

Strong energy dependence of transition-radiation intensity in a photonic crystal

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We report a striking E^4 energy dependence of transition-radiation (TR) intensity, observed at Brewster's angle in the optical region and emitted by ultrarelativistic charged particles traversing a specially engineered one-dimensional photonic crystal (PhC). This strong scaling—absent in conventional Cherenkov or TR—persists until saturation sets in at Lorentz factors $\gamma \sim \sqrt{N_s}$ (where N_s is the number of unit cells). The effect originates from a Dirac conelike band structure in the dispersion relation of the PhC, analogous to that of graphene, which emerges when the slab thicknesses within each period satisfy the proportion $a_b/a_e = (\sqrt{\varepsilon + b} - \varepsilon)/(b - \sqrt{\varepsilon + b})$. Remarkably, for both this periodic structure and a complementary class of disordered stacks exhibiting Brewster randomness [Z. Gong *et al.*, *Proc. Natl. Acad. Sci. USA* **122**, e2413336122 (2025)], the radiation problem admits an exact analytical solution at forward and backward Brewster observation angles. In the disordered case, a strong N_s^2 dependence of intensity on slab number N_s yielding pronounced directivity either in the forward or backward direction, depending on the type of randomness, was reproduced. The radiation formation zone remains limited ($\sim \lambda/2$) even for ultrarelativistic particles—unlike x-ray TR detectors—making this system a promising platform for compact, energy-sensitive detection of relativistic particles in the optical regime.

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I. INTRODUCTION

Recently, the interest in the radiation of a charged particle in layered media has resumed [1–4]. This is because of the possibility to fabricate necessary structures at nanoscale nowadays. Radiation from a charged particle can be used for many purposes including as a light source [5–11] in those wavelength regions where other methods are ineffective, detection of relativistic charged particles [12–16], and in biomedical applications [17,18]. In early works on transition radiation (TR) [19–23]—for a recent review see, Ref. [24]—it was believed that, to achieve directivity and high intensity, one needs to use regular structures. However, they had certain disadvantages: Since the observations were performed in the x-ray region, the maximal intensities were observed in the vicinity of the particle trajectory, and the efficiency of these detectors was limited while measuring large Lorentz factors because of fast growth of the radiation formation zone length.

Nevertheless, attention was also paid to radiation in irregular structures [25–27]. The main peculiarity in these random systems is the emergence of the diffusive mechanism of radiation [25,27] along with the ordinary TR. In Ref. [26], the conclusion was to investigate the aperiodic structures as possessing “more degrees of freedom to tailor the particle-interface interaction” compared with periodic

ones. Brewster's angle was highlighted in Ref. [1] as the one which allows us to achieve maximal intensity in observations because of the coherent interference of a particle radiation from each boundary trespassed. However, the other important quantity for the detector, the strong dependence on Lorentz factor of the trespassing particle, was not considered. In fact, no ultrarelativistic particle was considered.

In the periodical case [24], the possibility of generation of Cherenkov-like directed radiation in forward and backward directions was demonstrated. The radiation is concentrated inside the angles close to the particle trajectory and is sensitive to its energy. The energy dependence of varying the radiation angle is suggested to be implemented for detection of relativistic particles. However, Brewster's angle was out of the scope of the article; the main focus was on the observation directions, way closer to the particle trajectory. Additionally, the efficiency in measuring the Lorentz factor was limited even more compared with the traditional x-ray resonance TR (RTR) detectors.

Here, we suggest an alternative approach. Considering radiation from a charged particle at fixed Brewster observation angles, we get strong energy dependence of radiation intensity for a special type of one-dimensional (1D) photonic crystals (PhCs). It is well known that ordinary TR and Cherenkov radiation in the optical region slowly depend on particle energy. We find certain conditions which will allow us to tune the effective region of Lorentz factors for ultrarelativistic particles to be measured.

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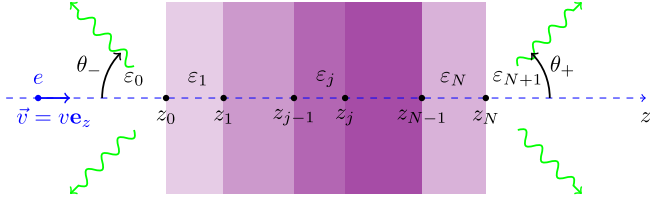


FIG. 1. Illustration of the problem.

II. FORMULATION OF THE PROBLEM

We consider a stack on N dielectric slabs with effective dielectric constants ε_i , $\forall i = \overline{1, N}$, separating the media with dielectric constants ε_0 from the left and ε_{N+1} from the right; see Fig. 1. A particle with charge e traverses the system normal to each slab and with constant velocity $v = c\beta$. We need to calculate the distribution of the radiated power.

We start from Maxwell's equations in the $(\omega, \mathbf{q}, z)$ domain, where ω is the cyclic frequency, and $\mathbf{q} = q_x \mathbf{e}_x + q_y \mathbf{e}_y$ is the transversal momentum of the electromagnetic field $\mathbf{F} = \mathbf{E}, \mathbf{H}, \mathbf{D}$,

$$\mathbf{F}(\mathbf{r}, t) = \int \frac{d\omega}{2\pi} e^{-i\omega t} \iint \frac{d^2 \mathbf{q}}{(2\pi)^2} e^{i\mathbf{q} \cdot \boldsymbol{\rho}} \mathbf{F}(z, \mathbf{q}, \omega), \quad (1)$$

[where $\boldsymbol{\rho} = (x, y) = x\mathbf{e}_x + y\mathbf{e}_y$] for $\mu = 1$ in case of negligible conductivities

$$\begin{aligned} (i\mathbf{q} + \mathbf{e}_z \times \partial_z) \times \mathbf{E} &= i\frac{\omega}{c} \mathbf{H}, \\ (i\mathbf{q} + \mathbf{e}_z \times \partial_z) \times \mathbf{H} &= \frac{4\pi}{c} \mathbf{j}_c - i\frac{\omega}{c} \varepsilon \mathbf{E}, \\ (i\mathbf{q} + \mathbf{e}_z \times \partial_z) \times \mathbf{D} &= 4\pi \rho_c, \\ (i\mathbf{q} + \mathbf{e}_z \times \partial_z) \times \mathbf{H} &= 0, \end{aligned} \quad (2)$$

with

$$\begin{aligned} \mathbf{j}_c(z, \mathbf{q}, \omega) &= \mathbf{e}_z \times e \iint d^2 \boldsymbol{\rho} e^{-i\mathbf{q} \cdot \boldsymbol{\rho}} \delta^{(2)}(\boldsymbol{\rho}) e^{i\frac{\omega}{v} z} = \mathbf{e}_z \times e \times e^{i\frac{\omega}{v} z}, \\ \rho_c(z, \mathbf{q}, \omega) &= \frac{e}{v} e^{i\frac{\omega}{v} z}, \end{aligned} \quad (3)$$

and

$$\varepsilon(z, \omega) = \sum_{i=0}^{N+1} \varepsilon_i(\omega) \times [\Theta(z - z_{i-1}) - \Theta(z - z_i)], \quad (4)$$

where $z_j, j = \overline{0, N}$ are the contact points of slabs with dielectric constants ε_j and ε_{j+1} , and by convention, we set $z_{-1} = -L_z/2$ and $z_{N+1} = L_z/2$. Note that, because for all fields $\mathbf{F}(\mathbf{r}, -\omega) = \mathbf{F}^*(\mathbf{r}, \omega)$, we have $\varepsilon(z, -\omega) = \varepsilon^*(z, \omega)$.

Acting by the operator $(i\mathbf{q} + \mathbf{e}_z \partial_z) \times$ on the second equation in the system in Eq. (2) and making use of all the remaining formulas of the system and the relation $\mathbf{D} \approx \varepsilon \mathbf{E}$, we get

$$\begin{aligned} \left[\frac{\omega^2}{c^2} \varepsilon(z, \omega) - q^2 + \partial_z^2 \right] \mathbf{H} \\ = -\frac{4\pi i e}{c} (\mathbf{q} \times \mathbf{e}_z) e^{i\frac{\omega}{v} z} + i\frac{\omega}{c} (\partial_z \varepsilon) (\mathbf{e}_z \times \mathbf{E}). \end{aligned} \quad (5)$$

Following the classical procedure [19–22], it is enough for us to consider the entire domain $z \in (-L_z/2, L_z/2)$ excluding

contact points z_i , which are the only places where $\partial_z \varepsilon \neq 0$. Note that considering the term $\sim \partial_z \varepsilon$ does not essentially affect the main results since it is enough for us to consider only the limiting values of the fields $\mathbf{F}$ while approaching each of the contact points. Furthermore, we can convert this equation to a 1D form because the source term here is collinear to the unit vector $\mathbf{e}_\Phi = (\mathbf{e}_z \times \hat{\mathbf{q}})$. This implies that the only projection of the magnetic field vector that survives is its projection along exactly that vector

$$\left[\frac{\omega^2}{c^2} \varepsilon(z, \omega) - q^2 + \partial_z^2 \right] H_\Phi = \frac{4\pi i e q}{c} e^{i\frac{\omega}{v} z}. \quad (6)$$

III. BOUNDARY CONDITIONS AND THE SOLUTION

We can find a general solution as a sum of a partial solution of the inhomogeneous equation and the general solution of the corresponding homogeneous one. Starting with the partial solution, we note that the source in the right-hand side of Eq. (6) is $\sim \exp(iz\omega/v)$, suggesting that one searches it in the respective auxiliary form

$$H_\Phi^e = \frac{4\pi i e q}{c} \frac{e^{i\frac{\omega}{v} z}}{\omega^2 \times [\varepsilon(z, \omega)/c^2 - 1/v^2] - q^2}. \quad (7)$$

Note that this partial solution is correct only if we exclude contact points z_i from the consideration domain. We need to add the general form of the solution of Eq. (6) without any sources

$$\begin{aligned} H_\Phi^r &= \sum_{i=0}^{N+1} [\Theta(z - z_{i-1}) - \Theta(z - z_i)] \\ &\times [A_i^+ e^{ik_{i,z}(z-z_i)} + A_i^- e^{-ik_{i,z}(z-z_i)}], \end{aligned} \quad (8)$$

where $k_{i,z} = \text{sgn}(\omega) \sqrt{(\omega/c)^2 \varepsilon_i(\omega) - q^2}$. Such a definition is required to ensure the reality of the fields $\mathbf{F}(\mathbf{r}, t)$.

The coefficients $A_i^\pm$ are determined from the need for electric and magnetic fields to satisfy the boundary conditions at the contact points $z_i, i = \overline{0, N}$.

Because

$$\mathbf{E} = \frac{c}{i\omega \varepsilon(z, \omega)} \left[\mathbf{e}_z \left(\frac{4\pi}{c} j_z - iq H_\Phi \right) + \hat{\mathbf{q}} \partial_z H_\Phi \right], \quad (9)$$

we find that the quantities H_Φ and $\varepsilon^{-1} \partial_z H_\Phi$ need to be continuous. We can write this mathematically as a system of equations

$$\begin{aligned} H_\Phi(z_j^-) &= H_\Phi(z_j^+), \\ \varepsilon^{-1} \partial_z H_\Phi(z_j^-) &= \varepsilon^{-1} \partial_z H_\Phi(z_j^+). \end{aligned} \quad (10)$$

Since

$$\begin{aligned} \Theta(z_j^- - z_{i-1}) - \Theta(z_j^- - z_i) &= \delta_{ij}, \\ \Theta(z_j^+ - z_{i-1}) - \Theta(z_j^+ - z_i) &= \delta_{i,j+1}, \end{aligned} \quad (11)$$

we can write this in matrix form

$$\begin{pmatrix} A_j^+ \\ A_j^- \end{pmatrix} = \hat{B}_j + \hat{M}_j \begin{pmatrix} A_{j+1}^+ \\ A_{j+1}^- \end{pmatrix}, \quad \forall j = \overline{0, N}, \quad (12)$$

with $\hat{M}_j, \hat{B}_j$ presented in Eq. (A1).

Applying the recurrent relation Eq. (12) for the column in the right-hand side over and over, we finally get

$$\begin{pmatrix} A_j^+ \\ A_j^- \end{pmatrix} = \sum_{i=j}^N \left[\prod_{k=j}^{i-1} \hat{M}_k \right] \hat{B}_i + \prod_{k=j}^N \hat{M}_k \begin{pmatrix} A_{N+1}^+ \\ 0 \end{pmatrix}, \quad (13)$$

with the use of the absence of backpropagation in the rightmost medium. For $j = 0$, we use the absence of forward propagating modes in the leftmost medium to obtain the following:

$$A_{N+1}^+ = -B_1/M_{11}, \quad \text{and} \quad A_0^- = B_2 - \frac{M_{21}}{M_{11}} B_1, \quad \text{with} \\ \hat{B} = \sum_{i=0}^N \left[\prod_{k=0}^{i-1} \hat{M}_k \right] \hat{B}_i, \quad \text{and} \quad \hat{M} = \prod_{k=0}^N \hat{M}_k. \quad (14)$$

Note that the fields we obtained so far correspond to so-called pseudophotons [23] rather than the real ones. In general, the sources make the generated total field nontransversal, which means they cannot be associated with real photons. To find the radiation intensities, we need to calculate the energy transferred by the Poynting vector first.

Before proceeding, some preliminary reasoning is required. First, to observe any radiation far enough from the given radiating system, the medium inside which the electromagnetic waves traverse should have a little enough absorption. We will start with the assumption that, for any material, the absorption, described by $\varepsilon_i'' = \Im m(\varepsilon_i)$, is small enough to perform a respective Taylor decomposition, and at the end, we will get a more specific mathematical form of that expression.

The Poynting vector

$$\mathbf{S}(\mathbf{r}, t) = \frac{c}{4\pi} \mathbf{E}(\mathbf{r}, t) \times \mathbf{H}(\mathbf{r}, t) \quad (15)$$

assists us in defining the energy transferred through a given surface Σ with a definite normal $\mathbf{n} = d\mathbf{\Sigma}/d\Sigma$ everywhere on it. Since we assume a negligible absorption of radiation—thus, it just traverses through the matter without causing any changes inside it—the transferred energy should not depend on the surface that encloses the given radiation source

$$W = \int_{-\infty}^{+\infty} dt \int_{\Sigma} d\mathbf{\Sigma} \cdot \mathbf{S} \\ = \frac{c}{4\pi} \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi} \frac{d\omega'}{2\pi} \int_{-\infty}^{+\infty} dt e^{-i(\omega+\omega')t} \\ \times \int_{\Sigma} d\mathbf{\Sigma} \cdot [\mathbf{E}(\mathbf{r}, \omega) \times \mathbf{H}(\mathbf{r}, \omega')]. \quad (16)$$

Since

$$\int_{-\infty}^{+\infty} dt e^{-i(\omega+\omega')t} = 2\pi \delta(\omega + \omega'), \quad (17)$$

we get

$$W = \frac{c}{4\pi} \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi} \int_{\Sigma} d\mathbf{\Sigma} \cdot [\mathbf{E}(\mathbf{r}, \omega) \times \mathbf{H}(\mathbf{r}, -\omega)] \\ = \frac{c}{4\pi^2} \int_0^{+\infty} d\omega \int_{\Sigma} d\mathbf{\Sigma} \cdot \Re[\mathbf{E}(\mathbf{r}, \omega) \times \mathbf{H}(\mathbf{r}, -\omega)], \quad (18)$$

where we used the fact that the fields were real quantities $\mathbf{F}(\mathbf{r}, t) = \mathbf{F}^*(\mathbf{r}, t)$.

Because $\mathbf{H} = \mathbf{H}^e + \mathbf{H}^r$ and using Eq. (9), we get the same decomposition for the electric field $\mathbf{E} = \mathbf{E}^e + \mathbf{E}^r$; we can write the total transmitted energy as the sum of the following three components [19]: $W = W^e + W^r + W^{\text{int}}$, where

$$W^e = \frac{c}{4\pi^2} \int_0^{+\infty} d\omega \int_{\Sigma} d\mathbf{\Sigma} \Re[\mathbf{E}^e(\mathbf{r}, \omega) \times \mathbf{H}^e(\mathbf{r}, -\omega)], \\ W^r = \frac{c}{4\pi^2} \int_0^{+\infty} d\omega \int_{\Sigma} d\mathbf{\Sigma} \Re[\mathbf{E}^r(\mathbf{r}, \omega) \times \mathbf{H}^r(\mathbf{r}, -\omega)], \\ W^{\text{int}} = \frac{c}{4\pi^2} \int_0^{+\infty} d\omega \int_{\Sigma} d\mathbf{\Sigma} \Re[\mathbf{E}^e(\mathbf{r}, \omega) \times \mathbf{H}^r(\mathbf{r}, -\omega) \\ + \mathbf{E}^r(\mathbf{r}, \omega) \times \mathbf{H}^e(\mathbf{r}, -\omega)]. \quad (19)$$

The first component represents the Cherenkov radiation, once the particle velocity exceeds the phase velocity of light propagating in the medium. In the classical literature [28], it can be calculated as an energy flow through the infinite cylinder, with its central axis directed along the trajectory of the particle. It can also be proved that there is no energy transfer from this component if the particle velocity is below the threshold. The formula is as follows:

$$W^e = \frac{e^2}{c^2} \int_{-\infty}^{+\infty} dz \int_0^{+\infty} d\omega \omega \left(1 - \frac{1}{\beta^2 \varepsilon'(z, \omega)} \right) \\ \times \Theta[\beta^2 \varepsilon'(z, \omega) - 1], \quad (20)$$

where $\beta = v/c$.

Consider now the contribution of the TR, i.e., the second term; in the right- and leftmost media, choose Σ as a plane, perpendicular to the z axis, inside the medium and with the normal $\mathbf{n} = \mathbf{e}_z$ and $\mathbf{n} = -\mathbf{e}_z$, respectively. Along with the Fourier decomposition with respect to the transverse momentum, using the identity

$$\int d^2 \boldsymbol{\rho} e^{i(\mathbf{q}+\mathbf{q}')\boldsymbol{\rho}} = (2\pi)^2 \delta(\mathbf{q} + \mathbf{q}') \quad (21)$$

leads to

$$W^{r,\pm} = \pm \frac{c}{4\pi^2} \int \frac{d^2 \mathbf{q}}{(2\pi)^2} \int_0^{+\infty} d\omega \\ \times \Re \left[\frac{c}{i\omega \varepsilon(z, \omega)} H_{\Phi}^{r,*} \partial_z H_{\Phi}^r \right]. \quad (22)$$

Denote $\varepsilon_0 = \varepsilon_-$ and $\varepsilon_{N+1} = \varepsilon_+$ and the z coordinates of the fixed observation point through $z_{\pm}^{\text{obs}}$, respectively, to arrive at

$$W^{r,\pm} = \frac{c}{4\pi} \int \frac{d^2 \mathbf{q}}{(2\pi)^2} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{c}{\omega} |H_{\Phi}^r(z_{\pm}^{\text{obs}}, \mathbf{q}, \omega)|^2 \\ \times \Re \left[\frac{1}{\varepsilon_{\pm}} \sqrt{\frac{\omega^2}{c^2} \varepsilon_{\pm} - q^2} \right]. \quad (23)$$

From $\mathbf{H}(\mathbf{r}, t) = \mathbf{H}^*(\mathbf{r}, t)$, it is easy to prove that

$$A_i^{+*}(\omega, \mathbf{q}) = -A_i^+(-\omega, -\mathbf{q}) \Theta(\sqrt{\varepsilon'_i} |\omega|/c - q) \\ - A_i^-(-\omega, -\mathbf{q}) \Theta(q - \sqrt{\varepsilon'_i} |\omega|/c),$$

$$A_i^{-*}(\omega, \mathbf{q}) = -A_i^{-}(-\omega, -\mathbf{q})\Theta(\sqrt{\varepsilon'_i}|\omega|/c - q) \\ - A_i^{+}(-\omega, -\mathbf{q})\Theta(q - \sqrt{\varepsilon'_i}|\omega|/c), \quad (24)$$

which implies that the Fourier components of the magnetic field outside the PhC are nonzero only for $q < \omega\sqrt{\varepsilon'_\pm}/c$, which allows us to convert the integration over momentum to the integration over the solid angle in the following way: Since the momentum inside each media can be represented as $\mathbf{q} = \sqrt{\varepsilon'_\pm}\omega/c \sin\theta_\pm(\mathbf{e}_x \cos\phi_\pm + \mathbf{e}_y \sin\phi_\pm)$ and since the solid angle is $d\Omega = \sin\theta d\theta d\phi$, we can write

$$\int_{q < \omega\sqrt{\varepsilon'_\pm}/c} \frac{d^2\mathbf{q}}{(2\pi)^2} = \left(\frac{\omega\sqrt{\varepsilon'_\pm}}{2\pi c}\right)^2 \int_{(2\pi)^+} d\Omega_\pm \cos\theta_\pm. \quad (25)$$

Note that, for the reflected waves, the count of the polar θ_- angle starts from the negative direction along the z axis. This allows us to write the energy flux as an integral over the solid angle and all the possible frequencies

$$W^{r,\pm} = \int_0^\infty d\omega \int_{(2\pi)^+} d\Omega_\pm \left(\frac{\omega}{4\pi^2 c}\right)^2 \\ \times \sqrt{\varepsilon'_\pm} \cos^2\theta_\pm |A^\pm(\omega, \theta_\pm)|^2 e^{\mp 2\Im[k_z^\pm](z_\pm^{\text{obs}} - z_\pm)}. \quad (26)$$

Note the exponential decay term. It is tailored to the absorption properties of the medium, while it can be neglected if $|z_\pm^{\text{obs}} - z_\pm| \ll 1/2\Im[k_z^\pm]$.

We omitted the subindex for each of the coefficients $A^\pm$ since it is apparent that the $+$ sign refers to A_{N+1}^+ and the $-$ sign refers to A_0^- . Thus, the spectral angular intensity

$$\frac{\partial W^{r,\pm}}{\partial \omega \partial \Omega_\pm} \equiv I^{r,\pm}(\omega, \theta_\pm) \\ = c \left(\frac{\omega}{4\pi^2 c}\right)^2 \sqrt{\varepsilon'_\pm} \cos^2\theta_\pm |A^\pm(\omega, \theta_\pm)|^2. \quad (27)$$

The third component represents the interference between the solutions of homogeneous and inhomogeneous wave equation. It can be proved [19] that this component contains the exponential term $\exp[-iz/z^\pm(\theta)]$, where $z^\pm(\theta) = \pi/|\omega/v \mp k_z(\theta)|$ is called the correlation length or the radiation formation zone length, which becomes highly oscillating for a fixed but large $z \gg z^\pm(\theta)$, once we try to integrate over a range of angles and frequencies; thus, its contribution to the total transmitted energy vanishes.

Thus, the distance of the observation point from the right- or leftmost slab along the z axis should satisfy the relation

$$\frac{\pi}{|\omega/v \mp k_z^\pm|} \ll |z - z_\pm| \ll \frac{1}{2\Im[k_z^\pm]}, \quad (28)$$

which can be satisfied only if

$$\varepsilon''(\omega) \ll \sqrt{\varepsilon'_\pm} \cos\theta_\pm |\beta^{-1} \mp \sqrt{\varepsilon'_\pm} \cos\theta_\pm|. \quad (29)$$

Apparently, for both forward and backward propagating waves, this can be violated at the observation almost perpendicular to the z axis, but for the forward propagating waves, this can be so also near the Cherenkov angle.

Such a general statement of the problem as in Eq. (14) allows us to consider various particular cases. For example,

by solving the problem in the case of two media, separated by a plane interface ($N = 0$), we can reproduce the results, derived by Garibyan and Shi [19] for the intensity, integrated over angles and frequencies $W^+ \sim \gamma$ and concentrated along the velocity of the particle with angular width $\theta \sim \gamma^{-1}$, and for the length of the radiation formation zone, we get $\sim \gamma$ since the cutoff frequency $\omega_{\min} \sim \omega_p \gamma$, where ω_p is the plasma frequency, which characterizes the dispersion of the medium.

Observables at Brewster's angle

Once the condition $\varepsilon^{-1}k_z^\varepsilon = b^{-1}k_z^b = \omega/(c\sqrt{\varepsilon + b})$ is satisfied (by the very definition of Brewster's angle), the intensities, observed at $\theta_\pm = \theta_{Br}$, could be drastically simplified: Since we have Eq. (A2), we get the amplitudes as $A^+ = -\exp[i \sum_{k=0}^{N_s-1} (k_z^b d_{2k+1} + k_z^\varepsilon d_{2k+2})](\hat{B})_1$ and $A^- = (\hat{B})_2$.

Apparently, the maximum for the intensity is achieved in the case of Brewster randomness, when the thickness of each slab is an odd multiplier of the respective correlation length

$$(\omega/v \mp k_z^\varepsilon) d_{2j+2} = \pi(2m_j^\varepsilon + 1), \\ (\omega/v \mp k_z^b) d_{2j+1} = \pi(2m_j^b + 1). \quad (30)$$

The spectral-angular intensity becomes $\sim N_s^2$.

IV. PHOTONIC CRYSTAL

Consider a particular case of a stratified system. For a system of N_s stacks with the dielectric constant b (we take the imaginary part to be zero for any dielectric constant) and arbitrary thicknesses and arbitrary distances between each other, inside the medium with the dielectric constant ε , we have the following:

$$\varepsilon_k = \varepsilon \delta_{0,k \bmod 2} + b \delta_{1,k \bmod 2}, \quad \forall k = \overline{0, 2N_s}. \quad (31)$$

With this

$$\hat{B} = \sum_{j=0}^{N_s-1} \left[\prod_{k=0}^{j-1} \hat{m}_k \right] \hat{C}_j, \\ \hat{M} = \prod_{k=0}^{N_s-1} \hat{m}_k, \quad \text{where} \\ \hat{m}_k = \hat{M}_{2k} \hat{M}_{2k+1} = \begin{pmatrix} (m_k)_{11} & (m_k)_{12} \\ (m_k)_{12}^* & (m_k)_{11}^* \end{pmatrix}, \\ \hat{C}_j = \hat{B}_{2j} + \hat{M}_{2j} \hat{B}_{2j+1} = \begin{pmatrix} (C_j)_1 \\ (C_j)_2 \end{pmatrix}, \quad (32)$$

with more details in Eq. (A3).

When these stacks are configured in a periodic structure, so that $d_{2j+1} = a_b$ and $d_{2j+2} = a_\varepsilon$ for $\forall j = \overline{0, N_s-1}$ (we can truncate the infinite rightmost medium to a slab with thickness a_ε , where only the forward propagating mode survives), with a period $l = a_b + a_\varepsilon$, we can write from Eq. (A3)

$$\hat{C}_j = e^{ijl\omega/v} \hat{C}_0, \quad \forall j = \overline{0, N_s-1}, \\ \hat{C}_0 \equiv \hat{C} = \begin{pmatrix} C_1 \\ C_2 \end{pmatrix},$$

$$\hat{m}_k = \hat{m} = \begin{pmatrix} m_{11} & m_{12} \\ m_{12}^* & m_{11}^* \end{pmatrix}, \quad \forall k = \overline{0, N_s - 1}, \quad (33)$$

thus,

$$\begin{aligned} \hat{B} &= \sum_{j=0}^{N_s-1} \hat{m}^j \hat{C}_j, \\ \hat{M} &= \hat{m}^{N_s}. \end{aligned} \quad (34)$$

To calculate the powers of matrix $\hat{m}$, the easiest way is to check whether it can be diagonalized. To do this, solve the eigenvalue problem $\hat{m}\bar{u} = \lambda\bar{u}$ or

$$\begin{pmatrix} m_{11} - \lambda & m_{12} \\ m_{12}^* & m_{11}^* - \lambda \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad (35)$$

which can be easily solved

$$\begin{aligned} \lambda &= \Gamma \pm \sqrt{\Gamma^2 - 1}, \\ \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} &= \begin{pmatrix} m_{12} \\ \lambda - m_{11} \end{pmatrix}, \end{aligned} \quad (36)$$

where we used $\det \hat{m} = 1$ and $\Gamma = \Re[m_{11}] = \text{Tr}[\hat{m}]/2$. Denote both eigenvalues and their respective eigenvectors by $\lambda_{\pm}$ and $\bar{u}_{\pm}$ and consider the matrix

$$\begin{aligned} \hat{P} &= (\bar{u}_+, \bar{u}_-) = \begin{pmatrix} m_{12} & m_{12} \\ \lambda_+ - m_{11} & \lambda_- - m_{11} \end{pmatrix}, \\ \hat{m}\hat{P} &= \hat{P} \begin{pmatrix} \lambda_+ & 0 \\ 0 & \lambda_- \end{pmatrix}, \\ \det \hat{P} &= m_{12}(\lambda_- - \lambda_+) \sim m_{12}\sqrt{\Gamma^2 - 1}. \end{aligned} \quad (37)$$

The matrix can be inverted if and only if it is not degenerate, which is achieved for $\hat{P}$ if neither $m_{12} = 0$ nor $\Gamma = \pm 1$. In this case, we have

$$\begin{aligned} \hat{m} &= \hat{P} \begin{pmatrix} \lambda_+ & 0 \\ 0 & \lambda_- \end{pmatrix} \hat{P}^{-1}, \\ \hat{m}^j &= \hat{P} \begin{pmatrix} \lambda_+^j & 0 \\ 0 & \lambda_-^j \end{pmatrix} \hat{P}^{-1}, \end{aligned} \quad (38)$$

which gives a compact expression for

$$\begin{aligned} \hat{B} &= \hat{P} \begin{pmatrix} f_{N_s}(\lambda_+) & 0 \\ 0 & f_{N_s}(\lambda_-) \end{pmatrix} \hat{P}^{-1}, \\ \hat{M} &= \hat{P} \begin{pmatrix} \lambda_+^{N_s} & 0 \\ 0 & \lambda_-^{N_s} \end{pmatrix} \hat{P}^{-1}, \\ \hat{D} &= \hat{P}^{-1} \hat{C}, \\ f_n(\lambda) &= \sum_{j=0}^{n-1} [\lambda e^{i\ell\omega/v}]^j = \frac{1 - \lambda^n \exp(in\ell\omega/v)}{1 - \lambda \exp(i\ell\omega/v)}, \end{aligned} \quad (39)$$

and for the amplitudes,

$$\begin{aligned} A^+ &= -\frac{P_{1\alpha} f_{N_s}(\lambda_{\alpha}) D_{\alpha}}{P_{1\beta} \lambda_{\beta}^{N_s} (P^{-1})_{\beta 1}}, \\ A^- &= \left[P_{2\alpha} - \frac{P_{2\beta} \lambda_{\beta}^{N_s} (P^{-1})_{\beta 1}}{P_{1\gamma} \lambda_{\gamma}^{N_s} (P^{-1})_{\gamma 1}} P_{1\alpha} \right] f_{N_s}(\lambda_{\alpha}) D_{\alpha}. \end{aligned} \quad (40)$$

Here, we will focus on their behavior for $N_s \gg 1$. Since $\lambda_+ \lambda_- = 1$, we have only two cases to consider: (1) $|\lambda_+| \neq |\lambda_-|$ or (2) $|\lambda_+| = |\lambda_-|$.

Case 1: That means that there is an index M so that $|\lambda_M| > 1$, while the other one $\lambda_m = \lambda_M^{-1}$ emerges when $|\Gamma| > 1$.

Expanding in powers of λ_M , it is easy to get

$$\begin{aligned} A^+ &\approx \frac{\exp(iN_s \ell\omega/v)}{1 - \lambda_M \exp(i\ell\omega/v)} \frac{D_M}{(P^{-1})_{M1}} + O(\lambda_M^{-N_s}), \\ A^- &\approx \left(P_{2m} - \frac{P_{2M}}{P_{1M}} P_{1m} \right) \frac{D_m}{1 - \lambda_M^{-1} \exp(i\ell\omega/v)} + O(\lambda_M^{-N_s}). \end{aligned} \quad (41)$$

Case 2: Denote $\lambda_{\pm} = \exp(\mp iQ_z \ell)$, where Q_z is determined using the equation $\cos(Q_z \ell) = \Gamma$. It represents the quasimomentum of the Bloch eigenmodes of the electromagnetic waves inside the PhC with the respective periodic structure.

Considering that, for $N_s \gg 1$, we have

$$\begin{aligned} f_{N_s}(\lambda_{\pm}) &= e^{i(N_s-1)\ell(\omega/v \mp Q_z)/2} \frac{\sin[(\omega/v \mp Q_z)\ell N_s/2]}{\sin[(\omega/v \mp Q_z)\ell/2]} \\ &\approx (2\pi/\ell) \delta([\omega/v \mp Q_z] \bmod 2\pi/\ell), \end{aligned} \quad (42)$$

we get Eq. (A4), which explains sharp peaks near the crossings with the isofrequency curve, observed in Ref. [24].

So far, we assumed $\det \hat{P} \neq 0$. Now consider the case $\det \hat{P} = 0$. Also, two cases are possible: (1) $m_{12} = 0$ or (2) $\Gamma = \pm 1$.

Case 1: This means the transfer matrix $\hat{m}$ is already diagonal. This case emerges when either (1) there is an observation along Brewster's angle, i.e., $\varepsilon^{-1} k_z^e = b^{-1} k_z^b$, or (2) for the given observation angle, the slab thickness satisfies $\sin(k_z^b a_b) = 0$.

Applying steps similar to the previous case, we get

$$\begin{aligned} A^+ &\approx -\frac{2\pi}{\ell} e^{iQ_z \ell N_s} C_1 \delta([\omega/v - Q_z] \bmod 2\pi/\ell), \\ A^- &\approx \frac{2\pi}{\ell} C_2 \delta([\omega/v + Q_z] \bmod 2\pi/\ell). \end{aligned} \quad (43)$$

Finally, case 2: The transfer matrix is nondiagonalizable. When $\Gamma = \Re[m_{11}] = \pm 1 \equiv (-1)^A$, using $\det \hat{m} = |m_{11}|^2 - |m_{12}|^2 = 1$ and $m_{11} = (-1)^A + i\Im[m_{11}]$, it is easy to show that the following matrix

$$\begin{aligned} \hat{N} &\equiv \hat{m} - (-1)^A \mathbf{1} = \begin{pmatrix} i(-1)^B |m_{12}| & m_{12} \\ m_{12}^* & -i(-1)^B |m_{12}| \end{pmatrix}, \\ \hat{N}^2 &= \hat{0} \end{aligned} \quad (44)$$

is nilpotent. This fact allows us to find the powers of $\hat{m}$ as follows:

$$\hat{m}^j = [(-1)^A \mathbf{1} + \hat{N}]^j = (-1)^{Aj} \mathbf{1} + j(-1)^{A(j-1)} \hat{N}. \quad (45)$$

It remains to denote $\xi \equiv \exp(i\eta) \equiv (-1)^A \exp(i\ell\omega/v)$ to get

$$\begin{aligned} \hat{B} &= \left[\frac{1 - \xi^{N_s}}{1 - \xi} \mathbf{1} + (-1)^A \frac{\xi}{1 - \xi} \left(\frac{1 - \xi^{N_s}}{1 - \xi} - N_s \xi^{N_s-1} \right) \hat{N} \right] \hat{C}, \\ \hat{M} &= (-1)^{AN_s} [\mathbf{1} + N_s (-1)^A \hat{N}]. \end{aligned} \quad (46)$$

It is easy to show that, even for $N_s|m_{12}| \gg 1$, the amplitudes $A^\pm$ cannot become of the order $\sim N_s^2$.

Thus, in all cases, except for the first one, $|A^\pm| \sim N_s$, and hence, the spectral-angular intensity $I \sim N_s^2$.

Assume our PhC consists of materials that are nondispersive in the frequency region of interest. With this,

$$\hat{C} = \frac{2\pi i e}{\omega} e^{iz_0\omega/v} \hat{\mathbf{C}}(\tilde{a}_b, \tilde{a}_\varepsilon, b, \varepsilon, \theta),$$

$$\hat{B} = \sum_{j=0}^{N_s-1} (e^{il\omega/v} \hat{m})^j \hat{C} = \frac{2\pi i e}{\omega} e^{iz_0\omega/v} \hat{\mathbf{B}}(\tilde{a}_b, \tilde{a}_\varepsilon, b, \varepsilon, \theta), \quad (47)$$

where details are presented in Eq. (A5), and we used $l = a_b + a_\varepsilon$ and $\tilde{a}_X = a_X/\lambda$, $X = b, \varepsilon$. This leads to

$$|A^\pm|^2 = \frac{4\pi^2 e^2}{\omega^2} |\alpha^\pm(\tilde{a}_b, \tilde{a}_\varepsilon, b, \varepsilon, \theta)|^2,$$

$$\frac{1}{\hbar} I^{r,\pm}(\omega, \theta) = \alpha \frac{\sqrt{\varepsilon'} \cos^2 \theta}{4\pi^2} |\alpha^\pm(\tilde{a}_b, \tilde{a}_\varepsilon, b, \varepsilon, \theta)|^2, \quad (48)$$

where $\alpha = e^2/\hbar c \approx 1/137$ is the fine-structure constant.

The following results lead to specific scaling rules for the angular intensity $\mathbf{I}(\theta) = \int d\omega I(\omega, \theta)$. If we could extrapolate the integration over all possible (positive) frequencies, we would get for the crystal, scaled by the factor $x > 0$: $(a_b, a_\varepsilon) \rightarrow (xa_b, xa_\varepsilon)$; we get for the angular intensity

$$\begin{aligned} \mathbf{I}(\theta; xa_b, xa_\varepsilon) &= \int_0^\infty d\omega I[\omega xa_b/(2\pi c), \omega xa_\varepsilon/(2\pi c)] \\ &= \frac{1}{x} \int_0^\infty d\omega' I[\omega' a_b/(2\pi c), \omega' a_\varepsilon/(2\pi c)] \\ &= \frac{1}{x} \mathbf{I}(\theta; a_b, a_\varepsilon), \end{aligned} \quad (49)$$

where we performed the linear transformation of the integration variable $\omega \rightarrow \omega' = x\omega$. This suggests the implementation of PhC with possibly the smallest period l .

With this, we can define the following scaling-invariant quantities:

$$\begin{aligned} \frac{l}{2\pi \hbar c} \frac{dW}{d\Omega} &= \int_{y_{\min}}^{y_{\max}} dy I[y(a_b/l), y(a_\varepsilon/l)], \\ \frac{2\pi c}{l} \frac{dN}{d\omega d\Omega} &= y^{-1} I[y(a_b/l), y(a_\varepsilon/l)], \\ \frac{c}{l \sin \theta_{Br}} \frac{dN}{d\omega} &\approx \frac{2\pi c}{l \hbar \omega} \int d\theta \frac{dW}{d\omega d\Omega} \\ &= y^{-1} \int d\theta I[y(a_b/l), y(a_\varepsilon/l)], \end{aligned} \quad (50)$$

where $y = \omega l/(2\pi c) = \tilde{l}$ is a dimensionless quantity, and $dN/d\omega$ is the differential yield of photons, integrated over the solid angle along Brewster's observation angle, i.e., $\int d\Omega = 2\pi \int \sin \theta d\theta \approx 2\pi \sin \theta_{Br} \int d\theta$. Apparently, they all remain invariant with regard to simultaneous transformation $a_{\varepsilon,b} \rightarrow xa_{\varepsilon,b}$ and $\lambda \rightarrow x\lambda$.

For $N_s \gg 1$, highly oscillating terms can emerge solely from the part $\sum_{j=0}^{N_s-1} (e^{i2\pi \tilde{l} \beta^{-1}} \hat{m})^j$. If we exclude the special case $|\Gamma| \neq 1$, the matrix $\hat{m}$ becomes diagonalizable, and the term mentioned above reduces to a combination of $f_{N_s}(\lambda_\pm)$.

Thus, we proceed with considering the oscillatory behavior of these terms. We will consider the case of absence of any absorption since, in a more general case of a small absorption, the main results could be retrieved by replacement $\varepsilon \rightarrow \varepsilon'$ and $b \rightarrow b'$.

Consider the approximations for ultrarelativistic particles ($\gamma \gg 1$). In this case, along with $\tilde{Q}_z = Q_z/(\omega/c) = \tilde{Q}_z(\tilde{a}_b, \tilde{a}_\varepsilon, b, \varepsilon, \theta)$ and $\tilde{l} = l/\lambda$, we get

$$\begin{aligned} f_{N_s}[\lambda_+ = \exp(-iQ_z l)] \\ \approx \frac{1 - \exp(2\pi i N_s \tilde{l} [1 - \tilde{Q}_z + \gamma^{-2}/2])}{1 - \exp(2\pi i \tilde{l} [1 - \tilde{Q}_z + \gamma^{-2}/2])}, \end{aligned} \quad (51)$$

where we used $c/v = \beta^{-1} \approx 1 + \gamma^{-2}/2$. If PhC allows for the fulfillment of condition $\tilde{Q}_z \equiv 1$ for any frequency $\omega = 2\pi c/\lambda$ in the region of interest, the exponent in the denominator will stay close to 1, if $\tilde{l} \ll \gamma^2$ in the region of interest. This condition is automatically fulfilled since $\tilde{Q}_z \in (0, 1/2\tilde{l})$, and once $\tilde{Q}_z = 1$, it requires $\tilde{l} < 1/2 \ll \gamma^2$.

At Brewster's angle, $Q_z l = \pm(k_z^b a_b + k_z^\varepsilon a_\varepsilon) + 2\pi n$, where the sign in front of the brackets and the integer n should be chosen so that $Q_z l \in (0, \pi)$. With this,

$$\tilde{Q}_z = \pm \frac{ba_b + \varepsilon a_\varepsilon}{l\sqrt{b + \varepsilon}} + \frac{n}{\tilde{l}}. \quad (52)$$

Only the second term in the right-hand side is λ dependent, unless $n = 0$. The first term can be $=1$ only in case of the $+$ sign (satisfied automatically since it is required to be $\tilde{Q}_z > 0$ by the very definition), and

$$\frac{a_b}{a_\varepsilon} = \frac{\sqrt{\varepsilon + b} - \varepsilon}{b - \sqrt{\varepsilon + b}}. \quad (53)$$

The right-hand side of this equation can be positive only in two cases:

$$\begin{aligned} \text{either} \quad b &> \max\left(\frac{1}{2} + \sqrt{\varepsilon + \frac{1}{4}}, \varepsilon(\varepsilon - 1)\right), \\ \text{or} \quad b &< \min\left(\frac{1}{2} + \sqrt{\varepsilon + \frac{1}{4}}, \varepsilon(\varepsilon - 1)\right). \end{aligned} \quad (54)$$

Note that the corresponding radiation-formation zone lengths are

$$z_X^\pm \approx \frac{\lambda}{2(1 \mp \varepsilon_X/\sqrt{\varepsilon + b})}, \quad X = \varepsilon, b, \quad (55)$$

where we neglected the term $\gamma^{-2} \ll 1$ in the denominator. Thus, the correlation length stays limited for the ultrarelativistic particles, unlike the case of an x-ray TR detector (TRD), where $z^+ \sim \gamma$.

From now on, we consider the case $\varepsilon = 1$. Interestingly, this condition can be satisfied for the dielectrics above a threshold dielectric constant, given by $b_{\text{threshold}} = (1 + \sqrt{5})/2 \approx 1.618\dots$, the golden ratio. Also, the numerical simulation in Fig. 2 shows that the z component of the group velocity of the waves, corresponding to the eigenvalue $\exp(-iQ_z l)$, is nonnegative once $\tilde{l} < 1/2$. This shows that modes corresponding to the eigenvalue λ_+ indeed propagate in the positive z direction [29].

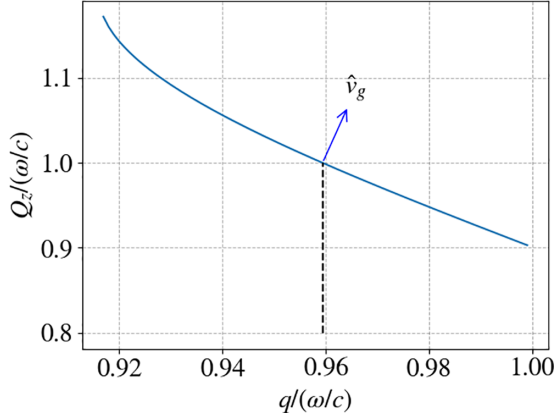


FIG. 2. Isofrequency curve for $\lambda = 2.4l$. The parameters of PhC are chosen as follows: $\varepsilon = 1$, $b = 3.4^2 = 11.56$, and $a_b/a_e = (\sqrt{\varepsilon + b} - \varepsilon)/(b - \sqrt{\varepsilon + b})$. The black vertical line shows the position of q_{Br} , and the blue vector shows the direction of the group velocity of the electromagnetic field.

Through Eq. (14) for the periodic system with the structure given in Eq. (53), we get the forward-propagating amplitude

$$A^+ \sim \frac{1 - \exp(iN_s \omega l [v^{-1} - c^{-1}])}{1 - \exp(i\omega l [v^{-1} - c^{-1}])}. \quad (56)$$

A. Saturation region

For ultrarelativistic particles $\gamma \gg 1$ and $v^{-1} - c^{-1} \approx \gamma^{-2}/(2c)$, we can write $A^+ \sim N_s$ once $N_s \omega l / (2c\gamma^2) \ll 1$ or $\gamma \gg \sqrt{N_s \omega l / 2c}$.

This threshold for the Lorentz factor increases along with N_s . Above the threshold, there is a saturation effect, where the spectral angular intensity is no longer γ dependent.

B. Below the saturation region

Consider the region below the threshold, where the particle is still ultrarelativistic $1 \ll \gamma \ll \sqrt{\pi N_s \tilde{l}}$. Then

$$\begin{aligned} \pi N_s \tilde{l} \gamma^{-2} &\gg 1 \rightarrow \text{oscillating numerator,} \\ \pi \tilde{l} \gamma^{-2} &\ll 1 \rightarrow \text{stable exponent in denominator,} \end{aligned} \quad (57)$$

in Eq. (51). Integral in the frequency region, $\omega \in (2\pi c/\lambda_{\max}, 2\pi c/\lambda_{\min})$, where $\lambda_{\min} > 2l$ and all materials should be nondispersive in the region between maximal and minimal frequencies.

Since we have

$$\begin{aligned} I^+(\omega, \theta_{Br}) &= \alpha \hbar \frac{\sqrt{\varepsilon} \cos^2 \theta_{Br}}{4\pi^2} |f_{N_s}[\exp(-iQ_z l)]|^2 |[\hat{C}(\theta_{Br})]_1|^2, \\ I^-(\omega, \theta_{Br}) &= \alpha \hbar \frac{\sqrt{\varepsilon} \cos^2 \theta_{Br}}{4\pi^2} |f_{N_s}[\exp(+iQ_z l)]|^2 |[\hat{C}(\theta_{Br})]_2|^2, \end{aligned} \quad (58)$$

and for estimation purposes, we can represent the integral over frequencies of a product of highly and slowly oscillating functions F and C , respectively, as $\int d\omega F(\omega) C(\omega) = C(\omega^*) \int d\omega F(\omega)$, where $\omega^* \in (\omega_{\min}, \omega_{\max})$. It remains for us to consider the properties of the integrals

$\int d\omega |f_{N_s}(\exp(\mp Q_z l))|^2$. Since $\tilde{l} < 1/2 \rightarrow \omega_{\max} < \pi c/l$,

$$\begin{aligned} \int_{\omega_{\min}}^{\omega_{\max}} d\omega |f_{N_s}[\exp(-iQ_z l)]|^2 &\approx 2 \int_{\omega_{\min}}^{\omega_{\max}} d\omega \frac{1 - \cos\left(N_s \frac{\omega l}{2c\gamma^2}\right)}{\left(\frac{\omega l}{2c\gamma^2}\right)^2} \\ &= \frac{4c\gamma^2}{l} \int_{y_{\min}}^{y_{\max}} dy \frac{1 - \cos N_s y}{y^2}, \end{aligned} \quad (59)$$

where $y = \omega l / 2c\gamma^2 = \pi \tilde{l} \gamma^{-2}$. Since we have

$$\begin{aligned} \frac{1 - \cos N_s y}{y^2} dy &= d\left(-\frac{1 - \cos N_s y}{y}\right) + N_s \frac{\sin N_s y}{y} dy, \\ \lim_{N_s \rightarrow \infty} \frac{\sin N_s y}{y} &= \pi \delta(y), \end{aligned} \quad (60)$$

and $y_{\min} > 0$, we get finally

$$\begin{aligned} \int_{\omega_{\min}}^{\omega_{\max}} d\omega |f_{N_s}[\exp(-iQ_z l)]|^2 &\approx \frac{4c}{\pi l} \gamma^4 \left[\frac{1 - \cos(N_s \pi \tilde{l}_{\min} \gamma^{-2})}{\tilde{l}_{\min}} - \frac{1 - \cos(N_s \pi \tilde{l}_{\max} \gamma^{-2})}{\tilde{l}_{\max}} \right]. \end{aligned} \quad (61)$$

Thus, for the waves propagating in the forward direction at Brewster's angle with regard to the z axis, we have for the angular intensity $I^+(\theta = \theta_{Br}) \sim \gamma^4$. Now for the second integral,

$$\begin{aligned} \int_{\omega_{\min}}^{\omega_{\max}} d\omega |f_{N_s}[\exp(+iQ_z l)]|^2 &\approx \int_{\omega_{\min}}^{\omega_{\max}} d\omega \left| \frac{1 - \exp(iN_s \omega 2l/c)}{1 - \exp(i\omega 2l/c)} \right|^2 \\ &= \frac{c}{2l} \int_{y_{\min}}^{y_{\max}} dy \left| \frac{1 - \exp(iN_s y)}{1 - \exp(iy)} \right|^2 \\ &\rightarrow \frac{c}{2l} N_s \int_{y_{\min}}^{y_{\max}} dy \delta(y \bmod 2\pi), \end{aligned} \quad (62)$$

where $y = \omega 2l/c$. Considering that $y_{\max} < 2\pi$ and $y_{\min} > 0$, the integral of the δ function vanishes $I^-(\theta = \theta_{Br}) \approx 0$.

C. Small angular deviations

Now consider the vicinity of Brewster's angle. We will neglect the corrections of order $O(\delta\theta^2)$, keeping only linear terms. Thus, with this approximation,

$$\delta\left(\frac{\varepsilon^{-1} k_z^\varepsilon}{b^{-1} k_z^b} + \frac{b^{-1} k_z^b}{\varepsilon^{-1} k_z^\varepsilon}\right) \approx 0, \quad (63)$$

since near Brewster's angle $\varepsilon^{-1} k_z^\varepsilon = b^{-1} k_z^b$. Hence, $\delta\Gamma = \delta[\cos(k_z^\varepsilon a_\varepsilon + k_z^b a_b)]$, and

$$\delta\tilde{Q}_z = -\delta\theta \sqrt{\frac{\varepsilon}{b}} [\sqrt{\varepsilon + b} - 1] = \tilde{Q}_z - 1. \quad (64)$$

With this, the highly oscillatory function

$$f_{N_s}[\exp(-iQ_z l)] \approx \frac{1 - \exp(i2\pi \tilde{l} N_s [-\delta\tilde{Q}_z + \gamma^{-2}/2])}{1 - \exp(i2\pi \tilde{l} [-\delta\tilde{Q}_z + \gamma^{-2}/2])} \quad (65)$$

achieves its maximum at $\delta\theta_\gamma = -\frac{1}{2\gamma^2} \frac{\sqrt{b/\varepsilon}}{\sqrt{\varepsilon+b}-1}$, and its width is independent of γ : Introducing $\delta\theta' = \delta\theta + \delta\theta_\gamma = \delta\theta - |\delta\theta_\gamma|$, we require

$$|\delta\theta'| \ll \frac{\sqrt{b/\varepsilon}}{\sqrt{\varepsilon+b}-1}, \quad (66)$$

for the denominator to be stable, as in previous cases.

D. Deviations from the proportion in Eq. (53)

If we consider deviations from the proportion $f_0^\varepsilon = a_\varepsilon^0/l = (b - \sqrt{b+\varepsilon})/(b-\varepsilon) \rightarrow f^\varepsilon = a_\varepsilon/l = f_0^\varepsilon - \delta f$ —hence, $f_b = 1 - f_\varepsilon = f_b^0 + \delta f$ —we will get

$$\tilde{Q}_z = 1 + \frac{b-\varepsilon}{\sqrt{\varepsilon+b}} \delta f. \quad (67)$$

This shifts the peak of the spectral-angular intensity to those γ values, which are determined by

$$\frac{\gamma^{-2}}{2} - \frac{b-\varepsilon}{\sqrt{\varepsilon+b}} \delta f = 0. \quad (68)$$

For $\delta f > 0$, it has a real solution, and the peak width is determined by

$$N_s \tilde{l} \left| \frac{\gamma^{-2}}{2} - \frac{b-\varepsilon}{\sqrt{\varepsilon+b}} \delta f \right| \sim 1. \quad (69)$$

Outside that region, determined by the relation

$$N_s \tilde{l} \left| \frac{\gamma^{-2}}{2} - \frac{b-\varepsilon}{\sqrt{\varepsilon+b}} \delta f \right| \gg 1 \gg \tilde{l} \left| \frac{\gamma^{-2}}{2} - \frac{b-\varepsilon}{\sqrt{\varepsilon+b}} \delta f \right|, \quad (70)$$

$$\int_{\theta_{Br}-\Delta\theta}^{\theta_{Br}+\Delta\theta} d\theta |f_{N_s}[\exp(-iQ_z l)]|^2 \approx \int_{-\Delta\theta}^{+\Delta\theta} d(\delta\theta) \left| \frac{1 - \exp\{iN_s 2\pi \tilde{l} [\gamma^{-2}/2 + \delta\theta \sqrt{\varepsilon/b}(\sqrt{\varepsilon+b}-1)]\}}{1 - \exp\{i2\pi \tilde{l} [\gamma^{-2}/2 + \delta\theta \sqrt{\varepsilon/b}(\sqrt{\varepsilon+b}-1)]\}} \right|^2. \quad (75)$$

The integration function has a sharp peak at $\delta\theta = \delta\theta_\gamma$ with a width $\delta\theta_{\text{peak}} = N_s^{-1} \tilde{l}^{-1} (\sqrt{\varepsilon+b}-1)^{-1} \sqrt{b/\varepsilon}$. This means that, if the integration window is too large to include $\delta\theta_\gamma \pm \delta\theta_{\text{peak}}$ for any γ inside the region of interest, the integrated intensity will become γ independent, which is not a preferable operation region for the particle energy detection.

V. ACCOUNTING THE LOSSES

If we consider the possibility of $\Im(\varepsilon) = \varepsilon'' > 0$, $\Im(b) = b'' > 0$, the transfer matrix takes a more complicated form, see Eq. (A6).

However, its eigenvalues are still determined by $\lambda = \Gamma \pm \sqrt{\Gamma^2 - 1}$, $\Gamma = (1/2)\text{Tr}[\hat{m}]$. Here, we will consider only the small losses which nonetheless can impose a restriction on total number of slabs in a stack.

we have a highly oscillating numerator and a slowly varying denominator in $I(\omega, \theta)$, which after integration over frequencies, leads to

$$\mathbf{I}^+ \sim \frac{\gamma^4}{\left| \frac{1}{2} - \frac{b-\varepsilon}{\sqrt{b+\varepsilon}} \delta f \gamma^2 \right|^2}. \quad (71)$$

E. Dealing with deviations δf

Considering the contributions of deviations from θ_{Br} and the ideal proportion $f_0^{\varepsilon,b}$ to the phase

$$\delta\tilde{Q}_z = \frac{b-\varepsilon}{\sqrt{\varepsilon+b}} \delta f - \sqrt{\frac{\varepsilon}{b}} [\sqrt{\varepsilon+b}-1] \delta\theta, \quad (72)$$

we immediately note that the linear terms could be canceled out once we choose

$$\delta\theta = \frac{b-\varepsilon}{\sqrt{\varepsilon+b}[\sqrt{\varepsilon+b}-1]} \sqrt{\frac{b}{\varepsilon}} \delta f. \quad (73)$$

This means that small deviations from the ideal proportion shift the angular position of the peak according to the formula above. Inside the detector, the influence of δf could be canceled out by slightly shifting the observation direction from θ_{Br} .

F. Angular integrated intensity

Consider the spectral-angular intensity, integrated over a window of angles that includes θ_{Br} (we consider here the case $\delta f = 0$ since other cases could be brought to the case of ideal proportion by the shift of observation angle, as shown in the previous section)

$$\begin{aligned} \frac{dW^+}{d\omega} &\sim \int_{\theta_{Br}-\Delta\theta}^{\theta_{Br}+\Delta\theta} d\theta \sin\theta |f_{N_s}[\exp(-iQ_z l)]|^2 \\ &\approx \sin\theta_{Br} \int_{\theta_{Br}-\Delta\theta}^{\theta_{Br}+\Delta\theta} d\theta |f_{N_s}[\exp(-iQ_z l)]|^2. \end{aligned} \quad (74)$$

Transforming the integration through $\delta\theta = \theta - \theta_{Br}$,

Keeping only the corrections of the first order, we get $\Gamma = \Gamma' + i\Gamma''$, with details in Eq. (A7).

Since we are primarily interested in the region $|\Gamma'| < 1$, we get the following for the eigenvalues:

$$\lambda \approx (\Gamma' \pm i\sqrt{1-\Gamma'^2}) \left[1 \pm \frac{\Gamma''}{\sqrt{1-\Gamma'^2}} \right], \quad (76)$$

which suggests that $f_{N_s}(\lambda) \approx N_s$ can be achieved only if

$$N_s \ll \frac{\sqrt{1-\Gamma'^2}}{|\Gamma''|}. \quad (77)$$

Thus, absorption puts certain restrictions on the maximum number of slabs. It can be proved that, in the case $|\Gamma'| = 1$, that number is restricted by $|\Gamma''|^{-1/2}$, which is much less than in the previous case.

VI. RESULTS

First, one of the ways to achieve powerful TR is to choose two materials with strongly differing dielectric constants. However, in the case when the PhC is composed of materials which differ significantly from the environment through their optical properties, this can prevent radiation propagating inside the PhC with Brewster's transversal momentum from escaping the system, once the following inequality $\sqrt{b^{-1} + \varepsilon^{-1}} \geq n_{\text{env}}^{-1}$ is violated (n_{env} is the refractive index of the environment). If we choose vacuum as an environment and one of the slabs with the highest dielectric constant to be made of Si $b \approx 3.4^2$, then the second material should satisfy $\varepsilon < 1.05$, which is too close to the properties of vacuum. Thus, it becomes feasible to consider the system "identical, optically dense slabs-environment" from the beginning.

Our proposal is to use the PhC, composed of two nondispersive materials with a particular structure inside a single period, given through the fraction of slab thickness and the gap between them according to Eq. (53). With this, the system now has a potential to expose a strong γ dependence. The spectral-angular intensity, observed at Brewster's angle along the forward direction,

$$I^+(\omega, \theta_{Br}) \sim \left| \frac{1 - \exp(iN_s \omega l \gamma^{-2}/2c)}{1 - \exp(i\omega l \gamma^{-2}/2c)} \right|^2 \quad (78)$$

achieves its peak as $\gamma \rightarrow \infty$. In Fig. 3, it is demonstrated through the behavior of the angular intensity, obtained after the integration of the spectral-angular intensity over the spectrum of wavelengths $\lambda > 2(a_b + a_e) = 2l$, denoted by $\mathbf{I}^+ = dW^F/d\Omega$. Please note that all integrations were performed using the Gauss-Kronrod rule, which introduces subtle errors during the integration of highly oscillating functions. The cutoff $\lambda = 50l$ is introduced for computational efficiency. From $\gamma \sim \sqrt{N_s}$, it achieves the saturation peak $\sim N_s^2$ (just as in the frame of the RTR effect [20]) and experiences no changes with the further increase of the particle energy, see Fig. 3(b), but way below the saturation threshold, though where $\gamma \gg 1$ still holds, the integrated intensity exposes $\mathbf{I}^+(\theta = \theta_{Br}) \sim \gamma^4$ dependence, see Fig. 3(a). Moreover, as it follows from Figs. 3(a) and 3(b), below the saturation region $\gamma \ll \sqrt{N_s}$, radiation intensity does not depend on N_s .

Note that the strong energy dependence and the peak radiation intensity at Brewster's angle have been observed in early experiments [27].

The dispersion equation $\cos(Q_z l) = \Gamma$ determines the range of frequencies, which can propagate along the PhC (allowed modes) and which are not (forbidden modes). Apparently, at Brewster's angle, there is a solution to the dispersion equation at any frequency. Once we determine the function $Q_z = Q_z(\omega, q) \in [0, \pi l^{-1}]$, we would like to express the frequency over Q_z , and this, in general, results in band structure since, for a given Q_z , there could be multiple corresponding frequencies ω , which can be expressed as $\omega_n(Q_z, \theta)$ with $n = 0, 1, 2, \dots$.

At Brewster's angle and considering that $\varepsilon a_e + b a_b = l\sqrt{\varepsilon + b}$, we get $\cos(Q_z l) = \cos(\omega l/c)$.

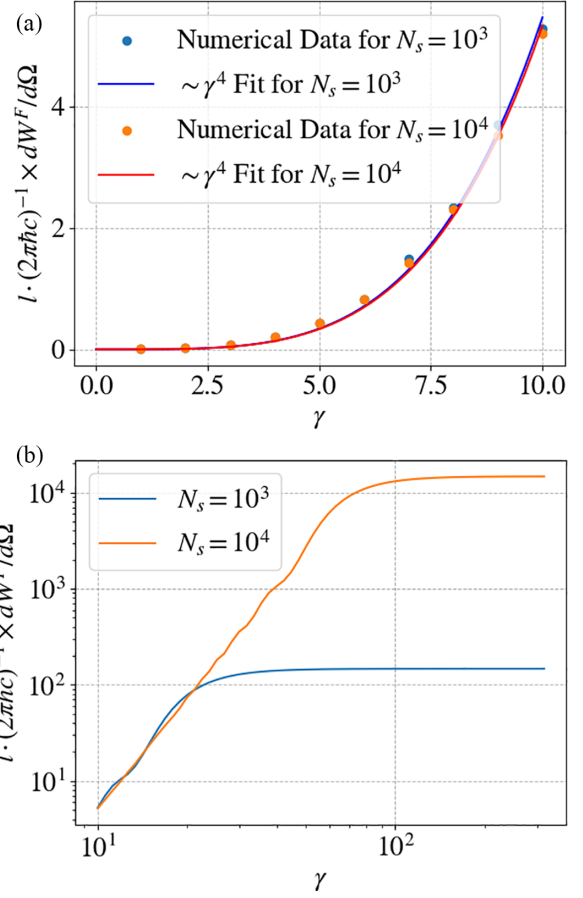


FIG. 3. (a) Illustration of the $\sim \gamma^4$ law, (b) demonstration of the saturation law for the forward-propagating radiation. The spectral-angular intensity was integrated over the frequencies, corresponding to $\lambda > 2l$ and $\lambda < 50l$. Here, we compare the graphs for $N_s = 10^3$ and 10^4 . The parameters of PhC are chosen as follows: $\varepsilon = 1$, $b = 3.4^2 = 11.56$, and $a_b/a_e = (\sqrt{\varepsilon + b} - \varepsilon)/(b - \sqrt{\varepsilon + b})$.

Solving it with respect to Q_z , we get

$$Q_z l = \begin{cases} \omega l/c - \pi \left\lfloor \frac{\omega}{\pi c l^{-1}} \right\rfloor, & \text{if } \left\lfloor \frac{\omega}{\pi c l^{-1}} \right\rfloor = \text{even}, \\ \pi \left(\left\lfloor \frac{\omega}{\pi c l^{-1}} \right\rfloor + 1 \right) - \omega l/c, & \text{if } \left\lfloor \frac{\omega}{\pi c l^{-1}} \right\rfloor = \text{odd}, \end{cases} \quad (79)$$

which suggests, after denoting $n = \lfloor \omega/\pi c l^{-1} \rfloor$ and solving the equation with respect to ω for even and odd values of n separately, the way to get the dispersion law

$$\omega_n(Q_z) = 2\pi c l^{-1} \left\lfloor \frac{n+1}{2} \right\rfloor + (-1)^n c \text{sign}(Q_z) Q_z, \quad (80)$$

where $n = 0, 1, 2, \dots$, and we require the quasimomentum $Q_z \in (-\pi l^{-1}, \pi l^{-1}]$, where the negative values emerge from the fact that the dispersion equation is symmetric with respect to the transformation $Q_z \rightarrow -Q_z$. Even small deviations from Brewster's angle cause an emergence of a gap in a band structure.

With this, the band structure of the PhC for the modes propagating at Brewster's angle resembles the graphene band structure near Dirac points; see Fig. 4(a). Deviations from Brewster's angle play a role, analogous to the

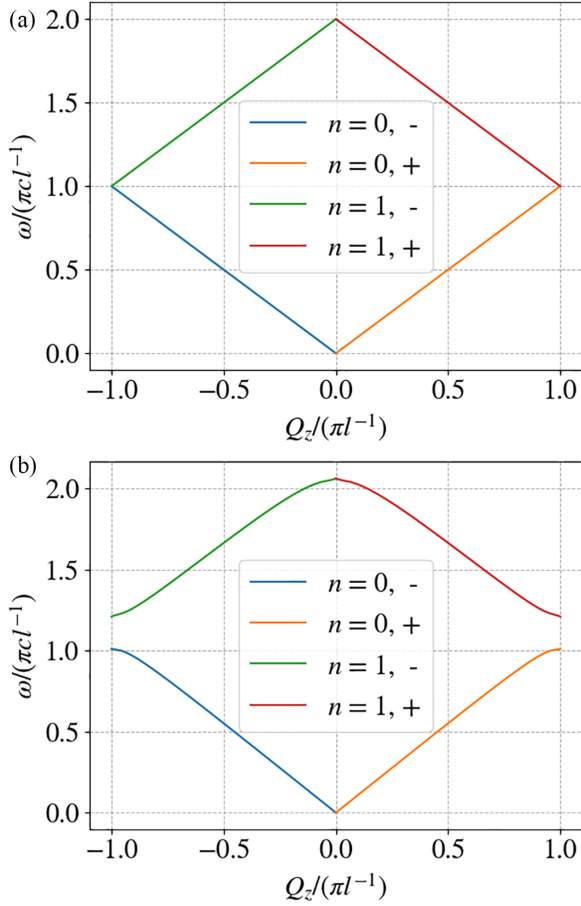


FIG. 4. Band structure of the described PhC for the modes propagating at (a) $\theta = \theta_{Br}$ and (b) $\theta = 1.013\theta_{Br}$. For labels, the number indicates n for a respective branch, and the symbol following the number indicates the sign of Q_z . The parameters of PhC are chosen as follows: $\varepsilon = 1$, $b = 3.4^2 = 11.56$, and $a_b/a_\varepsilon = (\sqrt{\varepsilon + b} - \varepsilon)/(b - \sqrt{\varepsilon + b})$.

hopping parameter between the graphene layers [30]; see Fig. 4(b). The first number indicates n for a respective branch, and the symbol following the number indicates the sign of Q_z .

VII. DISCUSSION

For ultrarelativistic particles, this radiation formation zone can increase significantly, leading to the distortions in observed intensity since the mixed component of the Poynting vector starts to give a nonvanishing contribution. The structure we propose can eliminate this issue, present in x-ray TRDs.

The regular structures possess an objective advantage over the irregular ones in the x-ray region, as proved in Ref. [19]. Additionally, the formulas for the regular structure could be considerably simplified, while for the irregular one, it can be done only in special cases. For example, Ref. [1] considers a structure called *Brewster randomness*, which allows the TR from each interface to interfere constructively to give a sharp peak $\sim N_s^2$, with the angular width $\sim N_s^{-1}$, where N_s is the number of slab pairs in that alternating structure.

Second, the dependence of the radiation intensity was not considered, contrary to the article [24], where PhCs were considered and where sharp peaks of the spectral angular intensity were demonstrated along the isofrequency curves, which allowed us to obtain a considerable γ dependence of the intensity peak angle and, hence, the potential use in modern TRDs. For ultrarelativistic particles, the only problem was to design a PhC in such a way that, in the given range of velocities of interest, the respective eigenmodes were generated in the PhC and showed a considerable angular correlation to γ .

It appears that the observation along Brewster's angle has considerable advantages over those along $\theta = 0$. First, any PhC has a respective Q_z along such a direction of propagation. Second, the quasimomentum of that eigenmode is reduced to a linear combination of the crystal parameters. This allows us to seek an optimal design of the latter to achieve strong γ dependence of the observables.

In addition, the efficiency of the detector requires a Dirac cone behavior. Since we need the factor $\sim \gamma^{-2}$ to become dominating even for high energies, the other terms in $\omega/v - Q_z(\omega, q)$ need to vanish, which demands $Q_z = \omega/c$ for some range of ω (linear behavior). On the other hand, $Q_z < \pi/l$, which imposes a physical limit on $\omega < \pi c/l$. Thus, any other possible design which utilizes $\tilde{Q}_z = 1$ can never overcome this constraint in frequencies. Since our detector satisfies that condition for any $\omega < \pi c/l$, it is possibly the most effective among those detectors which utilize $\tilde{Q}_z = 1$.

According to the numbers in Fig. 5, the yield of photons is typically $\sim 0.1\text{--}1\text{ eV}^{-1}$ for the PhC period $l \sim 1\text{ }\mu\text{m}$, which is much higher than that of the x-ray TRD [12]. This is due to the much lower energies for the single photons in the vicinity of the visible region. Additionally, in contrast with TRD, the radiation-formation zone remains limited to $\lambda/2|1 - \varepsilon/\sqrt{\varepsilon + b}|$ in the ultrarelativistic limit. This makes another advantage over the conventional x-ray TRD.

Moreover, these numbers apply to the case of $N_s \sim 10^3$, while $dN/d\omega \sim N_s$. This implies that the perspective of detectors composed of $N_s \sim 10^6$ or more slabs is beneficial also for increasing the differential yield of photons.

Feasibility and perspectives

First, we suggest checking the γ^4 law on easily realizable systems. One can suggest a setup composed of $N_s = 100$ Mylar slabs with dielectric constant $b = 2.7$ [31] and the thickness of $a_b = 110\text{ }\mu\text{m}$. From Eq. (53), we find $a_\varepsilon \approx 93.5\text{ }\mu\text{m}$ and $l = 203.5\text{ }\mu\text{m}$. For the optimal wavelength to obtain the highest possible yield, we use the numerical result $\lambda = 5l \approx 1\text{ mm}$ (see Fig. 5), which corresponds to the microwave radiation of frequency $\nu = 300\text{ GHz}$. With the electron beam of energies $E \leq 5\text{ MeV}$, we can cover the region $\gamma \leq 10 = \sqrt{N_s}$ up to saturation and experimentally verify the credibility of the analysis above. Brewster's angle is $\approx 58.6^\circ$. Here, $\delta\theta_{\text{peak}} \approx 0.05\text{ rad}$ or 3° is estimated, which allows us to tune the angular window easily.

As to the larger γ , the perspective of the detectors for registering particles with $\gamma \sim 10^3\text{--}10^4$ seems problematic

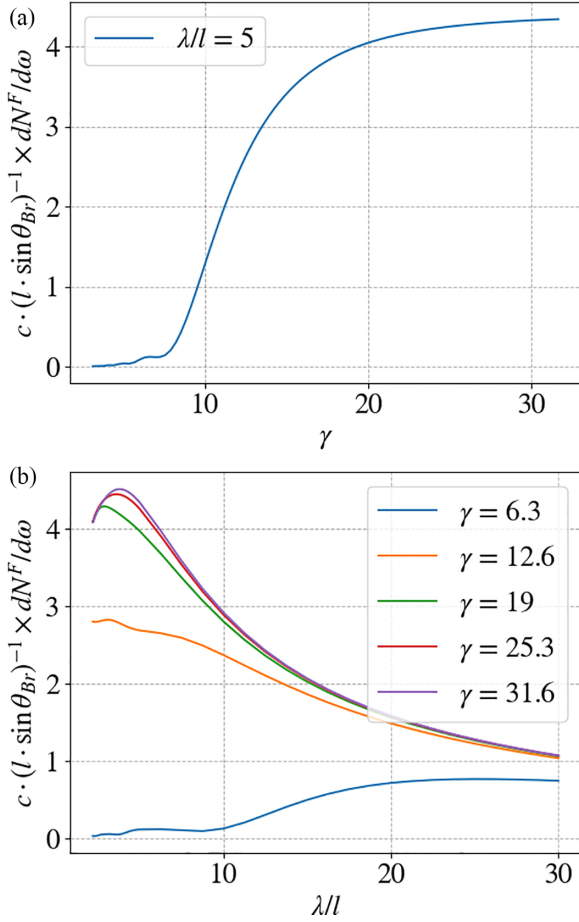


FIG. 5. Differential yield for the photons, radiated along Brewster's angle in the forward direction, integrated over a window $|\delta\theta| < 4 \times 10^{-3}$, (a) illustration of γ dependence for a fixed wavelength $\lambda = 5l$, (b) illustration of λ dependence for various fixed values of γ . Here, $N_s = 10^3$. The parameters of PhC are chosen as follows: $\varepsilon = 1$, $b = 3.4^2 = 11.56$, and $a_b/a_\varepsilon = (\sqrt{\varepsilon + b} - \varepsilon)/(b - \sqrt{\varepsilon + b})$.

to realize with state-of-the-art technologies since it would require the manufacture of periodic nanostructures with $N_s \sim 10^6 - 10^8$. However, some highly advancing methods like block copolymer lithography already allow achieving and even far exceeding the described period number [32,33]. It is worth mentioning the importance of the possibility of manufacturing nanostructures with high aspect ratio [34]. The TR field behaves itself as being locked inside the PhC, infinite in the transversal direction, once its characteristic lengths ($z_{\varepsilon,b}^\pm, \lambda$)

are much less than the sizes of PhC in the transversal direction $r_\perp$, but note that, at $\gamma \rightarrow \infty$, all these lengths are $\sim \lambda$. This means that optimal configuration of the system is $l < \lambda \ll r_\perp$. Technically, we can consider $l = 0.2\lambda$ and the aspect ratio $r_\perp/l = 50$ as representing the PhC as in the model considered above.

Thus, the only problem that remains is the precise angular spectroscopy, which will allow restricting the radiation to pass through a narrow enough range of angles ~ 0.1 arcseconds. Some articles have suggested the possibility of angular measurements with an error ~ 1 nanoradians, i.e., with even more accuracy [35].

Note that the possible size effects [36] do not introduce principal difficulties, once the desired slab thickness is predefined. This helps us to find the respective dielectric constant b and hence the required gap a_ε and Brewster's angle θ_{Br} beforehand.

VIII. SUMMARY

We considered the TR, induced by a particle traversing perpendicular to the layers of the stratified medium. In particular case of the bimaterial PhC with a specific proportion held for thicknesses of its components, the emitted radiation becomes highly dependent on the particle energy, which is absent in case of the other mechanisms, like Cherenkov radiation, bremsstrahlung, luminescence and scintillation. Along with the state of the art nanotechnologies and precision angular spectroscopy, we have the potential to manufacture compact TRDs, operating near the optical region for the detection of the ultrarelativistic energies.

ACKNOWLEDGMENT

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DATA AVAILABILITY

The data that support the findings of this article are openly available [37].

APPENDIX

In this appendix, we provide derivations of formulas presented in the main text. First, we define $\hat{B}_j$ and $\hat{M}_j$ found in Eq. (12), see section Boundary Conditions and the Solution,

$$\hat{B}_j = \frac{2\pi ieq}{ck_{j,z}\varepsilon_j^{-1}} e^{iz_j\omega/v} \left\{ \left[\frac{\varepsilon_j^{-1}k_{j,z} + \varepsilon_{j+1}^{-1}\omega/v}{k_{j+1,z}^2 - (\omega/v)^2} - \frac{\varepsilon_j^{-1}}{k_{j,z} - \omega/v} \right] \right\},$$

$$\hat{M}_j = \frac{1}{2k_{j,z}\varepsilon_j^{-1}} \begin{bmatrix} e^{-ik_{j+1,z}(z_{j+1}-z_j)} (k_{j,z}\varepsilon_j^{-1} + k_{j+1,z}\varepsilon_{j+1}^{-1}) & e^{ik_{j+1,z}(z_{j+1}-z_j)} (k_{j,z}\varepsilon_j^{-1} - k_{j+1,z}\varepsilon_{j+1}^{-1}) \\ e^{-ik_{j+1,z}(z_{j+1}-z_j)} (k_{j,z}\varepsilon_j^{-1} - k_{j+1,z}\varepsilon_{j+1}^{-1}) & e^{ik_{j+1,z}(z_{j+1}-z_j)} (k_{j,z}\varepsilon_j^{-1} + k_{j+1,z}\varepsilon_{j+1}^{-1}) \end{bmatrix}. \quad (A1)$$

Second, from their expressions at Brewster's angle, we have

$$\hat{m}_k = \begin{Bmatrix} \exp[-i(k_z^b d_{2k+1} + k_z^\varepsilon d_{2k+2})] & 0 \\ 0 & \exp[i(k_z^b d_{2k+1} + k_z^\varepsilon d_{2k+2})] \end{Bmatrix},$$

$$\hat{B} = \frac{2\pi ieq}{cb^{-1}k_z^b} e^{iz_0\omega/v} \left\{ \left(\frac{b^{-1}}{k_z^b - \omega/v} - \frac{\varepsilon^{-1}}{k_z^\varepsilon - \omega/v} \right) \sum_{j=0}^{N_s-1} e^{i \sum_{k=0}^{j-1} [(\omega/v - k_z^b) d_{2k+1} + (\omega/v - k_z^\varepsilon) d_{2k+2}]} [1 - e^{i(\omega/v - k_z^b) d_{2j+1}}] \right. \\ \left. \left(\frac{b^{-1}}{k_z^b + \omega/v} - \frac{\varepsilon^{-1}}{k_z^\varepsilon + \omega/v} \right) \sum_{j=0}^{N_s-1} e^{i \sum_{k=0}^{j-1} [(\omega/v + k_z^b) d_{2k+1} + (\omega/v + k_z^\varepsilon) d_{2k+2}]} [1 - e^{i(\omega/v + k_z^b) d_{2j+1}}] \right\}, \quad (A2)$$

which is used right before the derivation of Eq. (30). Third, we define $\hat{m}_k$ and $\hat{C}_j$ in Eq. (32), see section Photonic Crystal,

$$(m_k)_{11} = e^{-ik_z^\varepsilon d_{2k+2}} \left[\cos(k_z^b d_{2k+1}) - \frac{i}{2} \left(\frac{\varepsilon^{-1} k_z^\varepsilon}{b^{-1} k_z^b} + \frac{b^{-1} k_z^b}{\varepsilon^{-1} k_z^\varepsilon} \right) \sin(k_z^b d_{2k+1}) \right],$$

$$(m_k)_{12} = e^{ik_z^\varepsilon d_{2k+2}} \frac{i}{2} \left(\frac{\varepsilon^{-1} k_z^\varepsilon}{b^{-1} k_z^b} - \frac{b^{-1} k_z^b}{\varepsilon^{-1} k_z^\varepsilon} \right) \sin(k_z^b d_{2k+1}),$$

$$(C_j)_1 = \frac{2\pi ieq}{c} e^{iz_{2j}\omega/v} \left(\frac{1}{(k_z^b)^2 - \omega^2/v^2} \left\{ \left(1 + \frac{b^{-1}\omega/v}{\varepsilon^{-1}k_z^\varepsilon} \right) [1 - e^{id_{2j+1}\omega/v} \cos(k_z^b d_{2j+1})] + ie^{id_{2j+1}\omega/v} \left(\frac{b^{-1}k_z^b}{\varepsilon^{-1}k_z^\varepsilon} + \frac{\omega/v}{k_z^b} \right) \right. \right. \\ \times \sin(k_z^b d_{2j+1}) \left. \left. \right\} - \frac{1}{(k_z^\varepsilon)^2 - \omega^2/v^2} \left\{ \left(1 + \frac{\omega/v}{k_z^\varepsilon} \right) [1 - e^{id_{2j+1}\omega/v} \cos(k_z^b d_{2j+1})] + ie^{id_{2j+1}\omega/v} \left(\frac{b^{-1}k_z^b}{\varepsilon^{-1}k_z^\varepsilon} + \frac{\varepsilon^{-1}\omega/v}{b^{-1}k_z^b} \right) \right. \right. \\ \times \sin(k_z^b d_{2j+1}) \left. \left. \right\} \right),$$

$$(C_j)_2 = \frac{2\pi ieq}{c} e^{iz_{2j}\omega/v} \left(\frac{1}{(k_z^b)^2 - \omega^2/v^2} \left\{ \left(1 - \frac{b^{-1}\omega/v}{\varepsilon^{-1}k_z^\varepsilon} \right) [1 - e^{id_{2j+1}\omega/v} \cos(k_z^b d_{2j+1})] - ie^{id_{2j+1}\omega/v} \left(\frac{b^{-1}k_z^b}{\varepsilon^{-1}k_z^\varepsilon} - \frac{\omega/v}{k_z^b} \right) \right. \right. \\ \times \sin(k_z^b d_{2j+1}) \left. \left. \right\} - \frac{1}{(k_z^\varepsilon)^2 - \omega^2/v^2} \left\{ \left(1 - \frac{\omega/v}{k_z^\varepsilon} \right) [1 - e^{id_{2j+1}\omega/v} \cos(k_z^b d_{2j+1})] - ie^{id_{2j+1}\omega/v} \left(\frac{b^{-1}k_z^b}{\varepsilon^{-1}k_z^\varepsilon} - \frac{\varepsilon^{-1}\omega/v}{b^{-1}k_z^b} \right) \right. \right. \\ \times \sin(k_z^b d_{2j+1}) \left. \left. \right\} \right), \quad (A3)$$

Fourth, the magnetic field amplitudes in the case $|\lambda_+| = |\lambda_-| = 1$, obtained right after Eq. (42),

$$A^+ \approx -\frac{2\pi}{l} \left\{ \frac{P_{11}D_1}{M_{11}} \delta[(\omega/v - Q_z) \bmod 2\pi/l] + \frac{P_{12}D_2}{M_{11}} \delta[(\omega/v + Q_z) \bmod 2\pi/l] \right\},$$

$$A^- \approx -\frac{2\pi}{l} \left\{ \left(P_{21} - \frac{M_{21}}{M_{11}} P_{11} \right) D_1 \delta[(\omega/v - Q_z) \bmod 2\pi/l] + \left(P_{22} - \frac{M_{21}}{M_{11}} P_{12} \right) D_2 \delta[(\omega/v + Q_z) \bmod 2\pi/l] \right\}. \quad (A4)$$

Fifth, the current-induced terms $\hat{\mathbf{B}}$ and $\hat{\mathbf{C}}$ in Eq. (47)

$$\hat{\mathbf{C}} = \sqrt{\varepsilon'} \sin \theta \left\{ [1 - \exp(i2\pi \tilde{a}_b \beta^{-1}) \cos(2\pi \tilde{a}_b \sqrt{b - \varepsilon' \sin^2 \theta})] \right. \\ \times \left[\frac{1}{b - \varepsilon' \sin^2 \theta - \beta^{-2}} \left(1 + \frac{\beta^{-1} b^{-1}}{\varepsilon^{-1} \sqrt{\varepsilon - \varepsilon' \sin^2 \theta}} \right) - \frac{1}{\varepsilon - \varepsilon' \sin^2 \theta - \beta^{-2}} \left(1 + \frac{\beta^{-1}}{\sqrt{\varepsilon - \varepsilon' \sin^2 \theta}} \right) \right] \\ + i \exp(i2\pi \tilde{a}_b \beta^{-1}) \sin(2\pi \tilde{a}_b \sqrt{b - \varepsilon' \sin^2 \theta}) \\ \times \left[\frac{1}{b - \varepsilon' \sin^2 \theta - \beta^{-2}} \left(\frac{\beta^{-1}}{\sqrt{b - \varepsilon' \sin^2 \theta}} + \frac{b^{-1} \sqrt{b - \varepsilon' \sin^2 \theta}}{\varepsilon^{-1} \sqrt{\varepsilon - \varepsilon' \sin^2 \theta}} \right) - \frac{1}{\varepsilon - \varepsilon' \sin^2 \theta - \beta^{-2}} \left(\frac{\beta^{-1} \varepsilon^{-1}}{b^{-1} \sqrt{b - \varepsilon' \sin^2 \theta}} + \frac{b^{-1} \sqrt{b - \varepsilon' \sin^2 \theta}}{\varepsilon^{-1} \sqrt{\varepsilon - \varepsilon' \sin^2 \theta}} \right) \right] \left. \right\},$$

$$\hat{\mathbf{B}} = \sum_{j=0}^{N_s-1} (e^{i2\pi \tilde{l} \beta^{-1}} \hat{m})^j \hat{\mathbf{C}}. \quad (A5)$$

Sixth, see right before Eq. (76) in section Accounting the losses

$$\hat{m} = \begin{pmatrix} e^{-ik_z^e a_e} \left[\cos(k_z^b a_b) - \frac{i}{2} \left(\frac{b^{-1}k_z^b}{\varepsilon^{-1}k_z^e} + \frac{\varepsilon^{-1}k_z^e}{b^{-1}k_z^b} \right) \sin(k_z^b a_b) \right] & e^{+ik_z^e a_e} \left(-\frac{i}{2} \right) \left(\frac{b^{-1}k_z^b}{\varepsilon^{-1}k_z^e} - \frac{\varepsilon^{-1}k_z^e}{b^{-1}k_z^b} \right) \sin(k_z^b a_b) \\ e^{-ik_z^e a_e} \left(+\frac{i}{2} \right) \left(\frac{b^{-1}k_z^b}{\varepsilon^{-1}k_z^e} - \frac{\varepsilon^{-1}k_z^e}{b^{-1}k_z^b} \right) \sin(k_z^b a_b) & e^{+ik_z^e a_e} \left[\cos(k_z^b a_b) + \frac{i}{2} \left(\frac{b^{-1}k_z^b}{\varepsilon^{-1}k_z^e} + \frac{\varepsilon^{-1}k_z^e}{b^{-1}k_z^b} \right) \sin(k_z^b a_b) \right] \end{pmatrix}. \quad (\text{A6})$$

Finally, the real and imaginary parts of the trace of the transfer matrix [see near the same Eq. (76)] are

$$\begin{aligned} \Gamma' &\approx \cos(k_z^{\varepsilon'} a_e) \cos(k_z^{b'} a_b) - \frac{1}{2} \left(\frac{b'^{-1}k_z^{b'}}{\varepsilon'^{-1}k_z^{\varepsilon'}} + \frac{\varepsilon'^{-1}k_z^{\varepsilon'}}{b'^{-1}k_z^{b'}} \right) \sin(k_z^{\varepsilon'} a_e) \sin(k_z^{b'} a_b), \\ \Gamma'' &\approx \varepsilon'' \left\{ \frac{1}{2} \left(\frac{b'^{-1}k_z^{b'}}{\varepsilon'^{-1}k_z^{\varepsilon'}} - \frac{\varepsilon'^{-1}k_z^{\varepsilon'}}{b'^{-1}k_z^{b'}} \right) \sin(k_z^{\varepsilon'} a_e) \sin(k_z^{b'} a_b) \left(\varepsilon'^{-1} - \frac{1}{2\sqrt{\varepsilon' - (cq/\omega)^2}} \right) \right. \\ &\quad \left. - \frac{\omega a_e}{2c\sqrt{\varepsilon' - (cq/\omega)^2}} \left[\sin(k_z^{\varepsilon'} a_e) \cos(k_z^{b'} a_b) - \frac{1}{2} \left(\frac{b'^{-1}k_z^{b'}}{\varepsilon'^{-1}k_z^{\varepsilon'}} + \frac{\varepsilon'^{-1}k_z^{\varepsilon'}}{b'^{-1}k_z^{b'}} \right) \cos(k_z^{\varepsilon'} a_e) \sin(k_z^{b'} a_b) \right] \right\} \\ &\quad + b'' \left\{ \frac{1}{2} \left(\frac{b'^{-1}k_z^{b'}}{\varepsilon'^{-1}k_z^{\varepsilon'}} - \frac{\varepsilon'^{-1}k_z^{\varepsilon'}}{b'^{-1}k_z^{b'}} \right) \sin(k_z^{\varepsilon'} a_e) \sin(k_z^{b'} a_b) \left(\frac{1}{2\sqrt{b' - (cq/\omega)^2}} - b'^{-1} \right) \right. \\ &\quad \left. - \frac{\omega a_b}{2c\sqrt{b' - (cq/\omega)^2}} \left[\cos(k_z^{\varepsilon'} a_e) \sin(k_z^{b'} a_b) - \frac{1}{2} \left(\frac{b'^{-1}k_z^{b'}}{\varepsilon'^{-1}k_z^{\varepsilon'}} + \frac{\varepsilon'^{-1}k_z^{\varepsilon'}}{b'^{-1}k_z^{b'}} \right) \sin(k_z^{\varepsilon'} a_e) \cos(k_z^{b'} a_b) \right] \right\}. \quad (\text{A7}) \end{aligned}$$

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