

Spectroscopic footprints of quantum friction in nonreciprocal and chiral media

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We investigate how the quantum friction experienced by a polarizable atom moving with constant velocity parallel to a planar interface is modified when the latter consists of chiral or nonreciprocal media, with special focus on topological insulators. Macroscopic quantum electrodynamics is used to obtain the velocity-dependent Casimir-Polder frequency shift and decay rate. These results are a generalization to matter with time-reversal symmetry breaking. Our findings are illustrated by examining the nonretarded and retarded limits for five examples: a perfectly conducting mirror, a perfectly reflecting nonreciprocal mirror, a three-dimensional topological insulator, a perfectly reflecting chiral mirror, and an isotropic chiral medium. Different asymptotic power laws are found for all these materials. Interestingly, two bridges exist between chirality and nonreciprocity through the frequency shift, which arise as a consequence of the magnetoelectric coupling. Specifically, the position-dependent Casimir-Polder frequency shift for the nonreciprocal case depends on a geometric magnetic field associated with photoionization of chiral molecules, while the Casimir-Polder frequency shift depending on the velocity for the chiral case has the optical rotatory strength as the atomic response, and those for the nonreciprocal case depend on an analog of the optical rotatory strength.

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I. INTRODUCTION

Quantum friction is closely linked to macroscopic effects of quantum vacuum fluctuations such as the Casimir force between neutral bodies [1–4] and the Casimir-Polder force between an atom and a surface [5]. When two uncharged polarizable bodies move at constant velocity relative to each other, they experience a dissipative force opposing their motion due to the exchange of Doppler-shifted virtual photons at zero temperature, an effect known as quantum friction [6–10]. For flat, perfectly smooth surfaces, quantum friction resembles a van der Waals attraction, arising from quantum-fluctuation-induced electric dipoles on one surface and their lagging image dipoles on the other [6]. It has also been predicted in particle-surface (Casimir-Polder) interactions [11–14], with studies analyzing velocity- and distance-dependent drag forces in three-dimensional bulk materials and, more recently, in graphene [15].

Quantum friction has a quantum origin and the physics behind it is related to the quantum Cherenkov effect through the anomalous Doppler effect [16–19], where real photons are extracted from the vacuum at the cost of the object's

kinetic energy; they are absorbed and emitted by an atom, thus producing a fluctuating momentum recoil [20]. Considering only an atomic translational motion, spin-zero photons are absorbed and reemitted, which translates in an overall quantum frictional force that opposes the translational motion. In recent years, this process has been investigated in several scenarios [21–24] and its outstanding connection to nonequilibrium physics was recently figured out [25]. Generalizing for atomic rotational degrees of freedom results in what is called quantum rolling friction [26]. Note that the related phenomenon of Coulomb drag [27], in which a current in one plate induces a voltage bias in another via the fluctuating Coulomb field, has been successfully demonstrated [28].

In order to gain further insights into quantum friction, a different path has been taken in the works [9,24]. Particularly, in the article [24], by using the framework of macroscopic QED [29–32], the authors were able to provide testable quantities with direct experimental contrast in the context of the Casimir-Polder force leading to the prediction of a velocity-dependent quantum-vacuum effect. Specifically, this work reported atomic level shifts and rates of spontaneous decay for a zero-temperature neutral atom with dipole moment and nonrelativistic velocity moving next to a perfectly smooth macroscopic surface, which in principle can be measured. Also important efforts have recently been made to obtain a satisfactory description of quantum friction in all timescales regarding carefully all the typical approximations for this phenomenon, for example, Markovianity and linear response theory [33].

Besides the fundamental research of quantum friction, it has also been studied in innovative setups such as the flow

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of water in carbon nanochannels, where quantum friction is the dominant friction mechanism for water on carbon-based materials [34]. This phenomenon has been investigated in new kinds of materials such as bidimensional topological materials [10]. For instance, exposing quantum friction to a two-dimensional topological material enables an increase of two orders of magnitude in the quantum drag force with respect to conventional neutral graphene systems [10]. So the jump to three-dimensional topological insulators (TIs) will lead to the interplay between quantum friction generated by an atom with constant velocity and the topological magnetoelectric effect provided by the TI, to which this paper is devoted.

Three-dimensional TIs are materials with an insulating bulk and conducting surface states protected by time-reversal symmetry [35–38]. Breaking time-reversal symmetry at a TI-insulator interface, via magnetic coatings or doping, opens a gap in surface states, leading to exotic phenomena [35,39–49]. Topological insulators also show promise in hybrid systems, such as semiconductor quantum dots [50], highlighting their vast potential for future applications. These effects can also occur in axion insulators, whose topological properties are protected by inversion symmetry and arise in magnetically doped TI heterostructures with opposing magnetization at the top and bottom interfaces [51].

Due to the fact that TIs break time-reversal symmetry, they can be regarded as nonreciprocal media [52,53]. Consequently, the theory of macroscopic QED cannot be directly applied because it is based on Lorentz’s reciprocity principle (a particular case of the Onsager reciprocity from statistical physics [54]), which assumes the reversibility of optical paths. In terms of symmetry, it is equivalent to invariance under an exchange of positions and orientations of sources and fields. Thus, media respecting this principle are called reciprocal and preserve time-reversal symmetry. To deal with nonreciprocal media, which violate Lorentz’s principle relation [55], macroscopic QED was generalized to include the necessary features of nonreciprocal media in Ref. [52]. The aim of the present paper is to study the Casimir-Polder frequency shift and decay rate on a parallel moving atom to a nonreciprocal medium.

Another scenario where quantum friction has been investigated is in the presence of chiral media as in Ref. [56], to which this paper is devoted as well. A three-dimensional object that cannot be superimposed on its mirror image is said to be chiral; these distinct mirror images are called enantiomers. Spectroscopically, enantiomers have identical transition energies and distinguishing between the two is not trivial. The characteristic feature of chiral objects is the manner of their interactions with other chiral objects. For example, the refractive indices of left- and right-handed circularly polarized light are different in a chiral medium; therefore, the two polarizations will propagate at different speeds. The difference in velocity is related to the phenomenon of optical activity. Circular dichroism is closely related where the wave with the “slower” polarization is absorbed more strongly as it travels through the medium [57].

Many of the processes crucial to life involve chiral molecules whose chiral identity plays a central role in their chemical reactions. The undesired enantiomer can react differently, thereby not producing the required result. In nature, these chiral molecules predominantly, or even exclusively,

occur in one enantiomeric form, so their reactions proceed differently when different enantiomers are present. In contrast, artificial production without the addition of enriched enantiomers or other chiral agents creates both enantiomers in equal proportions. It is essential to distinguish between enantiomers and ultimately to separate a racemic mixture, i.e., one that contains both enantiomers, into an enantiomerically pure sample.

A frequently used method to separate enantiomers in an industrial setting is enantioselective chromatography. The initial racemic solution is passed through a column packed with a resolving agent, which by necessity has to be chiral. This either retards or stops the progress of one of the enantiomers passing through the column but crucially not for the other and thus allows the solution to be separated. From an optical perspective, it has been proposed that a racemic sample can be purified using coordinated laser pulses. This method involves a two-step process, where different transitions in the enantiomers are first driven, followed by a conversion of one enantiomer into the other. [58]. It has recently been calculated that in the presence of a chiral carbon nanotube the enantiomers of alanine possess different absorption energies and it was theorized that this could lead to a method of discrimination [59]. Furthermore, it has been shown that the van der Waals dispersion force between molecules can be enantiomer selective [60,61]. In fact, there has been much effort to provide a selection and distinguishing method such as the use of the Casimir-Polder force [62–64]. For all these reasons, we will also explore how the quantum friction is modified in these materials.

This paper is organized as follows. In Sec. II we calculate the time-dependent electric and magnetic fields in the framework of macroscopic QED for nonreciprocal media. This result is obtained through a direct quantization of the noise current. Section III is devoted to studying how the internal atomic dynamics is modified by the nonreciprocal media. To this end, we present modified equations for the frequency shift and decay rate that depend on the atom position as well as on its velocity. In Sec. IV we apply our results to a electrically neutral polarizable atom moving with constant velocity parallel to five different materials: a perfectly conducting mirror, a perfectly reflecting nonreciprocal mirror, a strong three-dimensional topological insulator, a perfectly reflecting chiral mirror, and an isotropic chiral medium. In Sec. V we provide numerical results for the frequency shift with electrical dipole transition via Rydberg states and discuss the viability of an experimental verification. Section VI comprises a concluding summary of our results.

II. TIME-DEPENDENT ELECTROMAGNETIC FIELD

Due to time-reversal symmetry violation in nonreciprocal media, the Lorentz reciprocity principle for the Green’s tensor [55] is no longer valid

$$\mathbb{G}(\mathbf{r}, \mathbf{r}'; \omega) \neq \mathbb{G}^\top(\mathbf{r}', \mathbf{r}; \omega), \quad (1)$$

where $\top$ denotes the matrix transpose. This requires us to introduce generalized definitions for the real and imaginary parts of the Green’s tensor $\mathbb{G}$,

$$\mathcal{R}[\mathbb{G}(\mathbf{r}, \mathbf{r}'; \omega)] = \frac{1}{2}[\mathbb{G}(\mathbf{r}, \mathbf{r}'; \omega) + \mathbb{G}^{*\top}(\mathbf{r}', \mathbf{r}; \omega)], \quad (2)$$

$$\mathcal{I}[\mathbb{G}(\mathbf{r}, \mathbf{r}'; \omega)] = \frac{1}{2i} [\mathbb{G}(\mathbf{r}, \mathbf{r}'; \omega) - \mathbb{G}^{*\top}(\mathbf{r}', \mathbf{r}; \omega)]. \quad (3)$$

For a reciprocal medium with

$$\mathbb{G}(\mathbf{r}, \mathbf{r}'; \omega) = \mathbb{G}^\top(\mathbf{r}', \mathbf{r}; \omega), \quad (4)$$

they reduce to ordinary, componentwise real and imaginary parts. The general expressions for the field operators read

$$\begin{aligned} \hat{\mathbf{E}}(\mathbf{r}) &= \int_0^\infty d\omega [\hat{\mathbf{E}}(\mathbf{r}; \omega) + \hat{\mathbf{E}}^\dagger(\mathbf{r}; \omega)], \\ \hat{\mathbf{B}}(\mathbf{r}) &= \int_0^\infty d\omega [\hat{\mathbf{B}}(\mathbf{r}; \omega) + \hat{\mathbf{B}}^\dagger(\mathbf{r}; \omega)], \end{aligned} \quad (5)$$

where the dagger denotes Hermitian conjugation in operator space which is distinct from $^*\top$, i.e., complex conjugation and matrix transpose in three-dimensional position space. The frequency components of the electric field (5) can be computed through the Green's tensor as

$$\begin{aligned} \hat{\mathbf{E}}(\mathbf{r}; \omega) &= i\mu_0\omega [\mathbb{G} \star \hat{\mathbf{j}}_N](\mathbf{r}; \omega) \\ &= i\mu_0\omega \int d^3r' \mathbb{G}(\mathbf{r}, \mathbf{r}'; \omega) \cdot \hat{\mathbf{j}}_N(\mathbf{r}'; \omega), \end{aligned} \quad (6)$$

where the star denotes spatial convolution and $\hat{\mathbf{j}}_N$ is the noise current density governed by the quantum fluctuations occurring in the medium and with a vanishing average $\langle \hat{\mathbf{j}}_N \rangle = \mathbf{0}$. The corresponding expression for the frequency components of the magnetic field (5) is obtained by using Faraday's law in the frequency space

$$\hat{\mathbf{B}}(\mathbf{r}; \omega) = \mu_0 \int d^3r' \nabla \times \mathbb{G}(\mathbf{r}, \mathbf{r}'; \omega) \cdot \hat{\mathbf{j}}_N(\mathbf{r}'; \omega). \quad (7)$$

We will directly quantize the noise current density $\hat{\mathbf{j}}_N$ following the noise-current-based schema, as we will show in Sec. II A. The Hamiltonian $\hat{H}$ for the atom-field system is composed of the atomic part $\hat{H}_A$, the field part $\hat{H}_F$, and a contribution for the atom-field interaction $\hat{H}_{AF}$, meaning that $\hat{H} = \hat{H}_A + \hat{H}_{AF} + \hat{H}_F$. The atomic part is given by

$$\hat{H}_A = \sum_n E_n \hat{A}_{nn}, \quad (8)$$

which incorporates the eigenenergies E_n for each atomic energy level and the atomic flip operators $\hat{A}_{mn} = |m\rangle\langle n|$. The Hamiltonian $\hat{H}_F$ comprises the integral over all the frequency-dependent number operators of the field-medium system

$$\hat{H}_F = \int d^3\mathbf{r} \int_0^\infty d\omega \hbar\omega \hat{\mathbf{f}}^\dagger(\mathbf{r}; \omega) \cdot \hat{\mathbf{f}}(\mathbf{r}; \omega), \quad (9)$$

which clearly resembles a quantum harmonic oscillator and can be cast through the quantization of the noise current (cf. Sec. II A). The interaction Hamiltonian $\hat{H}_{AF}$ for nonmagnetic atoms in multipolar coupling and long-wavelength approximation couples the atomic dipole to the electromagnetic field as

$$\begin{aligned} \hat{H}_{AF} &= -\hat{\mathbf{d}} \cdot \hat{\mathbf{E}}(\mathbf{r}_A) + \dot{\mathbf{r}}_A \cdot \hat{\mathbf{d}} \times \hat{\mathbf{B}}(\mathbf{r}_A), \\ &= -\sum_{m,n} \hat{A}_{mn} [\mathbf{d}_{mn} \cdot \hat{\mathbf{E}}(\mathbf{r}_A) + \mathbf{v} \cdot \mathbf{d}_{mn} \times \hat{\mathbf{B}}(\mathbf{r}_A)] \end{aligned} \quad (10)$$

and contains the electric dipole operator $\hat{\mathbf{d}} = \sum_{m,n} \mathbf{d}_{mn} \hat{A}_{mn}$, with $\mathbf{r}_A$ the position of the atom and $\dot{\mathbf{r}}_A = \mathbf{v}$ its velocity. As the atomic Hamiltonian $\hat{H}_A$ and the field operators commute, only the resulting commutation relations for these operators with the field Hamiltonian $\hat{H}_F$ and the interaction Hamiltonian $\hat{H}_{AF}$ deserve to be studied to find the expression for the electric field (5).

A. Electromagnetic field in terms of noise currents

In this section we solve the equations of motion for the time-dependent creation $\hat{\mathbf{f}}^\dagger$ and annihilation $\hat{\mathbf{f}}$ operators, enabling us to determine the time-dependent electromagnetic field. For this aim, the noise current is quantized directly by expressions for the field operators. In frequency space, Ohm's law reads

$$\hat{\mathbf{j}}(\mathbf{r}; \omega) = [\mathbb{Q} \star \hat{\mathbf{E}}](\mathbf{r}; \omega) + \hat{\mathbf{j}}_N(\mathbf{r}; \omega), \quad (11)$$

which describes the response of a linear medium with conductivity matrix $\mathbb{Q}$ to the electric field $\hat{\mathbf{E}}(\mathbf{r}; \omega)$. Therefore, we write the Helmholtz equations as

$$\begin{aligned} \left(\nabla \times \nabla - \frac{\omega^2}{c^2} \right) \mathbb{G}(\mathbf{r}, \mathbf{r}'; \omega) - i\mu_0\omega [\mathbb{Q} \star \mathbb{G}](\mathbf{r}, \mathbf{r}'; \omega) \\ = \delta(\mathbf{r} - \mathbf{r}') \mathbb{1}. \end{aligned} \quad (12)$$

We find the formal solution for this equation by employing the boundary condition $\mathbb{G}(\mathbf{r}, \mathbf{r}'; \omega) \rightarrow \mathbb{0}$ for $\|\mathbf{r} - \mathbf{r}'\| \rightarrow \infty$, which leads to Eq. (6). Next we quantize the noise-current density $\hat{\mathbf{j}}_N$ introduced in Eq. (6) simply by writing it in terms of creation $\hat{\mathbf{f}}^\dagger$ and annihilation $\hat{\mathbf{f}}$ operators

$$\hat{\mathbf{j}}_N(\mathbf{r}; \omega) = \sqrt{\frac{\hbar\omega}{\pi}} [\mathbb{R} \star \hat{\mathbf{f}}](\mathbf{r}; \omega), \quad (13)$$

where $\mathbb{R}$ is a square root of the positive-definite conductivity tensor and is related to it by [52]

$$[\mathbb{R} \star \mathbb{R}^{*\top}](\mathbf{r}, \mathbf{r}'; \omega) = \mathcal{R}[\mathbb{Q}(\mathbf{r}, \mathbf{r}'; \omega)]. \quad (14)$$

With all the above, we obtain the Heisenberg equation of motion for the annihilation operator $\hat{\mathbf{f}}$ as shown

$$\dot{\hat{\mathbf{f}}}(\mathbf{r}; \omega) = \frac{1}{i\hbar} [\hat{\mathbf{f}}(\mathbf{r}; \omega), \hat{H}], \quad (15)$$

where the three summands of $\hat{H}$ are given by Eqs. (8)–(10). Nevertheless, it proves convenient to rewrite the interaction Hamiltonian $\hat{H}_{AF}$ in terms of field operators,

$$\begin{aligned} \hat{H}_{AF} &= -i\mu_0 \sum_{m,n} \int_0^\infty d\omega \sqrt{\frac{\hbar\omega}{\pi}} \hat{A}_{mn} \\ &\quad \times \mathbf{d}_{mn} \cdot \{ [\mathbb{G} \star \mathbb{R} \star \hat{\mathbf{f}}](\mathbf{r}_A; \omega) - [\mathbb{G}^* \star \mathbb{R}^* \star \hat{\mathbf{f}}^\dagger](\mathbf{r}_A; \omega) \} \\ &\quad + \mu_0 \sum_{m,n} \int_0^\infty d\omega \frac{d\omega}{i\omega} \sqrt{\frac{\hbar\omega}{\pi}} \hat{A}_{mn} \\ &\quad \times \mathbf{v} \cdot \mathbf{d}_{mn} \times \{ \nabla \times [\mathbb{G} \star \mathbb{R} \star \hat{\mathbf{f}}](\mathbf{r}_A; \omega) \\ &\quad - \nabla \times [\mathbb{G}^* \star \mathbb{R}^* \star \hat{\mathbf{f}}^\dagger](\mathbf{r}_A; \omega) \}. \end{aligned} \quad (16)$$

In this way, the solution for the annihilation operator $\hat{\mathbf{f}}$ is given by

$$\begin{aligned} \hat{\mathbf{f}}(\mathbf{r}; \omega, t) = & e^{-i\omega(t-t_0)} \hat{\mathbf{f}}(\mathbf{r}; \omega) + \frac{\mu_0 \omega}{\hbar} \sqrt{\frac{\hbar \omega}{\pi}} \sum_{m,n} \int_{t_0}^t dt' \\ & \times \left(e^{-i\omega(t-t')} [\mathbb{G} \star \mathbb{R}]^{*\top}(\mathbf{r}_A(t'), \mathbf{r}; \omega) \cdot \mathbf{d}_{mn} \hat{A}_{mn}(t') \right. \\ & + \frac{e^{-i\omega(t-t')}}{\omega^2} \{ [\mathbb{G} \star \mathbb{R}]^{*\top}(\mathbf{r}, \mathbf{r}_A(t'); \omega) \times \overleftarrow{\nabla} \} \\ & \left. \times \mathbf{d}_{mn} \cdot \mathbf{v} \hat{A}_{mn}(t') \right), \end{aligned} \quad (17)$$

where one should notice that $\overleftarrow{\nabla}$ acts on the second argument of the Green's tensor. Here we have required that $\hat{\mathbf{f}}(\mathbf{r}; \omega, t_0) = \hat{\mathbf{f}}(\mathbf{r}; \omega)$, i.e., the Heisenberg-picture operator agrees with its time-dependent Schrödinger picture at initial time t_0 .

Assuming that the atom moves with nonrelativistic speed $v \ll c$ will enforce us to seek a solution to the atom-field dynamics that is correct within leading linear order of v/c . According to the Born-Oppenheimer approximation, we may assume the center-of-mass velocity to remain constant on the timescale relevant for the internal atom-field dynamics so that [32]

$$\mathbf{r}_A(t') = \mathbf{r}_A(t) + (t - t')\mathbf{v}. \quad (18)$$

Once this approximation is substituted into the solution for the annihilation operator (17), discarding quadratic terms in v/c , we find that

$$\begin{aligned} \hat{\mathbf{f}}(\mathbf{r}; \omega, t) = & e^{-i\omega(t-t_0)} \hat{\mathbf{f}}(\mathbf{r}; \omega) + \frac{\mu_0 \omega}{\hbar} \sqrt{\frac{\hbar \omega}{\pi}} \sum_{m,n} \int_{t_0}^t dt' \\ & \times \left(e^{-i\omega(t-t')} [\mathbb{G} \star \mathbb{R}]^{*\top}(\mathbf{r}_A(t), \mathbf{r}; \omega) \cdot \mathbf{d}_{mn} \hat{A}_{mn}(t') \right. \\ & - (t - t') e^{-i\omega(t-t')} [\mathbb{G} \star \mathbb{R}]^{*\top}(\mathbf{r}_A(t), \mathbf{r}; \omega) \\ & \cdot \mathbf{d}_{mn} (\overleftarrow{\nabla} \cdot \mathbf{v}) \hat{A}_{mn}(t') \\ & + \frac{e^{-i\omega(t-t')}}{\omega^2} \{ [\mathbb{G} \star \mathbb{R}]^{*\top}(\mathbf{r}, \mathbf{r}_A(t); \omega) \times \overleftarrow{\nabla} \} \\ & \left. \times \mathbf{d}_{mn} \cdot \mathbf{v} \hat{A}_{mn}(t') \right). \end{aligned} \quad (19)$$

After substituting Eq. (19) into Eq. (13), employing Eq. (14), and simplifying through the next integral relation for the Green's tensor in nonreciprocal media [52]

$$\mathcal{I}[\mathbb{G}(\mathbf{r}, \mathbf{r}'; \omega)] = \mu_0 \omega (\mathbb{G} \star \mathcal{R}[\mathbb{Q}] \star \mathbb{G}^{*\top})(\mathbf{r}, \mathbf{r}'; \omega), \quad (20)$$

we finally obtain expressions for the electric fields

$$\begin{aligned} \hat{\mathbf{E}}(\mathbf{r}; \omega, t) = & e^{-i\omega(t-t_0)} \hat{\mathbf{E}}(\mathbf{r}; \omega) + \frac{i\mu_0 \omega^2}{\pi} \sum_{m,n} \int_{t_0}^t dt' \\ & \times \left(e^{-i\omega(t-t')} \mathcal{I}[\mathbb{G}(\mathbf{r}, \mathbf{r}_A(t); \omega)] \cdot \mathbf{d}_{mn} \hat{A}_{mn}(t') \right. \\ & - (t - t') e^{-i\omega(t-t')} \mathcal{I}[\mathbb{G}(\mathbf{r}, \mathbf{r}_A(t); \omega)] \\ & \cdot \mathbf{d}_{mn} (\overleftarrow{\nabla} \cdot \mathbf{v}) \hat{A}_{mn}(t') \end{aligned}$$

$$\begin{aligned} & + \frac{e^{-i\omega(t-t')}}{i\omega} \{ \mathcal{I}[\mathbb{G}(\mathbf{r}, \mathbf{r}_A(t); \omega)] \times \overleftarrow{\nabla} \} \\ & \times \mathbf{d}_{mn} \cdot \mathbf{v} \hat{A}_{mn}(t') \} \end{aligned} \quad (21)$$

and magnetic fields

$$\begin{aligned} \hat{\mathbf{B}}(\mathbf{r}; \omega, t) = & e^{-i\omega(t-t_0)} \hat{\mathbf{B}}(\mathbf{r}; \omega) + \frac{\mu_0 \omega}{\pi} \sum_{m,n} \int_{t_0}^t dt' \\ & \times \{ e^{-i\omega(t-t')} \nabla \times \mathcal{I}[\mathbb{G}(\mathbf{r}, \mathbf{r}_A(t); \omega)] \cdot \mathbf{d}_{mn} \hat{A}_{mn}(t') \} \\ & + O(v/c), \end{aligned} \quad (22)$$

where we only considered the zeroth-order approximation in v/c because the magnetic field appears in conjunction with a factor $\mathbf{v}$. In Eq. (22) we remark that ∇ acts only on the first argument of the Green's tensor.

Brief observations on Eqs. (21) and (22) should be given. First, Eq. (21) extends the result for an atom at rest in nonreciprocal media of the work [53] to a moving atom. Second, Eq. (22) generalizes the standard magnetic field given by Eq. (8.11) of Ref. [32] to the case of nonreciprocal media. Both expressions differ from the usual expressions for reciprocal media, for which Eq. (4) applies, only by the redefinition of the imaginary part of the Green's tensor defined in Eq. (3).

III. INTERNAL ATOMIC DYNAMICS: FREQUENCY SHIFT AND DECAY RATE

The internal atomic dynamics can be described by the Heisenberg equations of motion for the flip operator

$$\dot{\hat{A}}_{mn} = \frac{1}{i\hbar} [\hat{A}_{mn}, \hat{H}] = \frac{1}{i\hbar} [\hat{A}_{mn}, \hat{H}_A] + \frac{1}{i\hbar} [\hat{A}_{mn}, \hat{H}_{AF}], \quad (23)$$

which does not include the field Hamiltonian $\hat{H}_F$ because it commutes with the flip operator. Our approach follows the procedure for a reciprocal surface fully detailed in Ref. [32] and the one applied for a nonreciprocal surface with an atom at rest of Ref. [53], now extended to a moving atom parallel to the surface. This leads to

$$\begin{aligned} \dot{\hat{A}}_{mn} = & i\omega_{mn} \hat{A}_{mn} + \frac{i}{\hbar} \sum_k \int_0^\infty d\omega \\ & \times \{ (\hat{A}_{mk} \mathbf{d}_{nk} - \hat{A}_{kn} \mathbf{d}_{km}) \cdot [\hat{\mathbf{E}}(\mathbf{r}_A; \omega) + \mathbf{v} \times \hat{\mathbf{B}}(\mathbf{r}_A; \omega)] \\ & + [\hat{\mathbf{E}}^\dagger(\mathbf{r}_A; \omega) + \mathbf{v} \times \hat{\mathbf{B}}^\dagger(\mathbf{r}_A; \omega)] \cdot (\mathbf{d}_{nk} \hat{A}_{mk} - \mathbf{d}_{km} \hat{A}_{kn}) \}, \end{aligned}$$

which appears in normal ordering and does not show the time dependence for brevity.

To evaluate the required time integrals, we will make some approximations. First, we will assume weak atom-field coupling, which requires that the field spectrum $\omega^2 \mathcal{I}[\mathbb{G}(\mathbf{r}_A(t), \mathbf{r}_A(t); \omega)]$ is sufficiently flat [32]. Particularly, it must not exhibit any narrow peaks in the vicinity of any atomic transition frequency. Second, we will assume that $\hat{A}_{mn}$ is dominated by oscillations with frequencies $\tilde{\omega}_{mn}$, which are yet to be determined in a self-consistent manner. Finally, we will formally carry out the time integrals by means of the Markov approximation. Thus, we are allowed to neglect the slow nonoscillatory dynamics of the atomic flip operator $\hat{A}_{mn}$ during the time interval $t' \in [t_0, t]$, so we may set $\hat{A}_{mn}(t') \simeq$

$\exp[i\tilde{\omega}_{mn}(t' - t)]\hat{A}_{mn}(t)$, where we foresee that the property $\tilde{\omega}_{mn} = -\tilde{\omega}_{nm}$ must be verified. Then we send $t_0 \rightarrow -\infty$ in the time integral, obtaining

$$\begin{aligned} \hat{A}_{mn}(t) & \int_{t_0}^t dt' e^{-i(\omega - \tilde{\omega}_{mn})(t-t')} \\ & \simeq \hat{A}_{mn}(t) \int_{-\infty}^t dt' e^{-i(\omega - \tilde{\omega}_{mn})(t-t')} \\ & = \hat{A}_{mn}(t) \left(\pi \delta(\omega - \tilde{\omega}_{mn}) - i \frac{\mathcal{P}}{\omega - \tilde{\omega}_{mn}} \right). \end{aligned} \quad (24)$$

By taking the derivative with respect to ω of this result, we determine the remaining time integral

$$\begin{aligned} & \int_{t_0}^t dt' (t - t') e^{-i\omega(t-t')} \hat{A}_{mn}(t') \\ & \simeq \frac{d}{d\omega} \left(\frac{\mathcal{P}}{\omega - \tilde{\omega}_{mn}} + i\pi \delta(\omega - \tilde{\omega}_{mn}) \right) \hat{A}_{mn}(t). \end{aligned} \quad (25)$$

For both results $\mathcal{P}$ stands for the Cauchy principle value and the limits of the frequency integral of Eq. (29) will lead to the appearance of the Heaviside function ϑ .

By defining the coefficient

$$\mathbf{C}_{mn} = \mathbf{C}_{mn}(\mathbf{r}_A) + \mathbf{C}_{mn}(\mathbf{r}_A, \mathbf{v}), \quad (26)$$

which contains a pure position-dependent component [53]

$$\begin{aligned} \mathbf{C}_{mn}(\mathbf{r}_A) & = \frac{\mu_0}{\hbar} \vartheta(\tilde{\omega}_{mn}) \tilde{\omega}_{mn}^2 \mathcal{I}[\mathbb{G}(\mathbf{r}_A, \mathbf{r}_A; \tilde{\omega}_{mn})] \cdot \mathbf{d}_{mn} \\ & - \frac{i\mu_0}{\pi \hbar} \mathcal{P} \int_0^\infty \frac{\omega^2 d\omega}{\omega - \tilde{\omega}_{nk}} \mathcal{I}[\mathbb{G}(\mathbf{r}_A, \mathbf{r}_A; \omega)] \cdot \mathbf{d}_{mn} \end{aligned} \quad (27)$$

as well as a position- and velocity-dependent one

$$\begin{aligned} \mathbf{C}_{mn}(\mathbf{r}_A, \mathbf{v}) & = \frac{i\mu_0}{\hbar} \vartheta(\tilde{\omega}_{mn}) \{ \omega^2 \mathcal{I}[\mathbb{G}(\mathbf{r}_A, \mathbf{r}_A; \omega)]' \}_{\omega=\tilde{\omega}_{mn}} \cdot \mathbf{d}_{mn} (\hat{\nabla} \cdot \mathbf{v}) \\ & + \frac{\mu_0}{\pi \hbar} \mathcal{P} \int_0^\infty \frac{d\omega}{\omega - \tilde{\omega}_{nm}} \{ \omega^2 \mathcal{I}[\mathbb{G}(\mathbf{r}_A, \mathbf{r}_A; \omega)]' \} \cdot \mathbf{d}_{mn} (\hat{\nabla} \cdot \mathbf{v}) \\ & - \frac{i\mu_0}{\hbar} \vartheta(\tilde{\omega}_{mn}) \tilde{\omega}_{nm} \{ \mathcal{I}[\mathbb{G}(\mathbf{r}_A, \mathbf{r}_A; \tilde{\omega}_{nm})] \times \hat{\nabla} \} \times \mathbf{d}_{mn} \cdot \mathbf{v} \\ & - \frac{\mu_0}{\pi \hbar} \mathcal{P} \int_0^\infty \frac{\omega d\omega}{\omega - \tilde{\omega}_{nm}} \{ \mathcal{I}[\mathbb{G}(\mathbf{r}_A, \mathbf{r}_A; \omega)] \times \hat{\nabla} \} \times \mathbf{d}_{mn} \cdot \mathbf{v} \\ & - \frac{i\mu_0}{\hbar} \vartheta(\tilde{\omega}_{mn}) \tilde{\omega}_{nm} \mathbf{v} \times \{ \nabla \times \mathcal{I}[\mathbb{G}(\mathbf{r}_A, \mathbf{r}_A; \tilde{\omega}_{nm})] \} \cdot \mathbf{d}_{mn} \\ & - \frac{\mu_0}{\pi \hbar} \mathcal{P} \int_0^\infty \frac{\omega d\omega}{\omega - \tilde{\omega}_{nm}} \mathbf{v} \times \{ \nabla \times \mathcal{I}[\mathbb{G}(\mathbf{r}_A, \mathbf{r}_A; \omega)] \} \cdot \mathbf{d}_{mn}, \end{aligned} \quad (28)$$

where primes in the first and second terms denote derivatives with respect to the respective frequency arguments, we are able to cast Eq. (24) into the form

$$\begin{aligned} \dot{\hat{A}}_{mn}(t) & = i\omega_{mn} \hat{A}_{mn}(t) + \frac{i}{\hbar} \sum_k \int_0^\infty d\omega \\ & \times \{ e^{-i\omega(t-t_0)} [\hat{A}_{mk}(t) \mathbf{d}_{nk} - \hat{A}_{kn}(t) \mathbf{d}_{km}] \} \end{aligned}$$

$$\begin{aligned} & \cdot [\hat{\mathbf{E}}(\mathbf{r}_A; \omega) + \mathbf{v} \times \hat{\mathbf{B}}(\mathbf{r}_A; \omega)] \\ & + e^{i\omega(t-t_0)} [\hat{\mathbf{E}}^\dagger(\mathbf{r}_A; \omega) + \mathbf{v} \times \hat{\mathbf{B}}^\dagger(\mathbf{r}_A; \omega)] \\ & \cdot [\mathbf{d}_{nk} \hat{A}_{mk}(t) - \mathbf{d}_{km} \hat{A}_{kn}(t)] \\ & - \sum_{k,l} [\mathbf{d}_{nk} \cdot \mathbf{C}_{kl} \hat{A}_{ml}(t) - \mathbf{d}_{km} \cdot \mathbf{C}_{nl} \hat{A}_{kl}(t)] \\ & + \sum_{k,l} [\mathbf{d}_{nk} \cdot \mathbf{C}_{ml}^* \hat{A}_{lk}(t) - \mathbf{d}_{km} \cdot \mathbf{C}_{kl}^* \hat{A}_{ln}(t)], \end{aligned} \quad (29)$$

where we have used the identity $\mathcal{I}[\mathbb{G}^*(\mathbf{r}_A, \mathbf{r}_A; \omega)] = \mathcal{I}[\mathbb{G}^\top(\mathbf{r}_A, \mathbf{r}_A; \omega)]$, which is deduced from Eq. (3). The main difference between Eq. (29) and the equation for the reciprocal case, for which Eq. (4) applies, lies in the redefinition of the imaginary part of the Green's tensor (3) introduced by the coefficients (27) and (28), as can be contrasted with the coefficients for reciprocal media given by Eqs. (5.52) and (8.17) in Ref. [32].

Next we take expectation values of Eq. (29) assuming that the electromagnetic field is prepared in its ground state at initial time t_0 , which implies that $\hat{\mathbf{E}}(\mathbf{r}; \omega) | \{0\} \rangle = \mathbf{0} \forall \mathbf{r}, \omega$. This also means that the involved density matrix reads $\hat{\rho}_0 = | \{0\} \rangle \langle \{0\} |$. As a consequence of normal ordering, the free contributions of the electric and magnetic fields will be zero.

Now we need to solve Eq. (29). To this end, we assume the atom to be free of quasidegenerate transitions. Additionally, the atom is assumed to be unpolarized in each of its energy eigenstates, i.e., $\mathbf{d}_{nn} = \mathbf{0}$, as well as that its states of a degenerate manifold are not connected by electric dipole transitions $\mathbf{d}_{nn'} = \mathbf{0}$. Both conditions are guaranteed by atomic selection rules [32]. Under these conditions, the fast-oscillating off-diagonal flip operators decouple from the nonoscillating diagonal ones as well as from each other [32].

For $m \neq n$, under the above assumptions, we obtain that

$$\begin{aligned} \dot{\langle \hat{A}_{mn}(t) \rangle} & = i\omega_{mn} \langle \hat{A}_{mn}(t) \rangle \\ & - \sum_k (\mathbf{d}_{nk} \cdot \mathbf{C}_{kn} + \mathbf{d}_{km} \cdot \mathbf{C}_{km}^*) \langle \hat{A}_{mn}(t) \rangle. \end{aligned} \quad (30)$$

For $m = n$, the diagonal flip operators are nonoscillating and mutually coupled. After retaining only diagonal terms in Eq. (29), we find

$$\begin{aligned} \dot{\langle \hat{A}_{nn}(t) \rangle} & = - \sum_k (\mathbf{d}_{nk} \cdot \mathbf{C}_{kn} + \mathbf{d}_{kn} \cdot \mathbf{C}_{kn}^*) \langle \hat{A}_{nn}(t) \rangle \\ & + \sum_k (\mathbf{d}_{kn} \cdot \mathbf{C}_{nk} + \mathbf{d}_{nk} \cdot \mathbf{C}_{nk}^*) \langle \hat{A}_{kk}(t) \rangle. \end{aligned} \quad (31)$$

By virtue of Eq. (3), the two terms $\mathbf{d}_{nk} \cdot \mathcal{I}[\mathbb{G}(\mathbf{r}_A, \mathbf{r}_A; \omega)] \cdot \mathbf{d}_{kn} = \text{Im}[\mathbf{d}_{nk} \cdot \mathbb{G}(\mathbf{r}_A, \mathbf{r}_A; \omega) \cdot \mathbf{d}_{kn}]$ and $\mathbf{d}_{kn} \cdot \mathcal{I}[\mathbb{G}^\top(\mathbf{r}_A, \mathbf{r}_A; \omega)] \cdot \mathbf{d}_{nk} = \text{Im}[\mathbf{d}_{nk} \cdot \mathbb{G}(\mathbf{r}_A, \mathbf{r}_A; \omega) \cdot \mathbf{d}_{kn}]$ are equal and real. This will occur for the real parts too.

At this point it is worth mentioning that the Green's tensor $\mathbb{G}$ can be split into a bulk part $\mathbb{G}^{(0)}$ and a scattering part $\mathbb{G}^{(1)}$. Here we will assume that the Lamb shift associated with the free-space Green's tensor $\mathbb{G}^{(0)}$ is already included in the bare transition frequency ω_{mn} by making the replacement $\mathbb{G} \rightarrow \mathbb{G}^{(1)}$, which refers only to the atom and does not consider the material properties of surrounding matter [29]. Hence, the remaining frequency shift stems from the presence of electromagnetic bodies around the atom.

So, taking real and imaginary parts of the coefficients in Eqs. (30) and (31) according to

$$\sum_k \mathbf{d}_{nk} \cdot \mathbf{C}_{kn} = \frac{\Gamma_n}{2} + i\delta\omega_n, \quad (32)$$

$$\sum_k \mathbf{d}_{kn} \cdot \mathbf{C}_{kn}^* = \frac{\Gamma_n}{2} - i\delta\omega_n, \quad (33)$$

we are able to identify the decay rate

$$\begin{aligned} \Gamma_n &= \Gamma_n(\mathbf{r}_A) + \Gamma_n(\mathbf{r}_A, \mathbf{v}), \\ &= \sum_k \Gamma_{nk} = \sum_{k < n} \Gamma_{nk}(\mathbf{r}_A) + \sum_k \Gamma_{nk}(\mathbf{r}_A, \mathbf{v}) \end{aligned} \quad (34)$$

and the frequency shift

$$\begin{aligned} \delta\omega_n &= \delta\omega_n(\mathbf{r}_A) + \delta\omega_n(\mathbf{r}_A, \mathbf{v}), \\ &= \sum_k \delta\omega_{nk} = \sum_k \delta\omega_{nk}(\mathbf{r}_A) + \sum_{k < n} \delta\omega_{nk}(\mathbf{r}_A, \mathbf{v}), \end{aligned} \quad (35)$$

with the position-dependent contributions [53]

$$\begin{aligned} \Gamma_{nk}(\mathbf{r}_A) &= \frac{2\mu_0}{\hbar} \tilde{\omega}_{nk}^2 \text{Im}[\mathbf{d}_{nk} \cdot \mathbb{G}^{(1)}(\mathbf{r}_A, \mathbf{r}_A; \tilde{\omega}_{nk}) \cdot \mathbf{d}_{kn}], \quad (36) \\ \delta\omega_{nk}(\mathbf{r}_A) &= -\frac{\mu_0}{\pi\hbar} \mathcal{P} \int_0^\infty \frac{\omega^2 d\omega}{\omega - \tilde{\omega}_{nk}} \\ &\quad \times \text{Im}[\mathbf{d}_{nk} \cdot \mathbb{G}^{(1)}(\mathbf{r}_A, \mathbf{r}_A; \omega) \cdot \mathbf{d}_{kn}] \end{aligned} \quad (37)$$

and the position- and velocity-dependent contributions

$$\begin{aligned} \Gamma_{nk}(\mathbf{r}_A, \mathbf{v}) &= \frac{2\mu_0}{\pi\hbar} \mathcal{P} \int_0^\infty \frac{d\omega}{\omega - \tilde{\omega}_{nk}} \\ &\quad \times \{\omega^2 \mathcal{I}[\mathbf{d}_{nk} \cdot (\mathbf{v} \cdot \nabla) \mathbb{G}^{(1)}(\mathbf{r}_A, \mathbf{r}_A; \omega) \cdot \mathbf{d}_{kn}]\}' \\ &\quad - \frac{2\mu_0}{\pi\hbar} \mathcal{P} \int_0^\infty \frac{\omega d\omega}{\omega - \tilde{\omega}_{nk}} \\ &\quad \times \mathbf{d}_{nk} \cdot \{\mathcal{I}[\mathbb{G}^{(1)}(\mathbf{r}_A, \mathbf{r}_A; \omega)] \times \hat{\nabla}\} \times \mathbf{d}_{kn} \cdot \mathbf{v} \\ &\quad - \frac{2\mu_0}{\pi\hbar} \mathcal{P} \int_0^\infty \frac{\omega d\omega}{\omega - \tilde{\omega}_{nk}} \\ &\quad \times \mathbf{d}_{nk} \cdot \mathbf{v} \times \{\nabla \times \mathcal{I}[\mathbb{G}^{(1)}(\mathbf{r}_A, \mathbf{r}_A; \omega)]\} \cdot \mathbf{d}_{kn}, \end{aligned} \quad (38)$$

$$\begin{aligned} \delta\omega_{nk}(\mathbf{r}_A, \mathbf{v}) &= \frac{\mu_0}{2i\hbar} [\omega^2 \mathbf{d}_{kn} \cdot (\mathbf{v} \cdot \nabla') \mathbb{G}^{(1)\top}(\mathbf{r}, \mathbf{r}'; \omega) \cdot \mathbf{d}_{nk}]'_{\substack{\omega=\tilde{\omega}_{nk} \\ \mathbf{r}=\mathbf{r}'=\mathbf{r}_A}} \\ &\quad - \frac{\mu_0}{2i\hbar} [\omega^2 \mathbf{d}_{nk} \cdot (\mathbf{v} \cdot \nabla') \mathbb{G}^{(1)\top*}(\mathbf{r}, \mathbf{r}'; \omega) \cdot \mathbf{d}_{kn}]'_{\substack{\omega=\tilde{\omega}_{nk} \\ \mathbf{r}=\mathbf{r}'=\mathbf{r}_A}} \\ &\quad - \frac{\mu_0}{\hbar} \tilde{\omega}_{nk} \\ &\quad \times \text{Im}\{\mathbf{v} \cdot \mathbf{d}_{kn} \times [\nabla \times \mathbb{G}^{(1)\top}(\mathbf{r}', \mathbf{r}; \tilde{\omega}_{nk})] \cdot \mathbf{d}_{kn}\}_{\mathbf{r}=\mathbf{r}'=\mathbf{r}_A} \\ &\quad + \frac{\mu_0}{\hbar} \tilde{\omega}_{nk} \\ &\quad \times \text{Im}\{\mathbf{v} \cdot \mathbf{d}_{nk} \times [\nabla \times \mathbb{G}^{(1)}(\mathbf{r}, \mathbf{r}'; \tilde{\omega}_{nk})] \cdot \mathbf{d}_{kn}\}_{\mathbf{r}=\mathbf{r}'=\mathbf{r}_A}. \end{aligned} \quad (39)$$

From Eqs. (38) and (39) we observe that the second and third terms can be traced back to the last four terms of the coefficient $\mathbf{C}_{mn}(\mathbf{r}_A, \mathbf{v})$ defined in Eq. (28). In fact, for an atom

with a time-reversal-invariant internal Hamiltonian and hence real dipole matrix elements, such terms will cancel pairwise and will not contribute. However, in the present work we take into account all these terms of $\mathbf{C}_{mn}(\mathbf{r}_A, \mathbf{v})$, because we choose dipole moments such that the atom is sensitive to chiral media or the violated time-symmetry of the nonreciprocal media. Our choice is based on Curie's principle, which states that a system consisting of a crystal and an external influence, each having a specific symmetry, only maintains the symmetries that are shared by both the crystal and the external influence [65].

With the definitions (38) and (39), the equations of motion (30) and (31) for the expectation values of the atomic flip operators take the final form

$$\langle \hat{A}_{mn}(t) \rangle = \left[i\tilde{\omega}_{mn} - \frac{1}{2}(\Gamma_m + \Gamma_n) \right] \langle \hat{A}_{mn}(t) \rangle, \quad (40)$$

$$\langle \hat{A}_{nn}(t) \rangle = -\Gamma_n \langle \hat{A}_{nn}(t) \rangle + \sum_{k < n} \Gamma_{nk} \langle \hat{A}_{kk}(t) \rangle, \quad (41)$$

where we have identified the frequencies

$$\tilde{\omega}_{mn} = \omega_{mn} + \delta\omega_m - \delta\omega_n, \quad (42)$$

which verifies the property $\tilde{\omega}_{mn} = -\tilde{\omega}_{nm}$ previously mentioned. For the appearance of the shifted frequency $\tilde{\omega}_{nk}$ in the expression for $\delta\omega_{nk}$ through the contributions $\delta\omega_{nk}(\mathbf{r}_A)$ (37) and $\delta\omega_{nk}(\mathbf{r}_A, \mathbf{v})$ (39), the frequency shift is given as a self-consistent result from the implicit equation.

Henceforth we restrict our analysis to the frequency shifts (37) and (39), which can be further simplified by using the redefinition of the imaginary part of the Green's tensor (3), the Schwarz principle valid for nonreciprocal media too,

$$\mathbb{G}^*(\mathbf{r}_A, \mathbf{r}_A; \omega) = \mathbb{G}^*(\mathbf{r}_A, \mathbf{r}_A; -\omega^*), \quad (43)$$

and the substitution $\omega \rightarrow -\omega$ in their second integrals arising from Eq. (3). Each of the integral contours along the positive and negative axes has one real pole and is evaluated in the ω complex plane. The path along the quarter circle does not contribute because $\lim_{|\omega| \rightarrow 0} \omega^2 \mathbb{G}^{(1)}(\mathbf{r}_A, \mathbf{r}_A; \omega)/c^2 = \mathbb{0}$. The part along the the imaginary axis leads to the position-dependent nonresonant frequency shift [53]

$$\begin{aligned} \delta\omega_{nk}^{\text{res}}(\mathbf{r}_A) &= \frac{\mu_0}{\pi\hbar} \int_0^\infty \frac{\xi^3 d\xi}{\xi^2 + \tilde{\omega}_{nk}^2} \\ &\quad \times \text{Im}[\mathbf{d}_{nk} \cdot \mathbb{G}^{(1)}(\mathbf{r}_A, \mathbf{r}_A; i\xi) \cdot \mathbf{d}_{kn}] \\ &\quad - \frac{\mu_0}{\pi\hbar} \int_0^\infty \frac{\xi^2 \tilde{\omega}_{nk} d\xi}{\xi^2 + \tilde{\omega}_{nk}^2} \\ &\quad \times \text{Re}[\mathbf{d}_{nk} \cdot \mathbb{G}^{(1)}(\mathbf{r}_A, \mathbf{r}_A; i\xi) \cdot \mathbf{d}_{kn}], \end{aligned} \quad (44)$$

with a Green's function $\mathbb{G}^{(1)}$ with imaginary frequency $\omega \rightarrow i\xi$. The evaluation of the poles gives the resonant contribution associated with the real-photon emission and a real-frequency expression $\tilde{\omega}_{nk}$ [53],

$$\delta\omega_{nk}^{\text{res}}(\mathbf{r}_A) = -\frac{\mu_0}{\hbar} \tilde{\omega}_{nk}^2 \text{Re}[\mathbf{d}_{nk} \cdot \mathbb{G}^{(1)}(\mathbf{r}_A, \mathbf{r}_A; \tilde{\omega}_{nk}) \cdot \mathbf{d}_{kn}]. \quad (45)$$

From Eq. (44) we observe that the matrix-vector product of the Green's tensor and the dipole moments is real for a reciprocal medium and therefore only the second contribution remains in this case [53]. Additionally, in Eq. (39) we

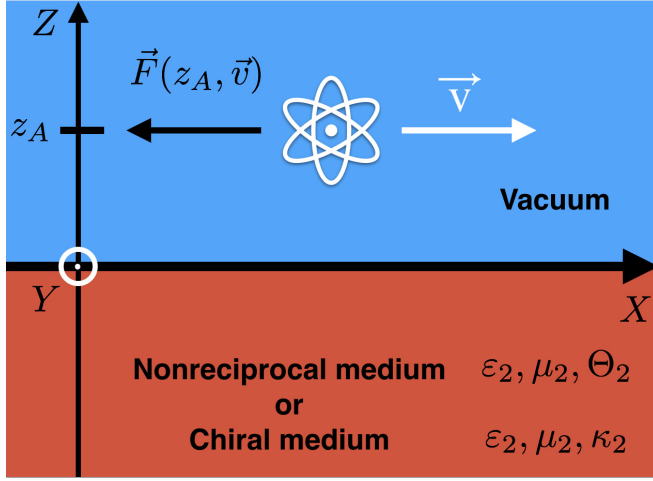


FIG. 1. Quantum friction experienced by an atom moving with constant velocity parallel to a planar interface made of vacuum and a nonreciprocal medium or of vacuum and a chiral medium. The experienced force is $\vec{F}(z_A, \vec{v})$, but is not treated in this work. Instead we study the frequency shift and decay rate induced in the atom.

appreciate another requirement for a nonzero contribution, which is that the Green's tensor of the corresponding material must have a curl different from zero. Remarkably, this last condition is satisfied for certain reciprocal media, opening the possibility to probe quantum friction in chiral materials, for example. The case of chiral media will be analyzed in detail in the following section.

IV. APPLICATIONS

Once we have derived expressions for the frequency shift (39), (44), and (45), we proceed to apply our findings for three ideal media: a perfectly conducting mirror, a perfectly reflecting nonreciprocal mirror, and a perfectly reflecting chiral mirror. Afterward, as more realistic examples, we illustrate our results for a strong three-dimensional (3D) TI and an isotropic chiral medium. In this work we consider the configuration depicted in Fig. 1, where we consider a bilayer configuration with the vacuum as the upper layer and the nonreciprocal medium or the chiral medium as the lower layer. For this selection of five media, we restrict ourselves to real and constant permittivity, permeability, and cross susceptibility unless the contrary is specified.

The scattering part of the Green's tensor $\mathbb{G}^{(1)}$ of a single planar surface has the form [66]

$$\mathbb{G}^{(1)}(\mathbf{r}, \mathbf{r}'; \omega) = \frac{i}{8\pi^2} \int \frac{d^2 \mathbf{k}_\parallel}{k_\perp} \sum_{\sigma=s,p} \sum_{\sigma'=s,p} r_{\sigma,\sigma'} \times \mathbf{e}_{\sigma+} \otimes \mathbf{e}_{\sigma'-} e^{i\mathbf{k}_\parallel \cdot (\mathbf{r}-\mathbf{r}')} e^{ik_\perp(z+z')}, \quad (46)$$

where $\mathbf{e}_{\sigma+}$ and $\mathbf{e}_{\sigma'-}$ are the polarization unit vectors for σ - and σ' -polarized waves in the upper layer of Fig. 1. The indices s and p refer to perpendicular or parallel polarization, respectively. Here $\mathbf{k}_\parallel$ represents the parallel component of the wave vector, $k_\perp$ stands for its perpendicular component, and z denotes the vertical distance to the interface.

As noted in Sec. III, Curie's principle [65] dictates that to probe an atom's sensitivity to the broken time-reversal symmetry of a nonreciprocal medium or the parity symmetry breaking of a chiral medium, its dipole moments must share the same symmetry violation. To be as general as possible, we choose the dipole moments

$$\mathbf{d}_{nk} = (d_{nk}^x, d_{nk}^y, d_{nk}^z)^\top, \quad \mathbf{d}_{kn} = \mathbf{d}_{nk}^*, \quad (47)$$

where $d_{nk}^x, d_{nk}^y, d_{nk}^z \in \mathbb{C}$, implying that such dipoles are not invariant if the direction of time is reversed $t \rightarrow -t$ as long as two of their components are not null. Actually, if $d_{nk}^x = d_{nk}/\sqrt{2}$, $d_{nk}^y = id_{nk}/\sqrt{2}$ and $d_{nk}^z = 0$ give circularly polarized dipoles, which were employed in the article [53], and their fields propagate in the z direction. Additionally, due to our interest in quantum friction closer to the interface between the materials, we will take the velocity of the atom with $v_z = \mathbf{v} \cdot \mathbf{e}_z = 0$, meaning that all the velocity-dependent quantities will depend only on $\mathbf{v}_\parallel = v_x \mathbf{e}_x + v_y \mathbf{e}_y$. Also we have to consider another constriction for the dipole moments and the velocity $\mathbf{v}_\parallel \cdot \mathbf{d}_{nk} = 0$, which is imposed by Maxwell equations.

A. Perfectly conducting mirror

The reflective coefficients for a perfectly conducting mirror are $r_{p,p} = 1$, $r_{s,s} = -1$, and $r_{p,s} = r_{s,p} = 0$. This set of coefficients is obtained from the reflective coefficients for a general material in the limit $\epsilon \rightarrow \infty$, which is explained in detail in Ref. [53]. For this mirror the Green's tensor (46) is only diagonal with

$$\mathbb{G}_{xx}^{(1)}(\mathbf{r}, \mathbf{r}; \omega) = \left(-\frac{1}{8\pi z} - \frac{ic}{16\pi \omega z^2} + \frac{c^2}{32\pi \omega^2 z^3} \right) e^{2i\omega z/c}, \quad (48)$$

$$\mathbb{G}_{yy}^{(1)}(\mathbf{r}, \mathbf{r}; \omega) = \mathbb{G}_{xx}^{(1)}(\mathbf{r}, \mathbf{r}; \omega), \quad (49)$$

$$\mathbb{G}_{zz}^{(1)}(\mathbf{r}, \mathbf{r}; \omega) = \left(-\frac{ic}{8\pi \omega z^2} + \frac{c^2}{16\pi \omega^2 z^3} \right) e^{2i\omega z/c}. \quad (50)$$

The contributions to the frequency shift (35) for the dipole (47) in the retarded limit ($\tilde{\omega}_{nk} z_A/c \gg 1$) read

$$\delta\omega_{nk}^{\text{res,ret}}(\mathbf{r}_A) = \frac{\mu_0 \tilde{\omega}_{nk}^2}{8\pi \hbar} \left[\frac{|\mathbf{d}_{nk}^\parallel|^2}{z_A} \cos\left(\frac{2\tilde{\omega}_{nk} z_A}{c}\right) - \frac{c|d_{nk}^z|^2}{\tilde{\omega}_{nk} z_A^2} \sin\left(\frac{2\tilde{\omega}_{nk} z_A}{c}\right) \right], \quad (51)$$

$$\delta\omega_{nk}^{\text{nres,ret}}(\mathbf{r}_A) = \frac{c|\mathbf{d}_{nk}|^2}{32\pi^2 \hbar \epsilon_0 \tilde{\omega}_{nk} z_A^4}, \quad (52)$$

$$\begin{aligned} \delta\omega_{nk}^{\text{ret}}(\mathbf{r}_A, \mathbf{v}) &= \frac{\mu_0 \tilde{\omega}_{nk}^2 \mathcal{S}_{nk}(\mathbf{v}_\parallel)}{4\pi \hbar c z_A} \sin\left(\frac{2\tilde{\omega}_{nk} z_A}{c}\right) \\ &= 0, \end{aligned} \quad (53)$$

where $\mathbf{d}_{nk}^\parallel = d_{nk}^x \mathbf{e}_x + d_{nk}^y \mathbf{e}_y$ and z_A is the distance of the atom to the interface. For the nonretarded limit ($\tilde{\omega}_{nk} z_A/c \ll 1$) we find

$$\delta\omega_{nk}^{\text{res,nret}}(\mathbf{r}_A) = -\frac{1}{16\pi \hbar \epsilon_0 z_A^3} \left(\frac{1}{2} |\mathbf{d}_{nk}^\parallel|^2 + |d_{nk}^z|^2 \right), \quad (54)$$

$$\delta\omega_{nk}^{\text{nres,nret}}(\mathbf{r}_A) = \frac{1}{32\pi^2 \hbar \epsilon_0 z_A^3} \left(\frac{1}{2} |\mathbf{d}_{nk}^\parallel|^2 + |d_{nk}^z|^2 \right), \quad (55)$$

TABLE I. Behavior under parity P and time reversal T of the five media considered in this work and their atomic electric dipole $\mathbf{d}_{nk}$ and its corresponding medium. The second and third columns display the behavior under P and T of the five materials. Each medium is coupled to an atomic response whose behavior under P and T is shown in the fourth and fifth columns. The last column shows that the coupling of each material with its atomic response gives a P - and T -even frequency shift. The $\mathcal{B}_z$ is given by Eq. (66), $T_{nk}(\mathbf{v}_{\parallel})$ is defined in Eq. (70), and $R_{nk}(\mathbf{v}_{\parallel})$ is in Eq. (93).

Medium	P	T	Atom	P	T	$\delta\omega_{nk}$
Perfectly conducting mirror	+	+	$ \mathbf{d}_{nk} ^2$	+	+	+
Nonreciprocal mirror	−	−	$\mathcal{B}_z, T_{nk}(\mathbf{v}_{\parallel})$	−	−	+
3D topological insulator	−	−	$\mathcal{B}_z, T_{nk}(\mathbf{v}_{\parallel})$	−	−	+
Chiral mirror	−	+	$R_{nk}(\mathbf{v}_{\parallel})$	−	+	+
Isotropic chiral medium	−	+	$R_{nk}(\mathbf{v}_{\parallel})$	−	+	+

$$\delta\omega_{nk}^{\text{nret}}(\mathbf{r}_A, \mathbf{v}) = \frac{\mu_0 \tilde{\omega}_{nk} S_{nk}(\mathbf{v}_{\parallel})}{8\pi \hbar z_A^2} = 0. \quad (56)$$

We observe that the relevant vectorial structure of the frequency shifts (51), (52), (54), and (55) depends only on the moduli of the dipole moments components, meaning that they will contribute to the whole quantities given in Eqs. (34) and (35) regardless of whether the dipole moments are real or complex. This reflects that the perfectly conducting mirror and its corresponding dipole moments do not exhibit parity breaking or time-reversal symmetry breaking, i.e., they are P even and T even. In Table I we summarize the behavior that a perfectly conducting mirror and its dipole moments $\mathbf{d}_{nk}$ must have under P and T transformations. Meanwhile, the shifts (53) and (56) are governed by

$$S_{nk}(\mathbf{v}_{\parallel}) = \text{Im}[(\mathbf{d}_{nk}^* \cdot \mathbf{e}_z)(\mathbf{d}_{nk} \cdot \mathbf{v}_{\parallel})] = 0, \quad (57)$$

which has the same units as the optical rotatory strength found in chiral molecules [61,62] but is a vanishing quantity because Maxwell equations require that $\mathbf{v}_{\parallel} \cdot \mathbf{d}_{nk} = 0$ so the P -even and T -even behavior of this material is respected. Finally, we remark that due to our approximation at first order in v/c , the sign of the velocity could lead to an increase or decrease in the velocity-dependent frequency shifts.

B. Perfectly reflecting nonreciprocal mirror

This mirror is characterized by vanishing polarization preserving reflection coefficients for incoming perpendicular or parallel polarization and outgoing perpendicular or parallel polarization $r_{s,s} = r_{p,p} = 0$ and therefore the mixing terms $r_{p,s}$ and $r_{s,p}$ can be chosen to be either ± 1 [53,67]. We will restrict ourselves to the case $r_{p,s} = r_{s,p} = -1$. Under these conditions the Green's tensor (46) results in the antisymmetric $\mathbb{G}^{(1)\top}(\mathbf{r}, \mathbf{r}'; \omega) = -\mathbb{G}^{(1)}(\mathbf{r}', \mathbf{r}; \omega)$, whose only nonzero components are [53]

$$\mathbb{G}_{xy}^{(1)}(\mathbf{r}, \mathbf{r}; \omega) = \left(-\frac{1}{8\pi z} - \frac{ic}{16\pi \omega z^2} \right) e^{2i\omega z/c}, \quad (58)$$

$$\mathbb{G}_{yx}^{(1)}(\mathbf{r}, \mathbf{r}; \omega) = -\mathbb{G}_{xy}^{(1)}(\mathbf{r}, \mathbf{r}; \omega). \quad (59)$$

The contributions to the frequency shift (35) for the dipole (47) in the retarded limit ($\tilde{\omega}_{nk} z_A/c \gg 1$) read

$$\delta\omega_{nk}^{\text{res,ret}}(\mathbf{r}_A) = -\frac{\mu_0 \tilde{\omega}_{nk}^2 \mathcal{B}_z}{4\pi \hbar z_A} \sin\left(\frac{2\tilde{\omega}_{nk} z_A}{c}\right), \quad (60)$$

$$\delta\omega_{nk}^{\text{nres,ret}}(\mathbf{r}_A) = -\frac{3c^2 \mathcal{B}_z}{32\pi^2 \hbar \varepsilon_0 \tilde{\omega}_{nk}^2 z_A^5}, \quad (61)$$

$$\delta\omega_{nk}^{\text{ret}}(\mathbf{r}_A, \mathbf{v}) = \frac{\mu_0 \tilde{\omega}_{nk}^2}{4\pi \hbar c z_A} T_{nk}(\mathbf{v}_{\parallel}) \cos\left(\frac{2\tilde{\omega}_{nk} z_A}{c}\right), \quad (62)$$

where the dominant contribution to the frequency shift (62) comes from the electric interaction $\mathbf{d}_{mn} \cdot \hat{\mathbf{E}}(\mathbf{r}_A)$. For the non-retarded limit ($\tilde{\omega}_{nk} z_A/c \ll 1$), we find

$$\delta\omega_{nk}^{\text{res,nret}}(\mathbf{r}_A) = -\frac{\mu_0 c \tilde{\omega}_{nk}}{8\pi \hbar z_A^2} \mathcal{B}_z, \quad (63)$$

$$\delta\omega_{nk}^{\text{nres,nret}}(\mathbf{r}_A) = -\frac{\mathcal{B}_z}{16\pi^2 \hbar \varepsilon_0 z_A^3}, \quad (64)$$

$$\delta\omega_{nk}^{\text{nret}}(\mathbf{r}_A, \mathbf{v}) = -\frac{T_{nk}(\mathbf{v}_{\parallel})}{16\pi \hbar \varepsilon_0 c z_A^3}. \quad (65)$$

For this mirror, we identify that the relevant vectorial structure of Eqs. (60), (61), (63), and (64) depends on

$$\mathcal{B}_z = \text{Im}(d_{nk}^x d_{nk}^{y*}) = \frac{1}{2i} (\mathbf{d}_{nk} \times \mathbf{d}_{nk}^*) \cdot \mathbf{e}_z, \quad (66)$$

which is T odd and shows that $\mathbf{d}_{nk}$, $\mathbf{d}_{nk}^*$, and $\mathbf{e}_z$ must not be coplanar. In this way, the time-reversal symmetry breaking for such position-dependent quantities becomes explicitly manifest in comparison with the previous study [53] because only circular or elliptical polarizations parallel to the xy plane will give a nonzero contribution. In other words, linear polarizations are ruled out because they give $\mathbf{d}_{nk} \times \mathbf{d}_{nk}^* = \mathbf{0}$ as a consequence of their invariance under time-reversal symmetry and collinearity. Important to highlight is the cross product of Eq. (66), which in the framework of chiral media can be understood as a geometric magnetic field $\mathcal{B}$ associated with photoionization of chiral molecules [68,69] and whose z component is only relevant here. This offers another first connection between the physics of chirality and nonreciprocity.

On the other hand, Eqs. (62) and (65) share the same vectorial structure for the atomic response

$$T_{nk}(\mathbf{v}_{\parallel}) = \text{Re}\{(\mathbf{d}_{nk}^* \cdot \mathbf{e}_z)[(\mathbf{v}_{\parallel} \times \mathbf{d}_{nk}) \cdot \mathbf{e}_z]\}, \quad (67)$$

which is T odd and does not vanish if $\mathbf{v}_{\parallel}$, $\mathbf{d}_{nk}$, and $\mathbf{e}_z$ are not coplanar and if $d_{nk}^z \neq 0$. These conditions imply the necessity of a superposition of linear polarizations to obtain a nonzero contribution for the velocity-dependent quantities opposed to what the position-dependent ones required via $\mathcal{B}_z$ (66). Besides $d_{nk}^z \neq 0$ this vectorial structure needs either $d_{nk}^x \neq 0$ or $d_{nk}^y \neq 0$, which results in the surprising possibility to turn off

only the position dependence of Eqs. (60), (61), (63), and (64). More specifically, if we choose the polarized dipole in the xz plane $\mathbf{d}_{nk} = d_{nk}/\sqrt{2}(0, 1, 1)^\top$, for which $d_{nk}^{x*} = 0$, then Eq. (66) is zero but $\text{Re}\{(\mathbf{d}_{nk}^* \cdot \mathbf{e}_z)[(\mathbf{v}_\parallel \times \mathbf{d}_{nk}) \cdot \mathbf{e}_z]\} \neq 0$, leading to nonzero velocity-dependent quantities.

Given the significance of this vectorial structure, as a next step we examine its origin. In this way, we recall the interaction Hamiltonian and regard only the Röntgen interaction, which can be rewritten as

$$\hat{H}_{\text{Röntgen}} = \mathbf{v} \cdot \hat{\mathbf{d}} \times \hat{\mathbf{B}}(\mathbf{r}_A) = -\hat{\mathbf{m}} \cdot \hat{\mathbf{B}}(\mathbf{r}_A), \quad (68)$$

where

$$\hat{\mathbf{m}} = -\mathbf{v} \times \hat{\mathbf{d}}. \quad (69)$$

This means that an effective magnetic moment is induced. In this way, the vectorial structure for the velocity-dependent quantities takes the form

$$\begin{aligned} \text{Re}\{(\mathbf{d}_{nk}^* \cdot \mathbf{e}_z)[(\mathbf{v}_\parallel \times \mathbf{d}_{nk}) \cdot \mathbf{e}_z]\} &= -\text{Re}(d_{nk}^{z*} m_{nk}^z) \\ &= T_{nk}(\mathbf{v}_\parallel), \end{aligned} \quad (70)$$

which resembles the optical rotatory strength $R_{nk} = -\text{Im}(d_{nk}^{z*} m_{nk})$ of chiral molecules [61,62], but this one needs the real part, implying the necessity of real dipoles. The form of $T_{nk}(\mathbf{v}_\parallel)$ is in agreement with the Casimir-Polder potential perspective studied in Ref. [67], where for nonreciprocal media it is required that $\mathbf{d}_{nk} \otimes \mathbf{m}_{kn}$ must be real as part of the crossed polarizability. In order to distinguish between the nonreciprocal and the ordinary rotatory strengths, we denote by $T_{nk}(\mathbf{v}_\parallel)$ the nonreciprocal optical rotatory strength. This constitutes the second connection found in this work between the physics of chirality and nonreciprocity. In Table I we summarize the behavior that a perfectly reflecting nonreciprocal mirror and its dipole moments $\mathbf{d}_{nk}$ must have under P and T transformations.

Our two final remarks for this section are as follows. First, without velocity the nonreciprocal optical rotatory strength T_{nk} is zero; thus there is no magnetic moment, so the whole velocity-dependent effect is zero. Second, due to our approximation at first order in v/c , the sign of the velocity could lead to an increase or decrease in the velocity-dependent frequency shifts, as happened to the perfectly conducting mirror discussed in the preceding section.

C. Three-dimensional topological insulator

For these materials, the electric and magnetic fields are connected with the displacement field $\hat{\mathbf{D}}(\mathbf{r}; \omega)$ and the magnetic field $\hat{\mathbf{H}}(\mathbf{r}; \omega)$ via

$$\begin{aligned} \hat{\mathbf{D}}(\mathbf{r}; \omega) &= \varepsilon_0 \varepsilon(\mathbf{r}; \omega) \hat{\mathbf{E}}(\mathbf{r}; \omega) + \frac{\alpha \Theta(\mathbf{r}; \omega)}{\pi \mu_0 c} \hat{\mathbf{B}}(\mathbf{r}; \omega) \\ &\quad + \hat{\mathbf{P}}_N(\mathbf{r}; \omega), \end{aligned} \quad (71)$$

$$\begin{aligned} \hat{\mathbf{H}}(\mathbf{r}; \omega) &= \frac{\hat{\mathbf{B}}(\mathbf{r}; \omega)}{\mu_0 \mu(\mathbf{r}; \omega)} - \frac{\alpha \Theta(\mathbf{r}; \omega)}{\pi \mu_0 c} \hat{\mathbf{E}}(\mathbf{r}; \omega) \\ &\quad - \hat{\mathbf{M}}_N(\mathbf{r}; \omega), \end{aligned} \quad (72)$$

where α is the fine-structure constant and $\varepsilon(\mathbf{r}; \omega)$, $\mu(\mathbf{r}; \omega)$, and $\Theta(\mathbf{r}; \omega)$ are the dielectric permittivity, magnetic permeability, and axion coupling, respectively. Here the

$\hat{\mathbf{P}}_N(\mathbf{r}; \omega)$ and $\hat{\mathbf{M}}_N(\mathbf{r}; \omega)$ terms are the noise polarization and magnetization, respectively. These are Langevin noise terms that model absorption within the material [31]. The topological nature of these materials provides a distinction between strong TIs (odd number of surface states per surface) from weak ones (even number on all but one surface) [37,70]. The axion coupling $\Theta(\mathbf{r}; \omega)$ takes even multiples of π in weak three-dimensional TIs and odd multiples of π in strong three-dimensional TIs, with the magnitude and sign of the multiple given by the strength and direction of the time-symmetry-breaking perturbation. Although the weak class of TIs is under active research, it provides more complications because lattice translation symmetry together with time-reversal symmetry protect the conducting states at the boundaries of a weak 3D TI, which requires one to take into account also disorder and anisotropies [37,71,72]. For these reasons, in this work we will only consider the strong class of them. When Θ is restricted to a nondynamical field, the theory is usually referred to as Θ electrodynamics [45,73,74], which has the necessary topological features to model the electromagnetic behavior of strong 3D TIs [43] and also naturally existing magnetoelectric media [75,76] such as Cr_2O_3 with $\Theta \simeq \pi/36$ [41,77].

The modified Fresnel reflective coefficients for nonreciprocal media $r_{\sigma, \sigma'}$ involved in Eq. (46) can be derived from the application of the boundary conditions to the above constitutive relations. These reflective coefficients can be found in Ref. [66] in their general form. However, we will particularize them for the case of an interface constituted by vacuum in the upper layer of Fig. 1 and a strong three-dimensional TI with real permittivity in the lower layer as shown by

$$r_{s,s} = \frac{(\mu_2 k_1^\perp - k_2^\perp) \mu_2 (\varepsilon_2 k_1^\perp + k_2^\perp) - \Delta^2 k_1^\perp k_2^\perp}{(\mu_2 k_1^\perp + k_2^\perp) \mu_2 (\varepsilon_2 k_1^\perp + k_2^\perp) + \Delta^2 k_1^\perp k_2^\perp}, \quad (73)$$

$$r_{s,p} = \frac{-2\mu_2 k_1^\perp k_2^\perp \Delta}{(\mu_2 k_1^\perp + k_2^\perp) \mu_2 (\varepsilon_2 k_1^\perp + k_2^\perp) + \Delta^2 k_1^\perp k_2^\perp}, \quad (74)$$

$$r_{p,p} = \frac{(\varepsilon_2 k_1^\perp - k_2^\perp) \mu_2 (\mu_2 k_1^\perp + k_2^\perp) + \Delta^2 k_1^\perp k_2^\perp}{(\mu_2 k_1^\perp + k_2^\perp) \mu_2 (\varepsilon_2 k_1^\perp + k_2^\perp) + \Delta^2 k_1^\perp k_2^\perp}, \quad (75)$$

$$r_{p,s} = r_{s,p}, \quad (76)$$

where the subscript 1 denotes the corresponding quantities in the vacuum and conversely the subscript 2 is for the strong 3D TI. The perpendicular parts of the wave vector are denoted by $k_1^\perp$ and $k_2^\perp$ for the corresponding medium. For the general case when two semi-infinite magnetoelectric media with different axion couplings constitute the planar interface, the topological parameter Δ takes the form [66]

$$\Delta = \alpha \mu_1 \mu_2 (\Theta_2 - \Theta_1) / \pi. \quad (77)$$

Nevertheless, for the selected interface of the strong 3D TI with $\Theta_1 = \pi$ and vacuum with $\Theta_2 = 0$, the topological parameter takes the next form

$$\Delta = \alpha(2m + 1), \quad (78)$$

where m is an integer depending on the details of the time-reversal symmetry breaking at the interface.

The contributions to the frequency shift (35) for the dipole (47) in the retarded limit ($\tilde{\omega}_{nk}z_A/c \gg 1$) read

$$\delta\omega_{nk}^{\text{res,ret}}(\mathbf{r}_A) = \frac{\mu_0\tilde{\omega}_{nk}^2}{4\pi\hbar} \left[\mathcal{B}_z \frac{r_{s,p}^{\text{ret}}}{z_A} \sin\left(\frac{2\tilde{\omega}_{nk}z_A}{c}\right) + |\mathbf{d}_{nk}^{\parallel}|^2 \frac{r_{p,p}^{\text{ret}}}{2z_A} \cos\left(\frac{2\tilde{\omega}_{nk}z_A}{c}\right) - \frac{cr_{p,p}^{\text{ret}}|d_{nk}^z|^2}{2\tilde{\omega}_{nk}z_A^2} \sin\left(\frac{2\tilde{\omega}_{nk}z_A}{c}\right) \right], \quad (79)$$

$$\delta\omega_{nk}^{\text{nres,ret}}(\mathbf{r}_A) = \frac{c}{32\pi^2\hbar\epsilon_0\tilde{\omega}_{nk}z_A^4} \left(\mathcal{B}_z \frac{3cr_{s,p}^{\text{ret}}}{\tilde{\omega}_{nk}z_A} + r_{p,p}^{\text{ret}}|\mathbf{d}_{nk}|^2 \right), \quad (80)$$

$$\delta\omega_{nk}^{\text{ret}}(\mathbf{r}_A, \mathbf{v}) = \frac{\mu_0\tilde{\omega}_{nk}^2}{4\pi\hbar c z_A} r_{s,p}^{\text{ret}} T_{nk}(\mathbf{v}_{\parallel}) \cos\left(\frac{2\tilde{\omega}_{nk}z_A}{c}\right), \quad (81)$$

where $S_{nk}(\mathbf{v}_{\parallel}) = 0$ from Eq. (57) was used and the nonreciprocal optical rotatory strength $T_{nk}(\mathbf{v}_{\parallel})$ was defined in Eq. (70); both of them will be subsequently employed. The retarded limit of the Fresnel coefficients is given in Appendix A. Once again, the dominant contribution to the frequency shift (81) arises from the electric interaction $\mathbf{d}_{mn} \cdot \hat{\mathbf{E}}(\mathbf{r}_A)$, similar to the frequency shift of the perfectly reflecting nonreciprocal mirror (62). For the nonretarded limit ($\tilde{\omega}_{nk}z_A/c \ll 1$), we obtain

$$\delta\omega_{nk}^{\text{res,nret}}(\mathbf{r}_A) = \frac{\mu_0\tilde{\omega}_{nk}^2}{8\pi\hbar} \left[\mathcal{B}_z \frac{r_{s,p}^{\text{nret}}c}{\tilde{\omega}_{nk}z_A^2} + |\mathbf{d}_{nk}^{\parallel}|^2 \left(\frac{r_{s,s}^{\text{nret}}}{2z_A} + \frac{r_{p,p}^{\text{nret}}c^2}{\tilde{\omega}_{nk}^2z_A^3} \right) + \frac{c^2r_{p,p}^{\text{nret}}|d_{nk}^z|^2}{2\tilde{\omega}_{nk}^2z_A^3} \right], \quad (82)$$

$$\delta\omega_{nk}^{\text{nres,nret}}(\mathbf{r}_A) = \frac{1}{16\pi^2\hbar\epsilon_0z_A^3} \left[\mathcal{B}_z r_{s,p}^{\text{nret}} + r_{p,p}^{\text{nret}} \pi \left(\frac{|\mathbf{d}_{nk}^{\parallel}|^2}{4} + \frac{|d_{nk}^z|^2}{2} \right) \right], \quad (83)$$

$$\delta\omega_{nk}^{\text{nret}}(\mathbf{r}_A, \mathbf{v}) = \frac{r_{s,p}^{\text{nret}} T_{nk}(\mathbf{v}_{\parallel})}{16\pi\hbar\epsilon_0c z_A^3}, \quad (84)$$

where the nonretarded limit of the Fresnel coefficients is given in Appendix A.

For the strong 3D TI, we observe that the resulting frequency shifts are a superposition of the corresponding quantities of the perfectly conducting mirror of Sec. IV A and the perfectly reflecting nonreciprocal mirror of Sec. IV B determined by the Fresnel coefficients $r_{\sigma,\sigma'}$. As a result, all the previous discussions and mainly the conditions over $\mathbf{v}_{\parallel}$, $\mathbf{d}_{nk}$, and $\mathbf{e}_z$ discussed previously apply. Moreover, the reflective coefficients are functions of the permittivity ϵ and the topological parameter Δ , which could help to modulate the behavior of the spectroscopical quantities. For instance, we could increase Δ by substituting the strong 3D TI by a magnetoelectric like TbPO₄ with $\epsilon_2 = 3.4969$, $\mu_2 = 1$, and $\Delta = 0.22$ [78,79]. Similarly to the sign of the velocity discussed for the perfectly conducting and the nonreciprocal mirror, the topological parameter Δ can acquire a different sign for certain strong 3D TIs. This can be easily seen from the crossed reflective

coefficients in Appendix A, which are proportional to Δ in the retarded and the nonretarded limits. This would lead to an increase or decrease in the corresponding contribution. However, the whole frequency shift is T even once T is applied to $T_{nk}(\mathbf{v}_{\parallel})$. In Table I we resume the behavior that a strong 3D TI and its dipole moments $\mathbf{d}_{nk}$ must have under P and T transformations, where both the atom and medium are falsely chiral. Furthermore, due to the internal connection between the frequency shift and the Casimir-Polder force, one is also able to switch from an attractive force to a repulsive force between the atom and medium. Notwithstanding this possibility, an extremely large axion coupling will approximate these results to those of the perfectly conducting mirror because in that limit $r_{p,s} = r_{s,p} \rightarrow 0$, $r_{s,s} \rightarrow -1$, and $r_{p,p} \rightarrow 1$, as discussed in Ref. [53].

D. Perfectly reflecting chiral mirror

Now we proceed towards chiral media. Although media of this type are reciprocal, meaning that they verify the Lorentz reciprocity principle for the Green's tensor given by Eq. (4) [62,80,81], our result for the velocity-dependent frequency shift (39) shows that the curl of the Green's tensor must not be zero. This feature is indeed satisfied by chiral media [62], justifying such a choice. At this point, we note that the application of our results for chiral media will only require the use of Eq. (4) in all the required formulas, which means that chiral media are a particular case of nonreciprocal media. The resulting expressions are

$$\begin{aligned} \delta\omega_{nk}(\mathbf{r}_A, \mathbf{v}) &= \frac{\mu_0}{\hbar} \{ \omega^2 \text{Im}[\mathbf{d}_{kn} \cdot (\mathbf{v} \cdot \nabla') \mathbb{G}^{(1)}(\mathbf{r}_A, \mathbf{r}_A; \omega) \cdot \mathbf{d}_{nk}] \}'_{\omega=\tilde{\omega}_{nk}} \\ &\quad - \frac{\mu_0}{\hbar} \tilde{\omega}_{nk} \text{Im}\{ \mathbf{v} \cdot \mathbf{d}_{kn} \times [\nabla \times \mathbb{G}^{(1)}(\mathbf{r}_A, \mathbf{r}_A; \tilde{\omega}_{nk})] \cdot \mathbf{d}_{nk} \} \\ &\quad + \frac{\mu_0}{\hbar} \tilde{\omega}_{nk} \text{Im}\{ \mathbf{v} \cdot \mathbf{d}_{nk} \times [\nabla \times \mathbb{G}^{(1)}(\mathbf{r}_A, \mathbf{r}_A; \tilde{\omega}_{nk})] \cdot \mathbf{d}_{kn} \} \end{aligned} \quad (85)$$

and the quantities given by Eqs. (44) and (45) remain the same.

Turning to the perfectly reflecting chiral mirror, which differs from the perfectly conducting mirror in Sec. IV A and the perfectly reflecting nonreciprocal mirror in Sec. IV B, it has the mixing terms $r_{p,s} = -r_{s,p} = \pm i$ and therefore $r_{s,s} = r_{p,p} = 0$ [56]. Here we adopt the convention that a left-handed mirror or a mirror of positive chirality rotates the polarization of an electromagnetic wave in a counterclockwise direction upon reflection when traveling alongside the wave defining the reflection coefficients $r_{p,s} = -r_{s,p} = i$, whereas a right-handed mirror of negative chirality rotates it in a clockwise direction specified by $r_{p,s} = -r_{s,p} = -i$. Under these conditions the Green's tensor (46) results [62]

$$\begin{aligned} \mathbb{G}^{(1)}(\mathbf{r}, \mathbf{r}'; \omega) &= \frac{ic}{8\pi^2\omega} \int \frac{d^2\mathbf{k}_{\perp}}{k_{\perp}} e^{i\mathbf{k}_{\perp} \cdot (\mathbf{r}-\mathbf{r}')} e^{ik_{\perp}(z+z')} \\ &\quad \times (r_{s,p}\mathbf{e}_s \otimes \mathbf{e}_p - r_{p,s}\mathbf{e}_p \otimes \mathbf{e}_s). \end{aligned} \quad (86)$$

The contributions to the frequency shift (35) for the dipole (47) in the retarded limit ($\tilde{\omega}_{nk}z_A/c \gg 1$) read

$$\delta\omega_{nk}^{\text{res,ret}}(\mathbf{r}_A) = 0, \quad (87)$$

$$\delta\omega_{nk}^{\text{res,ret}}(\mathbf{r}_A) = 0, \quad (88)$$

$$\delta\omega_{nk}^{\text{ret}}(\mathbf{r}_A, \mathbf{v}) = \pm \frac{\tilde{\omega}_{nk}^2 R_{nk}(\mathbf{v}_{\parallel})}{4\pi \hbar \varepsilon_0 c^2 z_A} \cos\left(\frac{2\tilde{\omega}_{nk} z_A}{c}\right), \quad (89)$$

where the dominant contribution to the frequency shift (89) arises from the electric interaction $\mathbf{d}_{mn} \cdot \hat{\mathbf{E}}(\mathbf{r}_A)$, as in the frequency shifts of the perfectly reflecting nonreciprocal mirror (62) and the 3D TI (81). For the nonretarded limit ($\tilde{\omega}_{nk} z_A/c \ll 1$) we find

$$\delta\omega_{nk}^{\text{res,nret}}(\mathbf{r}_A) = 0, \quad (90)$$

$$\delta\omega_{nk}^{\text{nres,nret}}(\mathbf{r}_A) = 0, \quad (91)$$

$$\delta\omega_{nk}^{\text{nret}}(\mathbf{r}_A, \mathbf{v}) = \mp \frac{R_{nk}(\mathbf{v}_{\parallel})}{8\pi \hbar \varepsilon_0 c z_A^3}. \quad (92)$$

For this mirror, we observe that all the position-dependent spectroscopical quantities are zero. This is explained by means of Curie's principle [65] mentioned in Sec. III, which applied for this situation establishes the necessity of a chiral source to probe the chirality of the mirror. This means that we need not only an electric dipole but also a magnetic dipole, for which our effective magnetic dipole of Eq. (69) does not work because velocities are not taken into account by the position-dependent frequency shifts (44) and (45). This can be traced back to the origin of these quantities, which arise only from the electric dipole interaction of the interaction Hamiltonian given in Eq. (10).

In spite of that, all velocity-dependent quantities are nonzero because now our effective magnetic dipole (69) and the electric dipole serve to probe the chirality of the mirror leading to nonzero results. All these quantities depend on

$$\begin{aligned} R_{nk}(\mathbf{v}_{\parallel}) &= \text{Im}\{(\mathbf{d}_{nk}^* \cdot \mathbf{e}_z)[(\mathbf{v}_{\parallel} \times \mathbf{d}_{nk}) \cdot \mathbf{e}_z]\} \\ &= -\text{Im}(d_{nk}^{z*} m_{nk}^z), \end{aligned} \quad (93)$$

where we identified in the last line the optical rotatory strength by means of the Röntgen interaction Hamiltonian (68) being a P -odd quantity appearing naturally for any chiral molecule [61,62]. Therefore, we can interpret an effective circular dichroism of the transition $n \leftarrow k$ between the electric dipole and the effective magnetic moment of Eq. (69). This means that we are observing motion-induced chirality.

Because the optical rotatory strength $R_{nk}(\mathbf{v}_{\parallel})$ (93) is a pure imaginary number, it needs circularly or elliptically polarized dipoles, which contrast strongly with the required ones for its nonreciprocal counterpart $T_{nk}(\mathbf{v}_{\parallel})$ (70). Due to its vectorial structure, the noncoplanarity condition over $\mathbf{v}_{\parallel}$, $\mathbf{d}_{nk}$, and $\mathbf{e}_z$ and the condition $\mathbf{v}_{\parallel} \cdot \mathbf{d}_{nk} = 0$ discussed previously are valid. The form of $R_{nk}(\mathbf{v}_{\parallel})$ (93) is in agreement with the Casimir-Polder potential perspective studied in Ref. [67], where for chiral media it is required that $\mathbf{d}_{nk} \otimes \mathbf{m}_{kn}$ must be a pure imaginary number as part of the crossed polarizability. In Table I we resume the behavior that a perfectly reflecting chiral mirror and its dipole moments $\mathbf{d}_{nk}$ must have under P and T transformations.

Similarly to our discussion for the previous nonreciprocal materials, we also observe that these quantities could increase or decrease the corresponding contribution depending on

which handedness for the mirror is used. This is a significant difference with the nonreciprocal mirror of Sec. IV B, whose quantities lack this freedom. The final comment for this section is that the frequency shift of Eq. (92) reproduces the z^{-3} dependence of Ref. [66] obtained through perturbation theory for the circularly polarized dipole $\mathbf{d}_{nk} = d_{nk}/\sqrt{2}(0, 1, i)^T$.

E. Isotropic chiral medium

For these materials, the electric and magnetic fields are connected with the displacement field $\hat{\mathbf{D}}(\mathbf{r}; \omega)$ and the magnetic field $\hat{\mathbf{B}}(\mathbf{r}; \omega)$ via the Boys-Post constitutive relationships [52,62,82,83]

$$\begin{aligned} \hat{\mathbf{D}}(\mathbf{r}; \omega) &= \varepsilon_0 \varepsilon(\mathbf{r}; \omega) \hat{\mathbf{E}}(\mathbf{r}; \omega) - \frac{i\kappa}{c} \hat{\mathbf{B}}(\mathbf{r}; \omega) \\ &\quad + \hat{\mathbf{P}}_N(\mathbf{r}; \omega) - \frac{i\kappa}{c} \hat{\mathbf{M}}_N, \end{aligned} \quad (94)$$

$$\begin{aligned} \hat{\mathbf{B}}(\mathbf{r}; \omega) &= \mu_0 \mu(\mathbf{r}; \omega) \hat{\mathbf{H}}(\mathbf{r}; \omega) + \frac{i\kappa}{c} \hat{\mathbf{E}}(\mathbf{r}; \omega) \\ &\quad + \mu_0 \mu(\mathbf{r}; \omega) \hat{\mathbf{M}}_N(\mathbf{r}; \omega), \end{aligned} \quad (95)$$

where κ is the chiral cross susceptibility, a magnetoelectric susceptibility that in general is a tensor. The chiral cross susceptibility of the medium has the effect of “rotating” a magnetic effect to contribute towards an electric response and vice versa. The similarities and differences between the above constitutive relations and those of the 3D TI, as given by Eqs. (71) and (72), can be examined from the perspectives of true and false chiralities [48,84,85]: In this context, the above chiral medium is truly chiral, whereas the 3D TI is falsely chiral [86].

The modified Fresnel reflective coefficients for nonreciprocal media $r_{\sigma,\sigma'}$ involved in Eq. (46) can be found in Ref. [80] in their general form. Nevertheless, we will particularize them for the case of an interface consisting of vacuum in the upper half space of Fig. 1 and an isotropic chiral medium in the lower half space. By employing the same notation as in Sec. IV C where subscript 1 denotes the corresponding quantities in the vacuum and conversely subscript 2 is for the chiral medium, we have

$$r_{s,s} = \frac{1}{\mathcal{D}}[(1 + a_R)(1 - b_L) + (1 + a_L)(1 - b_R)], \quad (96)$$

$$r_{s,p} = \frac{-2ik_1}{\mathcal{D}k_1^{\perp}} \left(\frac{k_2^{\perp,L}}{k_2^L} - \frac{k_2^{\perp,R}}{k_2^R} \right), \quad (97)$$

$$r_{p,p} = \frac{1}{\mathcal{D}}[(1 - a_R)(1 + b_L) + (1 - a_L)(1 + b_R)], \quad (98)$$

$$r_{p,s} = -r_{s,p}, \quad (99)$$

where $\mathcal{D} = (1 + a_R)(1 + b_L) + (1 + a_L)(1 + b_R)$ and

$$a_P = \frac{\sqrt{\varepsilon_2} k_1 k_2^{\perp,P}}{\sqrt{\mu_2} k_1^{\perp} k_2^P}, \quad b_P = \frac{\sqrt{\mu_2} k_1 k_2^{\perp,P}}{\sqrt{\varepsilon_2} k_1^{\perp} k_2^P}, \quad (100)$$

with $P = R, L$ referring to the right and left circular polarizations of the wave, respectively. Within the chiral medium the wave vectors are

$$(k_2^R)^2 = \frac{\omega^2}{c^2} (\sqrt{\varepsilon_2 \mu_2} + \kappa_2)^2, \quad (101)$$

$$(k_2^\perp)^2 = \frac{\omega^2}{c^2}(\sqrt{\varepsilon_2\mu_2} - \kappa_2)^2, \quad (102)$$

$$(k_2^{\perp,\text{P}})^2 = (k_2^\text{P})^2 - (k_2^\parallel)^2. \quad (103)$$

For the vacuum, which is at the achiral region 1, the wave vectors are the standard free-space ones

$$k_1 = \frac{\omega}{c}, \quad (k_1^\perp)^2 = k_1^2 - (k_1^\parallel)^2, \quad (104)$$

where $k^\parallel = k_x^2 + k_y^2$.

The contributions to the frequency shift (35) for the dipole (47) in the retarded limit ($\tilde{\omega}_{nk}z_A/c \gg 1$) read

$$\begin{aligned} \delta\omega_{nk}^{\text{res,ret}}(\mathbf{r}_A) = & \frac{\mu_0\tilde{\omega}_{nk}^2|\mathbf{d}_{nk}^\parallel|^2}{16\pi\hbar z_A} \left[\cos\left(\frac{2\tilde{\omega}_{nk}z_A}{c}\right) \text{Re}(r_{\text{p,p}}^{\text{ret}} - r_{\text{s,s}}^{\text{ret}}) \right. \\ & \left. + \sin\left(\frac{2\tilde{\omega}_{nk}z_A}{c}\right) \text{Im}(r_{\text{p,p}}^{\text{ret}} - r_{\text{s,s}}^{\text{ret}}) \right] \\ & - \frac{\mu_0\tilde{\omega}_{nk}|d_{nk}^z|^2}{8\pi\hbar c z_A^2} \left[\text{Re}(r_{\text{p,p}}^{\text{ret}}) \sin\left(\frac{2\tilde{\omega}_{nk}z_A}{c}\right) \right. \\ & \left. + \text{Im}(r_{\text{p,p}}^{\text{ret}}) \cos\left(\frac{2\tilde{\omega}_{nk}z_A}{c}\right) \right], \quad (105) \end{aligned}$$

$$\begin{aligned} \delta\omega_{nk}^{\text{nres,ret}}(\mathbf{r}_A) = & \frac{c^2}{32\pi^2\hbar\varepsilon_0\tilde{\omega}_{nk}^2z_A^5} \\ & \times \left(\frac{|\mathbf{d}_{nk}^\parallel|^2}{4} \text{Im}(r_{\text{s,s}}^{\text{ret}} - r_{\text{p,p}}^{\text{ret}}) - |d_{nk}^z|^2 \text{Im}(r_{\text{p,p}}^{\text{ret}}) \right) \\ & - \frac{c}{32\pi^2\hbar\varepsilon_0\tilde{\omega}_{nk}z_A^4} \left(\frac{|\mathbf{d}_{nk}^\parallel|^2}{2} \text{Re}(r_{\text{s,s}}^{\text{ret}} - r_{\text{p,p}}^{\text{ret}}) \right. \\ & \left. - |d_{nk}^z|^2 \text{Re}(r_{\text{p,p}}^{\text{ret}}) \right), \quad (106) \end{aligned}$$

$$\begin{aligned} \delta\omega_{nk}^{\text{ret}}(\mathbf{r}_A, \mathbf{v}) = & \frac{\tilde{\omega}_{nk}^2 R_{nk}(\mathbf{v}_\parallel)}{4\pi\hbar\varepsilon_0 c^2 z_A} \left[\text{Re}(r_{\text{s,p}}^{\text{ret}}) \sin\left(\frac{2\tilde{\omega}_{nk}z_A}{c}\right) \right. \\ & \left. + \text{Im}(r_{\text{s,p}}^{\text{ret}}) \cos\left(\frac{2\tilde{\omega}_{nk}z_A}{c}\right) \right], \quad (107) \end{aligned}$$

where $S_{nk}(\mathbf{v}_\parallel) = 0$ from Eq. (57) was used and the optical rotatory strength $R_{nk}(\mathbf{v}_\parallel)$ was defined in Eq. (93); both of them will be subsequently employed. In this limit, we find that for $\kappa_2 \in \mathbb{C}$, $r_{\text{s,p}}^{\text{ret}}$ will not vanish. For this reason, we consider the more general case of $\varepsilon_2, \kappa_2 \in \mathbb{C}$ to analyze the chiral effects in this retarded limit. The retarded limit of the Fresnel coefficients is given in Appendix B. Again, the dominant contribution to the frequency shift (107) comes from the electric interaction $\mathbf{d}_{mn} \cdot \hat{\mathbf{E}}(\mathbf{r}_A)$, as in the perfectly reflecting nonreciprocal mirror (62), the 3D TI (81), and the perfectly reflecting chiral mirror (89). For the nonretarded limit ($\tilde{\omega}_{nk}z_A/c \ll 1$) we obtain

$$\begin{aligned} \delta\omega_{nk}^{\text{res,nret}}(\mathbf{r}_A) = & \frac{\mu_0\tilde{\omega}_{nk}^2}{8\pi\hbar} \left[|\mathbf{d}_{nk}^\parallel|^2 \left(\frac{r_{\text{s,s}}^{\text{nret}}}{2z_A} + \frac{r_{\text{p,p}}^{\text{nret}} c^2}{\tilde{\omega}_{nk}^2 z_A^3} \right) \right. \\ & \left. + \frac{c^2 r_{\text{p,p}}^{\text{nret}} |d_{nk}^z|^2}{2\tilde{\omega}_{nk}^2 z_A^3} \right], \quad (108) \end{aligned}$$

$$\delta\omega_{nk}^{\text{nres,nret}}(\mathbf{r}_A) = \frac{r_{\text{p,p}}^{\text{nret}}}{16\pi\hbar\varepsilon_0 z_A^3} \left(\frac{|\mathbf{d}_{nk}^\parallel|^2}{4} + \frac{|d_{nk}^z|^2}{2} \right), \quad (109)$$

$$\delta\omega_{nk}^{\text{nret}}(\mathbf{r}_A, \mathbf{v}) = \frac{R_{kn} \text{Im}(r_{\text{s,p}}^{\text{nret}})}{8\pi\hbar\varepsilon_0 c z_A^3}, \quad (110)$$

where the nonretarded limit of the Fresnel coefficients is given in Appendix B.

For the isotropic chiral medium we appreciate that the resulting frequency shifts are a superposition of the corresponding quantities of the perfectly conducting mirror of Sec. IV A and the perfectly reflecting chiral mirror of Sec. IV D determined by the Fresnel coefficients $r_{\sigma,\sigma'}$. As a result of that, all the previous discussions and mainly the conditions over $\mathbf{v}_\parallel$, $\mathbf{d}_{nk}$, and $\mathbf{e}_z$ discussed previously are valid here too. In Table I we resume the behavior that an isotropic chiral medium and its dipole moments $\mathbf{d}_{nk}$ must have under P and T transformations, where both atom and medium exhibit true chirality. Moreover, the reflective coefficients are functions of the permittivity ε , the permeability μ , and the chirality κ , which could help to modulate the behavior of the spectroscopical quantities. In the same spirit as the role of the velocity's sign discussed for all the previous examples, the handedness of the crossed Fresnel coefficients could increase or decrease the corresponding contribution just as found for the chiral mirror, but now they are more complicated functions. Furthermore, in the nonretarded limit the contribution of the imaginary part of the crossed Fresnel coefficients in Eq. (110) becomes relevant because they could be more significant than the perfectly conductor behavior. Again, due to the internal connection between the frequency shift and the Casimir-Polder force, one can also switch from an attractive force to a repulsive force between the atom and medium.

V. DISCUSSION

As a first step towards observing motion-induced chirality and quantum friction with Rydberg atoms, we will discuss the magnitude of some key observables introduced above.

The central indicator of motion-induced chirality is the associated rotatory strength (93). Table II shows its magnitude for hydrogen in Rydberg states with a range of principal quantum numbers $\bar{n}$ for the transition from $|\bar{n}00\rangle$ to $|\bar{n}11\rangle$ and a velocity parallel to the plate of $v_\parallel = 300$ m/s.

It is interesting to compare this motion-induced rotatory strength with typical rotatory strengths of chiral molecules: For the S_0 - S_1 transition in the chiral carbonyl compounds fenchone and camphor, for instance, the rotatory strength corresponds to about 9.5×10^{-55} and 1.5×10^{-54} C² m³/s, respectively [87,88]. Helicenes and their derivatives are known for comparatively large rotatory strengths, which range, depending on the specific transition selected, between around 1×10^{-56} and 1×10^{-53} C² m³/s for [4]-helicene, [5]-helicene, and [6]-helicene [89–91]. The accompanying large isotropic optical rotatory dispersion of helicene derivatives qualifies them, for instance, as candidates for sensitive matter-wave interference studies on chiral molecules as proposed in Ref. [92]. For the Rydberg atoms, we find that the associated rotatory strength is very high compared with

TABLE II. Associated rotatory strength R_{nk} (93), bare and motion-corrected frequency shifts (64) and (65) for a nonreciprocal mirror, and motion-induced frequency shift (92) for a chiral mirror. The nonreciprocal frequency shift is given for an electrical dipole transition via Rydberg-type states $\mathbf{d}_{nk} = \langle \bar{n}_n l_n m_n | \hat{\mathbf{d}} | \bar{n}_k l_k m_k \rangle = \langle \bar{n} 11 | \hat{\mathbf{d}} | \bar{n} 00 \rangle$, where the bare value is quantized in the z direction, while for the motion-induced shift the quantization axis is chosen parallel to $\mathbf{v}_{\parallel}$. The chiral frequency shift is calculated for an electrical dipole transition via Rydberg-type states $\mathbf{d}_{nk} = \langle \bar{n}_n l_n m_n | \hat{\mathbf{d}} | \bar{n}_k l_k m_k \rangle = \langle \bar{n} 10 | \hat{\mathbf{d}} | \bar{n} 00 \rangle$, with the quantization axis along $\mathbf{e}_y + \mathbf{e}_z$ for $\mathbf{v}_{\parallel} = v_{\parallel} \mathbf{e}_x$. All values are in standard SI units; $[R] = \text{C}^2 \text{m}^3/\text{s}$ and $[\delta\omega] = \text{rad/s}$. Note that the velocity-dependent values are linear in v/c , with $v = |\mathbf{v}_{\parallel}|$, and the frequency shifts scale with z_A^{-3} ; the given values are calculated for the velocity $|\mathbf{v}_{\parallel}| = 300 \text{ m/s}$ and at a distance $z_A = 10^{-6} \text{ m}$.

Parameter	$\bar{n} \rightarrow$	20	40	60	80
R_{nk}		1.291×10^{-51}	2.069×10^{-50}	1.048×10^{-49}	3.312×10^{-49}
$\delta\omega_{nk}$		2.918×10^7	4.677×10^8	2.369×10^9	7.487×10^9
$\delta\omega_{nk}(v)$ (nonreciprocal)		9.173×10^1	1.470×10^3	7.446×10^3	2.354×10^4
$\delta\omega_{nk}(v)$ (chiral)		1.835×10^2	2.941×10^3	1.489×10^4	4.707×10^4

regular rotatory strengths in chiral molecules, as it scales with the fourth power of the principal quantum number. However, this scaling is rooted in the typical quadratic scaling of electrical dipoles in Rydberg systems, and thus, in relation to the electrical properties, the associated rotatory strength is still suppressed by a factor of v/c of the order of 10^{-6} .

To estimate the magnitude of motion-induced frequency shifts as evidence for quantum friction, we consider the bare and motion-corrected frequency shifts in Eqs. (64) and (65), respectively, of an atom at nonretarded distances $z_A = 1 \mu\text{m}$ from the surface of a perfectly reflecting nonreciprocal mirror. The results for a velocity $v = 300 \text{ m/s}$ are also displayed in Table II. Note that in (65) we need to choose the quantization axis in the $\mathbf{v}_{\parallel}$ direction parallel to the plate, which results in vanishing (64). In Table II we evaluate (64) therefore with respect to a quantization axis normal to the plate to allow for a useful comparison of the different frequency shifts.

Additionally, Table II also gives values for frequency shifts in the analog case of a chiral mirror (92), with adjusted transition states and quantization axis. A concrete scenario for the measurement of such frequency shifts would consist in directing a tightly focused atomic beam through a wedge-shaped gap, as done in pioneering measurements of the Casimir-Polder forces [93]. In order to implement chiral mirrors, the gap walls need to either consist of nanofabricated chiral metasurfaces [94] or be coated with chiral molecules [95]. Recent progress in the measurement of the transition frequency of Rydberg states has shown an accuracy of a few megahertz using electromagnetically induced transparency [96]; thus static dipole related energy shifts are resolvable starting from $\bar{n} \geq 60$. Motion-corrected frequency shifts for a nonreciprocal mirror and a chiral mirror should be observable starting from $\bar{n} \geq 40$, as kilohertz accuracy with Rydberg microwave Ramsey spectroscopy can be achieved when resorting to circular states [97]. Frequency shifts are advantageous for measurement because frequencies can be measured with very high accuracy [98]. In principle, laser excitation of Rydberg states is frequency selective to the width of the corresponding transition, where subkilohertz resolution in Rydberg states can be achieved for sufficiently high $\bar{n}$. In the current application, state-selective laser excitation might not be possible because of the proximity of the chiral mirror. Frequency measurement during free evolution of an atom excited to a Rydberg can be done via Ramsey interferometry. Interaction-induced energy shifts lead to accumulated phase shifts, which can be projected

to populations in a coherent excitation process, using other Rydberg states or a highly stable ground state as a reference. Measurements of interaction-induced frequency shifts in Rydberg atoms have been implemented [99] and proposed to be used in the application of chiral discrimination [56].

VI. CONCLUSION

In this work macroscopic QED was used to derive expressions for the Casimir-Polder frequency shift and spontaneous decay rate for a moving atom parallel to nonreciprocal (falsely chiral) and (truly) chiral media. The former violate Lorentz's reciprocity principle, meaning that time-reversal symmetry is broken, and the latter are reciprocal media breaking parity. Nonreciprocal media are included by using generalized real and imaginary parts of the Green's tensor which reduce to the standard definitions in the reciprocal case, including (truly) chiral media.

The basis of our study lies in the interaction Hamiltonian between the atom, the field, and the nonreciprocal or chiral medium, which enables us to obtain an expression for the electric and magnetic fields. To this end, noise currents were quantized directly, yielding one set of field operators for the combined electric and magnetic fields. The result for both fields allowed us to analyze the internal atomic dynamics. Through the redefinitions of the real and imaginary parts of a tensor, we provided general expressions for the atomic decay rate and the frequency shift, which can be split into two kinds of terms. One depends only on the position and the other one depends on the position and the velocity of the atom. Also, we found that such terms can be split into resonant and nonresonant contributions. These results reproduce and extend the previous results for a static atom of Ref. [53] to a polarizable atom moving parallel to nonreciprocal or chiral media.

By regarding only a moving nonmagnetic atom, we analyzed five different materials: a perfectly conducting mirror, a perfectly reflecting nonreciprocal mirror, a strong three-dimensional topological insulator, a perfectly reflecting chiral mirror, and an isotropic chiral medium. We found different power laws for all these materials in the nonretarded limit and oscillating functions whose amplitudes decay with the distance in the retarded one.

Let us begin by summarizing the results for the static frequency shift of nonreciprocal media. By employing an

arbitrary dipole moment, we were able to generalize the results of Ref. [53]. We found that for the perfectly conducting mirror there is no difference between any special polarizations because its frequency shift depends only on the magnitude of the dipole moments, i.e., all of them will always contribute. More interesting are the results for the nonreciprocal mirror, because its frequency shift depends on $\text{Im}(d_{nk}^x d_{nk}^{y*})$, which rewritten as Eq. (66) can be identified as the z component of the geometric magnetic field $\mathcal{B}_z$ associated with photoionization of chiral molecules [68,69]. This constitutes the first bridge between the physics of chirality and nonreciprocity obtained in this work: Nonreciprocal media couple to the falsely chiral response of the molecule. For the strong 3D TI we found that its frequency shift is a superposition of the perfectly conducting mirror and the nonreciprocal mirrors but modulated by the Fresnel coefficients. The latter provide the possibility to modulate the behavior of the spectroscopical quantities by adjusting the permittivity ϵ and mainly the topological parameter Δ because it can be negative for certain strong 3D TIs. All these results reproduce the same algebraic power laws and fading-out functions found in Ref. [53] for the nonretarded and retarded limits, respectively.

Now we discuss the results for the static frequency shift of the chiral media. For the perfectly reflecting chiral mirror and the isotropic chiral medium these spectroscopical quantities are identically zero. According to Curie's principle [65], these results are reasonable because our chosen dipole moments do not share the same symmetry. The square of the transition dipoles constitutes an achiral response that is unable to resolve the chirality present in both materials. So, to probe the chirality in this static case, one should employ a molecule with a magnetic moment as well as an electric one.

On the other hand, we have the velocity-dependent frequency shift of the nonreciprocal media. After employing again an arbitrary dipole moment, we found that for the perfectly conducting mirror the relevant vectorial structure is $S_{nk}(\mathbf{v}_{\parallel}) = \text{Im}[(\mathbf{d}_{nk}^* \cdot \mathbf{e}_z)(\mathbf{d}_{nk} \cdot \mathbf{v}_{\parallel})]$, which is zero from the Maxwell equation conditions leading to a vanishing shift and guarantees the P -even and T -even behavior of this material. For the nonreciprocal mirror the vectorial structure can be understood in terms of the nonreciprocal optical rotatory strength $T_{nk}(\mathbf{v}_{\parallel}) = \text{Re}\{(\mathbf{d}_{nk}^* \cdot \mathbf{e}_z)[(\mathbf{v}_{\parallel} \times \mathbf{d}_{nk}) \cdot \mathbf{e}_z]\}$ given by Eq. (70), which needs a superposition of linear polarized dipoles to not vanish. Because of its similarity to the optical rotatory strength of chiral media, this constitutes the second bridge found in this work between the physics of chirality and nonreciprocity. Then we analyzed the frequency shift of a strong 3D TI. In this case, we found again that its frequency shift is a superposition of the perfectly conducting mirror and the nonreciprocal mirrors but modulated by the Fresnel coefficients. However, the sign of the velocity could lead to an enhancement or decrease in the effect for the three materials, but for the strong 3D TI the topological parameter Δ is another quantity that has this feature. Algebraic power laws in the atom's distance z_A to the interface were provided for the three materials in the nonretarded limits, which could be useful for experimental verification. All these results are consistent with Curie's principle because T_{nk} is a T -odd

quantity and also the strong 3D TI breaks time-reversal symmetry via the topological parameter Δ being a T -odd quantity. For this reason, this study is significant for ongoing investigations into dark-matter detection, where axions emerge as a promising candidate [100]. In this pursuit, condensed-matter physics offers novel avenues through TIs [101], antiferromagnetically doped TIs [102,103], or multiferroics [104], utilizing the magnetoelectric effect which underpins our results.

Regarding the velocity-dependent frequency shift of the chiral media, let us first describe the results for the perfectly reflecting chiral mirror. Here we found that this spectroscopical quantity is determined by $R_{nk}(\mathbf{v}_{\parallel}) = \text{Im}\{(\mathbf{d}_{nk}^* \cdot \mathbf{e}_z)[(\mathbf{v}_{\parallel} \times \mathbf{d}_{nk}) \cdot \mathbf{e}_z]\}$, which is the optical rotatory strength [61,62] given by Eq. (93). Therefore, we can interpret an effective circular dichroism of the transition $n \leftarrow k$ between the electric dipole $\mathbf{d}_{nk}$ and the effective magnetic moment $\mathbf{m}_{nk} = -\mathbf{v}_{\parallel} \times \mathbf{d}_{nk}$. Although we did not employ a chiral source, the rectilinear uniform motion of our electric dipole induced an effective magnetic moment through its velocity as found in Eq. (69); therefore, a chiral source was indirectly used to probe the chirality of this mirror explaining the nonzero results, i.e., here occurs a motion-induced (true) chirality. Finally, for the isotropic chiral medium, its frequency shift is a superposition of the perfectly conducting mirror and the chiral mirrors but modulated by the Fresnel coefficients. For both chiral materials the sign of the velocity and the handedness of the media could lead to an enhancement or decrease in the effect. Algebraic power laws in the atom's distance z_A to the interface are provided for these two materials in the nonretarded limits, which could be useful for experimental verification. Nevertheless, for the isotropic chiral medium, the complex character of the Fresnel coefficient can enhance the power law of the distance to the surface. All these results are consistent with Curie's principle because R_{nk} is a P -odd quantity, if circular or elliptical polarizations are selected, and also the isotropic chiral medium breaks the parity symmetry via its chirality κ being a P -odd quantity.

It is worth making a few comments before closing this section. First, for the current interface Δ is proportional to the fine-structure constant α as mentioned in Eq. (78). Nevertheless, we could replace the strong 3D TI by a magnetoelectric such as TbPO_4 , whose $\Delta = 0.22$ [78,79] increases the influence of the nonreciprocal medium. However, we focused in the present work only on strong 3D TIs. Second, the present work provides general results for the frequency shifts either resonant or nonresonant, but we remark that all the concrete results for the five materials are valid only for constant permittivities and nonmagnetic strong 3D TIs, i.e., for the five materials their susceptibilities do not have frequency dependence. Third, due to a more exhaustive analysis of the frequency derivative involved in the velocity-dependent decay rates, we have left their study for these materials for future work. Fourth, of course, the study of the perpendicular motion ($v_z \neq 0$) can be investigated, but it has problems and we will leave it for further studies, which resembles the classical situation of transition radiation studied previously in Ref. [46]. Finally, all these results are obtained within the Markovian framework, meaning that they are only valid for times that verify the constraints described in Ref. [33].

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DATA AVAILABILITY

The data that support the findings of this article are not publicly available. The data are available from the authors upon reasonable request.

APPENDIX A: FRESNEL REFLECTION COEFFICIENTS FOR A VACUUM-STRONG 3D TI INTERFACE

To the purpose of this article being self-contained, in this Appendix we provide the necessary details of the reflective coefficients in the retarded and nonretarded limits for nonreciprocal media.

The Fresnel coefficients for an interface consisting of a 3D topological insulator with $\mu_2 = 1$ and vacuum in the retarded limit ($\tilde{\omega}_{nk}z_A/c \gg 1$) whose main contribution comes from the region due to small $k_{\parallel}$ are [53]

$$r_{s,s}^{\text{ret}} = \frac{1 - \varepsilon - \Delta^2}{(1 + \sqrt{\varepsilon_2})^2 + \Delta^2}, \quad (\text{A1})$$

$$r_{p,s}^{\text{ret}} = \frac{-2\Delta}{(1 + \sqrt{\varepsilon_2})^2 + \Delta^2}, \quad (\text{A2})$$

$$r_{p,p}^{\text{ret}} = -r_{s,s}^{\text{ret}}, \quad (\text{A3})$$

$$r_{s,p}^{\text{ret}} = r_{p,s}^{\text{ret}}, \quad (\text{A4})$$

where Δ is given by Eq. (78). On the other hand, these coefficients in the nonretarded limit ($\tilde{\omega}_{nk}z_A/c \ll 1$) whose leading contribution comes from the region due to large $k_{\parallel}$ are [53]

$$r_{s,s}^{\text{nret}} = \frac{-\Delta^2}{2(\varepsilon_2 + 1) + \Delta^2}, \quad (\text{A5})$$

$$r_{s,p}^{\text{nret}} = \frac{-2\Delta}{2(\varepsilon_2 + 1) + \Delta^2}, \quad (\text{A6})$$

$$r_{p,p}^{\text{nret}} = \frac{2(\varepsilon_2 - 1) + \Delta^2}{2(\varepsilon_2 + 1) + \Delta^2}, \quad (\text{A7})$$

$$r_{s,p}^{\text{nret}} = r_{p,s}^{\text{nret}}. \quad (\text{A8})$$

APPENDIX B: FRESNEL REFLECTION COEFFICIENTS FOR THE VACUUM-ISOTROPIC CHIRAL MEDIUM INTERFACE

The corresponding Fresnel coefficients for an interface consisting of an isotropic chiral medium and vacuum in the retarded limit ($\tilde{\omega}_{nk}z_A/c \gg 1$) whose main contribution comes from the region due to small $k_{\parallel}$ are [105]

$$\begin{aligned} r_{s,p}^{\text{ret}} &= \frac{2i(k_L|k_R| - k_R|k_L|)}{2\left(\frac{\varepsilon+\mu}{n_r}\right)(k_Lk_R + |k_Rk_L|) + (k_L|k_R| + k_R|k_L|)}, \\ r_{p,s}^{\text{ret}} &= -r_{s,p}^{\text{ret}}, \\ r_{s,s}^{\text{ret}} &= \frac{2(k_Lk_R - |k_Rk_L|) - \left(\frac{\varepsilon-\mu}{n_r}\right)(k_L|k_R| + k_R|k_L|)}{2(k_Lk_R + |k_Rk_L|) + \left(\frac{\varepsilon+\mu}{n_r}\right)(k_L|k_R| + k_R|k_L|)}, \\ r_{p,p}^{\text{ret}} &= \frac{2(k_Lk_R - |k_Rk_L|) + \left(\frac{\varepsilon-\mu}{n_r}\right)(k_L|k_R| + k_R|k_L|)}{2(k_Lk_R + |k_Rk_L|) + \left(\frac{\varepsilon+\mu}{n_r}\right)(k_L|k_R| + k_R|k_L|)}, \end{aligned} \quad (\text{B1})$$

where we recall the definitions for k_R and k_L given in Eqs. (101) and (102). To obtain these coefficients, the approximations $k_{\parallel}^{\perp} = 0$ and

$$k^{\perp} \simeq n_r\omega/c = \sqrt{\varepsilon(\omega)\mu(\omega)}\omega/c \quad (\text{B2})$$

were applied to their general expressions given in Ref. [80]. On the other hand, these coefficients in the nonretarded limit ($\tilde{\omega}_{nk}z_A/c \ll 1$) whose leading contribution comes from the region due to large $k_{\parallel}$ are [62]

$$r_{s,s}^{\text{nret}} = \frac{\varepsilon_2\mu_2 - \kappa_2^2 - \varepsilon_2 + \mu_2 - 1}{\varepsilon_2\mu_2 - \kappa_2^2 + \varepsilon_2 + \mu_2 + 1}, \quad (\text{B3})$$

$$r_{p,s}^{\text{nret}} = \frac{-2i\kappa_2}{\varepsilon_2\mu_2 - \kappa_2^2 + \varepsilon_2 + \mu_2 + 1}, \quad (\text{B4})$$

$$r_{p,p}^{\text{nret}} = \frac{\varepsilon_2\mu_2 - \kappa_2^2 + \varepsilon_2 - \mu_2 - 1}{\varepsilon_2\mu_2 - \kappa_2^2 + \varepsilon_2 + \mu_2 + 1}, \quad (\text{B5})$$

$$r_{s,p}^{\text{nret}} = -r_{p,s}^{\text{nret}}. \quad (\text{B6})$$

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