

Thermal screening and critical scaling of quantum energy teleportation in a harmonic chain

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We develop a finite-temperature Gaussian-state formulation of quantum energy teleportation in the one-dimensional harmonic chain. For Gibbs states, the optimized measurement-feedback protocol reduces to thermal two-point functions. In the single-site protocol, the extracted energy is governed by a single correlator, which makes the thermodynamic and near-critical limits analytically tractable. At fixed finite temperature, the extracted energy is exponentially screened with distance and is controlled by a thermal correlation length for both critical and noncritical α . In the zero-temperature critical limit taken after the thermodynamic limit, we derive the exact asymptotic law $E_{\text{ext}} \sim d^{-4}$. Numerical results confirm both regimes and resolve the crossover responsible for the apparent drift of the effective decay exponent at intermediate distances. We also analyze squeezed Gaussian measurements and show that they modify the extraction prefactor: p squeezing enhances the extracted energy, whereas q squeezing suppresses it without changing the large-distance scaling.

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I. INTRODUCTION

Quantum energy teleportation (QET) is a protocol in which a local measurement injects energy into a correlated many-body system, classical communication transfers the measurement outcome, and a distant conditional local operation extracts positive energy from another site [1–3]. The protocol does not transport usable energy during the classical-communication step. Instead, it uses local information together with preexisting correlations of the state. In this way, QET links quantum information, quantum thermodynamics, and many-body physics.

A large part of the literature has focused on ground-state protocols [4–9]. At finite temperature and for passive mixed states, however, detailed studies have mainly concentrated on few-body systems. In particular, Gibbs-state QET was analyzed for two spins by Frey *et al.* [10] and for a three-spin Ising chain by Trevison and Hotta [11]. Alongside these studies, our earlier work [12] examined finite-temperature energy extraction in a minimal two-qubit XY model, focusing on its relation to entanglement, discord, and the passivity of the initial state. This line of work clarified that nonzero temperature does not immediately destroy the protocol and that useful resources can survive in passive mixed states. Later developments connected QET with strong local passivity, including the original strong-local-passivity formulation [13], the experimental activation of strong local passive states with quantum information [14], the construction of strong local passive states in the minimal QET model [15], and strong-QET protocols [16]. Recent reviews showed that QET is now being discussed in a broader quantum-thermodynamic and experimental context [17].

QET has also been developed as an operational probe of correlated quantum matter. In particular, it has been linked to critical behavior in quantum field theory and to global and topological order, including symmetry-protected topological phases [8,9]. Related studies have explored feedback-induced entropy change in the Heisenberg chain, impurity effects in Kondo settings, and nonlocal correlations in the Kitaev spin liquid [18–20].

The present work moves from these few-body spin settings to a spatially extended bosonic system, where both a controlled thermodynamic limit and analytically tractable long-distance behavior are available. The one-dimensional harmonic chain is a natural model for this purpose. It has a gap-closing critical point, a well-defined thermodynamic limit, and Gaussian Gibbs states that permit an explicit treatment of both the state and the local operations. The same quadratic Hamiltonian also describes, after canonical rescaling, a chain of coupled linear microwave resonators, with node flux and charge playing the roles of canonical coordinate and momentum [21]. In this language, the regime $\alpha \rightarrow 1$ corresponds to weak local pinning relative to intersite coupling, where the gap closes, the correlation length grows, and long-wavelength correlations become dominant.

Earlier work by Nambu and Hotta established ground-state QET in the harmonic chain and studied both single-site and block-measurement settings [22]. Their numerical analysis reported an apparent critical decay close to $|\langle H_B \rangle| \sim 2 \times 10^{-3} d^{-3.6}$ at large distance, and no analytic derivation of that exponent was given [22]. The same work also explored enlarged measured regions on Alice's side, again at zero temperature and mainly numerically [22].

The main results of the present work are as follows. For translationally invariant Gibbs states of the harmonic chain, the optimized Gaussian QET protocol reduces, in the single-site setting, to thermal two-point correlators. This reduction makes the ordered thermodynamic and critical limits analytically transparent. At fixed finite temperature, the

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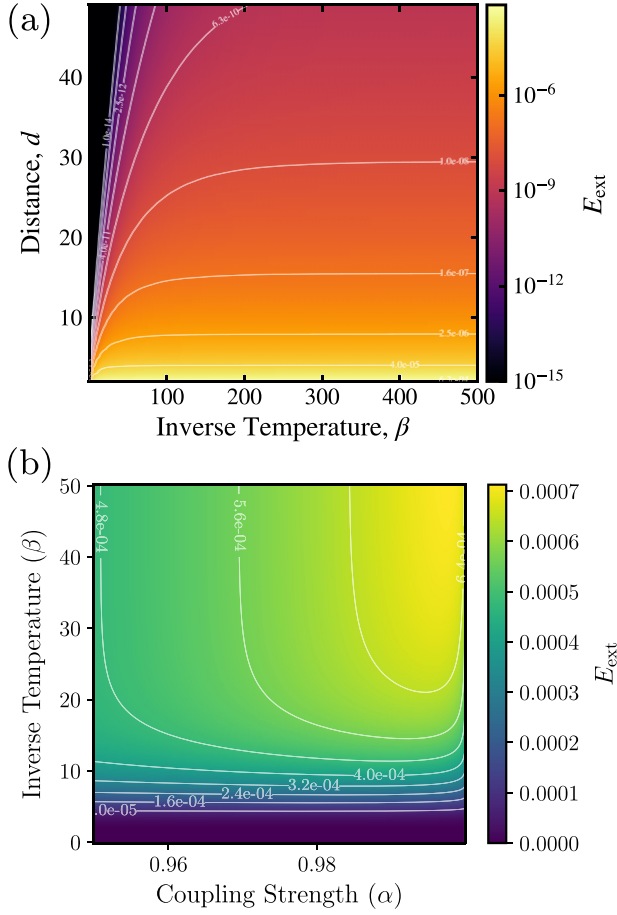


FIG. 1. Density plots of the optimized extracted energy E_{ext} in Eq. (21) as a function of the indicated control parameters. (a) Dependence on the inverse temperature β and Alice-Bob separation d , evaluated at $N = 300$, $\alpha = 0.9999$, and $\Omega = 1$. (b) Dependence on the coupling strength α and inverse temperature β at fixed separation $d = 2$, evaluated at $N = 300$ and $\Omega = 1$.

extracted energy is exponentially suppressed at large distance and is controlled by a thermal correlation length. In the zero-temperature critical limit taken after the thermodynamic limit, the relevant coupling function admits a closed form and yields the asymptotic behavior $E_{\text{ext}} \sim d^{-4}$. We further show that squeezed Gaussian measurements modify the extraction prefactor but not the asymptotic scaling. In this way, the harmonic-chain QET protocol becomes an analytically controlled probe of thermal Gaussian correlations.

This paper is organized as follows. Section II introduces the harmonic-chain Hamiltonian, its physical realizations, and the two-point correlators of the Gibbs state. Section III derives the optimized extracted energy for an arbitrary Gaussian block measurement and presents parameter surveys of the extracted energy (Fig. 1). Section IV focuses on the single-site protocol and establishes the key identity $J_q = h_d$. Section V derives the exact Matsubara series at finite temperature and the closed-form d^{-4} law at zero temperature, with Figs. 2 and 3 providing quantitative benchmarks. Section VI presents numerical diagnostics of the crossover between algebraic and exponential regimes (Fig. 4). Section VII analyzes squeezed Gaussian measurements and shows that squeezing modifies

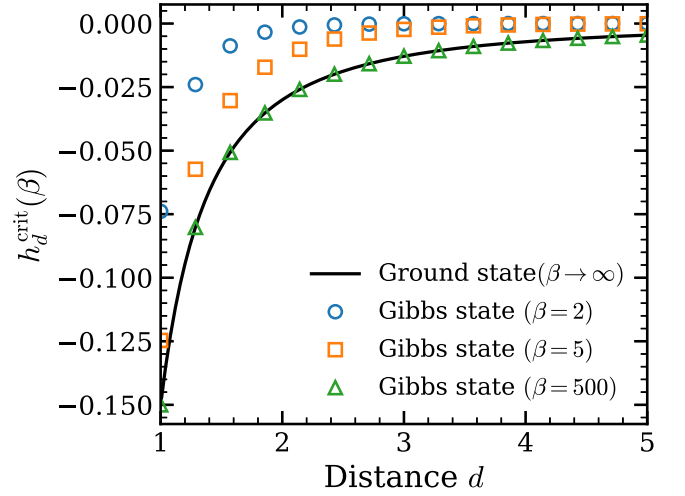


FIG. 2. Critical thermal coupling $h_d^{\text{crit}}(\beta)$ in the thermodynamic limit. Symbols: finite-temperature results from Eq. (31) at selected values of β . Solid line: exact zero-temperature result from Eq. (36). At finite β , thermal screening suppresses the coupling at large distance; as $\beta \rightarrow \infty$, the curves converge to the ground-state profile.

the extraction efficiency without changing the large-distance scaling class (Figs. 5 and 6). Section VIII discusses the results and outlines open directions.

II. MODEL AND PHYSICAL SETTING

A. Pinned phononic chain

We begin from a standard spring-mass chain with on-site pinning,

$$H_{\text{sm}} = \sum_j \left[\frac{P_j^2}{2M} + \frac{K_0}{2} u_j^2 + \frac{K}{2} (u_{j+1} - u_j)^2 \right]. \quad (1)$$

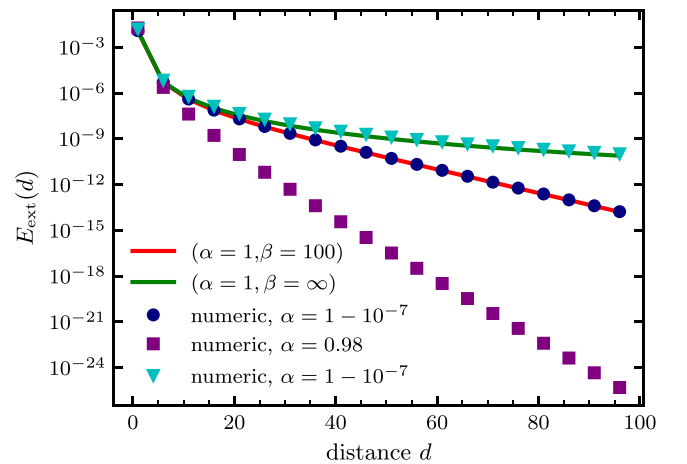


FIG. 3. Extracted energy $E_{\text{ext}}(d)$ in the thermodynamic regime. Solid curves: analytical critical results ($\alpha = 1$) for the Gibbs state at finite temperature and for the ground state. Symbols: numerical data for $N = 2^{15}$ in the near-critical regime ($\alpha = 1 - 10^{-7}$, circles) and in the gapped phase ($\alpha = 0.98$, squares). The near-critical data are nearly indistinguishable from the $\alpha = 1$ curves at this scale, whereas the gapped data exhibit a substantially faster decay.

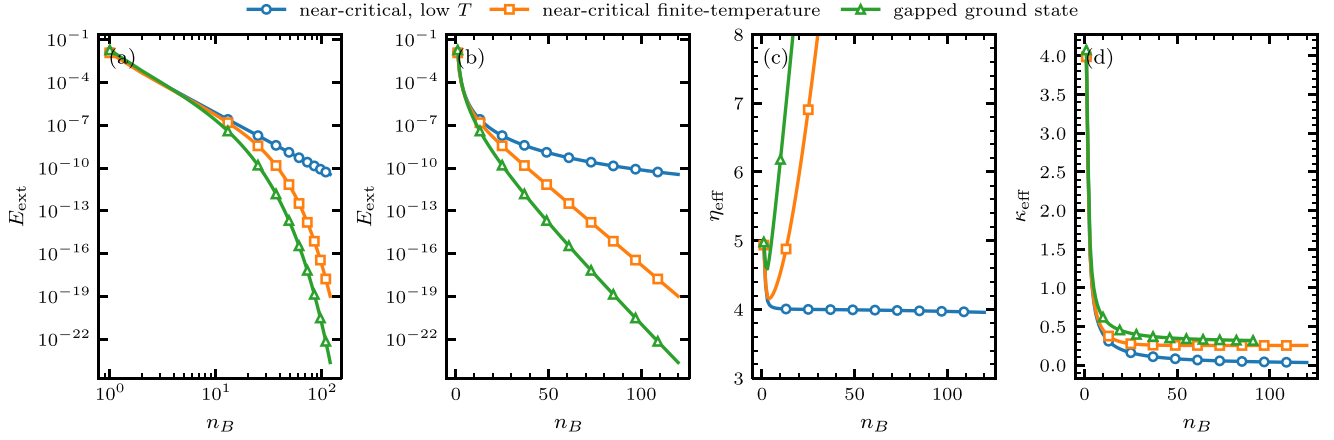


FIG. 4. Distance-scaling diagnostics for the extracted energy $E_{\text{ext}}(n_B)$. The three curves correspond to near-critical low temperature (blue), $(N, \alpha, \beta, \Omega) = (2000, 1 - 10^{-9}, 10^5, 1)$; near-critical finite temperature (orange), $(N, \alpha, \beta, \Omega) = (2000, 1 - 10^{-9}, 70, 1)$; and the gapped ground-state regime (green), $(N, \alpha, \beta, \Omega) = (2000, 0.99, 10^5, 1)$. (a) shows the data on logarithmic axes, testing algebraic decay, while (b) shows the same data on semilogarithmic axes, testing exponential decay. (c) shows the effective logarithmic exponent, and (d) shows the local attenuation rate $\kappa_{\text{eff}}(n_B) = -d \log E_{\text{ext}}/dn_B$. The log is defined for base e .

Expanding the elastic term gives

$$H_{\text{sm}} = \sum_j \left[\frac{P_j^2}{2M} + \frac{K_0 + 2K}{2} u_j^2 - K u_j u_{j+1} \right]. \quad (2)$$

Define the characteristic frequency

$$\Omega_0 := \sqrt{\frac{K_0 + 2K}{M}} \quad (3)$$

and dimensionless canonical variables

$$q_j := \sqrt{M\Omega_0} u_j, \quad p_j := \frac{P_j}{\sqrt{M\Omega_0}}, \quad [q_j, p_k] = i\delta_{jk}. \quad (4)$$

Then

$$H_{\text{sm}} = \frac{\Omega_0}{2} \sum_j [p_j^2 + q_j^2 - \alpha q_j q_{j+1}], \quad \alpha = \frac{2K}{K_0 + 2K}.$$

Measuring energies in units of Ω_0 yields the dimensionless Hamiltonian

$$H = \frac{1}{2} \sum_{j=1}^N (p_j^2 + q_j^2 - \alpha q_j q_{j+1}) \quad (5)$$

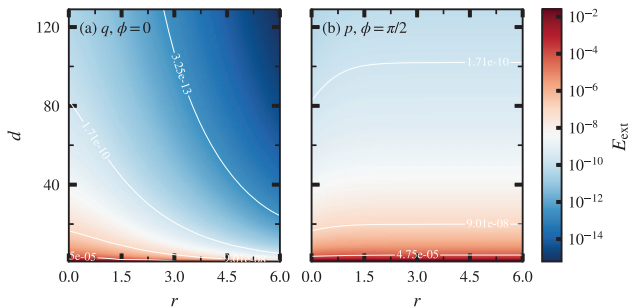


FIG. 5. Density plots of the extracted energy E_{ext} versus squeezing parameter r and separation d for $N_{\text{sys}} = 500$, $\beta = 10^5$, $\Omega = 1.0$, and $\alpha = 1 - 10^{-13}$. (a) corresponds to the q -squeezed measurement, and (b) corresponds to the p -squeezed measurement. The white lines indicate contours of constant extracted energy.

with periodic boundary conditions. The parameter range $0 < \alpha < 1$ corresponds to a gapped pinned chain. The critical point $\alpha = 1$ is the gap-closing limit.

B. Microwave-resonator interpretation

The same quadratic form arises in a chain of linear microwave resonators. For node fluxes Φ_j and conjugate charges Q_j , a capacitively or inductively coupled quadratic circuit has the generic form

$$H_{\text{res}} = \sum_j \left[\frac{Q_j^2}{2C} + \frac{\Phi_j^2}{2L} + \frac{(\Phi_{j+1} - \Phi_j)^2}{2L_c} \right]. \quad (6)$$

After the substitutions $u_j \leftrightarrow \Phi_j$, $P_j \leftrightarrow Q_j$, $M \leftrightarrow C$, $K_0 \leftrightarrow 1/L$, and $K \leftrightarrow 1/L_c$, Eq. (6) reduces by the same canonical rescaling to Eq. (5). The present QET setting can therefore be interpreted either as a universal bosonic low-energy theory or as an effective description of a circuit-QED bosonic lattice [21].

C. Gibbs state and local energy

The initial state is the Gibbs state

$$\rho_\beta = \frac{e^{-\beta H}}{\text{Tr}(e^{-\beta H})}. \quad (7)$$

For $0 < \alpha < 1$, H is a positive quadratic Hamiltonian. A linear canonical transformation diagonalizes it into independent harmonic oscillators, whose thermal states are Gaussian. Since Gaussianity is preserved under linear canonical transformations, the Gibbs state ρ_β is also Gaussian. Its first moments vanish, and the symmetrized q - p covariance vanishes in equilibrium. Consequently, the state is completely specified by its q - q and p - p covariance functions. See Appendixes A and B for complete derivations. In the thermodynamic limit, we denote these functions by

$$g_n(\beta, \alpha) = \langle q_j q_{j+n} \rangle, \quad (8)$$

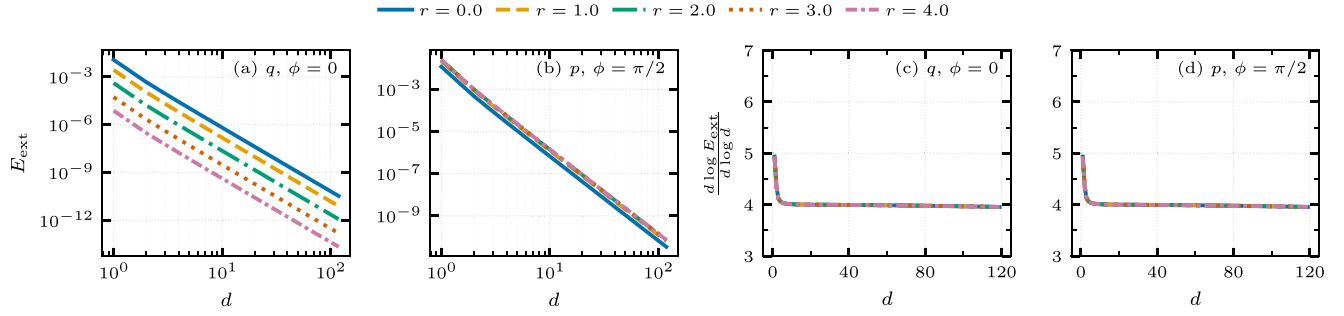


FIG. 6. Distance dependence of the extracted energy E_{ext} for squeezed Gaussian measurements at zero temperature. The system parameters are $N_{\text{sys}} = 2000$, $\beta = 10^5$, $\Omega = 1.0$, and $\alpha = 1 - 10^{-13}$. (a) and (b) show E_{ext} versus the separation d on a log-log scale for different squeezing parameters $r = 0, 1, 2, 3, 4$. (a) corresponds to the q -squeezed measurement, while (b) corresponds to the p -squeezed measurement. (c) and (d) display the local logarithmic slope, $d \log E_{\text{ext}} / d \log d$, for the same two measurement choices. In both cases the large-distance behavior approaches the same asymptotic exponent, close to -4 , while squeezing mainly changes the overall extraction amplitude. The log is defined for base e .

$$h_n(\beta, \alpha) = \langle p_j p_{j+n} \rangle, \quad (9)$$

which take the integral forms

$$g_n(\beta, \alpha) = \frac{1}{2\pi} \int_0^{2\pi} \frac{d\theta}{2\omega_\alpha(\theta)} \coth\left(\frac{\beta\omega_\alpha(\theta)}{2}\right) \cos(n\theta), \quad (10)$$

$$h_n(\beta, \alpha) = \frac{1}{2\pi} \int_0^{2\pi} \frac{\omega_\alpha(\theta)}{2} \coth\left(\frac{\beta\omega_\alpha(\theta)}{2}\right) \cos(n\theta) d\theta, \quad (11)$$

with $\omega_\alpha(\theta) = \sqrt{1 - \alpha \cos \theta}$.

Here g_n and h_n are, respectively, the position and momentum covariance functions at separation n . The nonlocal distance dependence of the single-site protocol is ultimately carried by h_d , while the on-site variances g_0 and h_0 determine the local noise and response prefactors. A normal-mode derivation of these covariance functions is given in Appendix B.

For later use we define the local Hamiltonian contribution probed by Bob's displacement,

$$H_B = \frac{1}{2} p_{n_B}^2 + \frac{1}{2} q_{n_B}^2 - \frac{\alpha}{2} q_{n_B} (q_{n_B-1} + q_{n_B+1}) - \frac{\epsilon}{2}, \quad (12)$$

where the constant shift ϵ cancels in the extracted-energy difference and will be left implicit. We also introduce

$$Q_B := q_{n_B} - \frac{\alpha}{2} (q_{n_B-1} + q_{n_B+1}). \quad (13)$$

With the equilibrium correlators and the operator H_B fixed, the QET problem reduces to how Alice's measurement noise and Bob's conditional displacement probe these thermal correlations. We now formulate the Gaussian measurement-and-feedback protocol for a general measured block.

III. GAUSSIAN QET PROTOCOL IN THE GIBBS STATE

Alice measures a block A of $m = 2\ell + 1$ sites with a coherent-state positive operator-valued measure (POVM). On each measured site she defines

$$\hat{b} = \sqrt{\frac{\Omega}{2}} \hat{q} + \frac{i}{\sqrt{2\Omega}} \hat{p}, \quad (14)$$

where $\Omega > 0$ is the measurement width. The coherent states $|X, P\rangle$ satisfy $\hat{b}|X, P\rangle = c|X, P\rangle$, with $c = \sqrt{\Omega/2} X + iP/\sqrt{2\Omega}$. The block POVM is

$$\Pi_A(\tilde{X}, \tilde{P}) = \frac{1}{(2\pi)^m} \prod_{j=1}^m |X_j, P_j\rangle \langle X_j, P_j|, \quad (15)$$

which satisfies $\int d\tilde{X} d\tilde{P} \Pi_A(\tilde{X}, \tilde{P}) = \mathbb{I}$. The associated Kraus operator is taken as $M_A = (2\pi)^{-m/2} \prod_j |X_j, P_j\rangle \langle X_j, P_j|$.

Alice then transmits the measurement outcomes $(\tilde{X}, \tilde{P})$ to Bob, and upon receiving the measurement outcome, he applies the local displacement

$$U_B(\tilde{X}, \tilde{P}) = \exp \{i[(\boldsymbol{\theta} \cdot \tilde{\mathbf{P}})q_{n_B} - (\boldsymbol{\varphi} \cdot \tilde{\mathbf{X}})p_{n_B}]\}, \quad (16)$$

where $\boldsymbol{\theta}, \boldsymbol{\varphi} \in \mathbb{R}^m$ are optimized real vectors. To distinguish QET from ordinary energy transport by propagating excitations, Bob must carry out the local operation before the injected energy can propagate from Alice's region to Bob's site.

Because U_B shifts Bob's quadratures as

$$U_B^\dagger p_{n_B} U_B = p_{n_B} + \boldsymbol{\theta} \cdot \tilde{\mathbf{P}}, \quad U_B^\dagger q_{n_B} U_B = q_{n_B} + \boldsymbol{\varphi} \cdot \tilde{\mathbf{X}}, \quad (17)$$

we obtain, after averaging over outcomes,

$$\langle H_B \rangle_{\text{post}} = \langle H_B \rangle_\beta + \frac{1}{2} \boldsymbol{\theta}^T T_p^{(\beta)} \boldsymbol{\theta} + (J_p^{(\beta)})^T \boldsymbol{\theta} + \frac{1}{2} \boldsymbol{\varphi}^T T_q^{(\beta)} \boldsymbol{\varphi} + (J_q^{(\beta)})^T \boldsymbol{\varphi}, \quad (18)$$

with

$$(T_p^{(\beta)})_{jk} = \langle p_j p_k \rangle_\beta + \frac{\Omega}{2} \delta_{jk}, \quad (J_p^{(\beta)})_j = \langle p_j p_{n_B} \rangle_\beta, \quad (19)$$

$$(T_q^{(\beta)})_{jk} = \langle q_j q_k \rangle_\beta + \frac{1}{2\Omega} \delta_{jk}, \quad (J_q^{(\beta)})_j = \langle q_j Q_B \rangle_\beta. \quad (20)$$

The matrices $T_p^{(\beta)}$ and $T_q^{(\beta)}$ are the covariance matrices of Alice's classical outcomes $\tilde{\mathbf{P}}$ and $\tilde{\mathbf{X}}$, respectively. The diagonal terms $\Omega/2$ and $1/(2\Omega)$ are the measurement noise added by the coherent-state POVM. The vectors $J_p^{(\beta)}$ and $J_q^{(\beta)}$ contain the corresponding cross-correlations with Bob's momentum p_{n_B} and local force Q_B . Equation (21) therefore weights the useful Alice-Bob correlations by the inverse covariance of Alice's noisy measurement record.

The baseline term satisfies $\langle H_B \rangle_{\text{pre}} = \langle H_B \rangle_\beta$, so minimizing Eq. (18) gives the optimized extracted energy; Appendix A contains the full derivation.

$$E_{\text{ext}}(\beta) = \frac{1}{2} (J_p^{(\beta)})^T (T_p^{(\beta)})^{-1} J_p^{(\beta)} + \frac{1}{2} (J_q^{(\beta)})^T (T_q^{(\beta)})^{-1} J_q^{(\beta)}. \quad (21)$$

Equation (21) is exact for an arbitrary Gaussian block measurement. To obtain analytic control over the temperature and distance dependence, we now specialize to the single-site protocol, where the matrix expression collapses to scalar thermal correlators.

Thus, the energy-extraction problem reduces to the preexisting thermal two-point functions. We now use Eq. (21) to map the extracted energy over the main control parameters. Figure 1(a) shows the dependence on inverse temperature β and separation d in the near-critical regime, while Fig. 1(b) shows the dependence on coupling α and inverse temperature β at fixed separation. These plots reveal the same physical trend from two complementary viewpoints: Finite temperature suppresses long-distance extraction, whereas lowering the temperature and approaching criticality both enhance the energy that Bob can extract.

IV. EXACT THERMODYNAMIC LIMIT

Single-site protocol

The clearest analytic results arise for a single measured site $A = \{n_A\}$ and a single-site B at distance $d = |n_B - n_A|$. In that setting,

$$E_{\text{ext}}(\beta, \alpha; d) = \frac{1}{2} \frac{h_d(\beta, \alpha)^2}{h_0(\beta, \alpha) + \Omega/2} + \frac{1}{2} \frac{J_q(\beta, \alpha; d)^2}{g_0(\beta, \alpha) + 1/(2\Omega)}, \quad (22)$$

with

$$J_q(\beta, \alpha; d) = g_d(\beta, \alpha) - \frac{\alpha}{2} [g_{d-1}(\beta, \alpha) + g_{d+1}(\beta, \alpha)]. \quad (23)$$

A useful identity holds before any critical limit is taken:

$$J_q(\beta, \alpha; d) = h_d(\beta, \alpha), \quad \alpha < 1, \quad 0 < \beta \leq \infty. \quad (24)$$

Indeed, inserting Eq. (10) into Eq. (23) and using $\cos[(d-1)\theta] + \cos[(d+1)\theta] = 2\cos(d\theta)\cos\theta$ yield

$$\begin{aligned} J_q(\beta, \alpha; d) &= \frac{1}{2\pi} \int_0^{2\pi} d\theta \frac{1 - \alpha \cos\theta}{2\omega_\alpha(\theta)} \coth\left(\frac{\beta\omega_\alpha(\theta)}{2}\right) \cos(d\theta) \\ &= \frac{1}{2\pi} \int_0^{2\pi} d\theta \frac{\omega_\alpha(\theta)}{2} \coth\left(\frac{\beta\omega_\alpha(\theta)}{2}\right) \cos(d\theta) \\ &= h_d(\beta, \alpha), \end{aligned} \quad (25)$$

which proves Eq. (24). This reduction is central because it isolates the entire protocol dependence in a single thermal coupling function. Therefore, Eq. (24) reduces the single-site problem to the asymptotics of one function, h_d . The rest of the analysis therefore asks how this coupling behaves under the ordered thermodynamic and near-critical limits relevant for the harmonic chain.

V. LARGE-DISTANCE AND NEAR-CRITICAL ASYMPTOTICS

For mathematical consistency, we now enforce the ordered limits appropriate to periodic boundary conditions. We first keep $\alpha < 1$, take the thermodynamic limit, then choose the temperature regime, and only afterward take the gap-closing limit $\alpha \uparrow 1$. Setting $\alpha = 1$ at finite N would make the zero mode singular and would obscure the ordered critical behavior.

A. Fixed finite β , then $\alpha \uparrow 1$

For $\beta \in (0, \infty)$ and $\alpha < 1$, a Mittag-Leffler expansion of coth gives

$$\frac{1}{2\omega} \coth\left(\frac{\beta\omega}{2}\right) = \frac{1}{\beta\omega^2} + \frac{2}{\beta} \sum_{m=1}^{\infty} \frac{1}{\omega^2 + v_m^2}, \quad (26)$$

$$\frac{\omega}{2} \coth\left(\frac{\beta\omega}{2}\right) = \frac{1}{\beta} + \frac{2\omega^2}{\beta} \sum_{m=1}^{\infty} \frac{1}{\omega^2 + v_m^2}, \quad (27)$$

with bosonic Matsubara frequencies $v_m = 2\pi m/\beta$. The remaining cosine integrals are rational and can be evaluated by contour integration. Defining $a_m := 1 + v_m^2$, $\Delta_m(\alpha) := \sqrt{a_m^2 - \alpha^2}$, $r_m(\alpha) := \frac{a_m - \Delta_m(\alpha)}{\alpha}$, and, for the $m = 0$ term of g_n , $\Delta_0(\alpha) := \sqrt{1 - \alpha^2}$ and $r_0(\alpha) := \frac{1 - \Delta_0(\alpha)}{\alpha}$, we find

$$g_n(\beta, \alpha) = \frac{1}{\beta} \frac{r_0(\alpha)^n}{\Delta_0(\alpha)} + \frac{2}{\beta} \sum_{m=1}^{\infty} \frac{r_m(\alpha)^n}{\Delta_m(\alpha)}, \quad (28)$$

$$h_n(\beta, \alpha) = -\frac{2}{\beta} \sum_{m=1}^{\infty} \frac{v_m^2}{\Delta_m(\alpha)} r_m(\alpha)^n, \quad n \geq 1, \quad (29)$$

$$h_0(\beta, \alpha) = \frac{1}{\beta} + \frac{2}{\beta} \sum_{m=1}^{\infty} \left(1 - \frac{v_m^2}{\Delta_m(\alpha)}\right). \quad (30)$$

The derivation is given in Appendix B.

In the ordered critical limit $\alpha \uparrow 1$, the on-site position correlator diverges as $g_0(\beta, \alpha) \sim \frac{1}{\beta\sqrt{1-\alpha^2}}$, $\alpha \uparrow 1$, whereas h_d remains finite. Hence, the q -sector contribution in Eq. (22) vanishes. Writing $\kappa_m := \text{arsinh}(\frac{v_m}{\sqrt{2}}) = \text{arsinh}(\frac{\sqrt{2}\pi m}{\beta})$ so that $\tanh \kappa_m = v_m/\sqrt{v_m^2 + 2}$ and $r_m^{\text{crit}} = e^{-2\kappa_m}$, we obtain

$$h_d^{\text{crit}}(\beta) = -\frac{2}{\beta} \sum_{m=1}^{\infty} \tanh \kappa_m e^{-2d\kappa_m}, \quad d \geq 1, \quad (31)$$

$$h_0^{\text{crit}}(\beta) = \frac{1}{\beta} + \frac{2}{\beta} \sum_{m=1}^{\infty} (1 - \tanh \kappa_m). \quad (32)$$

Therefore,

$$E_{\text{ext}}^{\text{crit}}(\beta; d) = \frac{1}{2} \frac{[h_d^{\text{crit}}(\beta)]^2}{h_0^{\text{crit}}(\beta) + \Omega/2}, \quad 0 < \beta < \infty. \quad (33)$$

At large d the $m = 1$ term dominates, so $h_d^{\text{crit}}(\beta) \sim -\frac{2}{\beta} \tanh \kappa_1 e^{-2d\kappa_1}$, and thus,

$$E_{\text{ext}}^{\text{crit}}(\beta; d) \sim \frac{2 \tanh^2 \kappa_1}{\beta^2 [h_0^{\text{crit}}(\beta) + \Omega/2]} e^{-4d\kappa_1}. \quad (34)$$

The thermal correlation length controlling h_d is therefore

$$\xi_\beta = \frac{1}{2\kappa_1} = \left[2 \operatorname{arsinh} \left(\frac{\sqrt{2}\pi}{\beta} \right) \right]^{-1}. \quad (35)$$

Here ξ_β is the thermal correlation length of $h_d^{\text{crit}}(\beta)$. Since Eq. (33) is quadratic in this correlator, the extracted energy decays on the shorter length $\xi_E = \xi_\beta/2 = (4\kappa_1)^{-1}$.

Before turning to the zero-temperature limit, we examine the critical thermal coupling $h_d^{\text{crit}}(\beta)$ that enters the single-site protocol directly. Figure 2 plots this quantity in the thermodynamic critical limit for several values of β , alongside the exact ground-state result. Two features are immediately apparent. First, $|h_d^{\text{crit}}(\beta)|$ decays rapidly with distance at any finite temperature, consistent with the thermal screening length set by Eq. (35). Second, as β increases, the finite-temperature curves converge toward the ground-state profile, anticipating the algebraic long-distance behavior derived in the following section.

B. $\beta \rightarrow \infty$, then $\alpha \uparrow 1$

At zero temperature we first set $\coth(\beta\omega/2) \rightarrow 1$ at fixed $\alpha < 1$. The position correlator again diverges in the ordered critical limit, now logarithmically, and the q sector vanishes (see Appendix B). For the momentum coupling we can evaluate the critical integral exactly:

$$h_d^{\text{crit}}(\infty) = \frac{1}{2\pi} \int_0^{2\pi} d\theta \frac{\sqrt{1 - \cos \theta}}{2} \cos(d\theta). \quad (36)$$

Using $\sqrt{1 - \cos \theta} = \sqrt{2} \sin(\theta/2)$ on $[0, 2\pi]$ gives $h_0^{\text{crit}}(\infty) = \frac{\sqrt{2}}{\pi}$ and $h_d^{\text{crit}}(\infty) = -\frac{\sqrt{2}}{\pi(4d^2 - 1)}$, with $d \geq 1$. Hence,

$$E_{\text{ext}}^{\text{crit}}(\infty; d) = \frac{1}{\pi^2(4d^2 - 1)^2(\sqrt{2}/\pi + \Omega/2)}. \quad (37)$$

For large separation, $h_d^{\text{crit}}(\infty) \sim -\frac{\sqrt{2}}{4\pi}d^{-2}$. Therefore,

$$E_{\text{ext}}^{\text{crit}}(\infty; d) \sim \frac{1}{16\pi^2(\sqrt{2}/\pi + \Omega/2)}d^{-4}. \quad (38)$$

This resolves the long-distance critical law analytically.

We benchmark the asymptotic formulas against thermodynamic-limit numerics across the critical, near-critical, and gapped regimes. Figure 3 compares analytical critical curves at $\alpha = 1$ with numerical data from a finite chain of $N = 2^{15}$ sites. The near-critical data at $\alpha = 1 - 10^{-7}$ track the critical predictions closely over the full distance range, confirming that the critical formulas remain accurate even slightly away from the exact critical point. In contrast, the gapped ground-state data at $\alpha = 0.98$ decay far more steeply, as expected once a finite correlation length enters the picture. Notably, finite-temperature QET at criticality can exceed gapped ground-state QET over a broad window of separations, indicating that proximity to a critical point can outweigh the thermal penalty. The residual deviation between the near-critical numerical points and the exact analytical curves is indiscernible at this scale and is examined separately in Fig. 7 in Appendix B.

VI. CROSSOVER DIAGNOSTICS

The asymptotic formulas above identify the limiting decay laws. We next show how these regimes appear in finite-size numerics and how the apparent exponent crosses over before the true asymptotic behavior becomes visible.

The physical distinction between these regimes is set by the gap and thermal correlation lengths. From the factor r_m^d in Eq. (29), we have $r_m = e^{-\operatorname{arccosh}[(1+v_m^2)/\alpha]}$. At fixed finite β , the $m = 1$ term has the longest range and defines $\xi_h(\alpha, \beta) = \{\operatorname{arccosh}[(1+v_1^2)/\alpha]\}^{-1}$, where $v_1 = 2\pi/\beta$. In the ordered critical limit this reduces to the thermal length ξ_β in Eq. (35), whereas the zero-temperature limit at fixed $\alpha < 1$ gives the gap length $\xi_\Delta = [\operatorname{arccosh}(1/\alpha)]^{-1}$. With the low-momentum dispersion in Eq. (11), the latter satisfies $\xi_\Delta \simeq v/\Delta_{\text{gap}}$ near criticality, with $\Delta_{\text{gap}} = \sqrt{1 - \alpha}$ and $v = \sqrt{\alpha/2}$. Thus, a nonzero gap limits the range of ground-state correlations, while finite temperature introduces an additional cutoff through the lowest Matsubara frequency. The near-critical low-temperature regime has both ξ_Δ and ξ_β larger than the displayed distances; the near-critical finite-temperature regime is instead limited mainly by ξ_β , while the gapped ground state is limited by ξ_Δ . For the gapped ground-state regime, this behavior is qualitatively consistent with general exponential decaying bound by QET energy extraction in finite-range gapped lattice systems [23].

Figure 4 examines the spatial decay of the extracted energy $E_{\text{ext}}(n_B)$ for three representative regimes: a near-critical low-temperature case, a near-critical finite-temperature case, and a gapped ground-state case.¹ The distinction between algebraic and exponential suppression is most transparent from Figs. 4(a) and 4(b). In the near-critical low-temperature regime, the curve is closest to linear on the log-log scale over an extended intermediate range of n_B , consistent with an approximate algebraic decay before eventual finite-correlation-length effects set in. By contrast, the finite-temperature near-critical case and the gapped ground state bend strongly on the log-log plot and are substantially closer to linear on the semilog plot, which is the expected signature of exponential suppression at large separation. Figures 4(c) and 4(d) provide local diagnostics of this behavior. The effective logarithmic slope $\eta_{\text{eff}}(n_B)$ remains nearly constant for the near-critical low-temperature case over the accessible window, whereas it grows steadily in the thermally suppressed and gapped cases. Similarly, the local attenuation rate $\kappa_{\text{eff}}(n_B)$ tends toward a small value in the near-critical low-temperature regime but approaches a finite nonzero scale in the other two cases, consistent with exponential decay. More explicitly, the numerics agree with the critical form $E_{\text{ext}} \propto n_B^{-4}$ from Eq. (38) and give $\eta_{\text{eff}} \simeq 4$ and $\kappa_{\text{eff}} \simeq 4/n_B \rightarrow 0$. An exponential envelope $E_{\text{ext}} \propto \exp(-n_B/\xi_E)$ instead gives $\eta_{\text{eff}} \simeq n_B/\xi_E$ and $\kappa_{\text{eff}} \simeq 1/\xi_E$. Hence, the growing η_{eff} and finite κ_{eff} in the thermally screened and gapped cases directly measure the finite length that cuts off the critical correlations.

¹To avoid confusion between the derivative and distance, we used $n_B := d$.

VII. SQUEEZED GAUSSIAN POVM

Since the large-distance behavior is fixed by the thermal correlators, the natural next question is whether Gaussian measurement design can improve the extraction prefactor without changing the scaling law. We therefore examine squeezed coherent-state POVMs.

The coherent-state POVM is not the only Gaussian choice. Let Alice instead measure squeezed coherent states with squeeze parameter $r \geq 0$ and squeeze angle ϑ . The only change relevant to the averaged energy is the measurement noise added to the second moments. For a q -squeezed measurement, $\vartheta = 0$ (see Appendix C),

$$v_X(r) = \frac{e^{-2r}}{2\Omega}, \quad v_P(r) = \frac{\Omega e^{2r}}{2}, \quad (39)$$

whereas for a p -squeezed measurement, $\vartheta = \pi/2$,

$$v_X(r) = \frac{e^{2r}}{2\Omega}, \quad v_P(r) = \frac{\Omega e^{-2r}}{2}. \quad (40)$$

The optimized single-site extracted energy becomes

$$E_{\text{ext}} = \frac{1}{2} \frac{h_d(\beta, \alpha)^2}{h_0(\beta, \alpha) + v_P(r, \vartheta)} + \frac{1}{2} \frac{J_q(\beta, \alpha; d)^2}{g_0(\beta, \alpha) + v_X(r, \vartheta)}. \quad (41)$$

In the ordered critical limit the q sector again drops out, so the only surviving effect of squeezing is the replacement $\Omega/2 \mapsto v_P$ in the denominator. Thus, p squeezing lowers the effective momentum noise and enhances the extracted energy, while q squeezing raises it and suppresses the yield. Figure 5 illustrates this point exactly. Importantly, the numerators are untouched. Squeezing therefore changes the efficiency but not the large-distance scaling. This is also visible in Fig. 6.

VIII. DISCUSSION AND CONCLUSION

In this work we placed quantum energy teleportation in the harmonic chain on a controlled finite-temperature footing. For translationally invariant Gibbs states, the optimized Gaussian protocol reduces to thermal two-point functions, and in the single-site setting the full distance dependence collapses to one correlator, h_d . This reduction makes the ordered limits transparent. At fixed finite temperature, the leading Matsubara contribution sets a thermal correlation length ξ_β and produces exponential suppression of the extracted energy. In the zero-temperature critical limit taken after the thermodynamic limit, the coupling admits a closed form, which yields the asymptotic law $E_{\text{ext}} \sim d^{-4}$.

These results sharpen the status of the harmonic chain as an analytically tractable QET platform. In the present setting, the protocol is controlled less by protocol-specific details than by the correlation structure of the underlying Gaussian state. Finite temperature screens the useful correlations, while proximity to criticality restores long-range tails. The squeezed-measurement analysis supports the same conclusion: Changing the Gaussian measurement modifies the effective noise and therefore the extraction efficiency, but it does not alter the large-distance scaling class.

The present analysis is limited to Gaussian measurements, primarily to single-site extraction. Natural extensions include

finite-temperature block protocols and multisite feedback on Bob's side, which may improve the extraction prefactor while preserving the same asymptotic scaling. Because the same quadratic Hamiltonian also describes coupled microwave resonators after canonical rescaling, the present results may also serve as a benchmark for bosonic QET implementations in circuit-QED architectures.

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DATA AVAILABILITY

The data that support the findings of this article are openly available [24].

APPENDIX A: DERIVATION OF THE GAUSSIAN BLOCK-MEASUREMENT FORMULA

In this appendix we derive the optimized extracted-energy formula quoted in Eq. (21) of the main text for an arbitrary Gaussian block measurement. The derivation proceeds in three steps. We first establish the coherent-state POVM identities needed for outcome averaging. We then compute the transformed local energy under Bob's conditional displacement. Finally, we minimize the resulting quadratic form over Bob's feedback parameters.

1. Coherent-state POVM on Alice's block

Alice measures a block A consisting of $m = 2\ell + 1$ sites. On each measured site she introduces the operator

$$\hat{b} = \sqrt{\frac{\Omega}{2}} \hat{q} + \frac{i}{\sqrt{2\Omega}} \hat{p}, \quad \Omega > 0. \quad (A1)$$

The corresponding coherent states $|X, P\rangle$ are defined by the eigenvalue equation

$$\hat{b} |X, P\rangle = c |X, P\rangle, \quad c = \sqrt{\frac{\Omega}{2}} X + \frac{i}{\sqrt{2\Omega}} P. \quad (A2)$$

The one-site POVM element is

$$\Pi(X, P) = \frac{1}{2\pi} |X, P\rangle \langle X, P|. \quad (A3)$$

To verify completeness, we begin from the standard coherent-state identity

$$\int \frac{d^2 c}{\pi} |c\rangle \langle c| = \mathbb{I}. \quad (A4)$$

Writing $c = u + iv$ and comparing the results with Eq. (A2), we obtain

$$u = \sqrt{\frac{\Omega}{2}} X, \quad v = \frac{1}{\sqrt{2\Omega}} P. \quad (A5)$$

Hence, the Jacobian is

$$du dv = \sqrt{\frac{\Omega}{2}} \frac{1}{\sqrt{2\Omega}} dX dP = \frac{1}{2} dX dP. \quad (A6)$$

Substituting Eq. (A6) into Eq. (A4), we find

$$\int dX dP \Pi(X, P) = \mathbb{I}. \quad (\text{A7})$$

For the full block measurement we define

$$\Pi_A(\tilde{X}, \tilde{P}) = \frac{1}{(2\pi)^m} \prod_{j=1}^m |X_j, P_j\rangle\langle X_j, P_j|, \quad (\text{A8})$$

where $\tilde{X} = (X_1, \dots, X_m)$ and $\tilde{P} = (P_1, \dots, P_m)$. Since the one-site POVM is complete, the block POVM satisfies

$$\int d\tilde{X} d\tilde{P} \Pi_A(\tilde{X}, \tilde{P}) = \mathbb{I}. \quad (\text{A9})$$

The associated Kraus operator is taken as

$$M_A(\tilde{X}, \tilde{P}) = \frac{1}{(\sqrt{2\pi})^m} \prod_{j=1}^m |X_j, P_j\rangle\langle X_j, P_j|, \quad (\text{A10})$$

so that

$$M_A^\dagger M_A = \Pi_A. \quad (\text{A11})$$

2. First and second moments of the POVM

To evaluate the averaged local energy, we require the first and second moments of the measurement outcomes. These moments can be derived from the coherent-state identities

$$\int \frac{d^2c}{\pi} c |c\rangle\langle c| = \hat{b}, \quad \int \frac{d^2c}{\pi} c^* |c\rangle\langle c| = \hat{b}^\dagger. \quad (\text{A12})$$

The proof is standard. Expanding $|c\rangle$ in the number basis and using Gaussian integration gives the matrix elements of $\hat{b}$ and $\hat{b}^\dagger$ directly.

Using Eq. (A2), we may rewrite the classical variables as

$$X = \frac{c + c^*}{\sqrt{2\Omega}}, \quad P = -i\sqrt{\frac{\Omega}{2}}(c - c^*). \quad (\text{A13})$$

Substituting Eq. (A12) into Eq. (A13), we obtain the one-site first moments

$$\int dX dP X \Pi(X, P) = q, \quad \int dX dP P \Pi(X, P) = p. \quad (\text{A14})$$

The second moments follow similarly. Using

$$X^2 = \frac{c^2 + (c^*)^2 + 2|c|^2}{2\Omega}, \quad P^2 = -\frac{\Omega}{2}(c - c^*)^2, \quad (\text{A15})$$

together with the standard identity

$$\int \frac{d^2c}{\pi} |c|^2 |c\rangle\langle c| = \hat{b}\hat{b}^\dagger, \quad (\text{A16})$$

we obtain

$$\begin{aligned} \int dX dP X^2 \Pi(X, P) &= q^2 + \frac{1}{2\Omega} \mathbb{I}, \\ \int dX dP P^2 \Pi(X, P) &= p^2 + \frac{\Omega}{2} \mathbb{I}. \end{aligned} \quad (\text{A17})$$

Because the block POVM factorizes across Alice's sites, the multisite moments are immediate:

$$\int d\tilde{X} d\tilde{P} X_j \Pi_A = q_j, \quad \int d\tilde{X} d\tilde{P} P_j \Pi_A = p_j, \quad (\text{A18})$$

and

$$\begin{aligned} \int d\tilde{X} d\tilde{P} X_j X_k \Pi_A &= q_j q_k + \frac{1}{2\Omega} \delta_{jk} \mathbb{I}, \\ \int d\tilde{X} d\tilde{P} P_j P_k \Pi_A &= p_j p_k + \frac{\Omega}{2} \delta_{jk} \mathbb{I}. \end{aligned} \quad (\text{A19})$$

3. Bob's conditional displacement

After receiving the classical outcome $(\tilde{X}, \tilde{P})$, Bob applies the local displacement

$$U_B(\tilde{X}, \tilde{P}) = \exp \{ i[(\theta \cdot \tilde{P})q_{n_B} - (\phi \cdot \tilde{X})p_{n_B}] \}, \quad (\text{A20})$$

where $\theta, \phi \in \mathbb{R}^m$ are real control vectors. For fixed outcome, define

$$a := \theta \cdot \tilde{P} = \sum_{j=1}^m \theta_j P_j, \quad b := \phi \cdot \tilde{X} = \sum_{j=1}^m \phi_j X_j. \quad (\text{A21})$$

Then Eq. (A20) becomes $U_B = \exp[i(aq_{n_B} - bp_{n_B})]$.

Applying the Baker-Campbell-Hausdorff formula and noting that the relevant nested commutators vanish after the first step, we obtain

$$U_B^\dagger p_{n_B} U_B = p_{n_B} + a, \quad U_B^\dagger q_{n_B} U_B = q_{n_B} + b. \quad (\text{A22})$$

It is convenient to introduce the operator

$$Q_B := q_{n_B} - \frac{\alpha}{2}(q_{n_B-1} + q_{n_B+1}) \quad (\text{A23})$$

since this is precisely the linear combination appearing in the local energy around Bob. Substituting Eq. (A22) into Eq. (12) of the main text, we find

$$U_B^\dagger H_B U_B = H_B + a p_{n_B} + \frac{1}{2}a^2 + b Q_B + \frac{1}{2}b^2. \quad (\text{A24})$$

4. Outcome averaging and quadratic form

Alice's un-normalized postmeasurement state is

$$\rho(\tilde{X}, \tilde{P}) = M_A(\tilde{X}, \tilde{P}) \rho_\beta M_A(\tilde{X}, \tilde{P}). \quad (\text{A25})$$

The averaged pre-Bob and post-Bob states are

$$\begin{aligned} \rho_{\text{pre}} &= \int d\tilde{X} d\tilde{P} M_A \rho_\beta M_A, \\ \rho_{\text{post}}(\theta, \phi) &= \int d\tilde{X} d\tilde{P} U_B M_A \rho_\beta M_A U_B^\dagger. \end{aligned} \quad (\text{A26})$$

Now let O be any operator supported outside Alice's measured block. Since $[O, \Pi_A] = 0$, the cyclicity of the trace gives

$$\text{Tr}(M_A \rho_\beta M_A O) = \text{Tr}(\rho_\beta O \Pi_A). \quad (\text{A27})$$

Applying Eq. (A27) to $O = U_B^\dagger H_B U_B$ and using Eqs. (A18) and (A19), we obtain

$$\begin{aligned} \langle H_B \rangle_{\text{post}}(\theta, \phi) &= \langle H_B \rangle_\beta + \sum_{j=1}^m \theta_j \langle p_j p_{n_B} \rangle_\beta \\ &+ \frac{1}{2} \sum_{j,k=1}^m \theta_j \left(\langle p_j p_k \rangle_\beta + \frac{\Omega}{2} \delta_{jk} \right) \theta_k \\ &+ \sum_{j=1}^m \phi_j \langle q_j Q_B \rangle_\beta \\ &+ \frac{1}{2} \sum_{j,k=1}^m \phi_j \left(\langle q_j q_k \rangle_\beta + \frac{1}{2\Omega} \delta_{jk} \right) \phi_k. \end{aligned} \quad (\text{A28})$$

After we introduce the matrices and vectors already used in the main text,

$$\begin{aligned} (T_p^{(\beta)})_{jk} &:= \langle p_j p_k \rangle_\beta + \frac{\Omega}{2} \delta_{jk}, \\ (T_q^{(\beta)})_{jk} &:= \langle q_j q_k \rangle_\beta + \frac{1}{2\Omega} \delta_{jk}, \end{aligned} \quad (\text{A29})$$

$$(J_p^{(\beta)})_j := \langle p_j p_{n_B} \rangle_\beta, \quad (J_q^{(\beta)})_j := \langle q_j Q_B \rangle_\beta, \quad (\text{A30})$$

Eq. (A28) becomes

$$\begin{aligned} \langle H_B \rangle_{\text{post}}(\theta, \phi) &= \langle H_B \rangle_\beta + \frac{1}{2} \theta^T T_p^{(\beta)} \theta + (J_p^{(\beta)})^T \theta \\ &+ \frac{1}{2} \phi^T T_q^{(\beta)} \phi + (J_q^{(\beta)})^T \phi. \end{aligned} \quad (\text{A31})$$

5. Optimization over Bob's feedback parameters

Since $T_p^{(\beta)}$ and $T_q^{(\beta)}$ are positive definite, Eq. (A31) is minimized at the stationary point. Differentiating with respect to θ and ϕ gives

$$T_p^{(\beta)} \theta_{\text{opt}} + J_p^{(\beta)} = 0, \quad T_q^{(\beta)} \phi_{\text{opt}} + J_q^{(\beta)} = 0.$$

Hence,

$$\theta_{\text{opt}} = -(T_p^{(\beta)})^{-1} J_p^{(\beta)}, \quad \phi_{\text{opt}} = -(T_q^{(\beta)})^{-1} J_q^{(\beta)}. \quad (\text{A32})$$

Substituting these expressions back into Eq. (A31), we obtain

$$\begin{aligned} \langle H_B \rangle_{\text{post, opt}} &= \langle H_B \rangle_\beta - \frac{1}{2} (J_p^{(\beta)})^T (T_p^{(\beta)})^{-1} J_p^{(\beta)} \\ &- \frac{1}{2} (J_q^{(\beta)})^T (T_q^{(\beta)})^{-1} J_q^{(\beta)}. \end{aligned} \quad (\text{A33})$$

If Bob does nothing, then $\theta = \phi = 0$, and therefore, $\langle H_B \rangle_{\text{pre}} = \langle H_B \rangle_\beta$. Comparing the Eq. (A33) with the definition of extracted energy in the main text, the baseline term cancels, and we recover Eq. (21):

$$E_{\text{ext}}(\beta) = \frac{1}{2} (J_p^{(\beta)})^T (T_p^{(\beta)})^{-1} J_p^{(\beta)} + \frac{1}{2} (J_q^{(\beta)})^T (T_q^{(\beta)})^{-1} J_q^{(\beta)}. \quad (\text{A34})$$

APPENDIX B: THERMODYNAMIC CORRELATORS AND ORDERED CRITICAL LIMITS

In this appendix we derive the thermodynamic correlators used in Eqs. (28)–(30) of the main text and then analyze their ordered critical limits. The crucial point is that for periodic

boundary conditions the gap-closing limit $\alpha \uparrow 1$ must be taken only after the thermodynamic limit.

1. Normal modes and Gibbs covariances

Introduce the Fourier angles

$$\theta_k = \frac{2\pi k}{N}, \quad k = 0, 1, \dots, N-1, \quad (\text{B1})$$

and the Fourier modes

$$q_j = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} Q_k e^{i\theta_k j}, \quad p_j = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} P_k e^{i\theta_k j}. \quad (\text{B2})$$

Applying the periodic boundary condition to Eq. (B2), we find that the Hamiltonian diagonalizes as

$$H = \frac{1}{2} \sum_{k=0}^{N-1} (P_k P_{-k} + \omega_k^2 Q_k Q_{-k}), \quad \omega_k^2 = 1 - \alpha \cos \theta_k. \quad (\text{B3})$$

For $\alpha < 1$, all ω_k are strictly positive, including the zero mode. At $\alpha = 1$, however, the zero mode becomes gapless, and the periodic Gibbs state is no longer trace class. That is why the order of limits matters.

Define the ladder operators

$$a_k = \sqrt{\frac{\omega_k}{2}} Q_k + \frac{i}{\sqrt{2\omega_k}} P_k, \quad a_k^\dagger = \sqrt{\frac{\omega_k}{2}} Q_{-k} - \frac{i}{\sqrt{2\omega_k}} P_{-k}.$$

Then

$$H = \sum_{k=0}^{N-1} \omega_k \left(a_k^\dagger a_k + \frac{1}{2} \right).$$

In the Gibbs state the Bose occupation number is

$$n_k = \langle a_k^\dagger a_k \rangle_{\beta, \alpha} = \frac{1}{e^{\beta\omega_k} - 1}, \quad n_k + \frac{1}{2} = \frac{1}{2} \coth \left(\frac{\beta\omega_k}{2} \right).$$

Therefore, the nonvanishing mode covariances are

$$\begin{aligned} \langle Q_k Q_{-k} \rangle_{\beta, \alpha} &= \frac{1}{2\omega_k} \coth \left(\frac{\beta\omega_k}{2} \right), \\ \langle P_k P_{-k} \rangle_{\beta, \alpha} &= \frac{\omega_k}{2} \coth \left(\frac{\beta\omega_k}{2} \right). \end{aligned}$$

2. Thermodynamic correlators

Define the translation-invariant correlators

$$g_n(\beta, \alpha; N) := \langle q_j q_{j+n} \rangle_{\beta, \alpha}, \quad h_n(\beta, \alpha; N) := \langle p_j p_{j+n} \rangle_{\beta, \alpha}. \quad (\text{B4})$$

Substituting the Fourier expansion and using the thermal covariances above yield

$$g_n(\beta, \alpha; N) = \frac{1}{N} \sum_{k=0}^{N-1} \frac{1}{2\omega_k} \coth \left(\frac{\beta\omega_k}{2} \right) \cos(n\theta_k) \quad (\text{B5})$$

and

$$h_n(\beta, \alpha; N) = \frac{1}{N} \sum_{k=0}^{N-1} \frac{\omega_k}{2} \coth \left(\frac{\beta\omega_k}{2} \right) \cos(n\theta_k). \quad (\text{B6})$$

Now, fix $\alpha < 1$, and take the thermodynamic limit. The Riemann sums become

$$g_n(\beta, \alpha) = \frac{1}{2\pi} \int_0^{2\pi} d\theta \frac{1}{2\omega_\alpha(\theta)} \coth\left(\frac{\beta\omega_\alpha(\theta)}{2}\right) \cos(n\theta) \quad (\text{B7})$$

and

$$h_n(\beta, \alpha) = \frac{1}{2\pi} \int_0^{2\pi} d\theta \frac{\omega_\alpha(\theta)}{2} \coth\left(\frac{\beta\omega_\alpha(\theta)}{2}\right) \cos(n\theta), \quad (\text{B8})$$

where

$$\omega_\alpha(\theta) = \sqrt{1 - \alpha \cos \theta}. \quad (\text{B9})$$

3. Single-site reduction and the identity $J_q = h_d$

For the single-site protocol discussed in Sec. IV of the main text, the extracted energy takes the scalar form,

$$E_{\text{ext}}(\beta, \alpha; d) = \frac{1}{2} \frac{h_d(\beta, \alpha)^2}{h_0(\beta, \alpha) + \Omega/2} + \frac{1}{2} \frac{J_q(\beta, \alpha; d)^2}{g_0(\beta, \alpha) + 1/(2\Omega)}, \quad (\text{B10})$$

with

$$J_q(\beta, \alpha; d) = g_d(\beta, \alpha) - \frac{\alpha}{2} [g_{d-1}(\beta, \alpha) + g_{d+1}(\beta, \alpha)]. \quad (\text{B11})$$

A central identity is

$$J_q(\beta, \alpha; d) = h_d(\beta, \alpha), \quad \alpha < 1, \quad 0 < \beta \leq \infty. \quad (\text{B12})$$

To prove it, define

$$A_{\beta, \alpha}(\theta) := \frac{1}{2\omega_\alpha(\theta)} \coth\left(\frac{\beta\omega_\alpha(\theta)}{2}\right).$$

Then Eq. (B7) reads

$$g_n(\beta, \alpha) = \frac{1}{2\pi} \int_0^{2\pi} d\theta A_{\beta, \alpha}(\theta) \cos(n\theta).$$

Substituting this expression into Eq. (B11) and using

$$\cos[(d-1)\theta] + \cos[(d+1)\theta] = 2 \cos(d\theta) \cos \theta,$$

we obtain

$$J_q = \frac{1}{2\pi} \int_0^{2\pi} d\theta A_{\beta, \alpha}(\theta) \cos(d\theta) (1 - \alpha \cos \theta).$$

Since $1 - \alpha \cos \theta = \omega_\alpha(\theta)^2$, the integrand collapses immediately, yielding Eq. (B12).

4. Ordered limits

For periodic boundary conditions we enforce the limit order

$$\alpha < 1, \quad N \rightarrow \infty \quad (\text{B13})$$

and choose the temperature regime $\alpha \uparrow 1$ last. This ordering is mandatory. If we set $\alpha = 1$ before taking $N \rightarrow \infty$, the zero mode becomes singular, and the Gibbs state ceases to be well-defined.

5. Fixed finite temperature, then $\alpha \uparrow 1$

Fix $0 < \beta < \infty$. To evaluate the thermodynamic integrals exactly, we use the Mittag-Leffler expansion

$$\coth x = \frac{1}{x} + 2x \sum_{m=1}^{\infty} \frac{1}{x^2 + \pi^2 m^2}. \quad (\text{B14})$$

Setting $x = \beta\omega/2$ and introducing the bosonic Matsubara frequencies

$$\nu_m = \frac{2\pi m}{\beta}, \quad m \geq 1, \quad (\text{B15})$$

we obtain

$$\frac{1}{2\omega} \coth\left(\frac{\beta\omega}{2}\right) = \frac{1}{\beta\omega^2} + \frac{2}{\beta} \sum_{m=1}^{\infty} \frac{1}{\omega^2 + \nu_m^2} \quad (\text{B16})$$

and

$$\frac{\omega}{2} \coth\left(\frac{\beta\omega}{2}\right) = \frac{1}{\beta} + \frac{2\omega^2}{\beta} \sum_{m=1}^{\infty} \frac{1}{\omega^2 + \nu_m^2}. \quad (\text{B17})$$

The remaining angular integrals reduce to the rational cosine integral

$$J_n(a, \alpha) := \frac{1}{2\pi} \int_0^{2\pi} \frac{\cos(n\theta)}{a - \alpha \cos \theta} d\theta, \quad a > \alpha > 0.$$

Passing to the unit circle via $z = e^{i\theta}$ and evaluating the residue of the pole inside $|z| = 1$, we find

$$J_n(a, \alpha) = \frac{1}{\sqrt{a^2 - \alpha^2}} \left(\frac{a - \sqrt{a^2 - \alpha^2}}{\alpha} \right)^n. \quad (\text{B18})$$

Define

$$\begin{aligned} a_m &:= 1 + \nu_m^2, \\ \Delta_m(\alpha) &:= \sqrt{a_m^2 - \alpha^2}, \\ r_m(\alpha) &:= \frac{a_m - \Delta_m(\alpha)}{\alpha}, \end{aligned}$$

and, for the $m = 0$ contribution in g_n ,

$$\Delta_0(\alpha) := \sqrt{1 - \alpha^2}, \quad r_0(\alpha) := \frac{1 - \Delta_0(\alpha)}{\alpha}.$$

Substituting Eq. (B18) into the thermodynamic correlators yields

$$g_n(\beta, \alpha) = \frac{1}{\beta} \frac{r_0(\alpha)^n}{\Delta_0(\alpha)} + \frac{2}{\beta} \sum_{m=1}^{\infty} \frac{r_m(\alpha)^n}{\Delta_m(\alpha)}, \quad (\text{B19})$$

$$h_n(\beta, \alpha) = -\frac{2}{\beta} \sum_{m=1}^{\infty} \frac{\nu_m^2}{\Delta_m(\alpha)} r_m(\alpha)^n, \quad n \geq 1, \quad (\text{B20})$$

and

$$h_0(\beta, \alpha) = \frac{1}{\beta} + \frac{2}{\beta} \sum_{m=1}^{\infty} \left(1 - \frac{\nu_m^2}{\Delta_m(\alpha)} \right). \quad (\text{B21})$$

These are exactly the formulas we use in Eqs. (28)–(30) of the main text.

We now take the ordered critical limit. The divergence of g_0 is dominated by the $m = 0$ term:

$$g_0(\beta, \alpha) \supset \frac{1}{\beta\sqrt{1-\alpha^2}} \rightarrow +\infty \quad (\alpha \uparrow 1).$$

Therefore, the q sector of Eq. (B10) vanishes, while the numerator remains finite because of Eq. (B12).

To write the surviving correlator compactly, define

$$\kappa_m := \operatorname{arsinh}\left(\frac{v_m}{\sqrt{2}}\right) = \operatorname{arsinh}\left(\frac{\sqrt{2}\pi m}{\beta}\right). \quad (\text{B22})$$

Then

$$\tanh \kappa_m = \frac{v_m}{\sqrt{v_m^2 + 2}}, \quad e^{-2\kappa_m} = \frac{1}{1 + v_m^2 + v_m\sqrt{v_m^2 + 2}}.$$

Hence, the critical momentum correlators are

$$h_d^{\text{crit}}(\beta) = -\frac{2}{\beta} \sum_{m=1}^{\infty} \tanh(\kappa_m) e^{-2d\kappa_m}, \quad d \geq 1, \quad (\text{B23})$$

and

$$h_0^{\text{crit}}(\beta) = \frac{1}{\beta} + \frac{2}{\beta} \sum_{m=1}^{\infty} (1 - \tanh \kappa_m). \quad (\text{B24})$$

Substituting Eq. (B23) and Eq. (B24) into Eq. (B10), we obtain

$$E_{\text{ext}}^{\text{crit}}(\beta; d) = \frac{1}{2} \frac{[h_d^{\text{crit}}(\beta)]^2}{h_0^{\text{crit}}(\beta) + \Omega/2}, \quad 0 < \beta < \infty. \quad (\text{B25})$$

At large separation the $m = 1$ term dominates the sum. Therefore,

$$h_d^{\text{crit}}(\beta) \sim -\frac{2}{\beta} \tanh(\kappa_1) e^{-2d\kappa_1},$$

and the extracted energy decays as

$$E_{\text{ext}}^{\text{crit}}(\beta; d) \sim \frac{2 \tanh^2(\kappa_1)}{\beta^2 [h_0^{\text{crit}}(\beta) + \Omega/2]} e^{-4d\kappa_1}. \quad (\text{B26})$$

Thus, the useful energy is exponentially screened at any fixed finite temperature.

6. Zero temperature, then $\alpha \uparrow 1$

We now set $\beta = \infty$ at fixed $\alpha < 1$. In this case the hyperbolic cotangent reduces to unity, so

$$g_n(\infty, \alpha) = \frac{1}{2\pi} \int_0^{2\pi} \frac{\cos(n\theta)}{2\omega_\alpha(\theta)} d\theta,$$

$$h_n(\infty, \alpha) = \frac{1}{2\pi} \int_0^{2\pi} \frac{\omega_\alpha(\theta)}{2} \cos(n\theta) d\theta.$$

The position variance again diverges in the ordered critical limit, now logarithmically. Writing $\mu^2 := 1 - \alpha$, we have for small θ

$$\omega_\alpha(\theta) \sim \sqrt{\mu^2 + \theta^2/2},$$

and therefore,

$$g_0(\infty, \alpha) \sim \int_0^\Lambda \frac{d\theta}{\sqrt{\mu^2 + \theta^2/2}} \sim \ln \frac{1}{\mu}.$$

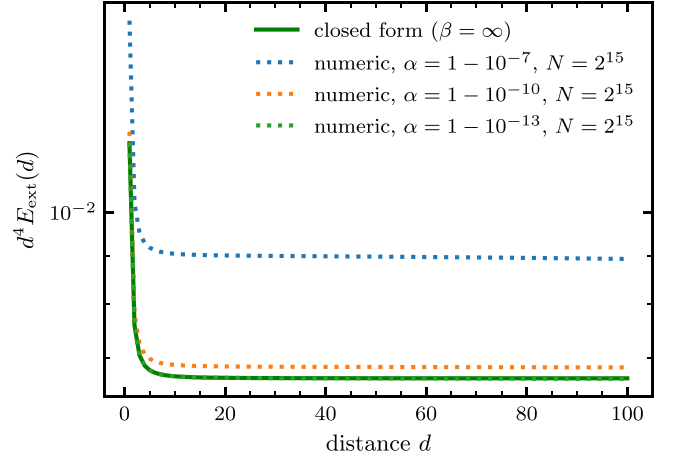


FIG. 7. Comparison of the rescaled quantity $d^4 E_{\text{ext}}(d)$ with the closed-form zero-temperature critical prediction. The finite-size numerical curves approach the analytic result as α is tuned closer to unity, consistent with the asymptotic behavior $E_{\text{ext}}(d) \sim Cd^{-4}$.

Thus, the q sector in Eq. (B10) again drops out.

For the momentum correlator, however, dominated convergence allows us to set $\alpha = 1$ directly in the integrand:

$$h_d^{\text{crit}}(\infty) = \frac{1}{2\pi} \int_0^{2\pi} \frac{\sqrt{1 - \cos \theta}}{2} \cos(d\theta) d\theta. \quad (\text{B27})$$

Using

$$\sqrt{1 - \cos \theta} = \sqrt{2} \sin\left(\frac{\theta}{2}\right), \quad 0 \leq \theta \leq 2\pi,$$

we obtain

$$h_d^{\text{crit}}(\infty) = \frac{\sqrt{2}}{4\pi} \int_0^{2\pi} \sin\left(\frac{\theta}{2}\right) \cos(d\theta) d\theta.$$

Expanding the product with $\sin a \cos b = \frac{1}{2}[\sin(a+b) + \sin(a-b)]$ and integrating term by term yield

$$\int_0^{2\pi} \sin\left(\frac{\theta}{2}\right) \cos(d\theta) d\theta = -\frac{4}{4d^2 - 1}.$$

Therefore,

$$h_0^{\text{crit}}(\infty) = \frac{\sqrt{2}}{\pi}, \quad h_d^{\text{crit}}(\infty) = -\frac{\sqrt{2}}{\pi(4d^2 - 1)}, \quad d \geq 1. \quad (\text{B28})$$

Substituting Eq. (B28) into Eq. (B10), we recover Eq. (37) of the main text:

$$E_{\text{ext}}^{\text{crit}}(\infty; d) = \frac{1}{\pi^2(4d^2 - 1)^2(\sqrt{2}/\pi + \Omega/2)}. \quad (\text{B29})$$

At large separation Eq. (B29) becomes

$$E_{\text{ext}}^{\text{crit}}(\infty; d) \sim \frac{1}{16\pi^2(\sqrt{2}/\pi + \Omega/2)} d^{-4}, \quad (\text{B30})$$

which is the analytic d^{-4} law stated in the main text.

As a numerical check of the zero-temperature asymptotics, Fig. 7 plots the rescaled quantity $d^4 E_{\text{ext}}(d)$. If the large-distance law is indeed $E_{\text{ext}}(d) \sim Cd^{-4}$, this rescaled combination should approach a constant at sufficiently large

d . Figure 7 shows precisely this behavior. The analytic $\beta = \infty$ closed form is compared with finite- N evaluations performed at progressively smaller gaps, $\alpha = 1 - 10^{-7}$, $1 - 10^{-10}$, and $1 - 10^{-13}$, all for $N = 2^{15}$. As α is tuned closer to unity, the numerical curves move toward the closed-form prediction over an extended range of distances, confirming that the ordered limit is captured correctly. In this way the plot serves as a direct benchmark of the analytic coefficient in the d^{-4} tail, rather than merely of the exponent.

APPENDIX C: SQUEEZED GAUSSIAN POVM

For completeness we also record the squeezed-POVM extension underlying Sec. VII. The only effect of squeezing is to modify the measurement-added noise in the second moments. Let $S(\zeta)$ be the single-mode squeeze operator with $\zeta = re^{i\vartheta}$ and $r \geq 0$. The squeezed coherent states are

$$|X, P; \zeta\rangle := D(X, P)S(\zeta)|0\rangle, \quad D(X, P) = e^{iPq - iXp}, \quad (\text{C1})$$

and the one-site POVM element is

$$\Pi_\zeta(X, P) = \frac{1}{2\pi} |X, P; \zeta\rangle \langle X, P; \zeta|. \quad (\text{C2})$$

For the block measurement we take the product over Alice's measured sites.

For a q -squeezed measurement, $\vartheta = 0$, the second moments become

$$v_X(r) = \frac{e^{-2r}}{2\Omega}, \quad v_P(r) = \frac{\Omega e^{2r}}{2}, \quad (\text{C3})$$

whereas for a p -squeezed measurement, $\vartheta = \pi/2$,

$$v_X(r) = \frac{e^{2r}}{2\Omega}, \quad v_P(r) = \frac{\Omega e^{-2r}}{2}. \quad (\text{C4})$$

Therefore, the covariance matrices in Appendix A are replaced by

$$\begin{aligned} (T_p^{(\beta, \zeta)})_{jk} &= \langle p_j p_k \rangle_\beta + v_P(r) \delta_{jk}, \\ (T_q^{(\beta, \zeta)})_{jk} &= \langle q_j q_k \rangle_\beta + v_X(r) \delta_{jk}, \end{aligned} \quad (\text{C5})$$

while the correlation vectors remain unchanged. Repeating the same Gaussian minimization as in Appendix A, we find

$$\begin{aligned} E_{\text{ext}}(\beta; r, \vartheta) &= \frac{1}{2} (J_p^{(\beta)})^T (T_p^{(\beta, \zeta)})^{-1} J_p^{(\beta)} \\ &\quad + \frac{1}{2} (J_q^{(\beta)})^T (T_q^{(\beta, \zeta)})^{-1} J_q^{(\beta)}. \end{aligned} \quad (\text{C6})$$

For the single-site protocol Eq. (C6) reduces to

$$\begin{aligned} E_{\text{ext}}(\beta, \alpha; d; r, \vartheta) &= \frac{1}{2} \frac{h_d(\beta, \alpha)^2}{h_0(\beta, \alpha) + v_P(r, \vartheta)} \\ &\quad + \frac{1}{2} \frac{\{g_d(\beta, \alpha) - \frac{\alpha}{2}[g_{d-1}(\beta, \alpha) + g_{d+1}(\beta, \alpha)]\}^2}{g_0(\beta, \alpha) + v_X(r, \vartheta)}. \end{aligned} \quad (\text{C7})$$

In the ordered critical limit the q sector again vanishes, so the only surviving effect of squeezing is the replacement

$$\frac{\Omega}{2} \mapsto v_P(r, \vartheta)$$

in the denominator of the momentum-sector contribution. Consequently,

$$E_{\text{ext}}^{\text{crit}}(\beta; d; r, \vartheta) = \frac{1}{2} \frac{[h_d^{\text{crit}}(\beta)]^2}{h_0^{\text{crit}}(\beta) + v_P(r, \vartheta)}. \quad (\text{C8})$$

Thus, squeezing changes the extraction prefactor but not the large-distance scaling class.

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