

Characterizing quantum measurement through environmental stochastic entropy production in a two-spin-1/2 system

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Quantum state diffusion is a framework within which measurement may be described as the continuous and gradual collapse of a quantum system to an eigenstate as a result of interaction with its environment. The irreversible nature of the quantum trajectories that arise may be characterized by the environmental stochastic entropy production associated with the measurement. We consider a system of two spin-1/2 particles undergoing either single-particle measurements or measurements of the total z -spin component S_z . The mean asymptotic rates of environmental stochastic entropy production associated with collapse can depend on the eigenstate of S_z selected and on the initial state of the system, offering an additional avenue for characterizing quantum measurement.

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I. INTRODUCTION

Entropy is a quantifier of the subjective uncertainty of the state of a system and its rate of production is a measure of the irreversibility of a process. It can arise when a system interacts with an underspecified complex environment according to dynamics characterized at a coarse-grained level [1,2]. The modern understanding of entropy production was influenced particularly by the development of fluctuation theorems [3–9]. Systems driven by external noise can be characterized by a stochastic entropy production that can be divided into a system stochastic entropy production and an environmental stochastic entropy production, each with particular behavior in a variety of circumstances [10–20]. The application of such ideas to quantum systems has contributed to the field of quantum thermodynamics, from which technologies such as quantum heat and measurement engines and refrigerators have emerged [21–23].

Stochastic entropy production in the processing of quantum systems has recently been explored, including studies of a two-level bosonic system undergoing time-dependent coupling to a harmonic environment, an open three-level quantum system in a nonequilibrium steady state, and entropy production associated with the quantum Zeno and quantum anti-Zeno effects [24–28].

The main focus of this paper is to consider the stochastic entropy production when a system of two spin-1/2 particles is subjected to measurement, extending work by Clarke and Ford on the stochastic entropy production associated with

measurements of a single spin-1/2 particle [25]. New features emerge, for example, requiring us to compute stochastic entropy production within a framework developed to handle singular diffusion matrices [26].

Since average system evolution obscures the irreversibility of measurement, we focus on single evolution pathways or quantum trajectories. A variety of methods for generating quantum trajectories exist such as stochastic Schrödinger equations, hierarchical equations of motion, and quantum jump trajectories comprised of piecewise deterministic dynamics interrupted by stochastic discontinuities [18,29–37]. The Liouville–von Neumann equation offers an alternative approach but does not produce physically meaningful trajectories since the evolution does not preserve the unit trace [24,38]. Many methods for generating quantum trajectories arise from unravelings of the Lindblad master equation [39]. A more recent development has been jump-time unravelings, whereby quantum trajectories are averaged at points where quantum jumps emerge [40].

Projective measurements on quantum systems are discontinuous and instantaneous, resulting in difficulties when attempting to find an associated stochastic entropy production. In contrast, quantum state diffusion (QSD) produces continuous stochastic quantum trajectories without jumps and it is this framework that we use to generate trajectories [10,36,41–43]. The quantum system is considered to interact with an environment which causes it to exhibit a diffusive evolution. A large environment requires knowledge of many degrees of freedom, so its state cannot be specified and instead its influence on the system is taken to be stochastic. The system's behavior is therefore akin to that of a classical Brownian particle. For the measurement process, we consider the environment to act as a measurement apparatus whose interaction with the system causes it to diffuse towards an eigenstate of the system observable being measured.

The coupling between the system and the environment characterizes the strength of measurement. In the limit of

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infinitely strong coupling, the system would seem to exhibit the conventional discontinuous quantum jumps, while at low coupling it undergoes weak measurement, resulting in ongoing extraction of partial information about the system. The evolution of the quantum system may in some circumstances then be described by a stochastic differential equation (SDE), or an Itô process, featuring a noise term that represents the influence of the environment on the system [36,41,44–46].

Utilizing QSD, we generate quantum trajectories for a system of two spin-1/2 particles undergoing either measurements of single-particle spin components or measurements of a component of the total spin of the two particles. We calculate the rate of environmental stochastic entropy production associated with each measurement process and compare the mean asymptotic rates of production for approach towards different eigenstates and for different initial conditions.

The plan for the paper is as follows. Section II introduces the framework of QSD, with Sec. III describing the key concepts relating to stochastic entropy production in a situation characterized by singular diffusion matrices. Section IV considers single-particle spin measurements of the particles, while Sec. V extends this to measurements of total spin. A summary is given and results are discussed in Sec. VI.

II. QUANTUM STATE DIFFUSION

In QSD, measurement is modeled as an interaction between a system and its environment that causes the system to evolve continuously and stochastically towards an eigenstate of the measured operator [44–46]. The evolution of ρ , the reduced density matrix of the system, in a time step dt , is described in terms of Kraus operators M_j [47,48],

$$\rho(t + dt) = \rho(t) + d\rho = \frac{M_j \rho(t) M_j^\dagger}{\text{Tr}[M_j \rho(t) M_j^\dagger]}, \quad (1)$$

and such a transition is considered to occur with probability $p_j = \text{Tr}[M_j \rho(t) M_j^\dagger]$ [27].

In order to generate continuous trajectories, Kraus operators that differ incrementally from a multiple of the identity are used,

$$M_j \equiv M_{k\pm} = \frac{1}{\sqrt{2}}(\mathbb{I} + A_{k\pm}), \quad (2)$$

with

$$A_{k\pm} = -iH_s dt - \frac{1}{2}L_k^\dagger L_k dt \pm L_k \sqrt{dt}, \quad (3)$$

where the system Hamiltonian is denoted by H_s and Lindblad operators by L_k and each Lindblad channel is associated with two Kraus operators labeled by $\pm$ [24,25,27,49–51]. These Kraus operators satisfy the completeness condition $\sum_j M_j^\dagger M_j = \mathbb{I}$. Since the future state of the system depends only on its current state $\rho(t)$, the evolution is Markovian.

A map with a Kraus operator of the form given in Eqs. (2) and (3) has been demonstrated to preserve positivity [27]. A

sequence of such transitions then yields a stochastic trajectory which corresponds to the solution to the SDE

$$d\rho = -i[H_s, \rho]dt + \sum_k \left(\left(L_k \rho L_k^\dagger - \frac{1}{2}\{L_k^\dagger L_k, \rho\} \right) dt + \{\rho L_k^\dagger + L_k \rho - \text{Tr}[\rho(L_k + L_k^\dagger)]\rho\} dW_k \right), \quad (4)$$

where $\hbar$ has been set to unity [10,36,41,42,44,45]. The evolution of ρ is thus an Itô process with Wiener increments dW_k , the index k denoting the Lindblad operator [45]. The stochasticity arises from system-environment interactions. In the absence of such interactions with the environment, the evolution of ρ reduces to the von Neumann equation.

It has been demonstrated that the average evolution over all possible system transitions and over all possible system states satisfies a Lindblad master equation [24,27,51]

$$d\bar{\rho} = -i[H_s, \bar{\rho}]dt + \sum_k \left(L_k \bar{\rho} L_k^\dagger - \frac{1}{2}\{L_k^\dagger L_k, \bar{\rho}\} \right) dt \quad (5)$$

for an ensemble-averaged reduced density matrix $\bar{\rho}(t)$. We employ an overbar as a reminder of the averaging, though this is not the usual notation in the literature. In spite of the invariance of the right-hand side of Eq. (5) under unitary transformations among the Kraus operators, the requirement that the underlying stochastic trajectories are continuous means that the Kraus operators are uniquely given by Eqs. (2) and (3) in this case.

III. ENVIRONMENTAL STOCHASTIC ENTROPY PRODUCTION

A. Dynamics of entropy production

The von Neumann entropy is an extension of the Gibbs entropy to quantum systems and provides a measure of the uncertainty over the possible outcomes associated with a projective measurement of a system starting in a given state. It is defined as $-\sum_i P_i \ln P_i$, where P_i is the probability of projection onto an eigenstate i of the measured observable [48]. It can also be written as $-\text{Tr} \rho \ln \rho$ in terms of the density matrix. The relative entropy or Kullback-Leibler divergence characterizes the difference between two probability density functions or density operators ρ [52,53]. These forms of entropy concern the average system evolution, but we wish to characterize the irreversibility of continuous single realizations of system behavior in Hilbert space, not projective measurement, so the von Neumann entropy is not suitable. It is stochastic entropy production that provides this: a change in subjective uncertainty of the quantum state of the system and its environment arising from our inability to make precise predictions when the state of the environment and its influence on the system is underspecified [13,16]. The von Neumann entropy has an upper limit of $\ln 2$ for a spin-1/2 system, since it represents the uncertainty of the state of a system with respect to the possible eigenstates available upon measurement, but the stochastic entropy production is unbounded since it concerns the uncertainty of adoption over a continuum of possible state vectors, not just those associated with eigenstates.

Moreover, as the system approaches an eigenstate, its stochastic entropy production will decrease, reflecting the reduction in uncertainty with regard to the adopted state. To adhere to the second law, the stochastic entropy production in the environment must therefore increase, reflecting an increasing uncertainty in its state. Under the QSD framework, the system only reaches an eigenstate in the limit of asymptotic time, and therefore the environmental stochastic entropy production will continue to increase even as the system state is refined.

The von Neumann entropy can be considered thermodynamic, since it represents uncertainty in the adoption of eigenstates, which are stationary states that could be associated with a Boltzmann distribution of probabilities, with a clear connection to energy. In contrast, stochastic entropy production characterizes uncertainty with respect to a continuum of state vectors which are for the most part nonstationary. There is no clear association with a Boltzmann distribution of probabilities and therefore with energy; thus the stochastic entropy production represents informational rather than thermodynamic entropy.

To be specific, the (total) stochastic entropy production Δs_{tot} associated with a trajectory is defined by the ratio of probabilities of the (forward) trajectory and the reverse sequence of events driven by a reverse protocol of external forces:

$$\Delta s_{\text{tot}} = \ln \left(\frac{\text{prob}(\text{forward trajectory})}{\text{prob}(\text{reverse trajectory})} \right). \quad (6)$$

This can be separated into two contributions, system Δs_{sys} and environmental Δs_{env} . The former depends on the probability density function (PDF) over the space of system microstates, while the latter may be derived from the stochastic dynamics that govern the system evolution.

We employ the following framework. A set of N coordinates $\mathbf{x} \equiv (x_1, x_2, \dots, x_N)$ specifies the configuration of a system, and their evolution is modeled using Markovian stochastic differential equations, or Itô processes, of the form

$$dx_i = A_i(\mathbf{x})dt + \sum_{j=1}^M B_{ij}(\mathbf{x})dW_j, \quad (7)$$

where the dW_j are M independent Wiener increments. Defining an $N \times N$ diffusion matrix $\mathbf{D}(\mathbf{x}) = \frac{1}{2}\mathbf{B}(\mathbf{x})\mathbf{B}(\mathbf{x})^T$, the Fokker-Planck equation for the PDF $p(\mathbf{x}, t)$ is given by

$$\begin{aligned} \frac{\partial p}{\partial t} &= - \sum_i \frac{\partial}{\partial x_i} (A_i p) + \sum_{ij} \frac{\partial}{\partial x_i \partial x_j} (D_{ij} p) \\ &= - \sum_i \frac{\partial}{\partial x_i} \left(C_i p - D_{ij} \frac{\partial p}{\partial x_j} \right), \end{aligned} \quad (8)$$

where

$$C_i = A_i - \sum_j \frac{\partial D_{ij}}{\partial x_j}. \quad (9)$$

The evolution of the system stochastic entropy production is given by $d\Delta s_{\text{sys}} = -d \ln p(\mathbf{x}, t)$, but for simplicity we focus our attention only on the environmental stochastic entropy production. We consider coordinates with even parity under

time reversal. The evolution of Δs_{env} is then governed by the SDE, given in Stratonovich form [12,15,16]

$$\begin{aligned} d\Delta s_{\text{env}} &= \sum_{ij} D_{ij}^{-1} C_j dx_i + \sum_{ijk} D_{ij} \frac{\partial (D_{ik}^{-1} C_k)}{\partial x_j} dt \\ &= \sum_{ij} D_{ij}^{-1} C_j \circ dx_i. \end{aligned} \quad (10)$$

A general expression suitable for systems with odd as well as even parity coordinates is given in Appendix A.

B. Procedure for singular diffusion matrices

Singular diffusion matrices arise for systems where there are more coupled Itô processes than independent Wiener noise increments. At points in the coordinate space of the system there are directions for which the rate of diffusion is zero. These directions are parallel to eigenvectors of the diffusion matrix with zero eigenvalues, referred to as null eigenvectors. The singularity in the diffusion matrix can be attributed to the existence of a time independent or a deterministically evolving function of the variables x_i [26]. The procedure for dealing with singular diffusion matrices involves focusing on a subset of the total phase space spanned by M coordinates, which we refer to as dynamical variables. The remaining $L = N - M$ coordinates will be regarded as spectator variables.

Such a distinction between dynamical and spectator variables is arbitrary since the total entropy production does not depend on which variable is chosen to be dynamical, but a careful choice can greatly simplify the computation. The reduced diffusion matrix defined in the coordinate subspace with dimensions $M \times M$ is nonsingular, so analysis using Eq. (10) can proceed. The spectator variables nevertheless still play a role in computing the entropy production. The main implication is that the derivatives dD_{ij}/dx_m are replaced by a new expression involving derivatives with respect to dynamical variables x_m and the spectator variables x_l ,

$$\frac{dD_{ij}}{dx_m} = \frac{\partial D_{ij}}{\partial x_m} + \sum_l \frac{\partial D_{ij}}{\partial x_l} \frac{dx_l}{dx_m} = \frac{\partial D_{ij}}{\partial x_m} + \sum_l \frac{\partial D_{ij}}{\partial x_l} R_{lm}, \quad (11)$$

where R_{lm} is a matrix formed of components of the null eigenvectors of the full $N \times N$ diffusion matrix. See Appendix B and [26] for further details.

IV. TWO SPIN-1/2 PARTICLES UNDERGOING SEPARATE MEASUREMENTS

A. System specification and dynamics

The density matrix describing a system of two spin-1/2 particles may be constructed as

$$\rho = \frac{1}{4}(\mathbb{I} + \mathbf{s} \cdot \mathbf{\Sigma}), \quad (12)$$

using the 15 component coherence vector $\mathbf{s} = (s_1, s_2, \dots, s_{15}) = \text{Tr}(\rho \mathbf{\Sigma})$ and the vector $\mathbf{\Sigma}$ of generators Σ_i , with $i = 1, \dots, 15$, built using Pauli matrices (details are given in Appendix C).

The density matrix may then be written as

$$\rho = \frac{1}{4} \begin{pmatrix} 1 + s_3 + s_{12} + s_{15} & -is_2 + s_{13} - is_{14} + s_1 & s_4 + s_7 - is_8 - is_{11} & s_5 - is_6 - is_9 - s_{10} \\ is_2 + s_{13} + is_{14} + s_1 & 1 - s_3 + s_{12} - s_{15} & s_5 + is_6 - is_9 + s_{10} & s_4 - s_7 - is_8 + is_{11} \\ s_4 + s_7 + is_8 + is_{11} & s_5 - is_6 + is_9 + s_{10} & 1 + s_3 - s_{12} - s_{15} & -is_2 - s_{13} + is_{14} + s_1 \\ s_5 + is_6 + is_9 - s_{10} & s_4 - s_7 + is_8 - is_{11} & is_2 - s_{13} - is_{14} + s_1 & 1 - s_3 - s_{12} + s_{15} \end{pmatrix}. \quad (13)$$

Stochastic trajectories of ρ are generated from SDEs for the components of the coherence vector, s_i , produced from

$$ds_i = \text{Tr}(d\rho \Sigma_i), \quad (14)$$

with $d\rho$ defined in Eq. (4). We now consider different scenarios of single-particle measurements performed on the two-spin-1/2 system, corresponding to particular choices of Lindblad operators.

B. Case 1: z -spins of each particle

We first consider simultaneous measurements of the z component of each of the two spins in the system. The system starts in the singlet state $|\Psi^-\rangle = \frac{1}{\sqrt{2}}(|1\rangle_z | -1\rangle_z - | -1\rangle_z | 1\rangle_z)$, where $|\pm 1\rangle_z$ are the eigenstates of the z -spin for each particle. We use Eq. (4) to model the evolution of the reduced density matrix ρ , with system Hamiltonian $H_s = 0$ and Lindblad operators $L_1 = a_1 \frac{1}{2} \sigma_{z,1} \otimes \mathbb{I}$ and $L_2 = \mathbb{I} \otimes a_2 \frac{1}{2} \sigma_{z,2}$, where the scalar coefficients a_1 and a_2 represent couplings between the system and the environment. Insertion into Eq. (4) yields 15 SDEs for the coherence vector components, given in Appendix D. The SDEs feature two noises, one for each measurement in the system. The so-called expectation values of the individual z -spin components may be considered to be physical attributes of the quantum state [27,51] and can be expressed as

$$\begin{aligned} \langle S_{z,1} \rangle &= \text{Tr}[(\tfrac{1}{2} \sigma_{z,1} \otimes \mathbb{I}) \rho] = \tfrac{1}{2} s_{12}, \\ \langle S_{z,2} \rangle &= \text{Tr}[(\mathbb{I} \otimes \tfrac{1}{2} \sigma_{z,2}) \rho] = \tfrac{1}{2} s_3. \end{aligned} \quad (15)$$

The SDEs take the form $dx_i = A_i dt + B_{i1} dW_1 + B_{i2} dW_2$, where dx_i is identified as the ds_i in Eq. (14) and the diffusion matrix elements are $D_{ij} = \frac{1}{2}(B_{i1}B_{j1} + B_{i2}B_{j2})$, with the expressions for B_{i1} and B_{i2} given in Appendix D.

There is only one nonzero eigenvalue, so we use the method outlined in Sec. III B, choosing s_{12} as the dynamical variable since its associated SDE has $A_{12} = 0$, which simplifies the calculation. The reduced (scalar) diffusion coefficient, for $a_1 = a_2 = 1$, is written

$$D = \tfrac{1}{2}(-1 + s_{12})^2 + \tfrac{1}{2}(s_3 s_{12} - s_{15})^2, \quad (16)$$

and Eq. (10) reduces to

$$d\Delta_{\text{Senv}} = -\frac{1}{D} \frac{dD}{ds_{12}} ds_{12} - \frac{d^2 D}{ds_{12}^2} dt + \frac{1}{D} \left(\frac{dD}{ds_{12}} \right)^2 dt, \quad (17)$$

with the derivative dD/ds_{12} given by

$$\frac{dD}{ds_{12}} = \frac{\partial D}{\partial s_{12}} + \sum_{l \neq 12} \frac{\partial D}{\partial s_l} R_{l,12}. \quad (18)$$

The diffusion coefficient only depends on the variables s_{12} , s_3 , and s_{15} , so from Eq. (11) the dynamical variable labeled m is s_{12} and the relevant spectator variables labeled l are s_3 and s_{15} ; thus the relevant R matrix components to consider are $R_{3,12}$ and $R_{15,12}$, which from Eq. (11) are expressed as follows, using $a_1 = a_2 = 1$:

$$\begin{aligned} R_{3,12} &= \frac{(s_3 s_{12} - s_{15}) dW_1 + (s_3^2 - 1) dW_2}{(s_{12}^2 - 1) dW_1 + (s_3 s_{12} - s_{15}) dW_2}, \\ R_{15,12} &= \frac{(s_{12} s_{15} - s_3) dW_1 + (s_3 s_{15} - s_{12}) dW_2}{(s_{12}^2 - 1) dW_1 + (s_3 s_{12} - s_{15}) dW_2}. \end{aligned} \quad (19)$$

Starting in $|\Psi^-\rangle = \frac{1}{\sqrt{2}}(|1\rangle_z | -1\rangle_z - | -1\rangle_z | 1\rangle_z)$, there are only two possible measurement outcomes $|1\rangle_z | -1\rangle_z$ and $| -1\rangle_z | 1\rangle_z$ such that $s_3 = -s_{12}$ throughout. Furthermore, it may be shown that s_{15} remains equal to -1 ; thus we find that $R_{3,12} = -1$ and $R_{15,12} = 0$. Hence, using Eqs. (16)–(18) and these values for $R_{3,12}$ and $R_{15,12}$, the evolution of the environmental stochastic entropy production may be written as

$$d\Delta_{\text{Senv}} = 4(s_{12}^2 + 1)dt + 4s_{12}(dW_1 - dW_2). \quad (20)$$

C. Case 2: z -spins of particle 1 and x -spins of particle 2

Next we consider the case where particle 1 undergoes a z -spin measurement and particle 2 an x -spin measurement. Using Lindblad operators $L_1 = a_1 \frac{1}{2} \sigma_{z,1} \otimes \mathbb{I}$ and $L_2 = \mathbb{I} \otimes a_2 \frac{1}{2} \sigma_{x,2}$, and once again $H_s = 0$, leads to 15 SDEs, shown in Appendix E. The SDEs contain two noises dW_1 and dW_2 relating to the measurements on the first and second particles. The expectation values of the z -spin of the first particle and the x -spin of the second particle are

$$\begin{aligned} \langle S_{z,1} \rangle &= \text{Tr}(\tfrac{1}{2} \sigma_{z,1} \rho) = \tfrac{1}{2} s_{12}, \\ \langle S_{x,2} \rangle &= \text{Tr}(\tfrac{1}{2} \sigma_{x,2} \rho) = \tfrac{1}{2} s_1. \end{aligned} \quad (21)$$

The diffusion matrix this time contains two nonzero eigenvalues, implying two dynamical variables that we choose to be s_1 and s_{12} . The reduced diffusion matrix in the basis of s_1 and s_{12} variables is

$$D = \frac{1}{2} \begin{pmatrix} (-1 + s_1^2)^2 + (s_1 s_{12} - s_{13})^2 & (-2 + s_1^2 + s_{12}^2)(s_1 s_{12} - s_{13}) \\ (-2 + s_1^2 + s_{12}^2)(s_1 s_{12} - s_{13}) & (-1 + s_{12}^2)^2 + (s_1 s_{12} - s_{13})^2 \end{pmatrix}. \quad (22)$$

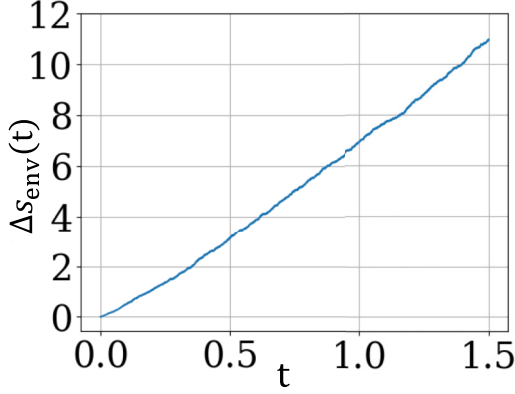


FIG. 1. Mean environmental stochastic entropy production for case 1: two spin-1/2 particles, each undergoing z -spin measurements starting at $t = 0$, with the initial state $|\Psi^-\rangle = \frac{1}{\sqrt{2}}(|1\rangle_z|1\rangle_z - |-1\rangle_z|1\rangle_z)$, measurement strengths $a_1 = a_2 = 1$, and an average over 326 trajectories using a time step of 0.0001.

Since s_1, s_{12} , and s_{13} form a closed group of equations, s_{13} is the only spectator variable we need to consider. We can express the derivative of the diffusion matrix inverse for use in Eq. (10) using

$$\frac{\partial(D^{-1}D)}{\partial x_k} = \frac{\partial D^{-1}}{\partial x_k}D + D^{-1}\frac{\partial D}{\partial x_k} = \frac{\partial \mathbb{I}}{\partial x_k} = 0, \quad (23)$$

and hence

$$\frac{\partial D^{-1}}{\partial x_k} = -D^{-1}\frac{\partial D}{\partial x_k}D^{-1}. \quad (24)$$

We do not present a full analytic expression for the environmental stochastic entropy production due to its complexity.

D. Environmental stochastic entropy production for cases 1 and 2

The mean environmental stochastic entropy production associated with case 1 is illustrated in Fig. 1. The mean rate associated with measurements of the z -spin of each particle is approximately double that arising from measurement of the z -spin of a single particle [25]. Since the particles start in the singlet state, there are only two possible measurement outcomes $|1\rangle_z|-1\rangle_z$ and $|-1\rangle_z|1\rangle_z$ such that $s_3 = -s_{12}$ and asymptotically $s_3 \rightarrow \pm 1$. The expression for the stochastic environmental entropy production in Eq. (20) suggests an asymptotic mean rate of environmental stochastic entropy production of $d\langle\Delta s_{\text{env}}\rangle/dt \rightarrow 8$, which is reflected in the slope in Fig. 1. Note that the mean rate of entropy production is considered to be asymptotic when the entropy production exhibits a linear increase with time and the system is close to an eigenstate.

Figure 2 illustrates a set of example trajectories for case 2, where a z -spin measurement is performed on particle 1 and an x -spin measurement is performed on particle 2. The initial state is $|\Psi^-\rangle = \frac{1}{\sqrt{2}}(|1\rangle_z|-1\rangle_z - |-1\rangle_z|1\rangle_z)$, which lies at $\langle S_{x,2} \rangle = \langle S_{z,1} \rangle = 0$. Particle 1 has two possible measurement

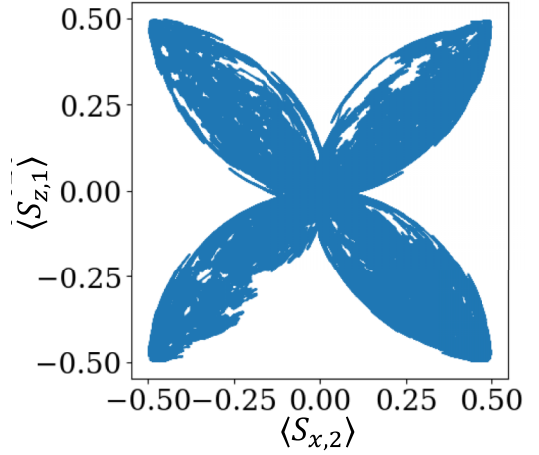


FIG. 2. Illustration of 500 example stochastic trajectories in the space of $\langle S_{x,2} \rangle$ and $\langle S_{z,1} \rangle$ for case 2: two spin-1/2 particles where a z -spin measurement is performed on particle 1 and an x -spin measurement is performed on particle 2. Trajectories begin at the state $|\Psi^-\rangle = \frac{1}{\sqrt{2}}(|1\rangle_z|-1\rangle_z - |-1\rangle_z|1\rangle_z)$, corresponding to $\langle S_{x,2} \rangle = \langle S_{z,1} \rangle = 0$, and proceed towards the tip of one of the "petals." Measurement strengths $a_1 = a_2 = 1$ and the time step is 0.0001.

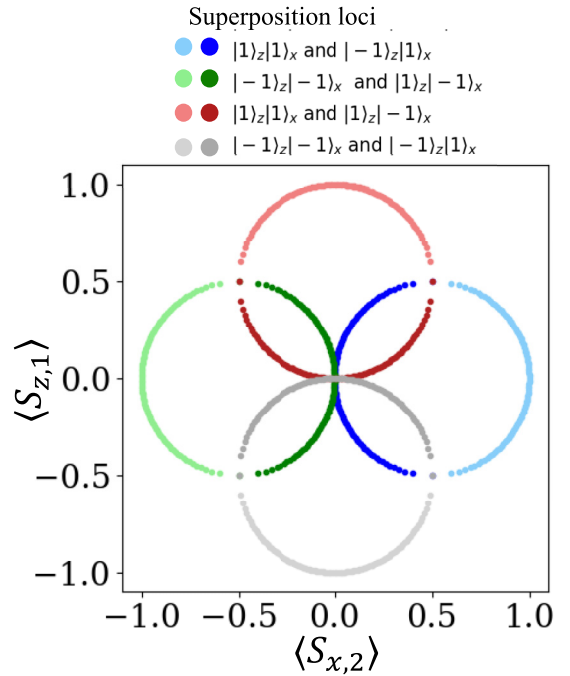


FIG. 3. Projections onto the $(\langle S_{x,2} \rangle, \langle S_{z,1} \rangle)$ space of superpositions (with real amplitudes) of two of the four measurement outcomes for case 2. Blue denotes $|1\rangle_z|1\rangle_x$ and $|-1\rangle_z|1\rangle_x$; green, $|-1\rangle_z|-1\rangle_x$ and $|1\rangle_z|-1\rangle_x$; red, $|1\rangle_z|1\rangle_x$ and $|1\rangle_z|-1\rangle_x$; and gray, $|-1\rangle_z|-1\rangle_x$ and $|-1\rangle_z|1\rangle_x$. Points represented by darker shades signify states that are dynamically accessible to the system under the specific measurement scheme. Points illustrated by lighter shades are not accessible but illustrate the broader pattern. The dynamics constrains the stochastic trajectories to lie in the overlap regions between the circles, forming the petal shapes in Fig. 2.

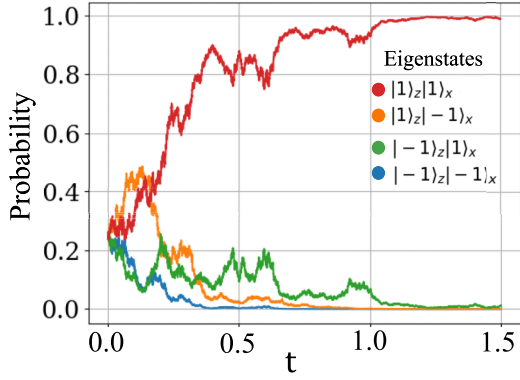


FIG. 4. Example evolution of the probability amplitudes of the possible measurement outcomes in case 2: the two-spin-1/2 system undergoing a z -spin measurement on particle 1 and an x -spin measurement on particle 2. Red denotes the probability amplitude of the $|1\rangle_z|1\rangle_x$ measurement outcome; orange, the $|1\rangle_z|-1\rangle_x$ measurement outcome; green, the $|-1\rangle_z|1\rangle_x$ measurement outcome; and blue, the $|-1\rangle_z|-1\rangle_x$ measurement outcome. The initial state $|\Psi^-\rangle = \frac{1}{\sqrt{2}}(|1\rangle_z|-1\rangle_z - |-1\rangle_z|1\rangle_z)$, measurements strengths $a_1 = a_2 = 1$, and the time step is 0.0001.

outcomes $|1\rangle_z$ and $|-1\rangle_z$ and similarly particle 2 has possible measurement outcomes $|1\rangle_x$ and $|-1\rangle_x$. Hence there are four possible joint outcomes $|1\rangle_z|1\rangle_x$, $|1\rangle_z|-1\rangle_x$, $|-1\rangle_z|1\rangle_x$, and $|-1\rangle_z|-1\rangle_x$ which lie at the tips of the petal shape in Fig. 2.

The pattern arises because the stochastic trajectories are constrained to lie between loci defined by superpositions (with real amplitudes) of pairs of measurement outcomes $|\pm 1\rangle_z|\pm 1\rangle_x$, depicted in Fig. 3. The blue circles illustrate a superposition of the $|1\rangle_z|1\rangle_x$ and $|-1\rangle_z|1\rangle_x$ measurement outcomes; green, $|-1\rangle_z|-1\rangle_x$ and $|1\rangle_z|-1\rangle_x$; red, $|1\rangle_z|1\rangle_x$ and $|1\rangle_z|-1\rangle_x$; and gray, $|-1\rangle_z|-1\rangle_x$ and $|-1\rangle_z|1\rangle_x$. Points on the circles represented by lighter shades in Fig. 3 cannot be reached by the dynamics of the system because they concern values of $\langle S_{x,2} \rangle$ and $\langle S_{z,1} \rangle$ with a magnitude greater than $\frac{1}{2}$. In contrast, the points represented by darker shades are accessible.

The petal pattern appears as a result of a multistage collapse cascade depicted in Fig. 4. The system is initiated with nonzero probability amplitudes for all four possible measurement outcomes, but this reduces to three, then two, and finally just one significant contribution. The system evolves towards one of the superposition circles depicted in Fig. 3 and eventually to the tip of the corresponding petal. For the probability amplitudes depicted in Fig. 4, the trajectory approaches the blue circle in Fig. 3 defined by nonzero probability amplitudes for the $|1\rangle_z|1\rangle_x$ and $|-1\rangle_z|1\rangle_x$ measurement outcomes. The system then travels along the circle and collapses at one of these measurement outcomes: in Fig. 4 it is the probability amplitude for the $|1\rangle_z|1\rangle_x$ measurement outcome that approaches unity. The regions outside the petals remain unexplored since the system begins at the center, $\langle S_{x,2} \rangle = \langle S_{z,1} \rangle = 0$, and cannot pass across a superposition circle.

The mean environmental stochastic entropy production associated with case 2 is depicted in Fig. 5. After an initially slow rate of entropy production, the mean rate rises to around three to four times that of a single particle undergoing a z -spin measurement and nearly double that of case 1. The rate of

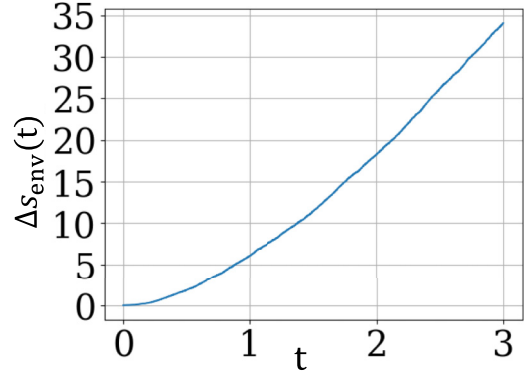


FIG. 5. Mean environmental stochastic entropy production for case 2: two spin-1/2 particles undergoing a z -spin measurement on particle 1 and an x -spin measurement on particle 2, with the initial state $|\Psi^-\rangle = \frac{1}{\sqrt{2}}(|1\rangle_z|-1\rangle_z - |-1\rangle_z|1\rangle_z)$, 314 trajectories, measurement strengths $a_1 = a_2 = 1$, and a time step of 0.0001.

entropy production for case 2 is higher than for case 1 possibly because there are four rather than two potential measurement outcomes the system could adopt. Furthermore, in case 1, the system already starts on a superposition circle, while in case 2 the system requires time to reach a superposition circle and complete the first stage of the measurement process. This could explain the initially slow rate of production exhibited by case 2.

V. MEASUREMENT OF TOTAL Z-SPINS OF TWO SPIN-1/2 PARTICLES

A. Dynamics

We now perform measurements of the z component of the total spin of the two-spin-1/2 system, $S_z = \frac{1}{2}\sigma_{z,1} \otimes \mathbb{I} + \mathbb{I} \otimes \frac{1}{2}\sigma_{z,2}$ where subscripts 1 and 2 label the two particles. The system Hamiltonian is once again zero $H_s = 0$. The Lindblad operator $L_z = aS_z$, where a is a coupling strength, is used in Eq. (4) to obtain SDEs for the 15 variables parametrizing the density matrix. The full set can be found in Appendix F. It is convenient, however, to consider the evolution in terms of fewer variables. Using $\langle X \rangle = \text{Tr}(X\rho)$, we obtain

$$\begin{aligned} \langle S_z \rangle &= \frac{s_3 + s_{12}}{2}, & \langle S_x \rangle &= \frac{s_4 + s_1}{2}, \\ \langle S_y \rangle &= \frac{s_2 + s_8}{2}, & \langle S^2 \rangle &= \frac{1}{2}(3 + s_5 + s_{10} + s_{15}), \\ \langle S_{z,1} \rangle &= \frac{1}{2}s_{12}, & \langle S_{z,2} \rangle &= \frac{1}{2}s_3, \end{aligned} \quad (25)$$

where $\langle S_x \rangle$ and $\langle S_y \rangle$ are defined in a similar manner to $\langle S_z \rangle$, $S^2 = S_x^2 + S_y^2 + S_z^2$, and $\langle S_{z,1} \rangle$ and $\langle S_{z,2} \rangle$ are the z -spins of particles 1 and 2, respectively.

By noting that $d\langle X \rangle = \text{Tr}(Xd\rho)$, we can write the SDE for $\langle S^2 \rangle$ as

$$d\langle S^2 \rangle = \frac{1}{2}a(s_3 + s_{12})(1 - s_5 - s_{10} - s_{15})dW_z. \quad (26)$$

Note that in three of the five cases that follow, we restrict the dynamics by starting the system in one of the triplet eigenstates of S^2 with $\langle S^2 \rangle = 2$, or a superposition of them. For a general superposition of triplet eigenstates (with real

amplitudes) we can compute the s_i using $s_i = \text{Tr}(\rho_{\text{trip}} \Sigma_i)$, where the state is represented by ρ_{trip} , and show that $s_5 + s_{10} + s_{15} = 1$. Thus the term in the second set of parentheses in Eq. (26) is zero and $d\langle S^2 \rangle = 0$ throughout, implying that $\langle S^2 \rangle$ remains constant and the system can never reach the singlet state at $\langle S^2 \rangle = 0$. We consider real amplitudes only since for the initial states we explore in this section, the variables $s_2 = s_6 = s_8 = s_9 = s_{11} = s_{14} = 0$ and remain so for all of time since $ds_2 = ds_6 = ds_8 = ds_9 = ds_{11} = ds_{14} = 0$. Therefore, only the real elements in the general expression for ρ [Eq. (13)] are nonzero. When starting in an initial state that is a triplet eigenstate of S^2 , the system of two-spin-1/2 particles is therefore equivalent to a single-spin-1 system, and the three possible eigenstates of S_z are $|1\rangle_z|1\rangle_z$ corresponding to $\langle S_z \rangle = 1$, $\frac{1}{\sqrt{2}}(|1\rangle_z|-1\rangle_z + |-1\rangle_z|1\rangle_z)$ corresponding to $\langle S_z \rangle = 0$, and $|-1\rangle_z|-1\rangle_z$ corresponding to $\langle S_z \rangle = -1$.

B. Environmental stochastic entropy production

The environmental stochastic entropy production is again calculated using an appropriate reduced diffusion matrix. We note that there is a closed set of three SDEs associated with

the protocol of S_z measurement,

$$\begin{aligned} ds_{12} &= a(-s_{12}^2 - s_3 s_{12} + s_{15} + 1)dW_z = -\kappa dW_z, \\ ds_3 &= a(-s_3^2 - s_3 s_{12} + s_{15} + 1)dW_z = -\nu dW_z, \\ ds_{15} &= a(-s_3 s_{15} + s_3 - s_{12} s_{15} + s_{12})dW_z = -\gamma dW_z, \end{aligned} \quad (27)$$

where compact notation in terms of κ , ν , and γ is introduced. The reduced diffusion matrix is given by

$$D_{S_z} = \frac{1}{2}a^2 \begin{pmatrix} \nu^2 & \nu\kappa & \nu\gamma \\ \nu\kappa & \kappa^2 & \nu\gamma \\ \nu\gamma & \kappa\gamma & \gamma^2 \end{pmatrix},$$

and since this matrix is singular, we follow the procedure outlined in Sec. III B. The diffusion matrix has only one nonzero eigenvalue, and so we require a single dynamical variable. For simplicity, we choose s_3 to be the dynamical variable with s_{12} and s_{15} acting as spectator variables. The scalar diffusion coefficient is then $D_{S_z} = \frac{1}{2}a^2\nu^2$. The environmental stochastic entropy production is found using the correction terms for the derivatives in Eqs. (11) and (17) and may be expressed as

$$\begin{aligned} d\Delta s_{\text{env}} &= a^2 \left(-\frac{1}{\nu} [-(4s_3 + 2s_{12})\gamma + (4s_3^2 + 4s_3 s_{12} - 2s_{15})\kappa] - (6s_3^2 + 8s_3 s_{12} + 2s_{12}^2 - 4s_{15} - 2) \right. \\ &\quad \left. + \frac{1}{\nu^2} (16s_3^3 + 32s_3^2 s_{12} + 16s_3 s_{12}^2 - 32s_3 s_{15} - 16s_3 - 16s_{12} s_{15})^2 \right) dt \\ &\quad + \frac{1}{\nu} a (4s_3^3 + 8s_3^2 s_{12} + 4s_3 s_{12}^2 - 8s_3 s_{15} - 4s_3 - 4s_{12} s_{15}) dW_z. \end{aligned} \quad (28)$$

Note that when the system is close to an eigenstate, the expression for $d\Delta s_{\text{env}}$ is vulnerable to exhibiting singularities due to numerical inaccuracies. The longer the duration of measurement, the more likely this will occur. Trajectories that gave rise to singularities were not considered in the calculation of mean environmental entropy production.

C. Results for various cases

Stochastic trajectories are generated as the two-spin-1/2 system undergoes a measurement of S_z and Fig. 6 depicts example trajectories in the $(\langle S_x \rangle, \langle S_z \rangle)$ coordinate space. We first initiate the system in one of the three triplet eigenstates of S_x , either the $|1\rangle_x|1\rangle_x$, $|-1\rangle_x|-1\rangle_x$ or the $\frac{1}{\sqrt{2}}(|1\rangle_x|-1\rangle_x + |-1\rangle_x|1\rangle_x)$ eigenstate (represented as the two purple and one gray crosses in Fig. 7) or any superposition of the three triplet eigenstates of S_z , such that the system remains in the triplet state space with $\langle S^2 \rangle = 2$.

The $|1\rangle_z|1\rangle_z$ and $|-1\rangle_z|-1\rangle_z$ eigenstates lie at the turquoise crosses at the top and bottom of Fig. 7. The $\frac{1}{\sqrt{2}}(|1\rangle_z|-1\rangle_z + |-1\rangle_z|1\rangle_z)$ and $\frac{1}{\sqrt{2}}(|1\rangle_x|-1\rangle_x + |-1\rangle_x|1\rangle_x)$ eigenstates lie at the center of the plot (the gray cross in Fig. 7), though they are orthogonal. The avoidance of the figure-8-like region in Fig. 6 can be understood using a measurement collapse cascade model [51] and the loci of superpositions (with real amplitudes) between pairs of eigenstates of S_z . The red and blue ellipses in Fig. 7 describe

a superposition between the $|1\rangle_z|1\rangle_z$ and $\frac{1}{\sqrt{2}}(|1\rangle_z|-1\rangle_z + |-1\rangle_z|1\rangle_z)$ eigenstates and the $|-1\rangle_z|-1\rangle_z$ and $\frac{1}{\sqrt{2}}(|1\rangle_z|-1\rangle_z + |-1\rangle_z|1\rangle_z)$ eigenstates, respectively. The green line represents a superposition between the $|-1\rangle_z|-1\rangle_z$ and $|1\rangle_z|1\rangle_z$ eigenstates. The black outer circle represents a further evolution pathway for collapse to the $|-1\rangle_z|-1\rangle_z$ and $|1\rangle_z|1\rangle_z$ eigenstates.

The empty regions result from a two-step measurement collapse cascade. Starting from the $|-1\rangle_x|-1\rangle_x$ eigenstate, for example, the system has nonzero probability amplitudes for collapse to any of the three eigenstates of S_z . The decrease of one of these probability amplitudes to zero is the first step in the collapse process such that the system arrives at one of the superposition loci, either the red or the blue ellipse. The system is not able to recover a nonzero probability amplitude for the third eigenstate, and it then moves along the ellipse towards one of the two eigenstates to complete the collapse. The empty regions cannot be reached from a starting point at one of the eigenstates of S_x since this would involve crossing a superposition locus. Further discussion of this pattern formation can be found in the work of Walls and Ford [51].

Note the similarities between the avoidance of regions in Fig. 6 for the system undergoing measurements of S_z and the pattern formation in Fig. 2 for simultaneous measurement of $S_{x,2}$ and $S_{z,1}$.

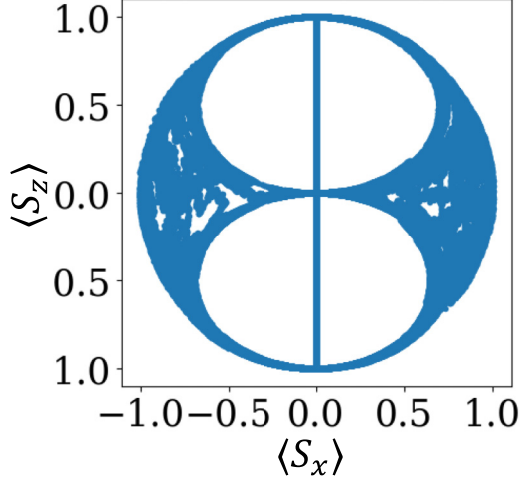


FIG. 6. Stochastic evolution of the system of two spin-1/2 particles undergoing measurement of total S_z . The 50 trajectories are illustrated in terms of the spin components $\langle S_x \rangle$ and $\langle S_z \rangle$. Each trajectory starts in one of the three triplet eigenstates of S_x , represented as the purple and gray crosses in Fig. 7, and terminates in the vicinity of one of the triplet eigenstates of S_z at the top, bottom, and center of the plot. The measurement strength $a = 1$ and the measurement duration is six time units with a time step of 0.0001.

We now consider the environmental stochastic entropy production associated with measurement of total S_z of the two-spin-1/2 system. In particular, we compare the mean rates of asymptotic environmental stochastic entropy production when

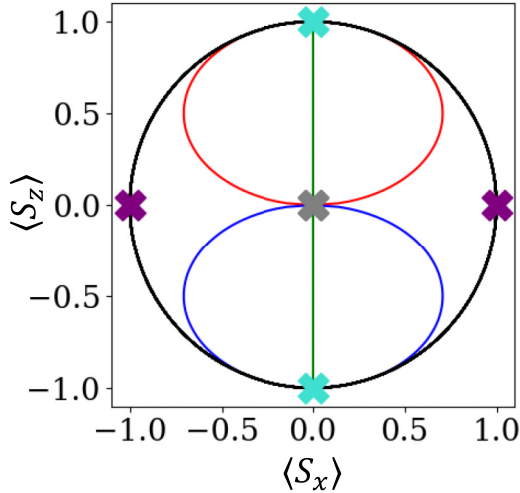


FIG. 7. Curves represent superpositions (with real amplitudes) between different eigenstates of S_z . The upper and lower ellipses describe a superposition between the $|1_z\rangle|1_z\rangle$ and $\frac{1}{\sqrt{2}}(|1_z\rangle|1_z\rangle + |-1_z\rangle|1_z\rangle)$ eigenstates (red) and between the $|-1_z\rangle|1_z\rangle$ and $\frac{1}{\sqrt{2}}(|1_z\rangle|1_z\rangle + |-1_z\rangle|1_z\rangle)$ eigenstates (blue), respectively. The vertical green line represents a superposition between the $|-1_z\rangle|1_z\rangle$ and $|1_z\rangle|1_z\rangle$ eigenstates [51]. The purple crosses represent the $|1_x\rangle|1_x\rangle$ and $|-1_x\rangle|1_x\rangle$ eigenstates of S_x and the turquoise crosses represent the $|1_z\rangle|1_z\rangle$ and $|-1_z\rangle|1_z\rangle$ eigenstates of S_z . The gray cross represents both the $\frac{1}{\sqrt{2}}(|1_x\rangle|1_x\rangle + |-1_x\rangle|1_x\rangle)$ and $\frac{1}{\sqrt{2}}(|1_z\rangle|1_z\rangle + |-1_z\rangle|1_z\rangle)$ eigenstates, noting they are orthogonal.

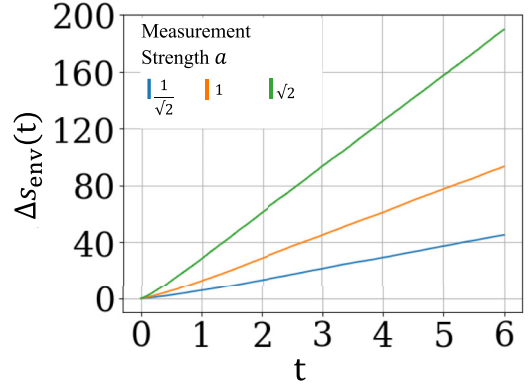


FIG. 8. Mean environmental stochastic entropy production for case A: the two-spin-1/2 system undergoing S_z measurement starting from the $\frac{1}{\sqrt{2}}(|1_x\rangle|1_x\rangle + |-1_x\rangle|1_x\rangle)$ eigenstate and approaching the $|1_z\rangle|1_z\rangle$ eigenstate. The measurement strengths are $a = \frac{1}{\sqrt{2}}$ (blue), $a = 1$ (orange), and $a = \sqrt{2}$ (green), with 600 trajectories and a time step of 0.0001.

the system is prepared in different initial states and collapses to various eigenstates of S_z .

Case A. We first consider the mean environmental stochastic entropy production when the system starts in the $\frac{1}{\sqrt{2}}(|1_x\rangle|1_x\rangle + |-1_x\rangle|1_x\rangle)$ eigenstate. Note that from this initial state, there is zero probability of collapsing to the $\frac{1}{\sqrt{2}}(|1_z\rangle|1_z\rangle + |-1_z\rangle|1_z\rangle)$ eigenstate of S_z but equal probabilities of collapsing to $|1_z\rangle|1_z\rangle$ and $|-1_z\rangle|1_z\rangle$. The system already lies on the superposition locus defined by the vertical green line in Fig. 7 and therefore does not need to undergo the first stage of the measurement collapse cascade. Figure 8 illustrates the mean environmental stochastic entropy production when approaching the $|1_z\rangle|1_z\rangle$ eigenstate, for different measurement strengths. A similar plot would be produced for approach to the $|-1_z\rangle|1_z\rangle$ eigenstate. The mean rate of environmental stochastic entropy production is roughly proportional to the square of the measurement strength a .

Starting from this initial state means that $s_{15} = 1$ throughout and, together with $s_3 = s_{12}$, the environmental stochastic entropy production in Eq. (28) simplifies to $d\Delta s_{\text{env}} = 8a^2(s_{12}^2 + 1)dt + 8as_{12}dW_z$. We would thus expect the asymptotic mean rate of stochastic environmental entropy production to be 16 for a measurement strength $a = 1$, which is consistent with the orange curve in Fig. 8. Note that this is double that of the single-particle z -spin measurements performed on the two-spin-1/2 system (case 1), depicted in Fig. 1.

Case B. We go on to consider a situation where the system takes an initial state $\frac{1}{\sqrt{2}}|1_z\rangle|1_z\rangle + \frac{1}{\sqrt{2}}(|1_z\rangle|1_z\rangle + |-1_z\rangle|1_z\rangle)$, hence with equal probabilities of collapsing at either the $|1_z\rangle|1_z\rangle$ or the $\frac{1}{\sqrt{2}}(|1_z\rangle|1_z\rangle + |-1_z\rangle|1_z\rangle)$ eigenstates of S_z . This is located at $(\langle S_x \rangle = \frac{1}{\sqrt{2}}, \langle S_z \rangle = \frac{1}{2})$ on the red ellipse in Fig. 7 and therefore the system proceeds without need for the first stage of the measurement collapse cascade. The mean environmental stochastic entropy production associated with collapse to the $\frac{1}{\sqrt{2}}(|1_z\rangle|1_z\rangle + |-1_z\rangle|1_z\rangle)$ and $|1_z\rangle|1_z\rangle$ eigenstates is depicted in Fig. 9, illustrated as the orange and blue lines, respectively. The mean asymptotic rate of

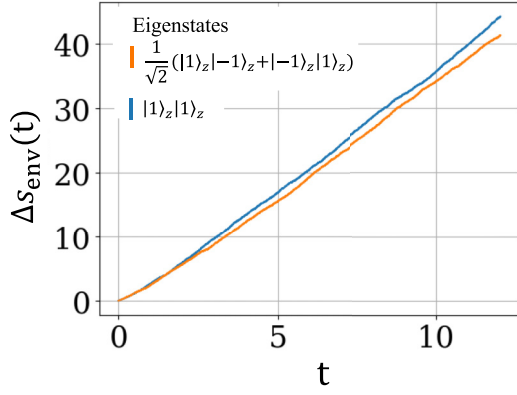


FIG. 9. Mean environmental stochastic entropy production for case B: the system undergoing a measurement of S_z starting in $\frac{1}{\sqrt{2}}|1_z|1_z + \frac{1}{\sqrt{2}}(|1_z| - 1_z) + |-1_z|1_z$, with an equal superposition (with real and positive amplitudes) of the $|1_z|1_z$ and $\frac{1}{\sqrt{2}}(|1_z| - 1_z) + |-1_z|1_z$ eigenstates. The production for collapse to the $\frac{1}{\sqrt{2}}(|1_z| - 1_z) + |-1_z|1_z$ eigenstate is shown in orange and for collapse to the $|1_z|1_z$ eigenstate in blue. Here there are 370 trajectories, $a = 1$, the measurement duration is 12, the time step is 0.0001.

stochastic entropy production for approach to both eigenstates is around four times smaller than that associated with collapse at the $|1_z|1_z$ and $|-1_z| - 1_z$ eigenstates in case A starting from $\frac{1}{\sqrt{2}}(|1_x| - 1_x) + |-1_x|1_x$ for $a = 1$, shown in Fig. 8.

We speculate that collapse to the $|1_z|1_z$ and $|-1_z| - 1_z$ eigenstates in case A (Fig. 8) occurs more quickly than collapse in case B because the system already begins with $s_{15} = 1$, the value required for the system to take either of the $|1_z|1_z$ and $|-1_z| - 1_z$ eigenstates, and greater speed is associated with a higher rate of mean asymptotic entropy production. In contrast, the system in case B (Fig. 9) is initiated with $s_{15} = 0$, while $s_{15} = 1$ is required for the system to take the $|1_z|1_z$ eigenstate and $s_{15} = -1$ for the $\frac{1}{\sqrt{2}}(|1_z| - 1_z) + |-1_z|1_z$ eigenstate. The system therefore requires more time to reach the desired value of s_{15} in case B, and the asymptotic mean rate of environmental stochastic entropy production could be slower than case A because of this.

Case C. The next initial state we consider is the $|1_x|1_x$ eigenstate from which the system can reach the $|1_z|1_z$, $|-1_z| - 1_z$, or $\frac{1}{\sqrt{2}}(|1_z| - 1_z) + |-1_z|1_z$ eigenstate. The system must now undergo the first stage of the measurement collapse cascade since it does not start on one of the superposition loci depicted in Fig. 7. Figure 10 illustrates the mean environmental stochastic entropy production associated with approach to the $\frac{1}{\sqrt{2}}(|1_z| - 1_z) + |-1_z|1_z$ eigenstate (green), the $|1_z|1_z$ eigenstate (blue), and the $|-1_z| - 1_z$ eigenstate (orange), starting from the $|1_x|1_x$ eigenstate.

There appears to be a lower mean rate of asymptotic environmental entropy production associated with collapse to the $\frac{1}{\sqrt{2}}(|1_z| - 1_z) + |-1_z|1_z$ eigenstate compared with the $|1_z|1_z$ and $|-1_z| - 1_z$ eigenstates. To investigate further, we consider the mean variation of the probability amplitudes of the triplet eigenstates of S_z , conditioned on collapse at a specific eigenstate. Figure 11(a) illustrates the mean variation of these amplitudes conditioned on system collapse at the $|-1_z| - 1_z$ eigenstate, while in Fig. 11(b) the trajectories

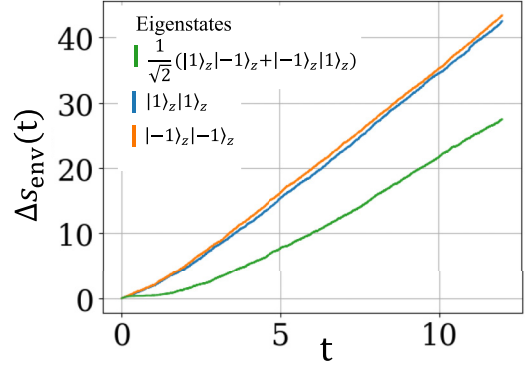


FIG. 10. Mean environmental stochastic entropy production for case C: approach towards the $\frac{1}{\sqrt{2}}(|1_z| - 1_z) + |-1_z|1_z$ eigenstate (green), the $|1_z|1_z$ eigenstate (blue), and the $|-1_z| - 1_z$ eigenstate (orange) during an S_z measurement starting from the $|1_x|1_x$ eigenstate. The measurement strength $a = 1$ and the measurement duration is 12, with 370 trajectories and a time step of 0.0001.

are conditioned on the system collapsing at the $\frac{1}{\sqrt{2}}(|1_z| - 1_z) + |-1_z|1_z$ eigenstate and the $|1_z|1_z$ eigenstate being the first probability amplitude to go to zero. The green line represents the probability amplitude of the $\frac{1}{\sqrt{2}}(|1_z| - 1_z) + |-1_z|1_z$ eigenstate, the blue line the $|-1_z| - 1_z$ eigenstate, and the orange line the $|1_z|1_z$ eigenstate.

In both Figs. 11(a) and 11(b), the $|1_z|1_z$ eigenstate is the first probability amplitude to fall to zero; thus we can compare how long it takes for this first stage of the collapse cascade to be completed. In Fig. 11(a) the probability amplitude of the $|1_z|1_z$ eigenstate reaches zero at around one time unit, whereas in Fig. 11(b) it only reaches zero at around six time units. Moreover, in Fig. 11(b) the probability amplitude of the $|-1_z| - 1_z$ eigenstate initially rises before falling towards zero, revealing perhaps that the system experiences a greater attraction towards the $|\pm 1_z| \pm 1_z$ eigenstates of S_z and more difficulty in collapsing to the $\frac{1}{\sqrt{2}}(|1_z| - 1_z) + |-1_z|1_z$ eigenstate. Perhaps this greater competition arises because the $\frac{1}{\sqrt{2}}(|1_z| - 1_z) + |-1_z|1_z$ eigenstate lies in the middle of the ladder of eigenstates of S_z and at the center of the circle in the $(\langle S_x \rangle, \langle S_z \rangle)$ coordinate space in Fig. 6. In contrast, the $|\pm 1_z| \pm 1_z$ eigenstates lie at the termini of the ladder and at the top or bottom of the circle in the $(\langle S_x \rangle, \langle S_z \rangle)$ coordinate space; therefore, they only face competition from the $\frac{1}{\sqrt{2}}(|1_z| - 1_z) + |-1_z|1_z$ eigenstate and not from each other.

The asymptotic mean rates of environmental stochastic entropy production for the approach to the $\frac{1}{\sqrt{2}}(|1_z| - 1_z) + |-1_z|1_z$ eigenstate differ when the system starts in the $|1_x|1_x$ eigenstate (case C) or in the $\frac{1}{\sqrt{2}}(|1_z|1_z + \frac{1}{\sqrt{2}}(|1_z| - 1_z) + |-1_z|1_z)$ state (case B). This is not the case for the $|\pm 1_z| \pm 1_z$ eigenstates, which exhibit the same asymptotic mean rates in cases B and C. This could again arise because the $\frac{1}{\sqrt{2}}(|1_z| - 1_z) + |-1_z|1_z$ eigenstate lies between the other two. In case B the $\frac{1}{\sqrt{2}}(|1_z| - 1_z) + |-1_z|1_z$ eigenstate only has to compete in the collapse with the $|1_z|1_z$ eigenstate, but in case C the $\frac{1}{\sqrt{2}}(|1_z| - 1_z) + |-1_z|1_z$ eigenstate competes for selection with both $|1_z|1_z$ and

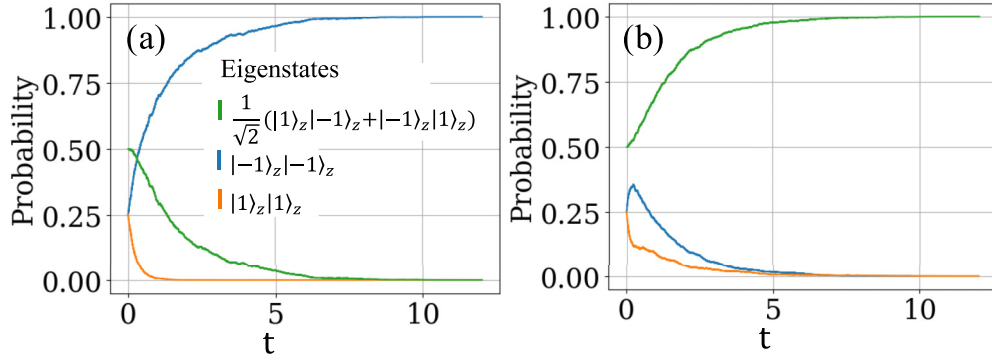


FIG. 11. Mean variation of the probability amplitudes of the triplet eigenstates of S_z , conditioned on collapse at a specific eigenstate. Their variation is shown over the duration of a measurement for case C: $a = 1$ and 370 trajectories. Green represents the probability amplitude for the $\frac{1}{\sqrt{2}}(|1\rangle_z|-1\rangle_z + |-1\rangle_z|1\rangle_z)$ eigenstate, blue the $|-1\rangle_z|-1\rangle_z$ eigenstate, and orange the $|1\rangle_z|1\rangle_z$ eigenstate. Both (a) and (b) share the same legend. (a) The system is conditioned on collapse at the $|-1\rangle_z|-1\rangle_z$ eigenstate and the $|1\rangle_z|1\rangle_z$ eigenstate being the first probability amplitude to fall to zero, with 445 trajectories. (b) The system is conditioned on collapse at the $\frac{1}{\sqrt{2}}(|1\rangle_z|-1\rangle_z + |-1\rangle_z|1\rangle_z)$ eigenstate and the $|1\rangle_z|1\rangle_z$ eigenstate being the first probability amplitude to fall to zero, with 445 trajectories.

$|-1\rangle_z|-1\rangle_z$ and thus it is more affected by a nonzero probability amplitude of the $|-1\rangle_z|-1\rangle_z$ eigenstate in case C.

Case D. The next initial condition we consider is the mixed state

$$\rho_{\text{in}} = \frac{1}{4}(|1\rangle_z|1\rangle_z\langle 1|_z\langle 1|_z + |-1\rangle_z|-1\rangle_z\langle -1|_z\langle -1|_z + |1\rangle_z|-1\rangle_z\langle 1|_z\langle -1|_z + |-1\rangle_z|1\rangle_z\langle -1|_z\langle 1|_z),$$

where the system has equal probabilities to collapse to the four eigenstates of S_z , $|1\rangle_z|1\rangle_z$, $|-1\rangle_z|-1\rangle_z$, and $\frac{1}{\sqrt{2}}(|1\rangle_z|-1\rangle_z + |-1\rangle_z|1\rangle_z)$, as before and now including the singlet state $\frac{1}{\sqrt{2}}(|1\rangle_z|-1\rangle_z - |-1\rangle_z|1\rangle_z)$. As expected, the system can collapse to $|1\rangle_z|1\rangle_z$ and $|-1\rangle_z|-1\rangle_z$, each with probability $\frac{1}{4}$, but the system is now found to evolve with a probability of $\frac{1}{2}$ to the stationary mixed state $\rho_{\text{st}} = \frac{1}{2}(|1\rangle_z|-1\rangle_z\langle 1|_z\langle -1|_z + |-1\rangle_z|1\rangle_z\langle -1|_z\langle 1|_z)$, which has a purity of $\frac{1}{2}$. Such a state is an equal mixture of the two degenerate eigenstates: $\frac{1}{\sqrt{2}}(|1\rangle_z|-1\rangle_z \pm |-1\rangle_z|1\rangle_z)$. Since $S_z|j\rangle|k\rangle = \frac{1}{2}(j+k)|j\rangle|k\rangle$ we have $S_z\rho_{\text{st}} = \rho_{\text{st}}S_z = 0$, which makes ρ_{st} a stationary state of the dynamics in Eq. (4), just like the eigenstates of S_z . The mean environmental stochastic entropy productions associated with collapse to the mixed state $\rho_{\text{st}} = \frac{1}{2}(|1\rangle_z|-1\rangle_z\langle 1|_z\langle -1|_z + |-1\rangle_z|1\rangle_z\langle -1|_z\langle 1|_z)$ (green), as well as the eigenstates $|1\rangle_z|1\rangle_z$ (blue) and $|-1\rangle_z|-1\rangle_z$ (orange), are illustrated in Fig. 12.

The mean asymptotic rate of environmental entropy production associated with reaching the $|1\rangle_z|1\rangle_z$ and $|-1\rangle_z|-1\rangle_z$ eigenstates is higher than in the approach towards the mixed state $\rho_{\text{st}} = \frac{1}{2}(|1\rangle_z|-1\rangle_z\langle 1|_z\langle -1|_z + |-1\rangle_z|1\rangle_z\langle -1|_z\langle 1|_z)$. Most likely the lower rate of entropy production associated with adopting the mixed state can be attributed to its lower purity. Furthermore, since it lies between the $|1\rangle_z|1\rangle_z$ and $|-1\rangle_z|-1\rangle_z$ eigenstates in the $(\langle S_x \rangle, \langle S_z \rangle)$ coordinate space, it could suffer from the additional competition in selection, potentially leading to the initially slower mean rate of production.

In order to further purify the system, we could measure another operator which commutes with S_z , such as S^2 , for which the $\frac{1}{\sqrt{2}}(|1\rangle_z|-1\rangle_z \pm |-1\rangle_z|1\rangle_z)$ eigenstates are not degenerate. Indeed, introducing an additional Lindblad $L_2 = S^2$

into the dynamics can successfully collapse the system at one of the two $\frac{1}{\sqrt{2}}(|1\rangle_z|-1\rangle_z \pm |-1\rangle_z|1\rangle_z)$ eigenstates. In contrast, introducing an additional Lindblad $L_2 = S_{z,2}$ leads to the system purifying to $|1\rangle_z|-1\rangle_z$ or $|-1\rangle_z|1\rangle_z$.

Case E. In the cases considered thus far, the z -spin components of the first and second spin-1/2 particles have been equal such that $s_3 = s_{12}$ throughout the measurement of S_z . Here we instead consider an initial state $|\Psi\rangle = \frac{1}{2}(|1\rangle_z|1\rangle_z + |-1\rangle_z|-1\rangle_z) + \frac{1}{\sqrt{2}}|1\rangle_z|-1\rangle_z$, where the z -spin components of particles 1 and 2 initially are $\frac{1}{4}$ and $-\frac{1}{4}$, respectively. As a result, $s_3 \neq s_{12}$ throughout the measurement and unlike the previous initial states, the particles are distinguishable and their behavior is uncorrelated. The system has a probability of $\frac{1}{2}$ of collapsing at the $|1\rangle_z|-1\rangle_z$ state and a probability of $\frac{1}{4}$ of collapsing to the $|1\rangle_z|1\rangle_z$ and $|-1\rangle_z|-1\rangle_z$ eigenstates.

The mean environmental stochastic entropy production associated with collapse to each of these eigenstates is

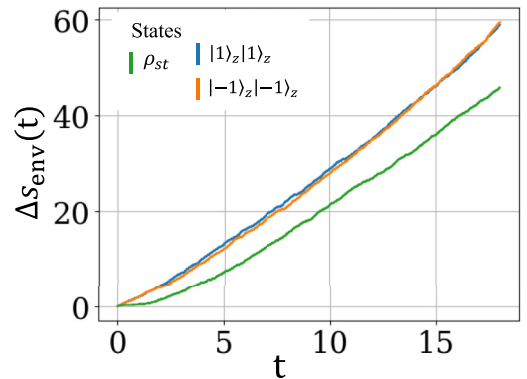


FIG. 12. Mean environmental stochastic entropy production for case D, where the system starts in a mixed state $\rho_{\text{in}} = \frac{1}{4}(|1\rangle_z|1\rangle_z\langle 1|_z\langle 1|_z + |-1\rangle_z|-1\rangle_z\langle -1|_z\langle -1|_z + |1\rangle_z|-1\rangle_z\langle 1|_z\langle -1|_z + |-1\rangle_z|1\rangle_z\langle -1|_z\langle 1|_z)$. The colors represent approaches to different states: green for the mixed state $\rho_{\text{st}} = \frac{1}{2}(|1\rangle_z|-1\rangle_z\langle 1|_z\langle -1|_z + |-1\rangle_z|1\rangle_z\langle -1|_z\langle 1|_z)$; blue, the $|1\rangle_z|1\rangle_z$ state; and orange, the $|-1\rangle_z|-1\rangle_z$ state. The measurement strength $a = 1$ and the duration is 16, with 137 trajectories.

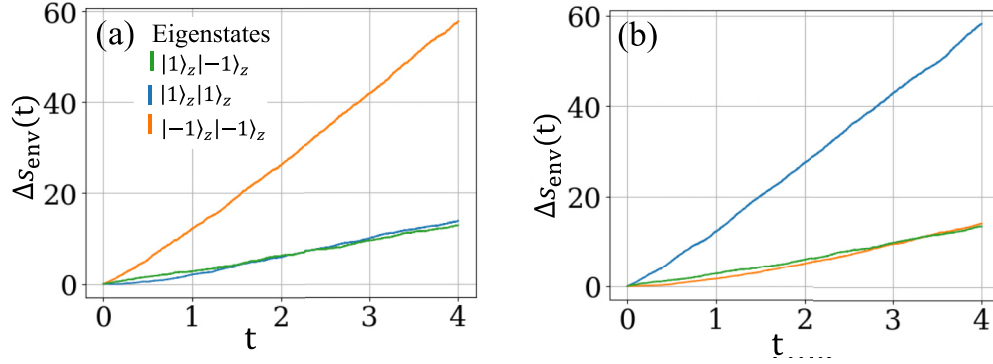


FIG. 13. Mean environmental stochastic entropy production of the two-spin-1/2 system starting in the state $|\Psi\rangle = \frac{1}{2}(|1_z|1_z\rangle + |-1_z| - 1_z\rangle) + \frac{1}{\sqrt{2}}|1_z| - 1_z\rangle$ for (a) s_3 as the dynamical variable, with 135 trajectories, and (b) s_{12} as the dynamical variable, with 200 trajectories. Different colors represent approaches to different states for $a = 1$: green, the $|1_z| - 1_z$ state; blue, the $|1_z|1_z$ eigenstate; and orange, the $|-1_z| - 1_z$ eigenstate. Both (a) and (b) share the same legend.

depicted in Fig. 13 for two different choices of the dynamical variable. In Fig. 13(a), s_3 is the dynamical variable, whereas in Fig. 13(b) it is s_{12} .

In Fig. 13(a) the mean asymptotic rate of environmental stochastic entropy production associated with collapse to the $|-1_z| - 1_z$ eigenstate (orange) is much greater than for the $|1_z| - 1_z$ state (green) or the $|1_z|1_z$ eigenstate (blue).

To produce Fig. 13(a) we choose s_3 as the dynamical variable, but if we instead pick s_{12} we obtain Fig. 13(b). The mean asymptotic rates of environmental entropy production associated with collapse to the $|-1_z| - 1_z$ eigenstate and the $|1_z|1_z$ state are now switched compared with Fig. 13(a).

At first sight this dependence is unexpected, but it can be resolved as follows. The total entropy production $\Delta s_{\text{tot}} = \Delta s_{\text{env}} + \Delta s_{\text{sys}}$ does not depend on the choice of dynamical variable, but Δs_{env} and Δs_{sys} can do so. In the case studied, choosing s_3 or s_{12} as the dynamical variable changes the asymptotic rate of environmental entropy production associated with approach to the $|-1_z| - 1_z$ eigenstate or the $|1_z|1_z$ eigenstate, and we would therefore expect the systemic stochastic entropy production to reflect this difference.

Recall that the system stochastic entropy production when s_3 is chosen as the dynamical variable evolves as $d\Delta s_{\text{sys}} = -d \ln p(s_3, t)$, where $p(s_3, t)$ is the PDF of s_3 , and similarly for s_{12} . Probability density functions for both s_3 and s_{12} after

a period of measurement are shown in Fig. 14. The initial state is such that the system has a probability of $\frac{1}{2}$ of collapsing to the $|1_z| - 1_z$ eigenstate and a probability of $\frac{1}{4}$ of collapsing to the $|1_z|1_z$ and $|-1_z| - 1_z$ eigenstates. Therefore, s_{12} has a probability of $\frac{3}{4}$ of collapsing at $+1$ and $\frac{1}{4}$ of collapsing at -1 , while s_3 has a probability of $\frac{3}{4}$ of collapsing at -1 and $\frac{1}{4}$ of collapsing at $+1$, i.e., the other way round. Since the system will not always have enough time to collapse precisely at an eigenstate, the PDFs in Fig. 14 are somewhat broader.

Nonetheless, from Fig. 14 we can infer that $-d \ln p(s_3, t) \neq -d \ln p(s_{12}, t)$ and the system entropy production is different according to whether s_3 or s_{12} is picked to be the dynamical variable. The asymmetry between the PDFs of the z -spins of the two spin-1/2 particles, $p(s_3, t) \neq p(s_{12}, t)$, in this case is the source of the dependence of the environmental stochastic entropy production on the choice of the dynamical variable.

VI. CONCLUSION

Calculating environmental stochastic entropy production associated with quantum measurement is problematic in the conventional framework of discontinuous and instantaneous wave-function collapse. In contrast, it is natural in the quantum state diffusion (QSD) formalism where measurement

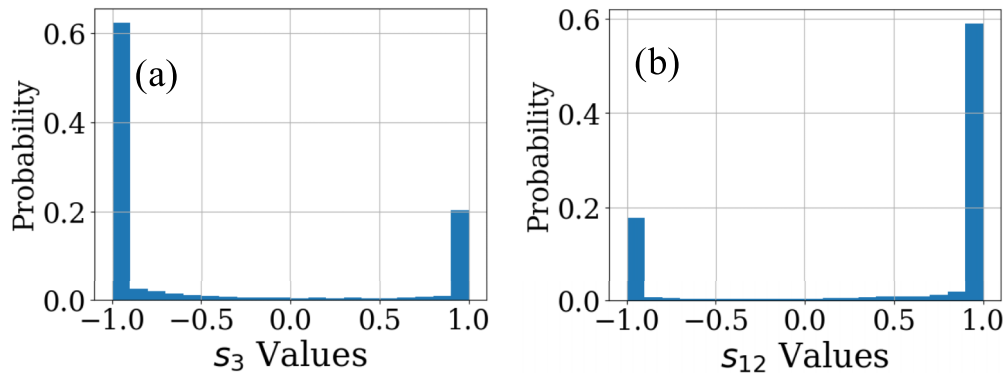


FIG. 14. Probability density functions of the dynamical variables (a) s_3 and (b) s_{12} after an S_z measurement performed on the two spin-1/2 particles in case E, with measurement strength $a = 1$ and a measurement duration of 12.

collapse is continuous and takes place over a finite period of time. Measurement is brought about by an interaction between a system and an environment (the measurement apparatus), causing the system to diffuse towards an eigenstate of the observable in question. The speed of collapse is related to the coupling strength between the system and the environment.

Using stochastic quantum trajectories generated in such a framework, we calculated the environmental stochastic entropy production associated with measurement of a system of two spin-1/2 particles, selected according to initial and final states and measured property. In particular, we compared the mean asymptotic rates of environmental stochastic entropy production when the system is close to an eigenstate. Such entropy production differs from changes in the von Neumann entropy in that it applies to single realizations of system behavior as opposed to average evolution and concerns the irreversibility of the approach towards a measurement outcome, as opposed to uncertainty in the choice of the measurement outcomes themselves. While the von Neumann entropy takes an upper limit of $\ln 2$, the environmental stochastic entropy production has no upper limit.

The type of measurement performed appears to affect the mean asymptotic rate of environmental stochastic entropy production. In case 1 single-particle z -spin measurements are performed on both particles, and in case 2 single-particle z -spin and x -spin measurements are performed on the first and second particles, respectively. The rate of production is almost two times greater in the latter case, as shown in Figs. 1 and 5.

Additionally, the mean asymptotic rate associated with collapse to the same eigenstate appears to depend on the initial state of the system. For example, in a situation where the total z -spin S_z of the two-particle system is measured, the mean asymptotic production associated with collapse to $|1\rangle_z|1\rangle_z$ is four times greater for case A (Fig. 8) than case B (Fig. 9). The system was prepared in different states but each with a probability of $\frac{1}{2}$ of collapsing to $|1\rangle_z|1\rangle_z$. The same can be said for cases B and C, where collapse to the $\frac{1}{\sqrt{2}}(|1\rangle_z - |1\rangle_z + |-1\rangle_z|1\rangle_z)$ eigenstate is associated with different mean asymptotic rates of production in Figs. 9 and 10. Again the system starts in different states but with the same probability of collapsing at $\frac{1}{\sqrt{2}}(|1\rangle_z - |1\rangle_z + |-1\rangle_z|1\rangle_z)$.

Case C demonstrates that an approach towards eigenstates with the highest and lowest eigenvalues appears to exhibit a higher asymptotic rate of production than collapse to the eigenstate in the middle, as shown in Figs. 10 and 12. This appears to be the case only when three or more eigenstates are involved since this distinction does not present itself in case B (see Fig. 9).

The way in which QSD handles degenerate eigenstates was demonstrated by case D, where the system collapsed to a state which was not pure but nevertheless a stationary state under the dynamics. Only by additionally measuring S^2 or a single-particle z -spin operator did the system finally purify.

Finally, case E illustrates an instance where the environmental stochastic entropy production depends on the choice of the dynamical variable. Such differences are reflected also in the system stochastic entropy production; thus, as we would expect, the total entropy production remains invariant to the choice of dynamical variable. The initial state brought about

an asymmetry in the PDFs of the two spin-1/2-particle z -spin components, unlike the other cases considered.

The environmental stochastic entropy production offers an additional way of characterizing measurement collapse that may uncover new insights into the process, and we have presented various examples in this work. One additional avenue to explore is to use environmental stochastic entropy production to distinguish two forms of mixed density matrices: one representing a single system entangled with its environment and one that merely represents an ensemble of pure quantum states. When the system is already in an eigenstate of the measured operator, it remains stationary and therefore its environmental stochastic entropy production under measurement would be zero. If however the system is in an entangled state, it will evolve towards an eigenstate and the environmental stochastic entropy production would be nonzero. The production of this quantity, and the irreversibility under measurement that it represents, can distinguish situations that are typically regarded as indistinguishable.

ACKNOWLEDGMENT

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DATA AVAILABILITY

The data that support the findings of this article are not publicly available. The data are available from the authors upon reasonable request.

APPENDIX A: GENERAL SDE FOR STOCHASTIC ENTROPY PRODUCTION

We consider a set of coordinates $\mathbf{x} = \{x_1, x_2, \dots, x_N\}$ whose dynamics are defined by Itô processes

$$dx_i = A_i(\mathbf{x})dt + \sum_j B_{ij}(\mathbf{x})dW_j, \quad (\text{A1})$$

where dW_j denotes independent Wiener increments. The $A_i(\mathbf{x})$ may be split into reversible $A_i^{\text{rev}}(\mathbf{x})$ and irreversible $A_i^{\text{irr}}(\mathbf{x})$ contributions as [16,26]

$$\begin{aligned} A_i^{\text{irr}}(\mathbf{x}) &= \frac{1}{2}[A_i(\mathbf{x}) + \epsilon_i A_i(\epsilon\mathbf{x})] = \epsilon_i A_i^{\text{irr}}(\epsilon\mathbf{x}), \\ A_i^{\text{rev}}(\mathbf{x}) &= \frac{1}{2}[A_i(\mathbf{x}) - \epsilon_i A_i(\epsilon\mathbf{x})] = -\epsilon_i A_i^{\text{rev}}(\epsilon\mathbf{x}), \end{aligned} \quad (\text{A2})$$

where $\epsilon_i = 1$ for variables x_i with even parity under time-reversal symmetry, while $\epsilon_i = -1$ for variables with odd parity. The form $\epsilon\mathbf{x}$ represents $(\epsilon_1 x_1, \epsilon_2 x_2, \dots)$. As the notation suggests, the A_i^{irr} terms are partially responsible for irreversible behavior, while the A_i^{rev} are time-reversal invariant and are not responsible. Defining an $N \times N$ diffusion matrix $\mathbf{D} = \frac{1}{2}\mathbf{B}\mathbf{B}^T$, the evolution of the environmental stochastic

entropy production is then governed by [16]

$$\begin{aligned} d\Delta s_{\text{env}} = & - \sum_i \frac{\partial A_i^{\text{rev}}}{\partial x_i} dt + \sum_{i,j} \left\{ D_{ij}^{-1} A_i^{\text{irr}} dx_j \right. \\ & - D_{ij}^{-1} \sum_m \frac{\partial D_{im}}{\partial x_m} dx_j - D_{ij}^{-1} A_i^{\text{rev}} A_j^{\text{irr}} dt \\ & + D_{ij}^{-1} A_i^{\text{rev}} \sum_n \frac{\partial D_{jn}}{\partial x_n} dt + \sum_k \left[D_{ik} \frac{\partial}{\partial x_k} (D_{ij}^{-1} A_j^{\text{irr}}) \right. \\ & \left. \left. - D_{ik} \frac{\partial}{\partial x_k} \left(D_{ij}^{-1} \sum_n \frac{\partial D_{jn}}{\partial x_n} \right) \right] dt \right\}. \end{aligned} \quad (\text{A3})$$

For a system described by even-parity variables with $\varepsilon_i = 1$, the reversible terms A_i^{rev} are zero and $A_i^{\text{irr}} = A_i$. Equation (A3) then simplifies to

$$\begin{aligned} d\Delta s_{\text{env}} = & \sum_{i,j} \left\{ D_{ij}^{-1} A_i dx_j - D_{ij}^{-1} \sum_m \frac{\partial D_{im}}{\partial x_m} dx_j \right. \\ & \left. + \sum_k \left[D_{ik} \frac{\partial}{\partial x_k} (D_{ij}^{-1} A_j) - D_{ik} \frac{\partial}{\partial x_k} \left(D_{ij}^{-1} \sum_m \frac{\partial D_{jm}}{\partial x_m} \right) \right] dt \right\}, \end{aligned} \quad (\text{A4})$$

which then takes the compact form of Eq. (10).

APPENDIX B: SINGULAR DIFFUSION MATRICES

Singularity in the diffusion matrix may be attributed to the existence of functions of the stochastic variables that either evolve deterministically or are constants of the motion. This implies that the irreversibility of the motion is captured by a subset of what we call dynamical variables. It is possible to establish a relationship between the dynamical and remaining spectator variables for the purposes of calculating an entropy production [26]. More specifically, the derivatives of the diffusion matrix elements in Eq. (A4) can be recast into a more complex expression.

We can define the relationship as

$$\sum_{l=1}^L \alpha_{kl} dx_l + \sum_{m=1}^M \alpha_{km} dx_m = G_k dt, \quad (\text{B1})$$

where α_{km} and α_{kl} are the dynamical and spectator variable components of the k th null eigenvector of the diffusion matrix (an eigenvector with an eigenvalue of zero) and G_k is a function of the stochastic variables whose form we need not discuss [26]. We write

$$dx_l = -P_{lk}^{-1} Q_{km} dx_m + P_{lk}^{-1} G_k dt = R_{lm} dx_m + S_l dt, \quad (\text{B2})$$

where the α_{km} have been arranged as elements of a rectangular $L \times M$ matrix Q_{km} and the α_{kl} as elements of a square $L \times L$ matrix P_{kl} . The $L \times M$ matrix R is defined as $R = -P^{-1}Q$.

The derivatives dD_{ij}/dx_m in Eq. (A4) are then given by a combination of derivatives with respect to dynamical variables x_m and the spectator variables x_l , corresponding to Eq. (11) in the main text [26].

APPENDIX C: GENERATOR MATRICES FOR A TWO-SPIN-1/2 PARTICLE SYSTEM

The $\text{SU}(2) \otimes \text{SU}(2)$ generators used to form the entangled two-spin-1/2-system density matrix in Eq. (12) are $\Sigma_1 = I_2 \otimes \sigma_x$, $\Sigma_2 = I_2 \otimes \sigma_y$, $\Sigma_3 = I_2 \otimes \sigma_z$, $\Sigma_4 = \sigma_x \otimes I_2$, $\Sigma_5 = \sigma_x \otimes \sigma_x$, $\Sigma_6 = \sigma_x \otimes \sigma_y$, $\Sigma_7 = \sigma_x \otimes \sigma_z$, $\Sigma_8 = \sigma_y \otimes I_2$, $\Sigma_9 = \sigma_y \otimes \sigma_x$, $\Sigma_{10} = \sigma_y \otimes \sigma_y$, $\Sigma_{11} = \sigma_y \otimes \sigma_z$, $\Sigma_{12} = \sigma_z \otimes I_2$, $\Sigma_{13} = \sigma_z \otimes \sigma_x$, $\Sigma_{14} = \sigma_z \otimes \sigma_y$, and $\Sigma_{15} = \sigma_z \otimes \sigma_z$, where I_2 is a 2×2 identity matrix. Explicitly,

$$\begin{aligned} \Sigma_1 &= \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, & \Sigma_2 &= \begin{pmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix}, \\ \Sigma_3 &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, & \Sigma_4 &= \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \\ \Sigma_5 &= \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, & \Sigma_6 &= \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}, \\ \Sigma_7 &= \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, & \Sigma_8 &= \begin{pmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & -i \\ i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{pmatrix}, \\ \Sigma_9 &= \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}, & \Sigma_{10} &= \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \\ \Sigma_{11} &= \begin{pmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & i \\ i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix}, & \Sigma_{12} &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \\ \Sigma_{13} &= \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, & \Sigma_{14} &= \begin{pmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \end{pmatrix}, \\ \Sigma_{15} &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \end{aligned} \quad (\text{C1})$$

APPENDIX D: SDES FOR CASE 1

The Itô processes for the variables parametrizing the two-spin-1/2 system undergoing z -spin measurements on each particle are the following, for $a_1 = a_2 = 1$:

$$\begin{aligned} ds_1 &= -\frac{1}{2}s_1 dt + (-s_1 s_{12} + s_{13})dW_1 - s_1 s_3 dW_2, \\ ds_2 &= -\frac{1}{2}s_2 dt + (-s_2 s_{12} + s_{14})dW_1 - s_2 s_3 dW_2, \\ ds_3 &= (-s_3 s_{12} + s_{15})dW_1 + (1 - s_3^2)dW_2, \\ ds_4 &= -\frac{1}{2}s_4 dt - s_4 s_{12}dW_1 + (-s_3 s_4 + s_7)dW_2, \end{aligned}$$

$$\begin{aligned}
ds_5 &= -s_5 dt - s_5 s_{12} dW_1 + (-s_3 s_5 + s_4) dW_2, \\
ds_6 &= -s_6 dt - s_6 s_{12} dW_1 - s_3 s_6 dW_2, \\
ds_7 &= -\frac{1}{2} s_7 dt - s_7 s_{12} dW_1 - (s_4 - s_3 s_7) dW_2, \\
ds_8 &= -\frac{1}{2} s_8 dt - s_8 s_{12} dW_1 + (-s_3 s_8 + s_{11}) dW_2, \\
ds_9 &= -s_9 dt - s_9 s_{12} dW_1 - s_3 s_9 dW_2, \\
ds_{10} &= -s_{10} dt - s_{10} s_{12} dW_1 - s_3 s_{10} dW_2, \\
ds_{11} &= -\frac{1}{2} s_{11} dt - s_{11} s_{12} dW_1 + (s_8 - s_3 s_{11}) dW_2, \\
ds_{12} &= (1 - s_{12}^2) dW_1 + (-s_3 s_{12} + s_{15}) dW_2, \\
ds_{13} &= -\frac{1}{2} s_{13} dt + (s_1 - s_{12} s_{13}) dW_1 - s_3 s_{13} dW_2, \\
ds_{14} &= -\frac{1}{2} s_{14} dt + (s_2 - s_{12} s_{14}) dW_1 - s_3 s_{14} dW_2, \\
ds_{15} &= (s_3 - s_{12} s_{15}) dW_1 + (s_{12} - s_3 s_{15}) dW_2. \quad (D1)
\end{aligned}$$

The diffusion matrix elements are defined as $D_{ij} = \frac{1}{2}(B_{i1}B_{j1} + B_{i2}B_{j2})$. When $a_1 = a_2 = 1$, $\mathbf{B}_1$ and $\mathbf{B}_2$ may be written as follows:

$$\mathbf{B}_1 = \begin{pmatrix} -s_{12}s_1 + s_{13} \\ s_2 s_{12} + s_{14} \\ -s_3 s_{12} + s_{15} \\ -s_4 s_{12} \\ -s_5 s_{12} \\ -s_6 s_{12} \\ -s_7 s_{12} \\ -s_8 s_{12} \\ -s_9 s_{12} \\ -s_{10} s_{12} \\ -s_{11} s_{12} \\ 1 + s_{12}^2 \\ -s_{12} s_{13} + s_1 \\ s_2 - s_{12} s_{14} \\ s_3 - s_{12} s_{15} \end{pmatrix}, \quad \mathbf{B}_2 = \begin{pmatrix} -s_3 s_1 \\ -s_2 s_3 \\ 1 - s_3^2 \\ -s_3 s_4 + s_7 \\ -s_3 s_5 \\ -s_3 s_6 \\ -s_3 s_7 + s_4 \\ -s_3 s_8 + s_{11} \\ -s_3 s_9 \\ -s_3 s_{10} \\ -s_3 s_{11} + s_8 \\ -s_3 s_{12} + s_{15} \\ -s_3 s_{13} \\ -s_3 s_{14} \\ -s_3 s_{15} + s_{12} \end{pmatrix}. \quad (D2)$$

APPENDIX E: SDES FOR CASE 2

The Itô processes for the variables parametrizing the two-spin-1/2 system undergoing a z -spin measurement on particle 1 and an x -spin measurement on particle 2 are the following, for $a_1 = a_2 = 1$:

$$\begin{aligned}
ds_1 &= (-s_1 s_{12} + s_{13}) dW_1 + (1 - s_1^2) dW_2, \\
ds_2 &= -\frac{1}{2} s_2 dt + (-s_2 s_{12} + s_{14}) dW_1 - s_1 s_2 dW_2, \\
ds_3 &= -\frac{1}{2} s_3 dt + (-s_3 s_{12} + s_{15}) dW_1 - s_1 s_3 dW_2, \\
ds_4 &= -\frac{1}{2} s_4 dt - s_4 s_{12} dW_1 + (-s_1 s_4 + s_5) dW_2,
\end{aligned}$$

$$\begin{aligned}
ds_5 &= -\frac{1}{2} s_5 dt - s_5 s_{12} dW_1 + (-s_1 s_5 + s_4) dW_2, \\
ds_6 &= -s_6 dt - s_6 s_{12} dW_1 - s_1 s_6 dW_2, \\
ds_7 &= -s_7 dt - s_7 s_{12} dW_1 - s_1 s_7 dW_2, \\
ds_8 &= -\frac{1}{2} s_8 dt - s_8 s_{12} dW_1 + (-s_1 s_8 + s_9) dW_2, \\
ds_9 &= -\frac{1}{2} s_9 dt - s_9 s_{12} dW_1 + (-s_1 s_9 + s_8) dW_2, \\
ds_{10} &= -s_{10} dt - s_{10} s_{12} dW_1 - s_1 s_{10} dW_2, \\
ds_{11} &= -s_{11} dt - s_{11} s_{12} dW_1 - s_1 s_{11} dW_2, \\
ds_{12} &= (1 - s_{12}^2) dW_1 + (-s_1 s_{12} + s_{13}) dW_2, \\
ds_{13} &= (s_1 - s_{12} s_{13}) dW_1 + (s_{12} - s_1 s_{13}) dW_2, \\
ds_{14} &= -\frac{1}{2} s_{14} dt + (s_2 - s_{12} s_{14}) dW_1 - s_1 s_{14} dW_2, \\
ds_{15} &= -\frac{1}{2} s_{15} dt + (s_3 - s_{12} s_{15}) dW_1 - s_1 s_{15} dW_2. \quad (E1)
\end{aligned}$$

APPENDIX F: SDES FOR MEASUREMENT OF THE TOTAL z COMPONENT OF SPIN

The Itô processes for the variables parametrizing the two-spin-1/2-system density matrix when the system is undergoing a measurement of total S_z are as follows:

$$\begin{aligned}
ds_1 &= -\frac{1}{2} a^2 s_1 dt + a[s_{13} - s_1(s_3 + s_{12})] dW_z, \\
ds_2 &= -\frac{1}{2} a^2 s_2 dt + a[s_{14} - s_2(s_3 + s_{12})] dW_z, \\
ds_3 &= a[1 - s_3(s_3 + s_{12}) + s_{15}] dW_z, \\
ds_4 &= \frac{1}{2} a^2 s_4 dt + a[s_7 - s_4(s_3 + s_{12})] dW_z, \\
ds_5 &= a^2(s_{10} - s_5) dt - a s_5(s_3 + s_{12}) dW_z, \\
ds_6 &= a^2(-s_6 - s_9) dt - a s_6(s_3 + s_{12}) dW_z, \\
ds_7 &= -\frac{1}{2} a^2 s_7 dt + a[s_4 - s_7(s_3 + s_{12})] dW_z, \\
ds_8 &= -\frac{1}{2} a^2 s_8 dt + a[s_{11} - s_8(s_3 + s_{12})] dW_z, \\
ds_9 &= -a^2(-s_6 - s_9) dt - a s_9(s_3 + s_{12}) dW_z, \\
ds_{10} &= a^2(s_5 - s_{10}) dt - a s_{10}(s_3 + s_{12}) dW_z, \\
ds_{11} &= -\frac{1}{2} a^2 s_{11} dt + a[s_8 - s_{11}(s_3 + s_{12})] dW_z, \\
ds_{12} &= a[1 - s_{12}(s_3 + s_{12}) + s_{15}] dW_z, \\
ds_{13} &= -\frac{1}{2} a^2 s_{13} dt - a(s_3 s_{13} + s_{12} s_{13} - s_1) dW_z, \\
ds_{14} &= -\frac{1}{2} a^2 s_{14} dt + a[s_2 - s_{14}(s_3 + s_{12})] dW_z, \\
ds_{15} &= -a(s_3 + s_{12})(-1 + s_{15}) dW_z. \quad (F1)
\end{aligned}$$

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