rung unit cell $\begin{smallmatrix} \bullet & \circ \\ \circ & \bullet \end{smallmatrix}$, with $\bullet$ ($\circ$) denoting occupied (empty) sites. Dominant exchange (large-enough t/V) leads to the XY1 phase, a Luttinger liquid (LL) corresponding to the superfluid (SF) phase of the bosons, with polynomially decaying $\langle \hat{b}_{i,\alpha}^{\dagger} \hat{b}_{j,\alpha} \rangle$. For intermediate t/V the system is in the Haldane phase, a gapped disordered phase with nonzero string order

$$\mathcal{O}_S = \lim_{|i-j| \rightarrow \infty} \langle \delta \hat{n}_i e^{i\pi \sum_{i < k < j} \delta \hat{n}_k} \delta \hat{n}_j \rangle, \quad (5)$$

with $\delta \hat{n}_j = 1 - \hat{n}_j$. This is a symmetry protected topological phase with a doubly-degenerate entanglement spectrum (ES) [50], characterized by $P_{\Lambda} = 0$, with $P_{\Lambda}^2 \equiv \sum_{n \geq 0} |\lambda_{2n} - \lambda_{2n+1}|^2$. The ES $\{\lambda_i\}$ is given by the eigenstates of the reduced density matrix $\rho_{L/2}$, obtained after cutting the ladder into two

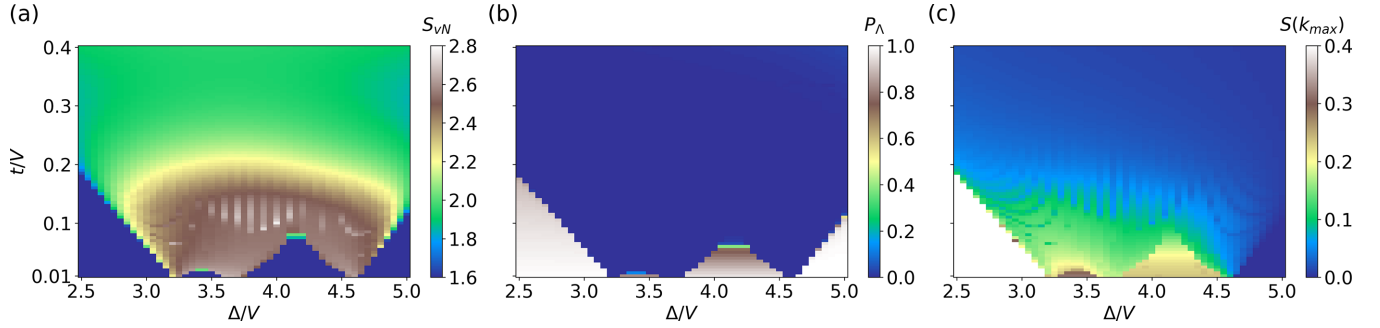


FIG. 2. (a) Entanglement entropy S_{vN} , (b) P_{Λ} and (c) the maximum of the structure factor $S(k)$ for $k \neq 0$. Results obtained by DMRG calculations for a ladder with $L = 96$ rungs.

across the legs at its center, and tracing the overall density matrix over one half.

The phases are adiabatically continued for $\Delta > 0$, see Fig. 1(b). For growing Δ , the 2DW phase shrinks to lower t/V values until disappearing at $\Delta/V = 4$. For larger Δ , the system enters the polarized (P) phase, with all particles in the lowest leg. The Haldane phase penetrates at $\Delta/V = 4$ down to $t = 0$.

(c) *Polar lattice gas.* The phase diagram of the actual polar lattice gas for $\theta = \pi/2$ is shown in Fig. 1(c). The $1/r^3$ tail results in new physics in the intermediate region in between 2DW and P, see Fig. 2. The system displays density waves with periods 3 (3DW) and 4 (4DW) sites, with unit cells at $t = 0$ $\bullet \circ \circ$ and $\bullet \circ \circ \bullet$, and peaks at $k = 2\pi/3$ and $\pi/2$ in $S(k)$, respectively.

(d) *Topological quantum floating phase.* In stark contrast to the NN model, the Haldane phase does not penetrate down to $t = 0$ through the interstitials in between the DW phases. For $t/V < 0.2$ there is a markedly different, gapless phase, characterized by a conformal central charge $c = 1$ (single component LL), see Fig. 2(b) and Fig. 4, which we have determined both from our DMRG and iDMRG calculations from the scaling of the entanglement entropy, $S_{vN} = -\text{Tr}[\rho_{L/2} \log \rho_{L/2}]$, with the system size L [51] and the correlation length [52], respectively. The presence of this phase is marked by a different entanglement entropy [Fig. 2(a)]. More relevantly, whereas the Haldane phase is a disordered phase with no peak in $S(k \neq 0)$, the observed phase presents well-defined peaks in $S(k)$ [Fig. 2(c)]. These peaks, however,

do not occur for a restricted set of discrete values linked to a well-defined density wave, but rather occur for a value of k that decreases continuously from $k = \pi$ when Δ/V increases, as shown in Fig. 2(c). This phase hence corresponds to a quantum floating phase, for which $S(k)$ presents a characteristic finite-size dependence [43], see Fig. 3. The floating phase, which characterizes the melting of the DWs at small hopping, fills the whole intermediate region between 2DW and P phases, surrounding the 3DW and 4DW phases. In contrast to the recent realization in QIM ladders using Rydberg atoms [43], in the dipolar XXZ ladder the floating phase transitions to a disordered but topological Haldane phase. Moreover, despite of being a gapless phase, the floating phase presents both a finite string order and a double-degenerate ES, see Fig. 4. The quantum floating phase constitutes hence an intriguing example of a gapless topological phase [44–47].

The origin of the floating phase for $t \ll V$ and $\Delta \simeq 4V$ is best understood by mapping the ladder into an effective 1D model with four possible states in a given site: $\bullet \equiv |\uparrow\rangle$, $\circ \equiv |\downarrow\rangle$, $\bullet \circ \equiv |\uparrow\downarrow\rangle$, $\bullet \bullet \equiv |0\rangle$ (denoted as an empty site). The 1D model is characterized by strong on-site $\uparrow\downarrow$ repulsion ($2V$), and hence for $t \ll V$ the system reduces to a $t - J$ -like model in which each site is either empty or with one particle in either $|\uparrow\rangle$ or $|\downarrow\rangle$. Moreover, strong $\uparrow\uparrow$ and $\downarrow\downarrow$ NN repulsion (V), results in antiferromagnetic order. Since the spin configuration is fixed, we focus on the distribution of empty sites, which we denote as holons. Considering for simplicity up to next-to-NN (NNN) interactions, the holons are given by the 1D hard-core

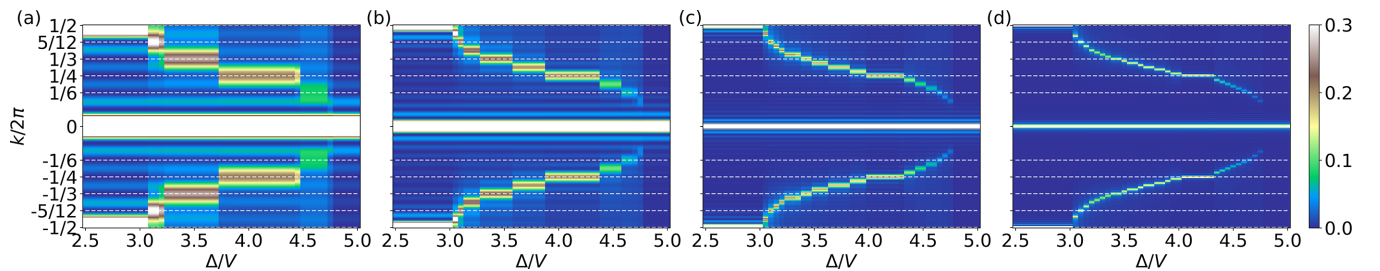


FIG. 3. Structure factor $S(k)$ for $t/V = 0.05$ as a function of Δ/V for $L = 12$ (a), 24 (b), 48 (c), and 96 (d) rungs calculated using DMRG. In addition to the plateaus at $k/\pi = 1$ (2DW) and $1/2$ (4DW), $S(k)$ displays peaks at intermediate incommensurate k values, which for growing L become a continuous function of Δ/V , a characteristic feature of the floating phase.

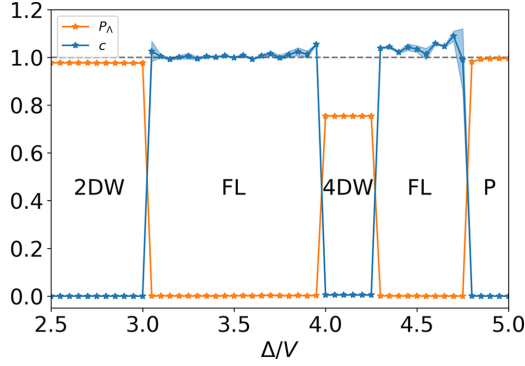


FIG. 4. P_A (see text) and central charge c obtained after a finite size scaling $\sim 1/L$ (error bars indicated in light blue). Both results are obtained for $t/V = 0.05$ from DMRG with $\chi = 600$ and $L = 96$ and $L \in \{60, 72, 84, 96\}$, respectively.

extended-Hubbard model [48]:

$$H = -t \sum_j (\hat{c}_j^\dagger \hat{c}_{j+1} + \text{H.c.}) - \mu \sum_j \hat{N}_j + V_{\text{NNN}} \sum_j \hat{N}_j \hat{N}_{j+1} - \frac{t^2}{V} \sum_j (\hat{c}_j^\dagger \hat{c}_{j+1}^\dagger + \text{H.c.}), \quad (6)$$

where $\hat{c}_j^\dagger$ creates a holon in site j , $\hat{N}_j = 0, 1$ is the holon occupation in rung j , $\mu = \frac{1}{2}(\Delta - 4V + 2V_{\text{NNN}})$, and $V_{\text{NNN}} = \frac{V}{2^{3/2}}$ is the NNN interaction. Note that we have included the second-order, but potentially relevant, term resulting from super-exchange involving a rung and a leg hopping that creates/destroys NN holons [48]. The NNN interactions prevent the creation/destruction of NN holon pairs for $t^2/V \ll V_{\text{NNN}}$. As a result, the system reduces to an extended Hubbard model with a well-defined holon filling, which presents four different ground states, a phase with unit filling of holons (the P phase), a phase with no holons (the 2DW), a density wave at half filling of holons (the 4DW), and an LL with delocalized holons (with up to NNN interactions there is no 3DW phase [48]). Doping the 2DW with delocalized holons following a commensurate-incommensurate transition into the LL phase, results in an effectively stretched separation between particles,

which translates into a peak in $S(k)$ at incommensurate k values. Hence, the LL of holons is the floating phase. When t/V increases, the second-order term eventually induces an efficient uncorrelated holon-pair creation/destruction, which results in exponentially decaying single-holon correlations. The system hence enters into an insulating phase, which as for the floating phase presents a diluted antiferromagnetic order, and hence finite $\mathcal{O}_S$. This insulating phase is thus the Haldane insulator. Finally, note that for the NN model, the chemical potential $\mu \simeq 0$ when $\Delta \simeq 4V$. In the absence of NNN interactions, even a vanishing t/V results in efficient uncorrelated creation/destruction of holon pairs, precluding the floating phase.

(e) *Dependence on the dipole orientation.* We considered above $\theta = \pi/2$. Figure 5(a) shows the results for $t = 0$ as a function of θ and Δ/V . For $\theta/\pi < 0.17$, a value close but not equal to the magic angle $\theta_M = \arcsin(1/\sqrt{3})$ at which $V_{0,1} = 0$, the 2DW transforms into a zigzag wave (ZZ2) [45,53], that corresponds to the Néel phase of XXZ spin ladders, which presents also period 2, but has a unit cell $\begin{smallmatrix} \bullet & \circ \\ \circ & \bullet \end{smallmatrix}$ at $t = 0$. Additional density waves occur, with period 3 (ZZ3) and 4 (ZZ4) with unit cell $\begin{smallmatrix} \bullet & \circ & \circ \\ \circ & \bullet & \bullet \end{smallmatrix}$ and $\begin{smallmatrix} \bullet & \circ & \circ & \circ \\ \circ & \bullet & \bullet & \bullet \end{smallmatrix}$, respectively. When θ decreases the Haldane phase shrinks in favor of the SF (see Fig. 3(b) for $\theta/\pi = 0.19$). The appearance of ZZ phases correlates with the disappearance of the Haldane and the floating phases [see Fig. 3(c)].

(f) *Conclusions.* Dipolar particles in optical ladders are an interesting novel platform for the realization of a quantum floating phase, which is gapless but topological, intermediate between a crystal and a gapped topological disordered Haldane phase. The floating phase results from the interplay between the $1/r^3$ tail of the dipolar interaction and the tilting of the ladder along the rungs. Observing the floating phase demands $V/t \gtrsim 5$, although if rung hopping is weaker than leg hopping, the floating phase may appear for significantly lower V/t values [48]. Since on-going experiments with erbium atoms have already achieved $V/t = 6$ [19], the topological floating phase in dipolar ladders is hence well within experimental reach. Our results open interesting questions concerning the phase transitions associated to

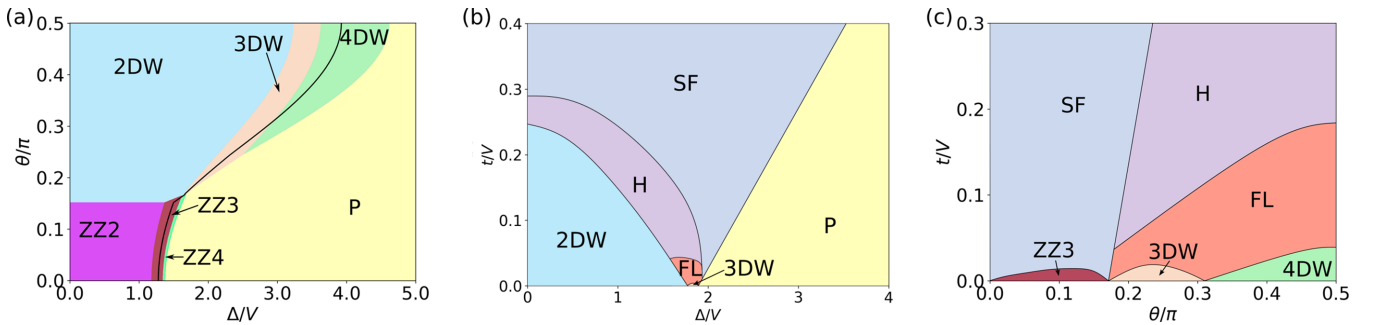


FIG. 5. (a) Phases for $t = 0$ as function of the dipole orientation θ and the tilting Δ (see text). The black solid curve indicates the values employed for the calculation of panel (c). (b) Phase diagram for $\theta/\pi = 0.19$. Contours drawn from iDMRG results. (c) Phase diagram along the black line panel (a). Contours drawn from DMRG results with $L = 96$ rungs.

crystal melting, which may present a highly nontrivial nature [34,38,39,42,54], the nonequilibrium dynamics of the dipolar ladder, and multileg ladders.

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