

Out-of-equilibrium behavior of quantum vortices: A comparison of point vortex dynamics and Fokker-Planck evolution

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(Received 21 March 2025; accepted 9 June 2025; published 22 July 2025)

An out-of-equilibrium two-dimensional superfluid relaxes towards equilibrium via a process of coarsening, driven by the annihilation of vortices with antivortices. Here we present a comparison of two different numerical models of this process, a dissipative point vortex model and Fokker-Planck evolution, across a wide range of initial configurations and levels of dissipation. We find that our dissipative point vortex model is well approximated by Fokker-Planck evolution only for very low initial energies per vortex, $E^0/N_v \lesssim -4$, when almost all vortices and antivortices are closely bound into dipoles. We observe that the dynamical critical exponent z in the dissipative point vortex model undergoes a crossover, from a roughly constant value close to 2 for $E^0/N_v \gtrsim -4$, to a value which depends explicitly on the initial conditions for $E^0/N_v \lesssim -5$.

DOI: [10.1103/PhysRevA.112.013313](https://doi.org/10.1103/PhysRevA.112.013313)

I. INTRODUCTION

When a system undergoes a rapid quench from a disordered to an ordered phase, it does not order instantly but instead relaxes towards equilibrium over time. During this relaxation, the dynamical scaling hypothesis predicts that the length scale of ordered regions increases, with later configurations statistically similar to earlier ones except for a change in global scale, characterized by a correlation length $L_c(t)$ [1]. Such phase-ordering kinetics is observed in a wide range of physical systems, including the Ising and XY magnetic models [2–5], decomposition of binary alloys [6], and the nonequilibrium behavior of quantum fluids [7]. Furthermore, the concept of universality suggests that this scaling behavior may be characterized by a limited number of exponents that reflect the dimensionality of the system and relevant conservation laws but are independent of the microscopic Hamiltonian [8]. In this paper we focus on the coarsening behavior of a two-dimensional superfluid following a quench and compare the results of two reduced models of the system across a range of initial conditions.

Following the creation of Bose-Einstein condensates (BECs) in ultracold atomic gases [9,10] and accelerated by more recent developments in imaging and manipulation of the atomic condensates [11–14], the nonequilibrium dynamics of two-dimensional superfluids has become an active area of experimental research. Studies include observations of scaling in the momentum distribution of a one-dimensional BEC [15]

and in the spatial correlations of spin excitations in a quasi-one-dimensional spinor BEC close to nonthermal fixed points [16]. There have also been signs of scaling behavior observed in vortex number decay for large-scale clusters formed in two-dimensional BECs trapped in highly oblate geometries [17]. Experiments using three-dimensional BECs have also demonstrated universal scaling in the momentum distribution far from equilibrium [18,19].

There is also a rich history of numerical investigations into coarsening behavior of a two-dimensional superfluid. Several recent studies have applied techniques based on the Gross-Pitaevskii equation (GPE) [20,21], in which the fluid is described by a single wave function ψ and scaling behavior is observed in the growth of spatial correlations in the phase following a quench [7,22,23]. Many studies have focused instead on the closely related planar XY model. In these, it is correlations between the angle of magnetic spins which demonstrate scaling over time [3–5]. In both GPE and XY models, vortices play a key role in the coarsening dynamics. In the GPE these are regions of zero density around which the phase has a wrapping of $+2\pi$ for a vortex or -2π for an antivortex, while in the XY model these are points around which the surrounding magnetic spins undergo a change in direction of $\pm 2\pi$ for one circuit. Dissipation tends to drive vortices and antivortices together, whereupon they annihilate. These annihilations drive the growth in regions of constant phase or angle, increasing the correlation length of the order parameter.

Numerical modeling of two-dimensional Bose gases has generally concluded that the dynamical scaling constant z , which characterizes the growth of the correlation length according to $L_c(t) \sim t^{1/z}$, is close to 2 when the microscopic dynamics includes some dissipation [7]. We note that it has been suggested that in some cases a logarithmic correction to the scaling equation of the form $L_c(t) \sim (t/\ln t)^{1/z}$ is required [24]. This is in accordance with the model A dynamical

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universality class [8] for a two-dimensional system with non-conserved order parameter. This type of universal scaling would be expected to hold regardless of the choice of, sufficiently nonequilibrium, initial conditions.

However, an alternative prediction has been made for quenches from below the Kosterlitz-Thouless (KT) transition. The KT transition in two-dimensional superfluids is a topological phase transition driven by vortex-antivortex unbinding [25,26] and has been observed, for example, in superfluid helium films [27,28] and trapped atomic gases [29]. Below the critical temperature, vortices and antivortices are bound in dipole pairs and the time dependence of the distribution of dipole lengths has been modeled using a Fokker-Planck equation [30–33]. This equation simulates the drift of dipoles towards shorter lengths, caused by dissipation, as well as thermally driven diffusion. A cutoff at one healing length takes the place of vortex-antivortex annihilation. Numerical and analytical studies using this method have found scaling behavior in the distribution of dipole lengths that depends explicitly on the temperature prior to the quench [31–33]. This change to the scaling laws may be a consequence of the long-range (power-law) spatial correlations below the KT transition [1,34].

The basic objects of the Fokker-Planck approach are vortex dipoles and it provides an effective model of two-dimensional superfluids at large scales, while not including compressible excitations such as phonons. An analogous, phonon-free, microscopic model is the dissipative point vortex model, in which simulating the fluid is reduced to calculating the motions and interactions of δ -function-like vortices and antivortices [35,36]. The dynamics of vortices in the GPE can be mapped onto the point vortex model [37], so long as the separation of adjacent vortices is much larger than the vortex core size, and a few studies have also observed evidence of scaling behavior using this simple model [38–40]. See Fig. 1 for an illustration of phase ordering as seen in both GPE simulations [7] (top row) and in simulation data found using the dissipative point vortex model employed in this paper (bottom row).

In this paper we compare the results of simulating an out-of-equilibrium two-dimensional superfluid using a dissipative point vortex model with the results of a Fokker-Planck equation for dipole lengths. We do this across a range of initial conditions and for different levels of dissipation. The point vortex model used in this paper does not account explicitly for thermal effects; nevertheless, the inclusion of dissipation ensures that vortices and antivortices are drawn together and, by removing any that get closer than one healing length, mimics the annihilations which drive coarsening. To test the effect of changing initial conditions, we make use of the energy per vortex. This depends on the spatial configuration of the vortices and antivortices and provides a useful characterization of the initial conditions [42]. For the lowest values, the system is below the KT transition, with the vortices and antivortices closely bound into dipoles, whereas at higher values we are above the transition and free vortices abound. At higher energies still, clusters of either vortices or antivortices would form. We find that for very low initial energies per vortex, the dipole length distributions extracted from point vortex simulations show good agreement with those found using the

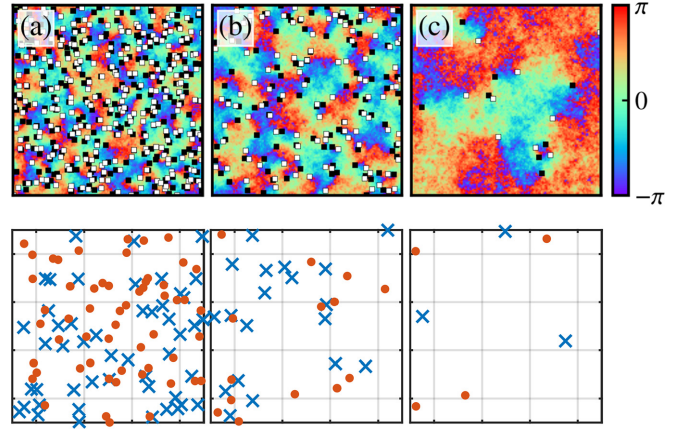


FIG. 1. Coarsening of a two-dimensional superfluid following a quench. The top row shows the results of one Gross-Pitaevskii simulation of the field ψ of a BEC in the case of zero dissipation, reproduced from [7] under CC-BY-4.0 international license [41]. The open white squares show the positions of vortices (positive circulation), the closed black squares show the positions of antivortices (negative circulation), and the background color indicates the quantum-mechanical phase. For a qualitative comparison, the bottom row shows snapshots from one evolution of the dissipative point vortex model used in this paper. Blue crosses mark the positions of vortices and closed red circles mark antivortices. Both rows show time progressing from left to right and in both models the annihilation of vortices and antivortices is the key mechanism driving the coarsening process.

Fokker-Planck equation, as expected for quenches from below the KT transition. For the same low-energy initial conditions, we also find evidence of a change in scaling behavior in the point vortex results, from an apparently universal value of $z \approx 2$ at higher energies to one which depends explicitly on the initial configuration.

The structure of this paper is as follows. In Sec. II we outline the theoretical background to the dynamical scaling hypothesis, the point vortex model, and the Fokker-Planck equation as applied to two-dimensional superfluids. In Sec. III we describe the numerical methodology for creating initial configurations of vortices and antivortices and compare them with expectations for a system of widely spaced, independent dipoles. We also simulate their dynamics using a dissipative point vortex model and calculate the evolution of the dipole length distribution using a Fokker-Planck equation. In Sec. IV we present results comparing the point vortex and Fokker-Planck methods. We also describe an analysis of the dynamical scaling constant z , which shows evidence of a change in behavior at low initial energies per vortex. In Sec. V we summarize our conclusions.

II. BACKGROUND

A. Scaling hypothesis

For a two-dimensional system undergoing a phase-ordering process, the scaling hypothesis [1] predicts that the correlation function $G(r, t)$, between two points separated by a distance r , is a scaling function. This means that changes

in time t can be replaced with a rescaling in length by $L_c(t)$, where $L_c(t)$ is a correlation length that depends on time only:

$$G(r, t) = G(r/L_c(t)). \quad (1)$$

Furthermore, when dynamical scaling holds, the dependence of correlation length on time is predicted to be a power law of the form

$$L_c \sim t^{1/z}, \quad (2)$$

where z is the dynamical critical exponent. The relevant dynamical universality class for a two-dimensional quantum fluid, with dissipation, is argued to be model A [8,43], with $z = 2$, resulting in $L_c \sim t^{1/2}$. If we make the assumption that the typical area per vortex is approximately L_c^2 , it follows that the number of vortices N_v should follow:

$$\begin{aligned} N_v &\sim t^{-2/z}, \\ &\rightarrow N_v \sim t^{-1} \text{ (for } z = 2). \end{aligned} \quad (3)$$

Logarithmic corrections have been proposed to the form of these scaling laws within the context of the planar XY model. One suggestion is that in the presence of free vortices the relation $L_c \sim t^{1/2}$ should be replaced with $L_c \sim (t/\ln t)^{1/2}$ [24,44], while another argues that the density of vortices ρ should follow $\rho \ln \rho \sim t^{-1}$ instead of $\rho \sim t^{-1}$ [5]. However, a comparison of fitting both a corrected and an uncorrected power law to data from GPE modeling [7] yielded equally close fits in both cases, but with slightly shifted values for z . In this paper we apply the uncorrected Eq. (2) when analyzing scaling behavior in our point vortex results.

B. Point vortex model description

The point vortex model is a classical model of fluid dynamics in two dimensions in which the vorticity is assumed to be concentrated in δ -function-like points and the fluid circulates around these points. Points with positive circulation are vortices and points with negative circulation are antivortices [35,36]. The dynamics of the fluid is then reduced to finding the motions of these vortices [45], each of which is advected by the local fluid velocity. The equations of motion for vortex i in an infinite plane are therefore

$$\begin{aligned} \frac{dx_i}{dt} &= -\frac{1}{2\pi} \sum_{j=1, j \neq i}^N \frac{\kappa_j (y_i - y_j)}{|\mathbf{r}_i - \mathbf{r}_j|^2}, \\ \frac{dy_i}{dt} &= \frac{1}{2\pi} \sum_{j=1, j \neq i}^N \frac{\kappa_j (x_i - x_j)}{|\mathbf{r}_i - \mathbf{r}_j|^2}, \end{aligned} \quad (4)$$

in which $\mathbf{r}_i = (x_i, y_i)$ are the coordinates of vortex i , $\mathbf{r}_j = (x_j, y_j)$ are the coordinates of all of the other vortices, and κ_j are the circulations of these other vortices. Note that for quantum fluids all circulations are either $+h/m$ or $-h/m$, where h is the Planck constant and m is the mass of an atom. The Hamiltonian for this system can be written as [46]

$$H = -\frac{1}{4\pi} \sum_{i=1, i \neq j} \kappa_i \kappa_j \ln |\mathbf{r}_i - \mathbf{r}_j|. \quad (5)$$

The point vortex model does not include compressible effects, such as vortex-sound interactions, and it also ignores the

structure of the vortex cores; however, if the separation of the vortices is much greater than the core size, it provides a good model of vortex dynamics [37]. Furthermore, by making calculations based on just vortex coordinates and signs and not on the wave function for the whole fluid, its relative computational simplicity compared to modeling with the Gross-Pitaevskii equation makes it attractive for simulating large systems.

One potential problem is that a pure point vortex model would not show coarsening, as there is no mechanism to remove vortices. We therefore mimic the annihilation of vortices with antivortices, as observed experimentally and in GPE simulations of superfluids, by removing any vortex-antivortex pair that come closer than some annihilation length. To choose this length we make a comparison with GPE simulations, such as those in Fig. 1. In the GPE the healing length ξ arises naturally as the distance over which the density recovers from zero to the bulk value; hence it is a good estimate both of the radius of a vortex core and of the annihilation length. When creating our initial conditions (see Sec. III A) we ensure that a unit distance in our point vortex simulations corresponds to ξ in the GPE simulations; we are then justified in removing any vortex-antivortex pairs that come closer than this length. By adding annihilations in this manner, we introduce into our model the most important qualitative feature of vortex-sound interactions.

For our simulations, we consider vortices in a doubly periodic square box of length L . The corresponding equations of motion and Hamiltonian then become [42]

$$\begin{aligned} \frac{d\tilde{x}_i}{dt} &= \sum_{j=1, j \neq i}^{N_v} \kappa_j \sum_{m=-\infty}^{\infty} \frac{-\sin(\tilde{y}_{ij})}{\cosh(\tilde{x}_{ij} - 2\pi m) - \cos(\tilde{y}_{ij})}, \\ \frac{d\tilde{y}_i}{dt} &= \sum_{j=1, j \neq i}^{N_v} \kappa_j \sum_{m=-\infty}^{\infty} \frac{\sin(\tilde{x}_{ij})}{\cosh(\tilde{y}_{ij} - 2\pi m) - \cos(\tilde{x}_{ij})}, \quad (6) \\ H_{\square} &= -\sum_{i=1}^{N_v} \sum_{j=1, j \neq i}^{N_v} \frac{\kappa_i \kappa_j}{2} \sum_{m=-\infty}^{\infty} \ln \left(\frac{\cosh \tilde{x}_{ij} - 2\pi m - \cos \tilde{y}_{ij}}{\cosh 2\pi m} \right) \\ &\quad - \frac{\tilde{x}_{ij}^2}{2\pi}, \end{aligned} \quad (7)$$

where κ_i is the circulation of vortex i and its coordinates (x_i, y_i) are rescaled by $2\pi/L$ to give $(\tilde{x}_i, \tilde{y}_i)$. Similarly, vortices j have circulations κ_j and their coordinates (x_j, y_j) are rescaled to $(\tilde{x}_j, \tilde{y}_j)$, so that in these units the box has a width of 2π . The $\tilde{x}_i - \tilde{x}_j$ and $\tilde{y}_i - \tilde{y}_j$ are written as $\tilde{x}_{ij}$ and $\tilde{y}_{ij}$. The time units are also scaled to give κ_i and $\kappa_j = \pm 1$. In all of these equations the term in the second sum falls rapidly with increasing m ; therefore, an excellent level of precision is achieved by truncating the limits of this sum from $m = \pm\infty$ to $m = \pm 5$.

C. Fokker-Planck description

Below the Kosterlitz-Thouless transition [25,26], vortices are bound in vortex-antivortex pairs known as dipoles. If Γ is the probability distribution of the dipole lengths, then its evolution over time may be subject to modeling using a Fokker-Planck equation [47], as explained in detail in Ref. [48], the general form of which, for a probability

distribution $\Gamma(\mathbf{R}, t)$, where $\mathbf{R}$ is a vector containing the state space variables, is

$$\frac{\partial \Gamma}{\partial t} = -\frac{\partial}{\partial R_i} (M_1^i \Gamma) + \frac{\partial^2}{\partial R_j \partial R_k} (M_2^{jk} \Gamma). \quad (8)$$

Here $M_1^i(\mathbf{R}, t)$ is called the drift vector and is given by

$$M_1^i(\mathbf{R}, t) = \lim_{\Delta t \rightarrow 0} \frac{\langle \Delta R_i \rangle(\mathbf{R}, t)}{\Delta t},$$

in which $\langle \Delta R_i \rangle$ is the change in the i th component of $\mathbf{R}$ in a time Δt , averaged over all of the particles being modeled; in our case these are the dipoles. The matrix $M_2^{jk}(\mathbf{R}, t)$ is the called the diffusion matrix and is given by

$$M_2^{jk}(\mathbf{R}, t) = \lim_{\Delta t \rightarrow 0} \frac{\langle \Delta R_i \Delta R_k \rangle(\mathbf{R}, t)}{2\Delta t}.$$

For a two-dimensional superfluid below the KT transition, the only relevant variable is the dipole length l . Therefore, by making the substitution $\mathbf{R} \rightarrow l$ it is possible to derive a one-dimensional form of the Fokker-Planck equation [31–33, 48, 49]

$$\frac{\partial \Gamma}{\partial t} = \left(\frac{1 + 2\pi K}{l} \right) \frac{\partial \Gamma}{\partial l} + \frac{\partial^2 \Gamma}{\partial l^2}. \quad (9)$$

In this form, t is the time measured in units of the diffusion time and l is the dipole length measured in units of the healing length ξ . In addition, K is the dimensionless superfluid density given by

$$K = \frac{\hbar^2 \sigma_s}{m^2 k_B T},$$

where $\hbar$ is the reduced Planck constant, σ_s is the dimensionful superfluid density, m is the atomic mass, k_B is the Boltzmann constant, and T is the temperature [50].

The first term in Eq. (9) corresponds to a drift towards smaller dipole lengths and is related to dissipation, while the second represents a random diffusion in dipole lengths. While diffusion is thermally driven in systems such as the XY model, our point vortex simulations do not explicitly include thermal effects. Nevertheless, we anticipate that the effect of many small interactions between different dipoles, as they move relative to each other, could manifest in similar stochastic changes in the dipole lengths.

Since one healing length is the distance at which a vortex and antivortex annihilate, we can write the normalization condition

$$\int_1^\infty \Gamma 2\pi l dl = \rho = \frac{N_d}{L^2}, \quad (10)$$

where N_d is the number of dipoles, L is the width of the box, and ρ is the dipole density (number of dipoles per unit area) in the system. The density ρ reduces in time as dipoles shrink to one healing length and their constituent vortices and antivortices annihilate.

In equilibrium we require $\partial \Gamma / \partial t = 0$ and so, if we have a collection of dipoles at equilibrium for some initial superfluid density K_i , it follows from Eq. (9) that the distribution of lengths must be of the form

$$\Gamma \sim l^{-2\pi K_i}. \quad (11)$$

In Ref. [32] an exact analytic solution to Eq. (9) is derived for a quench from a constant initial superfluid density K_i to a constant final density K_f in which the authors identify scaling behavior in Γ of the form $\Gamma(l, t) = F(l/t^{1/z})f(t)$, with $z = 2$. Using Eq. (10), they go on to show that, after a short time, the dipole density behaves as

$$\rho(t) \sim t^{(1-\pi K_i)}. \quad (12)$$

Rather than consider scaling in $\Gamma(l, t)$, in Sec. IV we look for scaling in the correlation function $G(r, t)$ as in Eq. (1), where we choose $G(r, t)$ to be a correlation function of the fluid velocity. This gives a scaling exponent z . If we now make the assumption that dipoles are the defects in the system, rather than free vortices, and if we further assume that the typical area per dipole is approximately L_c^2 , it follows from Eq. (2) that

$$\rho(t) \sim t^{-2/z}. \quad (13)$$

Note, however, that our scaling exponent z , measured from $G(r, t)$, is independent of these assumptions. A comparison of the above two expressions for $\rho(t)$ allows us to make the following prediction for the dynamical scaling constant for quenches below the KT transition, where we expect these assumptions to hold:

$$z = \frac{2}{\pi K_i - 1}. \quad (14)$$

This result depends explicitly on the initial conditions of the quench and differs from the model A prediction $z = 2$.

III. METHODOLOGY

A. Creating thermalized initial conditions at different vortex energies

Within the point vortex model the only energy is the incompressible kinetic energy of the fluid. This is entirely determined by the positions and circulations of the vortices through Eq. (7). Scaling by the number of vortices N_v gives the energy per vortex E/N_v ; when generating initial conditions, this is a useful parameter. Lowering E/N_v corresponds to moving vortices and antivortices closer together to form smaller dipoles and increasing it tends to form clusters of either only vortices or only antivortices. Consequently, two configurations with the same energy per vortex share similar levels of clustering and/or similar distributions of dipole lengths. Our aim is to create an ensemble of such configurations that is in microcanonical thermal equilibrium at the desired E/N_v , in the context of the point vortex model, subject to the constraint that the minimum separation of the vortices is ξ and so there are no annihilations.

Furthermore, we require these initial conditions, with specified values of E/N_v , to have the same vortex densities as the initial GPE simulations shown in the top row of Fig. 1. This ensures that one unit of distance in our model is equivalent to the healing length ξ from the GPE simulations. To create these configurations, we follow an approach similar to Refs. [51, 52] by first scattering 2069 vortices and 2069 antivortices in a doubly periodic square box of length $L = 2048$. These numbers give an initial vortex density similar to those found in the top row of Fig. 1, although our

much larger domain size increases the robustness of our later statistical analysis. We then employ a biased random-walk algorithm, in which the positions of vortices are updated at random and the updated positions accepted only if they bring E/N_v closer to the desired value. Once the desired value is achieved to within a $\pm 1\%$ tolerance, further “burn-in” steps are undertaken in order to thermalize the configuration. In each step, two vortices are randomly selected and moved by different random amounts. If E/N_v remains within the tolerance, the step is accepted. Throughout this process, changes that bring vortices closer than one healing length are rejected.

To ensure that sufficient burn-in steps are used to thermalize the configurations fully, we first test the effect of increasing the number of steps on the distribution of dipole lengths. To calculate the distribution of dipole lengths, we employ a recursive algorithm of the same form as that used in Ref. [53]. In this algorithm we perform the following steps.

- (1) We find the distance between every vortex and every antivortex.
- (2) We select the shortest distance between a vortex and an antivortex and save this value to the dipole length distribution.
- (3) We remove every distance associated with the vortex and antivortex selected in step 2 and keep the remaining distances.
- (4) We repeat steps 2 and 3 recursively until the distance between the last remaining vortex and antivortex is all that remains; this is the final value saved to the dipole length distribution.

It should be noted that this algorithm, which we use again in Sec. III B, systematically generates a distribution of dipole lengths, from shortest to longest, for any configuration of vortices. For low E/N_v the resulting distribution is easily interpreted as the lengths of the independent dipoles, but for higher E/N_v , where the vortices are randomly arranged or even clustered, its meaning is not clear. Once we have the distribution we plot it as a histogram for different numbers of burn-in steps. After a total of $O(10^5)$ steps it is clear that further steps make no difference to the distribution and we consider the configuration to be thermalized.

Using this method (with over 10^5 burn-in steps) we create ensembles of 100 sets of configurations at initial values of energy per vortex $E^0/N_v \approx \{-1, -2, -3, -4, -4.5, -5, -5.1, \dots, -5.6\}$. Figure 2 shows some examples of initial vortex configurations. Note that as E^0/N_v approaches -5.6 the vortices (blue crosses) and antivortices (red circles) are more closely bound into well-defined dipoles. For $E/N_v \approx -5.6$ the vortices and antivortices are so closely bound into dipoles that almost all proposed steps in the biased random-walk algorithm result in increasing the energy and are therefore rejected. This makes it impractical to create initial conditions at lower E^0/N_v .

Equation (11) describes a power-law relationship for the initial distribution of dipole lengths for a system of independent dipoles, derived from the condition for equilibrium of the Fokker-Planck equation. A similar relationship may also be derived using thermodynamic arguments for independent dipoles in a doubly periodic square box and we outline this argument below. A more detailed proof may be found in the Appendix.

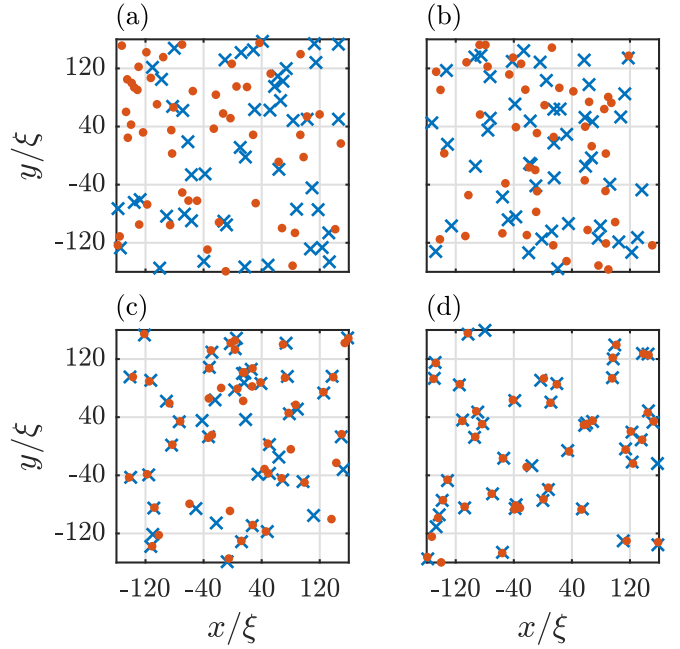


FIG. 2. Examples of initial configurations of point vortices at different energies per vortex, E^0/N_v . Blue crosses mark the positions of vortices and closed red circles mark antivortices. From (a) to (d) the chosen value of E^0/N_v falls further below zero (taking the values $-1, -3, -5$, and -5.6) and the vortices and antivortices become more closely bound into dipoles.

From Eq. (7) it can be shown numerically that a single isolated dipole of length l_i contributes an energy per vortex of $E/N_v \approx C + \ln l_i$, where C is a constant that depends on the box dimensions (in our case $C = -6.137$). By considering the different lengths, positions, and orientations of N_d dipoles that could yield an overall energy per vortex $E/N_v \leq \mathcal{E}$ (subject to the constraint that $l_i \geq 1 \forall i$), we derive an expression for the phase-space volume $\mathcal{V}(\mathcal{E})$ of such configurations. Using the relationship $\Omega(\mathcal{E}) = d\mathcal{V}/d\mathcal{E} = \beta = 1/k_B T$, where k_B is the Boltzmann constant and T the temperature, and the fact that we expect the dipole lengths to follow a Boltzmann distribution, we show that

$$p(l) \sim l^\alpha, \quad (15)$$

where $\alpha = -1 - \frac{1-1/N_d}{E/N_v + 6.137}$. In this expression $p(l)$ is the probability distribution of dipole lengths l and is related to $\Gamma(l)$ by $p(l)/l \sim \Gamma(l)$ [in order to fulfill the normalization condition in Eq. (10)]. Therefore, comparing Eq. (15) with Eq. (11), we expect

$$-2\pi K_i = \alpha - 1. \quad (16)$$

Figure 3 shows how well our thermalized initial conditions of vortices and antivortices compare with these predictions for independent dipoles. Figure 3(a) plots the exponent of the power-law fit to dipole lengths (obtained using the same recursive dipole allocation algorithm described above) against initial energy per vortex E^0/N_v . It is close to the prediction of Eq. (15) only for $E^0/N_v \lesssim -5$. Figure 3(b) shows the squared fractional error in fitting a power law to the initial dipole distribution. This likewise only shows a good fit for

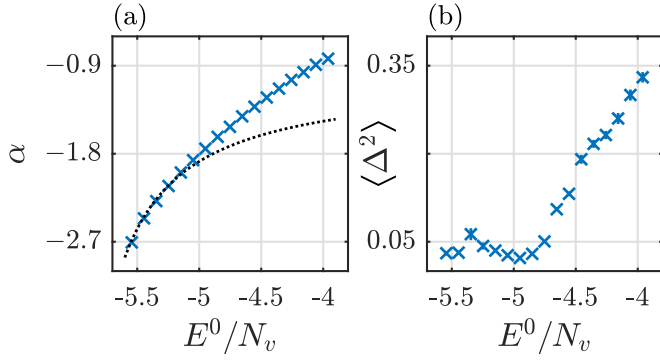


FIG. 3. Comparison of initial conditions with power-law predictions. (a) The crosses indicate the mean value of the exponent α from fitting a power law to 100 bootstrap resamplings [54] of the binned dipole lengths found from the initial conditions using a recursive dipole allocation algorithm. These are shown for a selection of values of initial energy per vortex E^0/N_v . The uncertainties in α , estimated from the bootstrap resampling, and the ranges in E^0/N_v , from the maximum to the minimum E^0/N_v in the ensemble, are too small to be visible in this plot. The dotted curve is the predicted value for α from Eq. (15) for fully independent dipoles. Our initial conditions match the prediction only for $E^0/N_v \lesssim -5$. (b) Mean-square fractional difference between the power-law fits and the numbers of dipoles in each bin. The average is taken over all of the bins and the crosses and the vertical error bars indicate the means and standard deviations from 100 bootstrap resamplings of the data.

$E^0/N_v \lesssim -5$. In short, the initial conditions we use for our point vortex modeling approximate a system of independent well-separated dipoles only for very low energies per vortex. Consequently, we do not expect the validity of the Fokker-Planck method to extend to cases with E^0/N_v much greater than -5 .

B. Dissipative point vortex dynamics

Once we have created the initial conditions, we simulate the ensuing dynamics of the vortices and antivortices using a dissipative point vortex model, with annihilations now included for vortices and antivortices that approach within one healing length of each other. In the context of this model, our thermalized initial conditions are no longer in equilibrium and instead now evolve towards a new equilibrium state in which all vortices and antivortices have annihilated.

Equation (6) gives the equations of motion for vortices in a doubly periodic square box. This gives us a nondissipative velocity $\tilde{\mathbf{v}}_i = (d\tilde{x}_i/dt, d\tilde{y}_i/dt)$ for each vortex. To include the effects of dissipation, we add a dissipative component at right angles to this to give the overall velocity $\tilde{\mathbf{v}}_i^{\text{diss}} = \tilde{\mathbf{v}}_i - \gamma \kappa_i \hat{\mathbf{z}} \times \tilde{\mathbf{v}}_i$, in which γ is the coefficient of dissipation [55]. We use a Dormand-Prince eighth-order Runge-Kutta algorithm [56] to solve these equations of motion, with an absolute tolerance of 1×10^{-8} . We simulate the annihilation of vortices and antivortices that come into too close proximity by removing those that come closer than one healing length, which in these units is $\tilde{\xi} = 2\pi/L$. Finally, we rescale lengths back to units of ξ and rescale times such that $\kappa = \pm 2\pi$. Note that in order to achieve the wholly dissipative case we set $\gamma = 1$ to

calculate $\tilde{\mathbf{v}}_i^{\text{diss}}$ and then remove the nondissipative component of velocity.

For a given initial energy per vortex E^0/N_v and dissipation γ , we carry out the above procedure for each of the 100 initial conditions in our ensemble, saving the vortex coordinates and signs at regular time steps. We then allocate all of the vortices and antivortices into dipoles, recursively from smallest to largest, using the algorithm described in Sec. III A. We do this at each time step and save the dipole lengths. The result, for each value of E^0/N_v and γ , is an ensemble of 100 sets of point vortex coordinates and corresponding values for the dipole lengths, as they evolve over time in accordance with the dissipative point vortex model.

C. Fokker-Planck evolution

To model Fokker-Planck evolution for a given E^0/N_v and γ , we first take the initial dipole lengths from the corresponding point vortex results and construct an initial probability distribution $\Gamma(l, 0)$ by binning the lengths from all 100 realizations in an ensemble into 480 equally sized bins,¹ starting at $l = 1$. We then normalize using the condition in Eq. (10), where N_d is the initial number of dipoles. Using this initial probability distribution of dipole lengths, we numerically model the Fokker-Planck evolution described by Eq. (9). We employ a second-order central-difference scheme for the spatial derivatives in the bulk, a forward-difference scheme at the lower $l = 1$ boundary, and at the upper boundary a central-difference scheme with a “ghost” point beyond the boundary calculated to ensure continuity of flux [31]. We combine this with a fourth-order Runge-Kutta method, with a relative tolerance of 1×10^{-3} , for the temporal evolution [56,57]. We use absorbing boundary conditions [47] to model the annihilation of dipoles whose length drifts below $l = 1$. Consequently, the number density of dipoles ρ , found using the integral in Eq. (10), decreases over time.

The final dimensionless superfluid density that we are quenching to in the Fokker-Planck method, K_f , is not a parameter in the point vortex model. Therefore, we treat it as an undetermined parameter to be determined by fitting. We test a range of values of K_f and select the one which gives the closest match to the point vortex results. This comparison is measured between $\Gamma_F(l, t)$, the distribution of dipole lengths over time from the Fokker-Planck evolution, and $\Gamma_P(l, t)$, the distribution of dipole lengths found by binning the lengths for the whole ensemble of 100 point vortex data sets, for a given E^0/N_v and γ . From these we calculate the squared fractional differences between the logarithms of $\Gamma_F(l, t)$ and $\Gamma_P(l, t)$ and average over all available dipole lengths and simulation times:

$$\langle \Delta_{F-P}^2 \rangle = \left\langle \left(\frac{\log_{10} \Gamma_P - \log_{10} \Gamma_F}{\log_{10} \Gamma_F} \right)^2 \right\rangle. \quad (17)$$

We then select the value of K_f that minimizes the comparison metric $\langle \Delta_{F-P}^2 \rangle$.

¹To determine a suitable number of bins, we increased the number used until numerical convergence was observed.

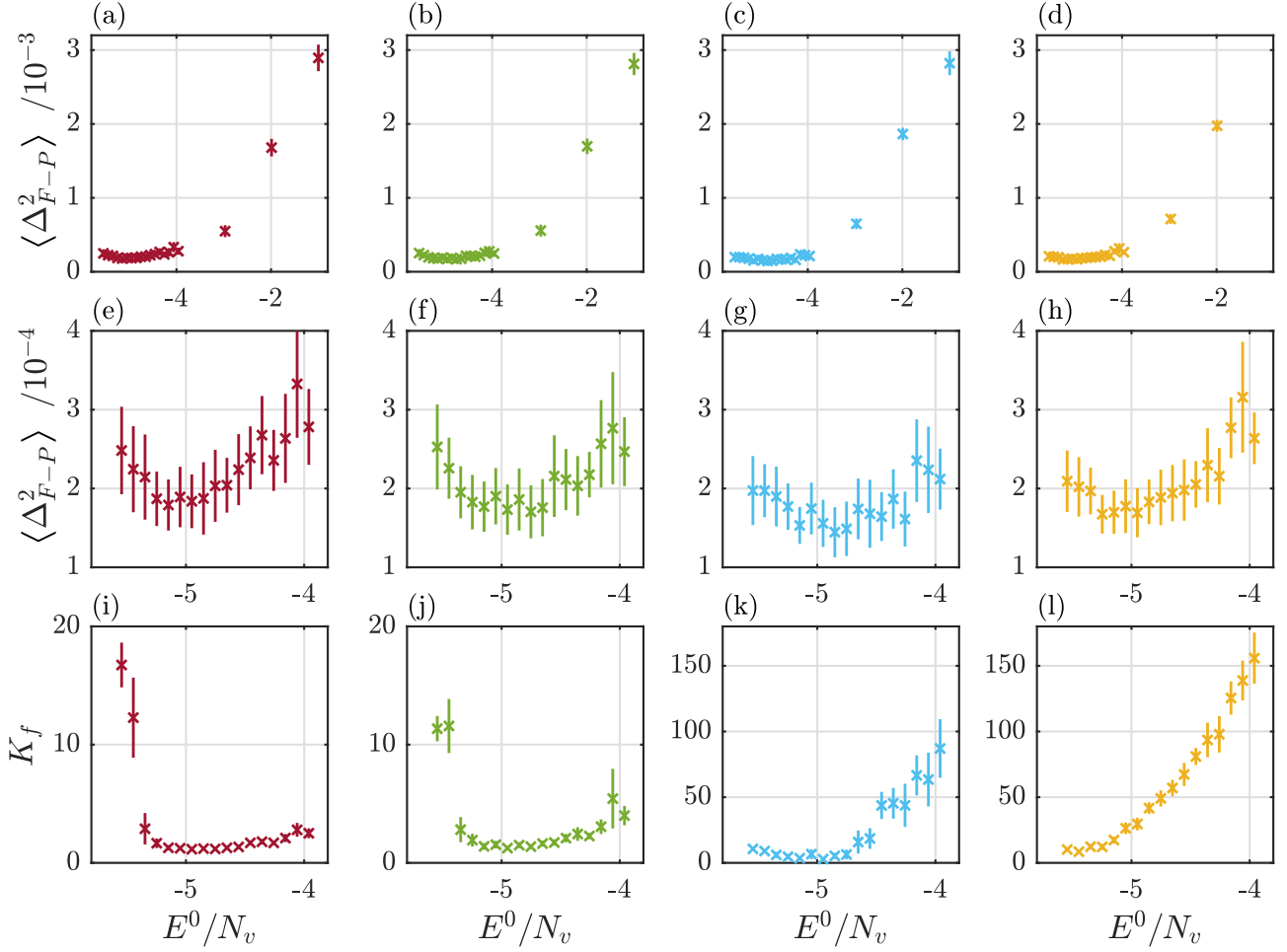


FIG. 4. Comparison of the probability distribution for dipole lengths using the Fokker-Planck equation, Γ_F , with the results of dissipative point vortex modeling, Γ_P . (a)–(d) Average over all lengths and times of the squared fractional differences between the logarithms of Γ_F and Γ_P , $\langle \Delta_{F-P}^2 \rangle$, plotted against initial energy per vortex E^0/N_v for four different levels of the dissipation $\gamma = \{1, 2^{-2}, 2^{-4}, 2^{-6}\}$, respectively. (e)–(h) Same as for (a)–(d) but just showing $E^0/N_v \lesssim -4$. (i)–(l) Values of K_f required in the Fokker-Planck equation to yield the minimum values of $\langle \Delta_{F-P}^2 \rangle$ plotted in (e)–(h). In all plots, the crosses indicate the means and the vertical error bars the standard deviations in values calculated across 100 bootstrap resamplings. Horizontal error bars (not visible) show the full range of E^0/N_v in each ensemble.

To estimate the variation in these values of K_f and $\langle \Delta_{F-P}^2 \rangle$, we employ a bootstrapping technique [54] when sampling the initial point vortex dipole lengths to create the Fokker-Planck initial distribution $\Gamma_F(l, 0)$. Rather than use each of the 100 sets of initial dipole lengths once, we instead sample these 100 sets randomly 100 times with replacement. We use this sample to create $\Gamma_F(l, 0)$. After completing the analysis described above, we then repeat, randomly resampling the initial point vortex dipole lengths to create another $\Gamma_F(l, 0)$ and so on. We then repeat this process 100 times, with a different resampling of the initial conditions each time, and take the mean and standard deviations of the values of K_f and $\langle \Delta_{F-P}^2 \rangle$ we find.

IV. RESULTS AND ANALYSIS

Figure 4 shows a selection of results comparing the evolution of the dipole length distributions using the dissipative point vortex model with the evolution using the Fokker-Planck equation. Four levels of dissipation are plotted, with

Figs. 4(a)–4(d) showing how the comparison metric defined in Eq. (17) depends on the initial energy per vortex E^0/N_v across a range from -1 down to -5.6 . It is clear that the Fokker-Planck approach closely matches the point vortex results only for a limited range of E^0/N_v , below about -4 , and that this is a consistent pattern across different levels of dissipation. Figures 4(e)–4(h) show the same data for just the region below $E^0/N_v \approx -4$ and suggest that $E^0/N_v \approx -5$ may give the closest comparison. It seems likely that part of the reason for the failure at higher E^0/N_v is because the point vortex configurations cannot be safely treated as a system of independent dipoles (see Fig. 3).

Figures 4(i)–4(l) show the value of K_f needed in the Fokker-Planck method to give the closest possible fit to the point vortex results. Here there is a difference in behavior at different levels of dissipation γ . For higher γ , K_f has a consistent, low value across a range of E^0/N_v from -4 down to about -5.3 . However, at lower dissipation K_f varies widely with a large increase in value as E^0/N_v increases above -5 .

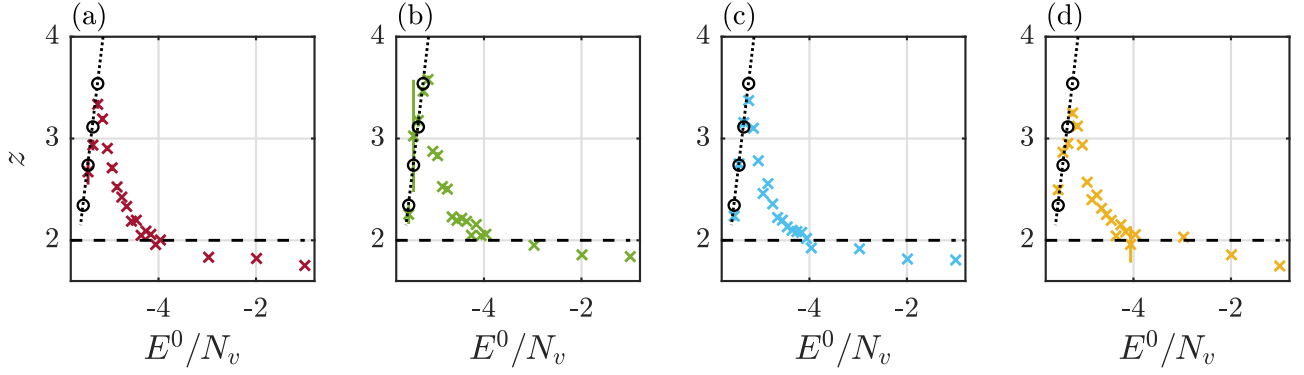


FIG. 5. (a)–(d) Dependence of the dynamical critical exponent z on initial energy per vortex E^0/N_v for four different levels of dissipation $\gamma = \{1, 2^{-2}, 2^{-4}, 2^{-6}\}$, respectively. The crosses and vertical error bars represent means and standard deviations in z found by bootstrap resampling $L_c(t)$ 100 times, conducting a power-law fit each time and finding the mean and standard deviation in z across the resamplings. This is done only using time steps for which scaling has been demonstrated by the collapse of the correlation function. In all plots the horizontal error bars (not visible) show the full range of E^0/N_v in each ensemble. The dashed horizontal line indicates $z = 2$ and the circles indicate the predicted value of z from Eq. (14), where $2\pi K_i$ is found from the magnitude of the exponent in a power-law fit to the initial dipole lengths (see Fig. 3). The dotted line shows the predicted value for z using Eq. (14), where $2\pi K_i$ is found using the exponent α from Eq. (15) and then using Eq. (16).

The role of K_f in the Fokker-Planck equation [see Eq. (9)] is to regulate the ratio of the drift term, associated with the shrinking of dipoles due to dissipation, to the diffusion term, associated with random variations in dipole length. High values of K_f imply that drift dominates diffusion. In our dissipative point vortex model there are no thermal effects; therefore, random variations in dipole lengths can only be due to interactions with the vortices in other dipoles. The very high values of K_f for some lower dissipation results suggest that a pure drift in dipole lengths is as good a fit to the dissipative point vortex model as can be achieved. Note that higher values of K_f are required for all dissipations at the very lowest E^0/N_v , suggesting that interdipole interactions become less significant as vortices are bound ever more tightly into their dipole pairs. In essence, a good fit between the Fokker-Planck and dissipative point vortex methods is seen at $E^0/N_v \lesssim -4$, but this comparison is less convincing at lower levels of dissipation. The initial configurations must be sufficiently dipole dominated for the Fokker-Planck method to apply and the dissipation must be high enough to shrink and annihilate the dipoles while interactions with other dipoles or free vortices remain relatively small and random in nature.

To test the comparison of the dissipative point vortex results with the Fokker-Planck model further, we investigate the evidence for dynamical scaling in the point vortex data and calculate the dynamical critical exponent z . Following a process detailed in Ref. [40], the fluid velocity $\mathbf{v}$ is reconstructed on a grid using the point vortex coordinates and signs at each time step. From this, the two-point correlation function $G(r, t)$ is found based on the corrected speed $v_c = |\mathbf{v}| - \langle |\mathbf{v}| \rangle$, where the angular brackets represent an average over all grid points. We then search for evidence of scaling in the time dependence of this function. Scaling is agreed to hold if the correlation functions at different time steps collapse to a single curve once rescaled by $L_c(t)$, as described in Eq. (1). In our case, we take $L_c(t)$ to be the distance at which the correlation function falls to 0.1 from a value of 1 at $r = 0$. A systematic test for this collapse, by measuring the similarity of the rescaled

correlation functions using the Mahalanobis distance [58], is used. Once the scaling region is identified in this manner, we conduct a least-squares fit of the power law $L_c \sim t^{1/z}$ to the data, for just the times within the scaling region. This approach has the benefit, over simply fitting a power law to the time dependence of vortex number (or density), in that we first verify that dynamical scaling holds for a given time window before conducting our fit to determine z . It also does not rely on the assumption that the area per dipole is approximately L_c^2 .

Figures 5(a)–5(d) show the values of z calculated in this manner for four different dissipations plotted against E^0/N_v . It is clear from these plots that above $E^0/N_v \approx -4$ the data broadly agree with the model A prediction of $z = 2$ [8]. However, as E^0/N_v falls further below this value, z increases rapidly until it meets, and then follows, the dotted line corresponding to Eq. (14). This crossover in behavior of the dynamical critical exponent is consistent across all values of dissipation tested. Note that in order to extract a value for K_i in Eq. (14) we use the value of α from a power-law fit to the initial conditions (see Fig. 3) and Eq. (16).

To investigate this crossover in more detail, we calculate the energy per vortex E/N_v at *all* time steps in the point vortex data. The results of this are plotted in Figs. 6(a)–6(d) against the number of vortices remaining, for four different levels of dissipation. These graphs show that point vortex simulations across a wide range of initial conditions, down to $E^0/N_v \approx -5.2$, converge towards similar energies per vortex as the vortex number decays. This “forgetting” of initial conditions is typical of universal scaling, with trajectories from multiple starting points converging on similar end points at late times. However, for $E^0/N_v \lesssim -5.2$, this convergence no longer occurs, with each trajectory remaining distinct throughout. This represents additional evidence of a change in scaling behavior of the dissipative point vortex model for very low initial energies per vortex.

To test if this lack of convergence in E/N_v is related to the prevalence of well-defined dipoles in the vortex

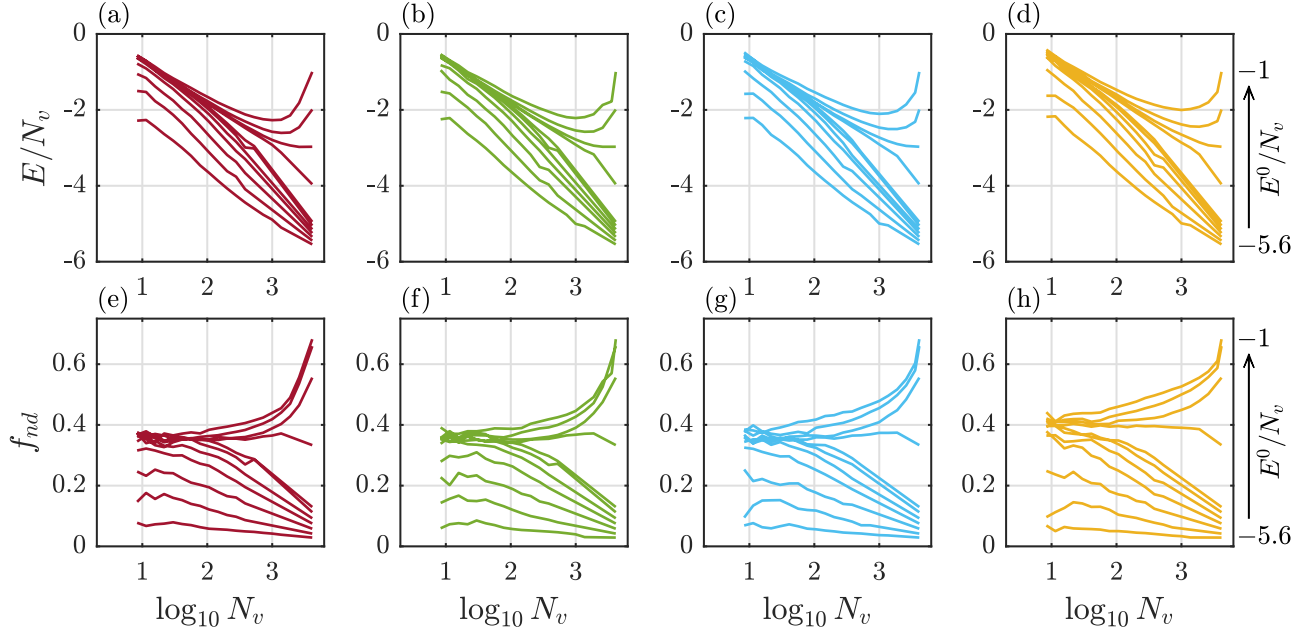


FIG. 6. Tracking the variation in vortex configurations as the number of vortices remaining decreases. (a)–(d) Variation in energy per vortex E/N_v with the number of vortices N_v . Each line plotted shows the variation in E/N_v from a given initial value, $E^0/N_v \approx \{-5.6, -5.5, -5.4, -5.3, -5.2, -5.1, -5, -4, -3, -2, -1\}$ on the right, to late times, when the number of vortices has fallen significantly, on the left. The lines show the mean value of E/N_v calculated across all time steps in all 100 runs in the ensemble which have N_v in a given range. The standard errors in E/N_v and N_v were calculated but were smaller than the thickness of the line. The plots in (a)–(d) correspond to the different levels of dissipation $\gamma = \{1, 2^{-2}, 2^{-4}, 2^{-6}\}$, respectively. (e)–(h) Variation in the fraction of vortices not allocated to dipoles, f_{nd} , calculated using the algorithm described in the final paragraph of Sec. IV. The same initial energies per vortex E^0/N_v are plotted as in (a)–(d) and start on the right of each plot in the same order (lower E^0/N_v also has lower initial f_{nd}). In all eight plots, results for $E^0/N_v > -5.2$ converge as N_v decreases while those for $E^0/N_v \lesssim -5.2$ remain distinct.

configurations, we employ an algorithm that allocates vortices and antivortices into either clusters or dipoles or neither. At each time step for each run of the point vortex results, this algorithm first searches for clusters of only vortices or only antivortices using a density-based scanning approach [59]. Like-signed vortices that are closer than the average inter-vortex distance, and have no other opposite-signed vortices nearby, are allocated to clusters so long as there are at least two of them. The remaining vortices are then searched for evidence of dipoles. A vortex and an antivortex are allocated to a dipole if they are closer than 0.8 times the average inter-vortex distance, are each others nearest-neighbor, and their next-nearest neighbors are no closer than 1.4 times the dipole length. All vortices not allocated to clusters or dipoles count as “neither.” The fraction of all vortices not allocated to dipoles, f_{nd} , is plotted against the number of vortices remaining N_v , in Figs. 6(e)–6(h), for four different levels of dissipation. The lower the initial energy per vortex E^0/N_v , the lower the initial value of f_{nd} on the right of each graph. Just as was seen in the E/N_v plots, initial conditions down to $E^0/N_v \approx -5.2$ converge (to $f_{nd} \approx 0.4$) at late times. However, those with $E^0/N_v \lesssim -5.2$ maintain higher dipole fractions throughout and do not converge.

V. CONCLUSION

In this paper we have presented numerical evidence that modeling coarsening behavior in two-dimensional superfluids

with a dissipative point vortex model tends towards the results of a Fokker-Planck equation at $E^0/N_v \lesssim -4$. We suggest that this is due to the vortices and antivortices being sufficiently closely bound into dipoles that the ensuing dynamics mimics the drift towards shorter dipole lengths, with some random variation, that is encoded in the Fokker-Planck equation. The comparison holds better for results at higher levels of dissipation, in which dipoles tend to shrink and annihilate before any more complex dynamics, caused by close interaction with other dipoles or vortices, can occur. For $E^0/N_v > -4$ the dissipative point vortex model and the Fokker-Planck equation no longer agree.

We also observed a crossover in the behavior of the dynamical critical exponent z determined from the point vortex data. For $E^0/N_v > -4$ we found a value of z that is close to 2 and independent of initial conditions, in line with expectations for model A universal dynamical scaling, whereas for $E^0/N_v \lesssim -5$, z depends explicitly on the initial conditions, as predicted using the Fokker-Planck method. This crossover is consistent across all levels of dissipation tested. Furthermore, we have observed that for $E^0/N_v \gtrsim -5.2$ all initial conditions converge to similar values of E/N_v and fraction of nondipole vortices at late times but those with $E^0/N_v \lesssim -5.2$ do not. This suggests that those configurations that are most tightly bound into dipoles at the beginning never fully “forget” their initial conditions.

Both the close agreement with the Fokker-Planck equation and the crossover in the behavior of the dynamical scaling

exponent are consistent with a KT transition in the initial vortex configurations in the region of $E^0/N_v \approx -5$. Below this transition vortices and antivortices are found only in dipole pairs. These shrink and annihilate under the effect of dissipation and the scaling behavior is dependent upon initial conditions. Above it, free vortices appear and proliferate as the energy increases, leading to more complex vortex dynamics and a tendency to universal scaling. This difference in scaling behavior below the KT transition was previously demonstrated in the context of the planar XY model [32] and we have now shown that it also occurs in the dissipative point vortex model. It may also be open to experimental verification using trapped Bose gases using techniques similar to those in Ref. [29]. Furthermore, it would be interesting to see if other systems that show KT physics, such as superconducting films [60] or two-dimensional melting [25,61], also show shifts in scaling behavior below the transition.

Note that the exact position of this crossover should not be considered to be universal. We had a fixed initial density of vortices for our simulations and it may be that the position of the crossover is sensitive to changes in this density. We have not investigated initial energies per vortex greater than zero, which tend towards clustering of like-signed vortices, as it is clear that the comparison with the Fokker-Planck description breaks down well before this. However, it would be interesting to know if the convergence of E/N_v , witnessed in Fig. 6, for $-5.2 \lesssim E^0/N_v \lesssim -1$ is maintained for highly clustered initial configurations with $E^0/N_v \gg 0$.

ACKNOWLEDGMENTS

This work was supported by a Lady Bertha Jeffreys Ph.D. Studentship. R.J.T. thanks G. A. Williams for valuable correspondence and input on dynamical scaling in quenched two-dimensional superfluids.

DATA AVAILABILITY

The data that support the findings of this article are openly available [62].

APPENDIX: PROOF OF THE POWER LAW FOR INITIAL DIPOLE LENGTHS BY THERMODYNAMIC ARGUMENTS

Here we present, in more detail, a proof of Eq. (15) from Sec. III A above. The system under consideration is one of independent, well-separated, vortex-antivortex dipoles.

Using Eq. (7), it can be demonstrated numerically that the energy per vortex of a single isolated dipole of length l in a doubly periodic square box of length L is well approximated by

$$\begin{aligned} E/N_v &\approx \ln \frac{2\pi a l}{L}, \\ E/N_v &\approx \ln \frac{2\pi a}{L} + \ln l, \end{aligned}$$

where a is determined to be 0.705 for a box of length $L = 2048$. This can be rewritten as

$$E/N_v \approx -6.137 + \ln l. \quad (\text{A1})$$

It follows that for a system of N_d well-separated dipoles of lengths l_i , the energy per vortex is given by

$$E/N_v \approx -6.137 + \frac{1}{N_d} \sum_{i=1}^{N_d} \ln l_i. \quad (\text{A2})$$

Note that Eq. (A2) implies that a surface of constant E/N_v , in an N_d -dimensional phase space, in which the dipole lengths define the coordinates of a point, obeys the condition

$$\sum_{i=1}^{N_d} \ln l_i = \text{const.} \quad (\text{A3})$$

For a given $E/N_v = \mathcal{E}$, any single configuration of dipoles can be entirely described by their lengths l_i , their orientations θ_i , and their center-of-mass coordinates (x_i, y_i) :

$$0 \leq x_i < L, \quad 0 \leq y_i < L, \quad 0 \leq \theta_i < 2\pi, \quad l_{\min} \leq l_i \leq l_{\max}.$$

Here $l_{\min}$ is the shortest dipole length and $l_{\max}$ is the longest possible dipole length, which occurs when every other dipole has length $l_{\min}$. Consequently, $l_{\max}$ is related to $\mathcal{E}$, N_d , and $l_{\min}$ via

$$\begin{aligned} \mathcal{E} &\approx -6.137 + \frac{N_d - 1}{N_d} \ln l_{\min} + \frac{1}{N_d} \ln l_{\max}, \\ \mathcal{E} &\approx -6.137 + \ln l_{\min} + \frac{1}{N_d} \ln \frac{l_{\max}}{l_{\min}}. \end{aligned} \quad (\text{A4})$$

Taking Khinchin's approach to microcanonical statistics [63], the phase-space volume $\mathcal{V}$ with $E/N_v \leq \mathcal{E}$ is given by

$$\begin{aligned} \mathcal{V}(\mathcal{E}) &= \int_{E/N_v \leq \mathcal{E}} dV \\ &= \int_{E/N_v \leq \mathcal{E}} \prod_{i=1}^{N_d} dx_i dy_i l_i dl_i d\theta_i \\ &= (2\pi)^{N_d} L^{2N_d} \int_{E/N_v \leq \mathcal{E}} \prod_{i=1}^{N_d} l_i dl_i. \end{aligned} \quad (\text{A5})$$

Consider the quantity

$$\mathcal{L}_i = \ln l_i / l_{\min} = \ln l_i - \ln l_{\min}.$$

It follows that

$$dl_i = l_i d\mathcal{L}_i, \quad l_i = l_{\min} \exp(\mathcal{L}_i).$$

Substituting these into Eq. (A5) gives

$$\mathcal{V}(\mathcal{E}) = (2\pi)^{N_d} L^{2N_d} l_{\min}^{2N_d} \int_{E/N_v \leq \mathcal{E}} \exp\left(2 \sum_{i=1}^{N_d} \mathcal{L}_i\right) \prod_{i=1}^{N_d} d\mathcal{L}_i. \quad (\text{A6})$$

Furthermore, the condition for a surface of constant E/N_v in Eq. (A3) becomes

$$\sum_{i=1}^{N_d} \mathcal{L}_i = \text{const} \quad (\text{A7})$$

and, using $\mathcal{L} = \ln l_{\max}/l_{\min}$, Eq. (A6) becomes

$$\mathcal{L}_{\max} \approx (\mathcal{E} + 6.137 - \ln l_{\min}) N_d. \quad (\text{A8})$$

For N_d dipoles the “volume” for the integral in Eq. (A6) is that of a simplex with an orthogonal corner and side length

$\mathcal{L}_{\max}$. Hence $V = \mathcal{L}_{\max}^{N_d}/N_d!$ and the “surfaces” correspond to $\sum_{i=1}^{N_d} \mathcal{L}_i = \text{const}$. It follows that

$$\frac{dV}{d\mathcal{L}_{\max}} = \frac{\mathcal{L}_{\max}^{N_d-1}}{(N_d - 1)!}.$$

If we introduce the dummy variable $\mathcal{L}' = \sum_{i=1}^{N_d} \mathcal{L}_i$, where $0 \leq \mathcal{L}' \leq \mathcal{L}_{\max}$ and each value of $\mathcal{L}'$ corresponds to a surface of constant E/N_v , then the volume element for the integral in Eq. (A6) can be replaced with

$$dV = \frac{\mathcal{L}'^{N_d-1} d\mathcal{L}'}{(N_d - 1)!}$$

and the integral can be rewritten as

$$\mathcal{V}(\mathcal{E}) = (2\pi)^{N_d} L^{2N_d} l_{\min}^{2N_d} \int_{\mathcal{L}'=0}^{\mathcal{L}'=\mathcal{L}_{\max}} \exp(2\mathcal{L}') \frac{\mathcal{L}'^{N_d-1} d\mathcal{L}'}{(N_d - 1)!}. \quad (\text{A9})$$

From the fundamental theorem of calculus, it follows that

$$\frac{d\mathcal{V}}{d\mathcal{L}_{\max}} = (2\pi L^2 l_{\min}^2)^{N_d} \exp(2\mathcal{L}_{\max}) \frac{\mathcal{L}_{\max}^{N_d-1}}{(N_d - 1)!}.$$

We are interested in the quantity $\Omega(\mathcal{E}) = d\mathcal{V}/d\mathcal{E}$, since $d \ln \Omega/d\mathcal{E} = \beta = 1/k_B T$. Using the chain rule

$$\Omega(\mathcal{E}) = \frac{d\mathcal{V}}{d\mathcal{L}_{\max}} \frac{d\mathcal{L}_{\max}}{d\mathcal{E}}$$

and noting that, from Eq. (A8), $d\mathcal{L}_{\max}/d\mathcal{E} = N_d$, we get

$$\Omega(\mathcal{E}) = N_d (2\pi L^2 l_{\min}^2)^{N_d} \exp(2\mathcal{L}_{\max}) \frac{\mathcal{L}_{\max}^{N_d-1}}{(N_d - 1)!}.$$

Hence

$$\begin{aligned} \ln \Omega &= \ln N_d + N_d \ln (2\pi L^2 l_{\min}^2) + 2\mathcal{L}_{\max} \\ &\quad + (N_d - 1) \ln \mathcal{L}_{\max} - \ln(N_d - 1)! \end{aligned}$$

and so

$$\beta = \frac{d \ln \Omega}{d\mathcal{E}} = \frac{d \ln \Omega}{d\mathcal{L}_{\max}} \frac{d\mathcal{L}_{\max}}{d\mathcal{E}} = \left(2 + \frac{N_d - 1}{\mathcal{L}_{\max}}\right) N_d. \quad (\text{A10})$$

Using Eq. (A8) to replace $\mathcal{L}_{\max}$ with an expression containing the energy per vortex $\mathcal{E}$ gives

$$\beta = 2N_d + \frac{N_d - 1}{\mathcal{E} + 6.137 - \ln l_{\min}}.$$

This equation gives an unambiguous definition of a thermodynamic temperature for a system of point vortex dipoles in a doubly periodic square box of length $L = 2048$. It simplifies further in our case as we take $l_{\min} = 1$ (since a vortex and antivortex annihilate at a distance of one healing length); hence

$$\beta = 2N_d + \frac{N_d - 1}{\mathcal{E} + 6.137}. \quad (\text{A11})$$

This definition fails if $\mathcal{E}$ falls to -6.137 ; this is because all of the dipoles would be of length 1 at this point and would annihilate.

To get a distribution of dipole lengths, we note that from Eq. (A1) each dipole contributes an energy per vortex of

$$\varepsilon_i = \frac{1}{N_d} (-6.137 + \ln l_i)$$

and has a degeneracy given by the product of the area its center could occupy with the circumference of the circle described by all 2π possible orientations, i.e.,

$$g_i = L^2 \times 2\pi l_i = 2\pi L^2 l_i.$$

We expect the probability distribution of dipole lengths to go as $p(l_i) \sim g_i \exp(-\beta \varepsilon_i)$. From this, following a few lines of algebra, it can be shown that

$$p(l_i) \sim l_i^{(1-\beta/N_d)}. \quad (\text{A12})$$

In other words, $p(l) \sim l^\alpha$, with the exponent α given by $1 - \beta/N_d$. Substituting the expression for β from Eq. (A11) gives

$$\alpha = 1 - 2 - \frac{N_d - 1}{N_d(\mathcal{E} + 6.137)} = -1 - \frac{1 - 1/N_d}{\mathcal{E} + 6.137}. \quad (\text{A13})$$

Note that the probability distribution $p(l)$ is related to Fokker-Planck distribution $\Gamma(l)$, from Eq. (9), by $p(l)/l \sim \Gamma(l)$, in order to fulfill the normalization condition in Eq. (10). Therefore, we expect

$$-2\pi K_i = \alpha - 1. \quad (\text{A14})$$

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