

Quantum walks: First hitting times with weak measurements

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We study the first detected recurrence time problem of continuous-time quantum walks on graphs. While previous works have employed projective measurements to determine the first return time, we implement a protocol based on weak measurements on a dilated system, enabling minimally invasive monitoring throughout the evolution. To achieve this, we implement a weak measurement protocol theoretically and complement it with both numerical simulations and investigations on an IBM quantum computer. Despite the implementation of a generalized measurement, the proposed protocol provides a description purely within the Hilbert space of the quantum system. Our results reveal that for rank-one monitored, pure initial states, the first hitting timescales inversely with the coupling parameter between the ancilla and the quantum system.

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I. INTRODUCTION

Quantum random walks are universal for quantum computation with demonstrated exponential speed-up over random walks [1–3]. Being the quantum counterpart of classical random walks, quantum walks find applications in stochastic processes [4], quantum search algorithms [5], quantum computation [6], and many more [7]. Quantum recurrence indicates the ability of a quantum walker to return to its initial location [8] with probability one. The first hitting time of a quantum walk characterizes the expected time that a quantum particle takes to evolve through a lattice and hit its initial site for the first time. The phrases hitting time, return time, and recurrence time are used synonymously. Classical first hitting times on graphs [9,10] are utilized, for instance, in cellular biology [11], and their quantum counterpart finds applications in quantum chemistry [12] and quantum networks [13].

To detect the position of a walker obeying the laws of quantum theory, one has to measure the system. Axiomatically, a quantum-mechanical measurement intercepts the evolution and hence changes the quantum state. Thus, a measurement-induced quantum walk refers to a protocol where a quantum system evolves unitarily for some time and is repeatedly measured. This repeated measurement procedure is called monitoring, and there is no unique way to specify such a protocol [14]. The resulting dynamics depends sensitively on the type of monitoring. There are two canonical choices

for choosing the rate at which the system is measured. First, one can measure stroboscopically at a constant rate. This means that the system is always measured at constant frequency. Within this protocol, any quantum particle in a finite-dimensional system will definitely return to its initial state after a finite time period [15]. Hence, the first hitting time provides information about the size, complexity, and connectivity of a quantum system. Second, after a fixed time interval, loosely speaking, one flips a coin whose outcome decides whether the quantum system is measured or is free to evolve [14,16]. We focus on the first protocol, i.e., we repeat the measurement of a tight-binding continuous-time quantum walk at a fixed, constant rate.

In quantum information science, an elementary algorithm consists of two basic components: unitary evolution and measurement. However, there are two conceptual approaches in designing quantum circuits: Static circuits, where the measurement operation is executed at the end of a unitary sequence, and dynamic circuits [17,18], which employ measurements within the middle of the sequence. In light of quantum circuit design, the monitored quantum walk represents a minimal version of a dynamic circuit. Quantum mechanically, a strong measurement refers to an idealized, projection-valued measurement, with no further disturbance to the system. Regarding a quantum algorithmic framework, where the unitary operator acts on a (finite) register of qubits, a measurement is often realized by additional ancilla qubits that are utilized for weak measurements. In practice, these measurements affect the quantum dynamics via less disturbance than strong, projection-valued measurements, which classifies these measurements as weak. Weak measurements indicate that the interaction between the system and the measurement apparatus is sufficiently small, resulting in limited information gain per measurement and minimal disturbance to the system. This may manifest as increased uncertainty in the

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measurement outcomes, which can resemble additional randomness. This increased uncertainty is induced by the parametrized coupling strength between the two systems, which affects the outcome of the measurement. Thus, the (weak) first hitting time indicates an expected number of (weak) measurements that it takes for the particle to be detected for the first time. This must not be confused with a physical time. Hence, the event of first detection is a sequence of binary outcomes dependent upon whether the quantum walker (i.e., a quantum particle) has been detected “yes” or not “no” [19]. In case the detector clicks (yes, click measurement), the protocol is terminated. If the detector does not click (no, no-click measurement), the protocol is continued, while for weak measurements, the ancilla qubit influences the system qubit [20].

First hitting times for the continuous-time quantum walk with generalized weak measurements were already studied by Varbanov *et al.* [14]. Their protocol differs from our weak measurement protocol in the way that we always measure (stroboscopic monitoring) once the unitary evolution is stopped. Their protocol includes a third event, which corresponds to letting the quantum system evolve freely; that is, no measurement at all. The fundamental theoretical framework, which is extensively utilized in this work, was developed by Grünbaum *et al.* [15]. There, the authors derived a topologically protected, spectral characterization of quantum recurrence for discrete-time unitaries based on the properties of Schur functions [21], which was extended by Bourgain *et al.* [22] and generalized to open quantum walks [23,24]. On quantum hardware, first hitting times were already evaluated through strong, projective measurement protocols in Refs. [25–27].

In this work, we compute the expected first return time of a weakly monitored continuous-time quantum walk through graphs by applying weak measurements stroboscopically. The measurement operation that we utilize is realized by a controlled rotation through one additional ancilla qubit around the Pauli-Y axis. The return time is computed as a function of the coupling strength between the ancilla qubit and the quantum system. Although such a protocol is rigorously described by generalized operator-valued measurements, our specific choice of the rotation enables a description within the Hilbert space of the quantum system. Hence, we can map our protocol to the Hilbert space framework of quantum recurrence [15,22] and analyze the effect of the coupling strength. Therefore, our description avoids the system’s dilation and builds upon the fundamental theory developed in Refs. [15,22–24,28–30]. In particular, this nondilated embedding is restricted to our individual weak measurement protocol. In general, other generalized quantum measurement operations require the system’s dilation. Moreover, we provide a proof-of-principle demonstration of our weakly monitored first hitting time protocol on an IBM quantum computer, which supports the recurrence theory’s resilience to weak couplings.

The structure of this paper is outlined as follows: First, we recapture monitored quantum walks through projection-valued measurements in Sec. II. Second, we propose our weak monitoring protocol in Sec. III. Third, the weakly monitored first hitting time is analyzed in Sec. IV. Moreover, we provide analytical and numerical examples for the weakly monitored

first hitting time within the specific phase-damping-equivalent protocol, considered here. We observe an inverse dependence on the coupling parameter and rigorously establish the exact scaling law for rank-one monitored states and pure initial states. Extensions beyond this setting, including broader classes of channels or more general mixed states, remain open and are discussed only at a preliminary level. Finally, we propose further research directions in Sec. V.

II. MONITORED QUANTUM WALKS

The stochastic process under investigation is a continuous-time quantum walk of a single quantum state evolving through a finite graph. The unitary evolution is generated by a time-independent Hamilton operator, is given by the adjacency matrix of the graph. Let H denote that adjacency matrix. The corresponding quantum walker evolves unitarily according to the Schrödinger equation via

$$U(t) = e^{-iHt}, \quad (1)$$

such that the state of the system at time t is given by $|\psi(t)\rangle = U(t)|\psi(0)\rangle$, for some known initial state $|\psi(0)\rangle$. The continuity of the map $t \mapsto e^{-iHt}$ gives the continuous-time quantum walk its name. If the unitary evolution is interspersed with measurements, the resulting quantum walk is called a monitored quantum walk (MQW) [14]. If the unitary is chosen to be of the form (1), one calls the resulting protocol a monitored continuous-time quantum walk (MCTQW).

A. Quantum recurrence

We consider the first detected return time problem [26,31]. Contrary to classical random walks, where the observer can detect the position of the walker without interfering with the system’s dynamics, a quantum walk must be subjected to repeated measurements. We monitor the quantum system stroboscopically for N times at a constant rate. The real-valued time interval at which the quantum system evolves unitarily between two consecutive measurements is denoted by t . The resulting stochastic process is a Markov chain [15]. For example, the event in which the particle is first detected after the third measurement, given that it was not detected twice before, is written “no-no-yes” or simply “001.”

Throughout, we follow the notation of Bourgain *et al.* [22] and define for an initial quantum state $|\psi\rangle$ a subspace $V \subset \mathcal{H}$, which will be repeatedly measured. The measurement operator, which describes a projection onto $|\psi\rangle$, is denoted by P . Hence, for $|\psi\rangle \in V$, $V := \text{Im}P$ denotes the image of the projection operator. The set of quantum states on V is defined as [22]

$$S_V := \{|\psi\rangle \in V : \|\psi\rangle\| = 1\}.$$

Let $Q := \mathbb{1} - P$ be the complementary projector onto $V^\perp := \text{Im}Q$. For shorter notation, we sometimes denote $U(t)$ as U_t . The operators PU_t and QU_t are referred to as termination and survival operators. This means that a successful measurement (i.e., with P) terminates the protocol, whereas an unsuccessful measurement lets the walker survive. To ensure that an initial state $|\psi\rangle$ is contained in V , Bourgain *et al.* [22] defined the

linear first return amplitude operator $\hat{a}_n(t)$ to be

$$\hat{a}_n(t) := PU_t \tilde{U}_t^{n-1} P, \quad \tilde{U}_t := QU_t. \quad (2)$$

First hitting time

For a stochastic process with initial state $|\psi\rangle \in S_V$, the detection probability $p_n \equiv p_n(t)$ is referred to the probability to detect the quantum walker after $n \in \mathbb{N}$ steps [22]:

$$p_n(t) := \|\hat{a}_n(t)|\psi\rangle\|^2. \quad (3)$$

Then, the overall return probability R_N is defined as the total probability that the process returns to its initial state after N steps, i.e.,

$$R_N := \sum_{n=1}^N p_n, \quad R_\infty := \lim_{N \rightarrow \infty} R_N.$$

Since the hardware implementation on a quantum computer requires a protocol with finite circuit depth N , we study any involved stochastic quantity also with regard to a finite sample space. Note that R_N technically depends on the time t and the choice of the initial state $|\psi\rangle \in S_V$. Here, $\|\cdot\| \equiv \sqrt{\langle \cdot, \cdot \rangle}$ denotes the usual Hilbert-space norm. The expected return time τ_N refers to the expected number of measurements that it takes for the quantum walker to return to its initial state. The quantity τ_N becomes a physical time, if we multiply it by t so that $t_N := \tau_N t$ represents the physical expected first return time. t_N and τ_N are proportional to t so that we restrict our analysis to τ_N .

For an initial state $|\psi\rangle \in S_V$, $\tau_N \equiv \tau_N(t)$ is the expectation value of the first return event, which is given by [22]

$$\tau_N(t) := \frac{\sum_{n=1}^N n p_n(t)}{R_N(t)}. \quad (4)$$

Note that $\tau_\infty := \lim_{N \rightarrow \infty} \tau_N = \lim_{N \rightarrow \infty} \sum_{n=1}^N n p_n$, because here $\lim_{N \rightarrow \infty} R_N = 1$. Under the assumption that $\mathcal{H}$ is finite-dimensional, any closed subspace V is automatically V -recurrent [22,23].

B. Generalized quantum measurements

In Sec. III, we consider the first hitting time with weak measurements. Here, we repeat the basic background for operator-valued measurements.

Generalizing the quantum measurement from projection-valued measurements (PVMs) on a Hilbert space to positive operator-valued measurements (POVMs) on a dilated system is the unique mathematical construction of modeling imperfect quantum measurements [32,33]. Those quantum walks, which are measured via an interaction with an ancilla system (as we consider it in this paper), are classified as open quantum walks [23]. General mixed quantum states are described by density operators, i.e., Hermitian, positive semidefinite operators with trace one. A subnormalized density operator is a density operator with trace less than or equal to one. In quantum information, any general CPTP channel, i.e., a completely positive (CP) and trace preserving (TP) linear map, is

of the form

$$\mathcal{E}(\rho) = \sum_{i=1}^n K_i \rho K_i^\dagger, \quad (5)$$

with K_i being i and the effect operators $E_i := K_i K_i^\dagger$ are positive with $0 \leq E_i \leq \mathbb{1}$ and satisfy the completeness relation $\sum_{i=1}^n E_i = \mathbb{1}$. The first return amplitude operator from (2) becomes a subnormalized first return channel, which is [23]

$$A_n(\rho) := \tilde{a}_n \rho \tilde{a}_n^\dagger \quad \forall n \geq 1, \quad (6)$$

with $\tilde{a}_n \equiv \tilde{a}_n(t)$ similar to (2), but now modified to a generalized measurement, which reads

$$\tilde{a}_n := K_{\text{yes}} U \tilde{U}^{n-1}, \quad \tilde{U} := K_{\text{no}} U, \quad n \geq 1. \quad (7)$$

The probability of detecting the walker at the n th measurement, given that the particle is not detected $n-1$ times, is hence

$$p_n = \text{Tr} A_n(\rho). \quad (8)$$

The Kraus operators K_{yes} and K_{no} indicate a binary measurement, which uniquely refers to the events of having detected the particle yes or not no. The evolution in (6) was already proposed by Krovi *et al.* [34]. Stinespring's dilation theorem states that any POVM is a PVM on a larger, dilated space [35]. However, the converse does not hold true. In Sec. III, we exactly proceed along such a converse direction.

The unitary evolution of a density operator $\rho \mapsto U \rho U^\dagger$ can be represented as a matrix-vector-multiplication on a space of squared dimension [24,36], as

$$\text{vec}(U \rho U^\dagger) = (U \otimes \bar{U}) \text{vec}(\rho). \quad (9)$$

For a complex-valued, square matrix $A = (a_{ij})_{i,j=1}^n$, the bijective vectorization mapping $\text{vec}(A) : M_n(\mathbb{C}) \rightarrow \mathbb{C}^{n^2}$ is $\text{vec}(a_{ij}) = a_{(i-1)n+j}$. Therefore, one can use the spectral recurrence formalism [15,22] subjected to operator-valued measurements of any quantum channel as discussed in Refs. [23,24].

III. MONITORING VIA WEAK MEASUREMENTS

A. Weak measurements

For simplicity, we describe first hitting times through weak measurements solely by using subnormalized pure states and projection-valued measurements. By doing so, we can utilize the spectral characterization of recurrence developed in Ref. [15] for our approach of measuring weakly. We restrict our analysis to the protocol in Fig. 1 applied to site measurements.

If we are able to track the action of the POVM back to a modified pseudo-PVM on the same (Hilbert) space of the quantum system (and not on the dilated system), we are able to describe the monitored protocol nevertheless through vectors on Hilbert spaces. For quantum recurrence, there already exists a comprehensive theory of both (scalar-valued) state recurrence [15] and (operator-valued) subspace recurrence [22,23]. As we will see, our choice of weak measurements can be mathematically embedded into the framework of quantum state recurrence [15].

Our choice of a weak measurement, which we use throughout this paper, is operationally equivalent to an evolution

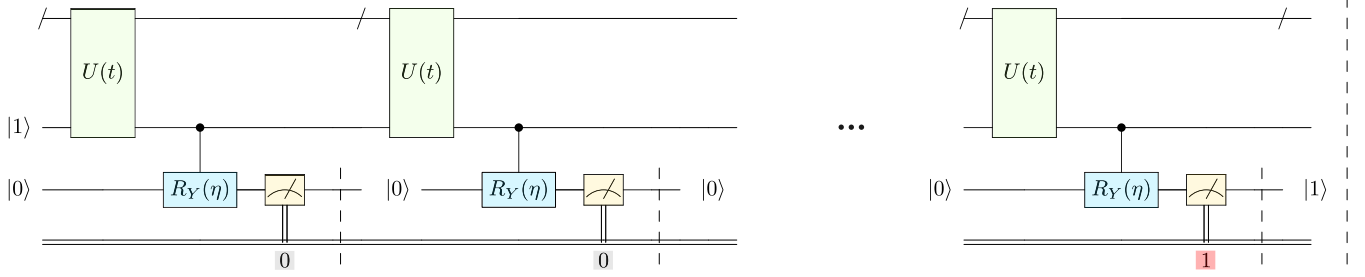


FIG. 1. Quantum circuit of the first return event via weak measurements, with a renormalized rotation angle $\eta \in (0, 1]$. The bit string of this measurement is $00 \dots 01$, where the 1 occurs at position n so that the circuit depth is $n \leq N$.

through a quantum phase damping channel [33]. This channel shears the projection operators as a function of the rotation angle between the ancilla and the quantum system. Physically speaking, such a channel describes the event (occurring with probability η) that a photon has been scattered without loss of energy [33]. The corresponding Kraus operators are

$$K_{\text{yes}}(\eta) = \begin{pmatrix} 0 & 0 \\ 0 & \sqrt{\eta} \end{pmatrix}, \quad K_{\text{no}}(\eta) = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1-\eta} \end{pmatrix}. \quad (10)$$

It is emphasized that these operators satisfy the subsequent pseudocompleteness relation

$$K_{\text{yes}}(\eta) + K_{\text{no}}(\eta) = \mathbb{1} + f(\eta) \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad (11)$$

$$f(\eta) := \sqrt{\eta} + \sqrt{1-\eta} - 1. \quad (12)$$

By pseudocompleteness we mean that K_{yes} and K_{no} resolve the identity plus an η -dependent perturbation by some diagonal operator. Here, that operator is just

$$|1\rangle\langle 1| \equiv \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

Relation (11) is essential for our protocol: in what follows, we map the operator-valued weak measurements (which are realized by a controlled R_Y rotation on the ancilla system) to the projection-valued recurrence theory through rank-one projections developed in Ref. [15] and extended in Ref. [22]. Our approach is built upon (11), as will become clear in Sec. IIIB 1. However, any POVM measurement that does not fulfill such a pseudocompleteness relation cannot be canonically embedded (without dilation) into a recurrence formalism for pure states. In general, the wider theoretical framework of open quantum walks [23,24] must be taken into account. We consider the Hilbert space $\mathcal{H}$ of two composite systems, i.e., $\mathcal{H} = \mathcal{H}_{\text{sys}} \otimes \mathcal{H}_{\text{anc}}$. The first subsystem represents a register of d qubits $\mathcal{H}_{\text{sys}} = \mathbb{C}^{2^d}$, which underlies a Hamiltonian evolution and the second subsystem is an ancilla qubit $\mathcal{H}_{\text{anc}} = \mathbb{C}^2$. We identify one qubit out of that register as the control qubit, which is measured. The remaining $d-1$ qubits refer to the rest of the system. For simplicity, let us start with one system qubit ($d=1$) and one ancilla qubit so that $\mathcal{H}_{2\text{-qub.}} = \mathbb{C}^2 \otimes \mathbb{C}^2$. For this minimal scenario, we recall a weak measurement as follows [37]:

Definition 1 (Two-qubit null-type weak measurement). Let $\eta \in (0, 1]$. With $g(\eta) := \arcsin(\sqrt{\eta})$, the unitary operator on

the ancilla qubit is defined by

$$R_Y(\eta) = \begin{pmatrix} \cos g(\eta) & -\sin g(\eta) \\ \sin g(\eta) & \cos g(\eta) \end{pmatrix}.$$

Thus, the two-qubit controlled R_Y rotation reads

$$CR_Y(\eta) = e^{-ig(\eta)\frac{(\mathbb{1}-\sigma_z)}{2} \otimes \sigma_y} = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & R_Y(\eta) \end{pmatrix} \quad (13)$$

$$\equiv \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos g(\eta) & -\sin g(\eta) \\ 0 & 0 & \sin g(\eta) & \cos g(\eta) \end{pmatrix}. \quad (14)$$

Applying projection operators

$$P := \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad Q := \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad (15)$$

to the ancilla qubit yields a weak measurement. The Kraus operators of this measurement are given in (10).

Note that $\eta \mapsto g(\eta)$ parametrizes the angle of the R_Y rotation, i.e., $g(\eta) \in (0, \frac{\pi}{2}]$. Thus, measuring the ancilla qubit strongly corresponds to measuring the control qubit weakly. The setup is visualized in Fig. 1. without loss of generality, we initialize the ancilla qubit in state $|0\rangle$ and the system qubit in state $|1\rangle$. One iteration in our weakly monitored quantum walk consists of three steps. At first, the unitary operator $U(t)$ is applied to the register of d qubits. Second, the controlled R_Y gate is coupled to the control qubit that is to be measured. The control qubit is part of the quantum system under investigation, where the ancilla qubit represents an environment through which the weak measurement is performed. Finally, the ancilla qubit is measured. If the measurement is successful (i.e., the system qubit was activated), the protocol is terminated. If the measurement was not successful, the ancilla qubit is reset to state $|0\rangle$ and the protocol continues in step one.

Hence, the measurement-induced quantum trajectory [38–40] on the ancilla qubit is a flip from $|0\rangle$ to $|1\rangle$, with conditional resets performed along each unsuccessful measurement. Equivalently, this transition event on the ancilla space $\mathcal{H}_{\text{anc}}$ corresponds to a return event on a $|\psi\rangle \in V \subset \mathcal{H}_{\text{sys}}$. To illustrate this, recall the two-qubit CNOT gate acting on the (product) states $|1\rangle|0\rangle$ and $|0\rangle|0\rangle$:

$$\text{CNOT}(|1\rangle|0\rangle) = |1\rangle|1\rangle,$$

$$\text{CNOT}(|0\rangle|0\rangle) = |0\rangle|0\rangle.$$

If the ancilla qubit is in state $|1\rangle$, the particle has been detected (because the control qubit is also in $|1\rangle$). If the ancilla qubit is in state $|0\rangle$, the particle has not been detected (control qubit is in $|0\rangle$). In our implementation, we use the parametrized, controlled R_Y rotation $CR_Y(\eta)$. Depending on $\eta \in (0, 1]$ this gate characterizes the entanglement between the control and the ancilla qubit. For $\eta = 1$, it follows the same logic as the CNOT gate for the states considered above, i.e.,

$$\begin{aligned} CR_Y(\eta = 1)(|1\rangle|0\rangle) &= |1\rangle|1\rangle, \\ CR_Y(\eta = 1)(|0\rangle|0\rangle) &= |0\rangle|0\rangle. \end{aligned}$$

For general $\eta \in (0, 1]$, the resulting states evolve as [37]

$$(\alpha|0\rangle + \beta|1\rangle)|0\rangle \rightarrow (\alpha|0\rangle + \beta\sqrt{1-\eta}|1\rangle)|0\rangle + \beta\sqrt{\eta}|1\rangle|1\rangle,$$

with details given in Proposition A2. Since the ancilla state remains always in $|0\rangle$ after every unsuccessful measurement, there is no additional mathematical constraint for the reset operation. By measuring the ancilla qubit, one gains the desired information about the control qubit, depending on the coupling strength η . $\eta = 0$ indicates a full separation between the two qubits, i.e., no information at all (we henceforth exclude $\eta = 0$ from the protocol). $\eta = 1$ means that the two qubits are maximally entangled. Thus, the $\eta = 1$ scenario is equivalent to taking strong measurements, as developed in Refs. [15,22]. In the subsequent mathematical formulation, we omit the ancilla qubit (and hence omit doubling the dimension of the Hilbert space), but work with linear shearings of the projection operators on the nondilated Hilbert space $\mathcal{H}_{\text{sys}}$.

These shearings are seen as follows. By defining

$$P_\eta := K^-(\eta), \quad Q_\eta := K^+(\eta), \quad (16)$$

and writing $P_\eta = \sqrt{\eta}P$, $Q_\eta = Q + \sqrt{1-\eta}P$, with P, Q , as in (15), the Kraus operators for phase damping (10) are the nondilated linear transformation of the projection operators:

$$\begin{pmatrix} P_\eta \\ Q_\eta \end{pmatrix} = \begin{pmatrix} \sqrt{\eta} & 0 \\ \sqrt{1-\eta} & 1 \end{pmatrix} \begin{pmatrix} P \\ Q \end{pmatrix}. \quad (17)$$

The conservation of probability is seen as follows:

$$1 = \|\psi\|^2 = \|P_\eta|\psi\rangle\|^2 + \|Q_\eta|\psi\rangle\|^2 = \eta + (1-\eta).$$

Note that such a one-to-one mapping between operator-valued and projection-valued measurements does not exist for other quantum channels [33]. Due to the nonprojection-valued measurement, it is emphasized that $P_\eta + Q_\eta \neq \mathbb{1}$, but rather [see Fig. 2(b)]

$$P_\eta + Q_\eta = \mathbb{1} + f(\eta)P, \quad (18)$$

with $f(\eta)$ as in (12). This two-qubit measurement can be embedded into any register of d qubits as long as one control qubit is marked.

Since such a construction is not true for generalized channels, we emphasize that the formalism (17) holds true only when specifically choosing the protocol as in Fig. 1.

Our protocol can be applied to arbitrary large (but finite) systems, as the graph's system size is encoded in the dimension of the unitary, not the measurement operation.

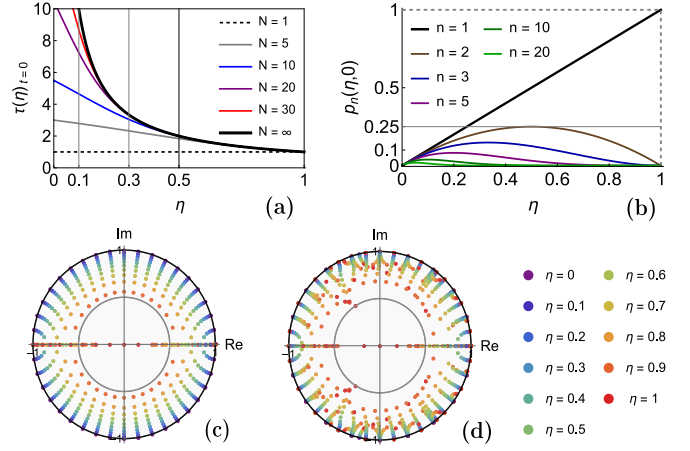


FIG. 2. (a) $1/\eta$ decay of the mean recurrence time for $t = 0$: The function $\tau_N(\eta, t = 0)$ from (40) is plotted against $\eta \in (0, 1]$ for several values of N . One can see that the convergence toward infinitely many measurements happens very fast. Even for a small number of measurements, one gets the dominating $1/\eta$ behavior. For $N = \infty$, the N -dependent corrections vanish, so that the black curve is exactly $\tau_\infty(\eta, t = 0) = 1/\eta$. (b) $p_n(\eta, 0)$ from Eq. (33) plotted against η for several values of n . (c) η -dependent spectrum of the survival operator $\tilde{U}_\eta(t)$ for the two-level system described in Sec. IV A 2. The spectrum is shown for 40 equidistant time steps between 0 and 2π , i.e., $\Delta t = \pi/20$. As η goes to zero, the eigenvalues move toward the edge of the unit disk. At $\eta = 0$, the values lie exactly on the edge, which means that no measurement intercepts at all [i.e., $\tau_N(\eta = 0, t) = \infty$ for any N]. The inner circle has radius $1/2$, which corresponds to the dimensional separation of the measurement. (d) η -dependent spectrum of the survival operator $\tilde{U}_\eta(t)$ for the benzene ring. Due to larger system size and more support points, the spectral regularity is more disturbed compared with panel (c).

B. Weak first return event

1. Weak first return amplitude operator

We analyze the first hitting time depending on $\eta \in (0, 1]$ by applying the spectral description of quantum recurrence [15,22–24] to weak measurements. It is emphasized that P_η and Q_η characterize a binary measurement, as Proposition A3 shows. The logical negation of this result indicates that whenever there is a η -dependent, nonzero overlap with S_V , we have detected the particle and the protocol is terminated. Otherwise, the measurement process is continued.

By applying the modified measurement operators from (16) to the survival and termination operator, we modify the first return amplitude operator from (2) as follows:

Definition 2. Let $|\psi\rangle \in S_V$. We define the survival operator as $\tilde{U}_\eta(t) := Q_\eta U_t \equiv (Q + \sqrt{1-\eta}P)U_t$ and the termination operator as $P_\eta U_t$. This allows us to define the weak first return amplitude operator as

$$\hat{a}_n^\eta(t) := P_\eta U(t) \tilde{U}_\eta(t)^{n-1} P. \quad (19)$$

Consequently, the first detection probability (3) is modified to an η -dependent first detection probability

$$p_n(\eta, t) := \|\hat{a}_n^\eta(t)|\psi\rangle\|^2, \quad (20)$$

with a total detection probability

$$R_N(\eta, t) = \sum_{n=1}^N p_n(\eta, t), \quad (21)$$

and with $\hat{a}_n^\eta(t)$ as in (19). Note that the projector P on the right side of Eq. (19) establishes that $|\psi\rangle \in \text{Im } P$. Do not confuse it with a weak measurement in the zeroth step. This total detection probability satisfies

$$\begin{aligned} R_N(\eta, t) &= \sum_{n=1}^N p_n(\eta, t) = \sum_{n=1}^N \|\hat{a}_n^\eta(t)|\psi\rangle\|^2 \\ &= 1 - \|\tilde{U}_\eta(t)^N |\psi\rangle\|^2, \end{aligned} \quad (22)$$

from which one concludes that

$$\lim_{N \rightarrow \infty} R_N(\eta, t) = 1. \quad (23)$$

Details are shown in Corollary A1, and further details can be found in Appendix A2.

2. First hitting time after N steps

Similarly to the above weak first return amplitude operator (19) and return probability (20), we have for the weak first hitting time after measuring maximally N times:

$$\tau_N(\eta, t) := \frac{\sum_{n=1}^N n p_n(\eta, t)}{R_N(\eta, t)}. \quad (24)$$

By employing the telescoping structure of the sum, we can compute a simpler expression of (24) similar to Ref. [22], which reads

$$\tau_N(\eta, t) = \frac{\sum_{k=0}^{N-1} \|\tilde{U}_\eta(t)^k |\psi\rangle\|^2 - N \|\tilde{U}_\eta(t)^N |\psi\rangle\|^2}{1 - \|\tilde{U}_\eta(t)^N |\psi\rangle\|^2}. \quad (25)$$

Details are presented in Lemma A1 and complemented by Corollaries A1 and A2. Intuitively, these statements might be obvious. The weaker we measure, the smaller is the return probability and consequently the longer it will take to detect the quantum walker.

3. First hitting time after infinitely many measurements

In the limit of infinitely many measurements, Eq. (23) yields $R_\infty(\eta, t) = 1$. Then, (24) becomes

$$\begin{aligned} \tau_\infty(\eta, t) &:= \sum_{n=1}^{\infty} n p_n(\eta, t) \stackrel{\text{def.}}{=} \sum_{n=1}^{\infty} n \|\hat{a}_n^\eta(t)|\psi\rangle\|^2 \\ &= \sum_{k=0}^{\infty} \|\tilde{U}_\eta(t)^k |\psi\rangle\|^2. \end{aligned} \quad (26)$$

For this limit, the Laplace transform of the first return operator provides a useful tool [15,22,23,41]. Due to the fact that $\|\hat{a}_n^\eta(t)\| \leq 1 \forall \eta \in [0, 1]$, one defines the (absolutely convergent) generating function

$$\hat{a}(\eta, z, t) := \sum_{k=0}^{\infty} \hat{a}_k^\eta(t) z^k, \quad z \in \mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}, \quad (27)$$

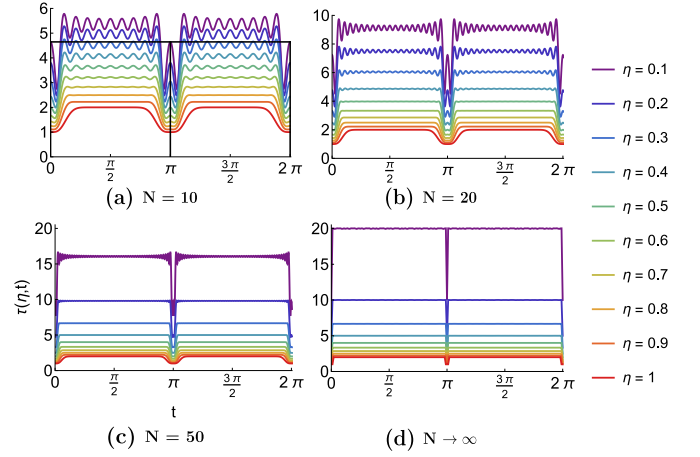


FIG. 3. Two-level system: expected first return time $\tau_N(\eta, t)$ of an MCTQW on a 2-vertex graph with one edge with (a) $N = 10$, (b) $N = 20$, (c) $N = 50$, and (d) $N = \infty$ measurements, plotted against $t \in [0, 2\pi]$ for several values of η . This minimal graph is illustrated in Fig. 4. We start the protocol in vertex 0 and compute the expected number of measurements, which it takes for the particle to return for the first time. $\eta = 1$ corresponds to a projection-valued measurement with $\tau(1, t) = 2$, except for $t = \pi$. Hence, apart from $t = \pi$, the average return time is expected to be between 2 and 20 [compare with (35)]. One obtains Gibbs oscillations of the return time, which increase in amplitude as η goes to zero. In panel (a), the vertical black lines indicate the positions of the peaks that arise at multiple values of π . The horizontal black line indicates $\tau_{10}(\eta = 0.1, t = 0)$ in the Zeno limit ($U_0 \equiv \mathbb{1}$). For $\eta = 1$, clearly $\tau_{10}(1, 0) = 1$. For other values of η , $\tau_N(\eta, 0)$ increases according to (40). We see oscillations for a small number of interspersed measurements ($N \simeq 10^1$), which decay for larger N . This so-called Gibbs phenomenon originates from the slow convergence of the respective Fourier coefficients.

By using the Green's function of the survival operator,

$$\tilde{U}_\eta(z, t) := (\mathbb{1} - z\tilde{U}_\eta(t))^{-1}, \quad (28)$$

the generating function in (27) becomes

$$\hat{a}(\eta, z, t) = zP_\eta U(t)\tilde{U}_\eta(z, t)P. \quad (29)$$

Thus, the complex analytic methods from Refs. [15,22] are directly applicable. In analogy to Ref. [22], we define $z := e^{i\theta}$ to rewrite (26) as the η -dependent integral, which is $-1/2\pi$ times a one-parameter family of Aharonov-Anandan phases [22,42–44], i.e.,

$$\tau_\infty(\eta, t) \equiv \frac{1}{2\pi i} \int_0^{2\pi} d\theta \left\langle \psi(\eta, \theta, t), \frac{\partial}{\partial \theta} \psi(\eta, \theta, t) \right\rangle, \quad (30)$$

with $|\psi(\eta, \theta, t)\rangle := \hat{a}(\eta, e^{i\theta}, t)|\psi\rangle$ and $\theta \in [0, 2\pi]$. The detailed derivation of (30) is presented in Appendix B 2 b. Note that we only consider a finite-dimensional Hilbert space. The spectral measure is the point spectrum of the survival operator [22,23]. An example of (30) is presented in Fig. 3(d).

Equation (30) is equivalent to computing [24]

$$\tau_\infty(\eta, t) = 1 + \lim_{x \rightarrow 1} \frac{\partial}{\partial x} \text{Tr}[\text{vec } \hat{a}(\eta, x, t) \text{vec } \rho], \quad (31)$$

with $\rho := |\psi\rangle\langle\psi|$ and $\text{vec } \hat{a}(\eta, x, t)$ being the vectorized generating function of (29), i.e.,

$$\begin{aligned} \text{vec } \hat{a}(\eta, x, t) &= (P_\eta \otimes P_\eta)(U_t \otimes \bar{U}_t) \\ &\times [\mathbb{1} \otimes \mathbb{1} - x(Q_\eta \otimes Q_\eta)(U_t \otimes \bar{U}_t)](P \otimes P). \end{aligned} \quad (32)$$

The integral representation in (30) is formulated in the quantum system's Hilbert space, whereas (31) requires squaring that space.

IV. WEAKLY MONITORED FIRST HITTING TIME

A. Analytical results

Equations (24) and (30) allow an analytical discussion for both the Zeno limit and the minimal, two-level system.

1. Zeno dynamics

If the rate at which a quantum walker is measured is too high, there is no unitary evolution, but the walker is frozen at its initial state. This freezing phenomenon is known as the quantum Zeno effect [45]. Mathematically, this limit is obtained by sending t to zero, so the unitary evolution time becomes infinitely small. If $|\psi\rangle$ is in the initial state, i.e., $P|\psi\rangle = |\psi\rangle$, then by Eq. (17), $Q_\eta|\psi\rangle = \sqrt{1-\eta}|\psi\rangle$, so the first detection probability (20) reads

$$p_n(\eta, 0) = \eta(1-\eta)^{n-1}. \quad (33)$$

For $\eta = 1$, $p_1(1, 0) = 1$ and $p_n(1, 0) = 0$ for all $n > 1$. Hence, the usual (strongly monitored) Zeno limit for the detection probability coincides with the weak protocol, i.e., when the dynamics is frozen, the detection probability in step one is 100%. For $\eta \in (0, 1)$ and $n > 1$, $0 \leq p_n(\eta, 0) < 1$, enabling a nonzero probability to detect the particle in the second, third, etc. measurement, even though the system was frozen at its initial conditions. Since any $\eta < 1$ causes a loss of information, one can interpret such a delocalization as a lack of accuracy during the measurement process.

For the Zeno limit, we can compute the strong first hitting time $\tau_N(1, 0)$, as well as the weak first hitting time $\tau_N(\eta, 0)$ analytically. Naturally, since the quantum walker does not move at all, $\tau_N(1, 0) = 1$, i.e., only one measurement is required to find the quantum walker. Due to the loss of information at each measurement step in the weak protocol, this number decays reciprocally with η as

$$\tau_N(\eta, 0) = \frac{1}{\eta} - \frac{N(1-\eta)^N}{1-(1-\eta)^N}. \quad (34)$$

Details are presented in Lemma A2. Coupling the quantum system to an environment, i.e., the ancilla qubit, corresponds to losing information, or equivalently, missing some first return events on average. Hence,

$$\tau_\infty(\eta, 0) = \frac{1}{\eta} > 1, \quad 0 < \eta < 1.$$

For infinitely many measurements, the projection-valued limit ($\eta = 1$) is clearly recovered, i.e., $\tau_\infty(1, 0) = 1$. Note that the obtained weak recurrence time corresponds to the expected value of the geometric distribution, which is in agreement

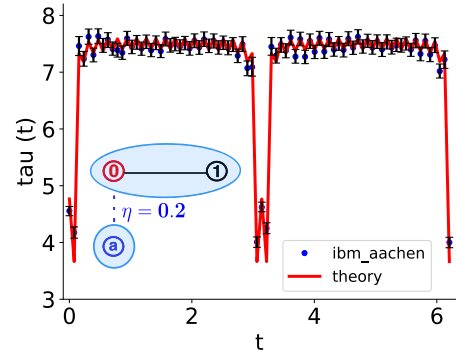


FIG. 4. The expected first return time $\tau_{20}(\eta = 0.2, t)$ through $N = 20$ weak measurements with coupling strength $\eta = 0.2$ for the minimal graph example, computed on IBM-Aachen (blue). The time interval $[0, 2\pi]$ is partitioned into equidistant values with $\Delta t = 2\pi/80$. Red curve: theoretical prediction for $\tau_{20}(\eta = 0.2, t)$, with $|\psi\rangle = |10\rangle$ via Eq. (25). The quantum circuit is displayed in Fig. 1 and executed on the hardware with 40 000 shots. The involved qubits correspond to qubits no. 59, 75 on IBM-Aachen. The quantum-computational demonstration supports the theoretical prediction. The error bars are obtained by statistical bootstrapping for 1000 repetitions and 2000 samples, uniformly drawn over the 40 000 shots. Details are presented in Appendix E.

with (33). The scaling law from the Zeno limit holds in more generality and is shown in Sec. IV C.

2. Two-level system

The minimal example consists of a graph with two vertices and one edge connecting them. Without loss of generality, we measure the vertex labeled with 0, as displayed in Fig. 4. The Hamilton operator is given by

$$H = -\sigma_x = -\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

The two nondegenerate eigenvalues are $E_\pm = \pm 1$, which determine the unitary evolution to be $U(t) = e^{-itH} = e^{-it}P_+ + e^{it}P_-$, where $P_\pm$ denote the projectors onto the respective eigenspaces. For a two-level system, we compute $\tau_\infty(t, \eta)$ by Eqs. (28), (29), and (31). We use the vectorization approach of Ref. [23] and use $\rho := |\psi\rangle\langle\psi|$,

$$\begin{aligned} U_t &= \begin{pmatrix} \cos t & i \sin t \\ i \sin t & \cos t \end{pmatrix}, \quad |\psi\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \\ P_\eta &= \begin{pmatrix} \sqrt{\eta} & 0 \\ 0 & 0 \end{pmatrix}, \quad Q_\eta = \begin{pmatrix} \sqrt{1-\eta} & 0 \\ 0 & 1 \end{pmatrix}, \end{aligned}$$

and

$$\tilde{U}_\eta(t) = \begin{pmatrix} \sqrt{1-\eta} \cos t & i\sqrt{1-\eta} \sin t \\ i \sin t & \cos t \end{pmatrix}$$

to evaluate $g(t, x, \eta) := \text{Tr}[\text{vec } \hat{a}(t, x, \eta) \text{vec } \rho]$. Then,

$$\begin{aligned} \tau_\infty(t, \eta) &= 1 + \lim_{x \rightarrow 1} \frac{\partial}{\partial x} g(t, x, \eta) \\ &= \frac{\tau_\infty(t)}{\eta} = \begin{cases} \frac{1}{\eta}, & \text{if } t = k\pi, k \in \mathbb{Z} \\ \frac{2}{\eta}, & \text{else.} \end{cases} \end{aligned} \quad (35)$$

$g(t, x, \eta)$ is written out in Appendix C.

Our findings coincide with the first hitting times calculated in Ref. [31]. There, it was shown that

$$\tau_{\infty}(t) = \begin{cases} 1, & \text{if } t = k\pi, k \in \mathbb{Z} \\ 2, & \text{else.} \end{cases}$$

Their result corresponds exactly to the projection-valued limit $\eta = 1$. Consequently, our weak protocol yields for a two-level system:

$$\tau_{\infty}(t, \eta) = \frac{\tau_{\infty}(t)}{\eta}. \quad (36)$$

More generally, consider a tight-binding Hamilton operator for rings of length L :

$$H = - \sum_{i=0}^{L-1} (|i\rangle\langle i+1| + |i+1\rangle\langle i|). \quad (37)$$

For this system, the mean first detection time τ_{∞} reads [see Eq. (57) in Ref. [31,38]]

$$\tau_{\infty}(\eta = 1) = \begin{cases} \frac{L+2}{2}, & \text{if } L \text{ is even} \\ \frac{L+1}{2}, & \text{if } L \text{ is odd.} \end{cases} \quad (38)$$

Hence, for all noncharacteristic times $t \in [0, 2\pi)$, we expected a mean value of two, which is confirmed by the red curves ($\eta = 1$) in Fig. 3. At those times $t \in [0, 2\pi)$, that belong to the spectral support of the system, one observes discontinuous jumps in the return time. The height of the plateaus contains both information about the size of the system and also the spectral degeneracy of the underlying graph. Rings of size L are studied in great generality in Ref. [31]. The exceptional sampling times t_e are given by [31]

$$t_e = \frac{2\pi k}{\Delta E} \bmod 2\pi \quad (39)$$

for $k \in \mathbb{Z}$ and energy difference $\Delta E \equiv E_i - E_j > 0$. In our minimal example, $L = 2$ so there is only one energy gap $\Delta E = +1 - (-1) = 2$. Thus, the exceptional sampling times occur at integer multiples of π , which is seen in Fig. 3.

We emphasize that the weak measurement procedure does not have any influence on the exceptional sampling times or the relative shape of the peaks. It is merely an offset in the first detection time τ due to a decreasing coupling strength between the ancilla qubit and the quantum system. The special sampling times are degeneracies in the spectrum of the unitary operator [46].

B. Numerical examples and quantum-computational demonstration

1. Two-level system on an IBM quantum computer

We implement the quantum circuits shown in Fig. 1 on a superconducting quantum processor for the two-level system. The hardware, used for this demonstration, is IBM-Aachen, an IBM Quantum Heron R2 processor featuring 156 superconducting transmon qubits [47].

The readout assignment error—which quantifies the inaccuracy in qubit state discrimination—is approximately 5×10^{-3} . The qubit relaxation times lie between 334 and 360 μs with resonance frequencies between 4.7 and 4.82 GHz. The qubit dephasing times range from 133 to 460 μs . The single-qubit gate error ranges between 1×10^{-4} and 3×10^{-4}

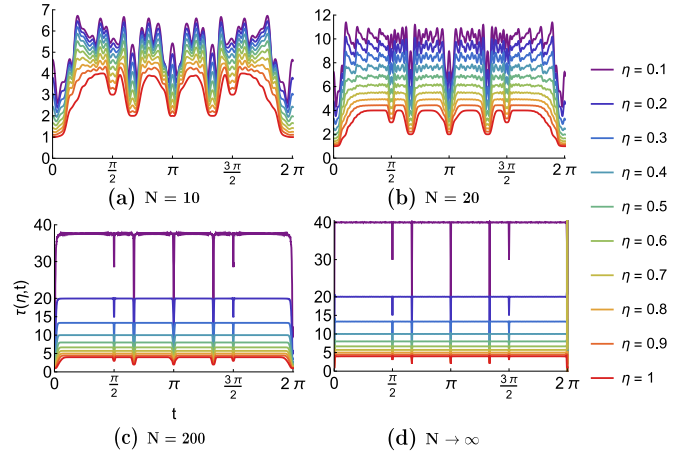


FIG. 5. Benzene ring: expected first return time of an MCTQW on Benzene with (a) $N = 10$, (b) $N = 20$, (c) $N = 200$, and (d) $N = \infty$ measurements for several values of η . We start the protocol in vertex 5 and compute the expected first return time $\tau(\eta, t)$ against t for several η . $\eta = 1$ corresponds to a projection-valued measurement with $\tau(1, t) = 4$, except for the exceptional points $\mathcal{T}_{\text{Benzene}}$ (as discussed in the main text). Hence, the average return time is expected to be between 4 and 40 [compare with (40)]. Again, there are Gibbs oscillations of the return time, which increase in the amplitude as η goes to 0. For $\eta = 1$, this is clearly at $\tau(1, 0) = 1$. For other values of η , that goes up according to (40). We see oscillations for a small number of interspersed measurements ($N \simeq 10^1$), which decay for larger N . This (Gibbs) phenomenon is caused by a slow convergence of the underlying Fourier coefficients.

with a gate runtime of approximately 75 to 76 ns and a readout time of 2600 ns. The error rates for two-qubit cross-resonance gates are approximately 1.5×10^{-3} . To mitigate decoherence effects, we employ dynamical decoupling [48], a technique that preserves quantum coherence by applying a sequence of control pulses to suppress environmental noise and system-bath interactions during measurement. The exemplary hardware results obtained from IBM-Aachen (blue dots) provide an illustrative proof-of-principle demonstration of the protocol and show qualitative agreement with the theoretical prediction (red curve) in Fig. 4. The standard deviation is obtained by statistical bootstrapping.

2. Benzene ring

The first return time problem for strongly monitored quantum walks on Benzene-type rings was extensively studied in Ref. [31]. The complete graph K_6 subjected to an open quantum dynamics was studied in Ref. [49]. To make an adequate comparison to our weak measurement protocol, we choose the Benzene ring of length $L = 6$ as the second example (see Fig. 6). The Hamiltonian is given in (37). The energy eigenvalues of the Hamiltonian are $\sigma(H) = \{-2, -1_2, 1_2, 2\}$, where the subscript 2 indicates double degeneracy. For $L = 6$, Eq. (38) predicts on average $\tau_{\infty} = 4$ for nearly all values of t , which corresponds to the expected first return time through strong measurements ($\eta = 1$, compare with red curves in Fig. 5). According to (39), we expect exceptional sampling times at (see Fig. 5) $t \in \mathcal{T}_{\text{Benzene}} := \{\frac{\pi}{2}, \frac{2\pi}{3}, \pi, \frac{4\pi}{3}, \frac{3\pi}{2}\}$ with discontinuous return time due to fractional revival of the wave

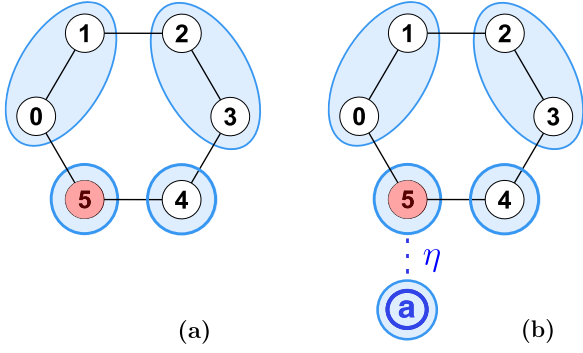


FIG. 6. (a) Benzene ring encoded into a four-qubit basis. This is the (quantum) system on which an MCTQW is performed. The blue areas indicate one qubit each. The initial vertex that is being measured is marked in red. Since benzene is symmetric, which vertex we measure does not play any role. (b) The benzene ring from panel (a) with an additional ancilla qubit “a.” The weak measurement procedure on the quantum system is implemented through the ancilla qubit with the coupling strength η . The encoding follows Proposition 4.

packet [31]. The resulting curves in Fig. 5 show qualitatively the same offset, which is proportional to $1/\eta$, similar to those in the two-level system (Fig. 3). Again, the exceptional sampling times $\mathcal{T}_{\text{Benzene}}$ are not affected by the type of measurement protocol: whether we measure strongly ($\eta = 1$) or weakly ($0 < \eta < 1$) makes no difference for the location of exceptional sampling times. This was to be expected, since the weakening of the measurement strength should not affect the spectral characteristics of the unitary operator [46]. As η goes to zero, the according loss of information causes an increase of the expected first return time. Figure 5(d) suggests that the first hitting time after $N \rightarrow \infty$ measurements for Benzene, $\tau_{\infty}^B(t, \eta)$, holds:

$$\tau_{\infty}^B(t, \eta) = \frac{\tau_{\infty}^B(t)}{\eta} = \frac{1}{\eta} \begin{cases} 1, & t = 0 \\ 2, & t \in \left\{ \frac{2\pi}{3}, \pi, \frac{4\pi}{3} \right\} \\ 3, & t \in \left\{ \frac{\pi}{2}, \frac{3\pi}{2} \right\} \\ 4, & \text{else,} \end{cases}$$

where every t is modulo 2π .

C. Analytical result: The $\frac{1}{\eta}$ -decay of the first hitting time

Numerical computation showed that the leading order in which the expected return time decays as η goes from 1 to 0, behaves like $1/\eta$. In the limit of $N \rightarrow \infty$, the $1/\eta$ behavior is very sharp, which indicates $\tau_{\infty}(\eta, t) = \frac{1}{\eta} \tau_{\infty}(t, 1)$, where $\tau_{\infty}(t, 1)$ denotes the expected return time through strong, projection-valued measurements, i.e., $\eta = 1$. Recall that the mean recurrence time for rank one projections on localized initial states—as shown by Grünbaum *et al.* [15]—is always an integer or infinite.

This raises the question: Does such a scaling law hold true in general? Restricting to recurrent pairs $(U, |\psi\rangle)$ with an expected first return time $\tau_{\infty} \in \mathbb{N}$, which are monitored via the protocol in Fig. 1, we can give a positive answer. Under our weak-monitoring protocol—the weak first detected return

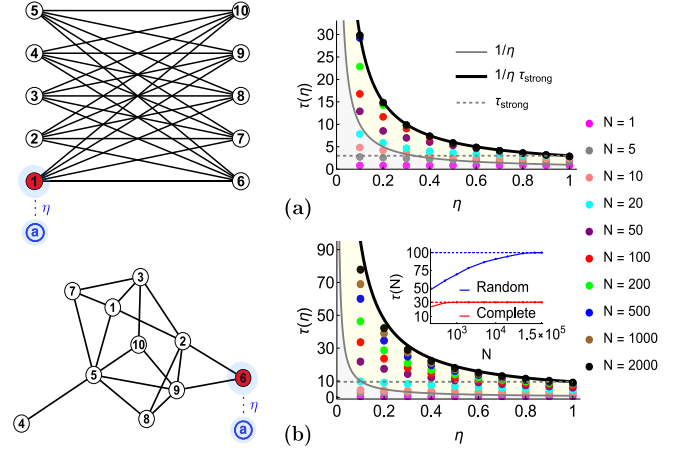


FIG. 7. More examples for the η^{-1} decay of the first hitting time with weak measurements: (a) complete bipartite graph with 10 vertices and (b) random graph with 10 vertices. The initial sites are marked in red. The blue surface indicates one qubit, which is coupled to one ancilla qubit. For $t = 0.9$, the projection-valued first hitting time τ_N^{strong} (horizontal dashed gray line) is compared with the weak first hitting time τ_N (25), according to Fig. 1, for several values of N up to $N = 2000$ for 10 values of η between $\eta = 0.1$ and $\eta = 1.0$. The strong hitting times are (a) $\tau_{\infty}^{\text{strong}} = 3$ and (b) $\tau_{\infty}^{\text{strong}} = 10$. The correlation in Eq. (40) is visible, although $\tau_N^{\text{weak}}(\eta)$ converges more slowly to $\frac{1}{\eta} \tau_N^{\text{strong}}$ for the random graph (b) compared with the complete bipartite graph (a). The yellow surface marks the range of expected first hitting times $\frac{1}{\eta} \leq \tau_{\infty}(\eta, t)$ (see Lemma A3). Inside (b), the convergence to $\frac{1}{\eta} \tau_{\infty}^{\text{strong}}$ is plotted for fixed $\eta = 0.1$.

time $\tau_{\infty}(\eta) = \frac{1}{\eta} \tau_{\infty}$ for $\eta \in (0, 1]$. Moreover, it is not relevant whether U is a continuous-time or a discrete-time unitary operator, the mere property of being unitary is sufficient. This is shown explicitly by the following Lemma:

Lemma 1. Let $\mathcal{H}$ be finite-dimensional and $(U_t, |\psi\rangle)$ be recurrent for $U \equiv U_t := e^{-itH}$ with recurrence time $\tau_{\infty} := \sum_{k=0}^{\infty} \|\tilde{U}^k |\psi\rangle\|^2 \in \mathbb{N}$. Here, the projection operators are $P := |\psi\rangle\langle\psi|$ and $Q := 1 - P$. Then, the weak first return time from (26), i.e., $\tau_{\infty}(\eta) := \sum_{k=0}^{\infty} \|\tilde{U}_{\eta}^k |\psi\rangle\|^2$, is given by

$$\tau_{\infty}(\eta) = \frac{1}{\eta} \tau_{\infty} \quad (40)$$

for all $\eta \in (0, 1]$.

The proof is in Appendix D. Several examples, which underscore Eq. (40), are shown in Figs. 7 and 8. Obviously, Eq. (40) is not restricted to fully localized initial states, as seen in Fig. 8.

In Fig. 7(a), we choose a complete graph (often applied in machine-learning setups) and a random graph in Fig. 7(b). For some fixed but arbitrary time $t = 0.9$ (which is not a fractional multiple of 2π), the first hitting time converges differently to the η^{-1} scaling for the more symmetric complete graph, rather than the random graph. In Fig. 7(a), the limiting scaling is already reached for $N = 500$, whereas in Fig. 7(b), for small values of η and $N = 2000$, the scaling is not yet met. Nevertheless, the lower bound $\frac{1}{\eta} \leq \tau_{\infty}(\eta, t)$ proved in Lemma A3 works well even for small N values ($N > 20$ for $\eta > 0.1$), as Figs. 7 and 8 reveal. Moreover, Fig. 8 shows

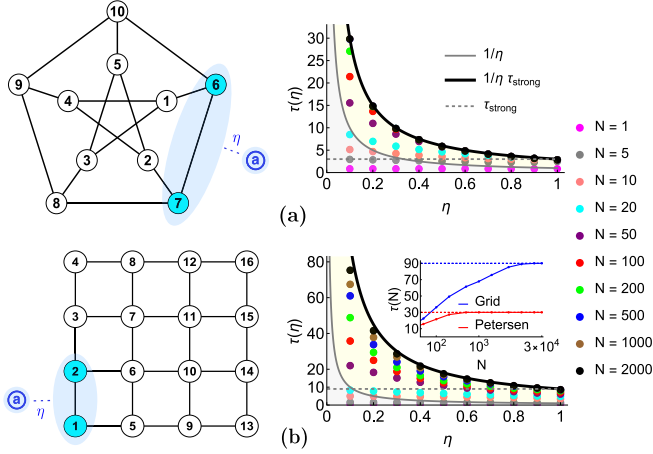


FIG. 8. Examples for the η^{-1} decay of the first hitting time with weak measurements for initial states that are a superposition of two sites: (a) Petersen graph with 10 vertices and (b) graph of a 4×4 grid. The sites that span the initial state are marked in cyan. The blue surface indicates one qubit, which is coupled to one ancilla qubit. The initial states are (a) $|\psi\rangle = \frac{1}{\sqrt{2}}(|6\rangle + |7\rangle)$ and (b) $|\psi\rangle = \frac{1}{\sqrt{2}}(|1\rangle + |2\rangle)$. For $t = 0.9$, the projection-valued first hitting time τ_N^{strong} (horizontal gray line) is compared with the weak first hitting time τ_N (25), following Fig. 1, for several values of N up to $N = 2000$ and for 10 values of η between $\eta = 0.1$ and $\eta = 1.0$. The strong hitting times are (a) $\tau_\infty^{\text{strong}} = 3$ and (b) $\tau_\infty^{\text{strong}} = 9$. The expected correlation from Eq. (40) is visible. The yellow surface marks the range of expected first hitting times $\frac{1}{\eta} \leq \tau_\infty(\eta, t)$ (see Lemma A3). In panel (b), the convergence to $\frac{1}{\eta} \tau_\infty^{\text{strong}}$ is plotted for fixed $\eta = 0.1$.

that the scaling of Eq. (40) is invariant under localized initial states: both the Petersen graph [Fig. 8(a)] (despite its role for several counterexamples in algebraic combinatorics [50]) and the grid graph [Fig. 8(b)] (commonly used for representing two-dimensional (2D) lattices in data science, molecular biology, or material science [51]) indicate convergence to $\eta^{-1} \tau_\infty$ for uniformly superposed initial states. As well as in Fig. 7, a smaller projection-valued first hitting time τ_N^{strong} in Fig. 8(a) causes faster convergence compared with a larger τ_N^{strong} in Fig. 8(b).

Numerical computations reveal that Eq. (40) is not restricted to pure quantum states but seem to hold true for arbitrary initial states—assumed that those are fully supported within the range of P . For a counterexample, consider a triangular graph consisting of three vertices, labeled— $i \in \{0, 1, 2\}$ and corresponding basis states $|i\rangle \in \mathbb{C}^3$. Consider the projection $P = |0\rangle\langle 0|$ onto site $|0\rangle$ and perform the weak measurement protocol with an initial state $\rho_0(\alpha, \beta) := \alpha|0\rangle\langle 0| + \beta|1\rangle\langle 1|$ such that $\text{Tr} \rho_0(\alpha, \beta) = 1$. Depending on the weight α , the obtained scaling is always larger than η^{-1} , but only up to slight variations. This observation is in correspondence with Lemma A3. Investigations toward the effect of the overlap of the initial state with the range of the measurement projector, as well as scaling laws for a broader class of quantum channels and quantum states, are part of ongoing research.

There are two physical intuitions behind Eq. (40). As already mentioned, $1/\eta$ is the expected value of a geometric

distribution arising from the events “detecting the quantum walker with probability η ” and “not detecting the quantum walker with probability $1 - \eta$.” However, the events do not decouple from the measurement observables, since there is also the event of measuring the quantum walker but losing it: there are states within the range of Q_η which have a $\sqrt{1 - \eta}$ overlap to P but only those states correspond to detected states, if and only if there is a nonzero overlap to Q (cf. Proposition A3).

Geometrically, $1/\eta$ corresponds to the curvature of the circle induced by the radius arising from the norm of the scaled success projector $\|P_\eta|\psi\rangle\|^2 = \eta$, which is one if and only if we measure strongly.

Preliminarily, we believe that the scaling law can be linked to the amount of entanglement between the control qubit and the ancilla. To illustrate, consider only one iteration and change the η values between the extreme points $\eta = 1$ and $\eta = 0$, respectively (cf. Proposition A2). $\eta = 1$ yields a Bell state (up to normalization) while $\eta = 0$ gives a separable state. Hence, the entanglement strength induced by η is inherited in the altered recurrence statistics. Conversely, provided that a universality of the hitting time under various initial states and quantum channels can be established, the recurrence statistics could reveal a computable connection to the underlying amount of entanglement between the ancilla and control qubit. Therefore, Fig. 1 could eventually be transferred to a possibly novel approach of entanglement determination, which raises an interesting question for future work.

The increase in the expected-first return time, which is proportional to $1/\eta$, gives rise to the question: which $\eta' \in (0, 1]$ characterizes the threshold between the quantum and the classical expected first return time? For the generic case of rings of size L , and for $N \rightarrow \infty$ measurements, we can give a simple answer:

$$\eta'(L) = \frac{1}{2} + \begin{cases} \frac{1}{L}, & \text{if } L \text{ is even} \\ \frac{1}{2L}, & \text{if } L \text{ is odd,} \end{cases} \quad (41)$$

which goes to $1/2$ as $L \rightarrow \infty$. This can be seen as follows: the classical return time for those rings was shown to be $\tau_\infty^{\text{cl}} = L$ [31]. The expected first return time through projection-valued measurements was derived in Ref. [31] and is given in (38). Based on (40), we can solve $\frac{1}{\eta} \tau_\infty^{\text{strong}} \stackrel{!}{=} \tau_\infty^{\text{cl}}$ for $\eta' \in (0, 1]$ and conclude (41). We emphasize that Eq. (40) remains valid, even if the time-reversal symmetry is broken, through the presence of a weak, homogeneous magnetic field [52]. For rings, these systems are modeled by a tight-binding Hamiltonian of the form [26] $H = -\sum_{j=0}^{L-1} (e^{i\alpha}|j\rangle\langle j+1| + e^{-i\alpha}|j+1\rangle\langle j|)$, where $e^{i\alpha}$, for $\alpha \neq 1$, is a Peierls phase [52] which represents the magnetic flux and that causes different degeneracies in the energy levels and hence affects the mean recurrence time [26].

The scaling law (40) begs the question of under what circumstances does its validity fail? As discussed, one restriction covers the support of the initial state: once the initial state cannot be fully recovered by the measurement operation, the scaling of τ_∞ is larger than η^{-1} . Moreover, its applicability is limited to systems that can be implemented according to Fig. 1. Specifically, if the unitary operator is not implementable, there are no recurrence properties observable independent of the type of monitoring.

Consequently, there are two canonical insights of our results. First, we have found a novel method for controlling the recurrence time of a quantum system by varying its measurement strength, while simultaneously preserving its recurrence properties. In this way, we are able to modify its dynamics in a controllable setting. Second, by employing a stochastic measurement method (coupling the quantum system to an environment and allowing for phase loss), we still find a deterministic recurrence property that scales inversely with the stochastic information loss, caused by the measurement. This second recognition especially underscores once more the beautifully stochastic yet controllable nature of quantum theory—an artifact of a supposedly never-ending journey of scientific discovery.

V. CONCLUSIONS

In this work, we analyzed the first hitting time problem for continuous-time quantum walks on graphs through weak measurements. The composite quantum system under investigation was chosen to be suitable for implementation on quantum computers. The graph is encoded into a finite qubit register and an ancilla qubit is connected to one of the system qubits. The weak measurement is implemented by a controlled rotation gate, whose reparameterized rotation angle indicates the measurement strength. This specific operator-valued measurement corresponds to quantum evolution through a phase-damping channel [33]. The monitoring process of the quantum walk is realized by an on-site measurement protocol at a constant rate. After an initially fixed time period, the quantum system is intercepted and weakly measured. Depending on the outcome of this measurement, the protocol either continues for no-click measurement, or is terminated for click measurement. By applying the theory of quantum state recurrence [15] and quantum subspace recurrence [22], we can compute the first hitting time depending on the rotation angle of the measurement. Both for finitely and infinitely many measurements, we can evaluate the weakly monitored first return time by utilizing the Hilbert space formalism without dilation [Eq. (26)]; a special case, which fails in general for operator-valued measurements.

Unlike in the seminal work of Grünbaum *et al.* [15], where the mean return time is strictly integer quantized, or infinite, in our setting, the mean return time is not quantized. Nevertheless, a quantized structure emerges in the obtained scaling law Eq. (40): $\tau_{\text{weak}} = \frac{1}{\eta} \tau_{\text{strong}}$, where $\eta = \sin^2(g)$ and g are the rotation angle inducing the strength of the measurement. For higher-dimensional pure quantum states, the η^{-1} scaling is also found to lower bound the weak return time: $1/\eta \leq \tau_{\text{weak}}$. However, the derived scaling (40) is not universal, but holds true only under weak monitoring through phase-damping channels, rank-one projections, and pure initial quantum states. Nevertheless, its universality remains to be precisely enclosed, and numerical computations suggest that similar scaling behavior may also arise for more general initial states. We emphasize, however, that this observation is preliminary and does not constitute a theorem. Relation (40) reveals a direct connection between recurrence properties under strong projective measurements and those under weak measurements through phase damping channels. While we

expect such a correspondence to generally hold for recurrence observables, its validity for other classes of observables remains an open question.

Moreover, a generalized scaling law enlightening recurrence properties for arbitrary mixed states would constitute a desirable result. Computational investigations reveal that the overlap from the statistical mixture of the initial states to the range of the measurement observable plays a relevant role.

The relation shows, as expected, that the weaker we measure, the longer it takes to detect the recurrence. Thus, on the one hand, weak measurements interfere less with the system. On the other hand, we need more measurements to detect the system in the target state.

Ultimately, the validity of the η^{-1} scaling law is restricted to either finite-dimensional Hilbert spaces (which naturally cover the scope of gate-based quantum computation) or abstract recurrent quantum systems, where a unitary operator and an initial state are known to be recurrent. On the contrary, its applicability to quantum mechanics in infinite-dimensional systems remains an open field of research. The recurrence assumption in infinite dimensions must be revisited so that a general scaling behavior might not be visible.

Consequently, several natural research questions arise. At first, can we find a similar dependence on the first hitting time by the measurement parameter if we use a different coupling, i.e., an XX coupling? If yes, is it also proportional to the inverse of the parameter? Second, a generalization to multiqubit weak measurements with various rotation angles provides a promising area of research with applications in designing measurement-induced quantum algorithms. Third, our implementation draws a bridge between recurrence and the rich literature on entanglement measures [53–55] and information theory of networks [56]. Studying the connection between the first hitting time and quantum nonlocality may provide further insights into the interplay between stochastic processes, entanglement theory, information theory, and foundational quantum mechanics. Finally, the analogous first transfer time problem may be studied within the same scope, both for phase-damping channels and more general quantum measurements.

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DATA AVAILABILITY

The data that support the findings of this article are not publicly available upon publication because it is not technically feasible and/or the cost of preparing, depositing, and

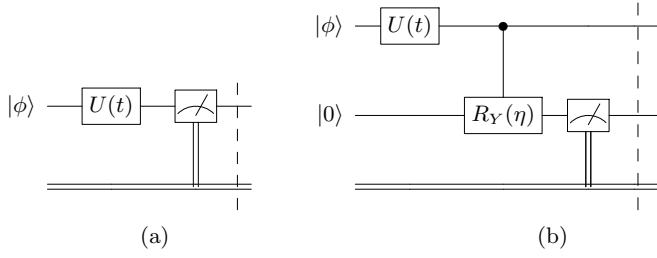


FIG. 9. (a) Strong and (b) weak measurements. (a) One iteration of the MCTQW with a projection-valued measurement on the system. We refer to this type of measurement as a *strong* measurement. (b) One iteration of the MCTQW with a weak measurement: projection-valued measurement on the ancilla system corresponds to a weak measurement on the quantum system. We refer to this type of measurement as a *weak* measurement.

hosting the data would be prohibitive within the terms of this research project. The data are available from the authors upon reasonable request.

APPENDIX A: RECURRENCE BY WEAK MEASUREMENTS

1. Encoding of the first return event

To implement monitored quantum walks on real quantum hardware, one must give a precise encoding of states into qubit-Hilbert spaces of dimension $d = 2^N$. The goal of such an embedding is to keep $N \in \mathbb{N}$ as small as possible in order to minimize computational resources. The following well-known block encoding ensures the usage of $\log d$ many qubits for computing first hitting times on rings.

$$|\Psi(\eta)\rangle = \begin{cases} |\phi'_\eta\rangle \otimes |1\rangle, & \text{if the particle has been detected, i.e., yes} \\ |\phi_\eta\rangle \otimes |0\rangle, & \text{if the particle has not been detected, i.e., no.} \end{cases} \quad (\text{A1})$$

Here, $|\phi'_\eta\rangle := (\frac{0}{\beta_t b(\eta)})$ and $|\phi_\eta\rangle := (\frac{\alpha_t}{\beta_t a(\eta)})$, where $a(\eta) := \sqrt{1-\eta}$ and $b(\eta) := \sqrt{\eta}$ originate from the weak measurement, for some $\eta \in (0, 1]$.

Proof. The weak measurement consists of two steps. First, we apply the controlled R_Y rotation on the ancilla qubit and second, we measure the ancilla in the computational basis. The controlled R_Y rotation applied to the state $|\psi_t\rangle \otimes |0\rangle_{\text{anc}}$ gives

$$\begin{aligned} CR_Y(\eta)(|\psi_t\rangle \otimes |0\rangle) \\ = (|0\rangle\langle 0| \otimes \mathbb{1} + |1\rangle\langle 1| \otimes R_Y(\eta))(|\psi_t\rangle \otimes |0\rangle) \\ = \alpha_t |0\rangle|0\rangle + \beta_t |1\rangle(a(\eta)|0\rangle + b(\eta)|1\rangle). \end{aligned}$$

We define the state after this rotation by $|\varphi_t\rangle := \alpha_t |0\rangle|0\rangle + \beta_t |1\rangle(a(\eta)|0\rangle + b(\eta)|1\rangle)$. Now, the measurement is applied to the ancilla qubit. Let $Q = |0\rangle\langle 0|$ and $P = |1\rangle\langle 1|$, so that we get

$$\begin{aligned} P|\varphi_t\rangle &= \beta_t b(\eta) |1\rangle \otimes |1\rangle, \\ Q|\varphi_t\rangle &= (\alpha_t |0\rangle + \beta_t a(\eta) |1\rangle) \otimes |0\rangle. \end{aligned}$$

Proposition A1. Consider the tight-binding Hamiltonian of the form (37) for rings of length $L \in \mathbb{N}$. If $k < L$ sites are to be weakly measured, then there exists an encoding which requires $N(L) = \lceil \log_2(L-k) \rceil + 2k$ many qubits.

Proof. Our goal is to keep the number of qubits $N \in \mathbb{N}$ as small as possible, where N is a function of the ring length L . Assuming we measure $k < L$ qubits weakly, we need an additional register of k ancilla qubits (one ancilla per site). We call the number of sites that we do not measure $n := L - k$, thus $L = n + k$. For the first n sites we can choose the canonical bit-to-qubit mapping to occupy n modes. That is, take the binary representation of each $i \in [n]$, i.e., $0 = |000 \dots 0_n 000 \dots 00\rangle$, $1 = |000 \dots 1_n 000 \dots 00\rangle$, $2 = |00 \dots 10_n 000 \dots 00\rangle$ and so on. To get the number of qubits needed for this, just compute $\lceil \log_2(n) \rceil = \lceil \log_2(L-k) \rceil$. Since there are $2k$ qubits left, the number of required qubits is $N(L) = \lceil \log_2(L-k) \rceil + 2k$. ■

State after the weak measurement

We consider a strong [Fig. 9(a)] and a weak measurement [Fig. 9(b)]. Let us consider the postmeasurement state. Let $|\psi(t)\rangle = U(t)|\phi\rangle_{\text{ctrl-qubit}}$ denote the t -dependent state after one iteration of $U(t)$ on the control qubit.

Proposition A2. Let

$$|\psi_t\rangle = \begin{pmatrix} \alpha_t \\ \beta_t \end{pmatrix}$$

be some arbitrary state on the control qubit after $U(t)$ has been applied. Then, the state after the null-type weak measurement on the ancilla qubit is a separable, subnormalized, two-qubit quantum state of the form

Note that the projector P corresponds to a YES measurement and Q to a NO measurement. By defining $|\phi'_\eta\rangle := \beta_t b(\eta) |1\rangle$ and $|\phi_\eta\rangle := \alpha_t |0\rangle + \beta_t a(\eta) |1\rangle$, we have the assertion. ■

The quantum circuits displaying the respective postmeasurement states are visualized in Fig. 10. The postmeasurement state is obtained by normalizing $P|\varphi_t\rangle$, respectively, $Q|\varphi_t\rangle$. Proposition 5 shows that there is no additional reset operation needed in the protocol. Mathematically, the ancilla remains in state $|0\rangle$ after every unsuccessful measurement.

2. Weak measurements: Subnormalized states and recurrence

a. Shear mapping of the projection operators

By Proposition A2, we can now say how the orthogonal projections are stretched to those linear mappings that represent the weak measurement. The transformation matrix from (17), $A_\eta := (\frac{\sqrt{\eta}}{\sqrt{1-\eta}}, 0; 1, 1)$ is singular if and only if $\eta = 0$. For $\eta = 1$, $A_1 = \mathbb{1}_2$ and all other values $0 < \eta < 1$ induce a shear mapping of the observables. Since the weak measurement is unitarily equivalent to a phase-flip channel, a picture of how

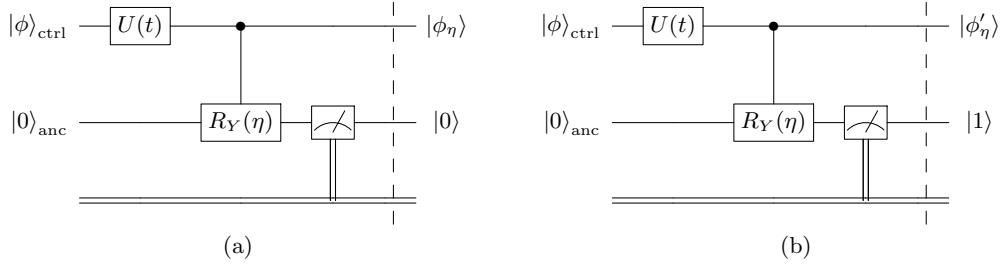


FIG. 10. Quantum state before and after it is measured weakly. (a) State after the “NO” event: $|\Psi(\eta)\rangle = |\phi_\eta\rangle \otimes |0\rangle$. (b) State after the “YES” event $|\Psi(\eta)\rangle = |\phi'_\eta\rangle \otimes |1\rangle$.

the above mapping acts on qubits is seen, i.e., in Chap. 8 from Nielsen and Chuang [33].

It is recaptured that the encoding is invariant under any unitary basis transformation. Let B denote a unitary complex 2×2 matrix, which transforms the computational basis, i.e.,

$$P \mapsto B^\dagger P B =: P', \quad Q \mapsto B^\dagger Q B =: Q'.$$

Then, using Eq. (17),

$$\begin{aligned} P_\eta &\mapsto B^\dagger P_\eta B = \sqrt{\eta} B^\dagger P B = \sqrt{\eta} P' \equiv P'_\eta, \\ Q_\eta &\mapsto B^\dagger Q_\eta B = B^\dagger (Q + \sqrt{1-\eta} P) B \\ &= B^\dagger Q B + \sqrt{1-\eta} B^\dagger P B \\ &= Q' + \sqrt{1-\eta} P' \equiv Q'_\eta. \end{aligned} \quad (\text{A2})$$

An important application is, when the diagonal basis $\{|+\rangle, |-\rangle\}$ is chosen. In this case, $B = H$, where H is the Hadamard gate so that $|0\rangle \mapsto H|0\rangle =: |+\rangle$ and $|1\rangle \mapsto H|1\rangle =: |-\rangle$. Explicitly, $H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$, yielding with Eq. (A2)

$$\begin{aligned} P'_\eta &= \frac{1}{2} \begin{pmatrix} \sqrt{\eta} & -\sqrt{\eta} \\ -\sqrt{\eta} & \sqrt{\eta} \end{pmatrix}, \\ Q'_\eta &= \frac{1}{2} \begin{pmatrix} 1 + \sqrt{1-\eta} & 1 - \sqrt{1-\eta} \\ 1 - \sqrt{1-\eta} & 1 + \sqrt{1-\eta} \end{pmatrix}. \end{aligned}$$

Definition A1. Consider $S_V^{\sqrt{\eta}} = \{|\psi\rangle \in V : \|\psi\| = \sqrt{\eta} \forall \eta \in (0, 1]\}$. The orthogonal projection $P : \mathcal{H} \rightarrow V$ gives a linear (but not projection-valued) mapping $P_\eta : \mathcal{H} \rightarrow V$, $P_\eta := \sqrt{\eta} P$. Similarly, the mapping Q_η is given by (17), i.e., $Q_\eta = Q + \sqrt{1-\eta} P$.

Proposition A3. Consider $S_V^{\sqrt{1-\eta}} = \{|\psi\rangle \in V : \|\psi\| = \sqrt{1-\eta}\}$ and $S_V^{\sqrt{\eta}} = \{|\psi\rangle \in V : \|\psi\| = \sqrt{\eta}\}$.

- (i) Let $|\psi\rangle \in S_V$. Then, $P_\eta|\psi\rangle \in S_V^{\sqrt{\eta}}$ and $Q_\eta|\psi\rangle \in S_V^{\sqrt{1-\eta}}$.
- (ii) Let $|\psi\rangle \in S_{V^\perp}$. Then, $P_\eta|\psi\rangle = 0$ and $Q_\eta|\psi\rangle \in S_{V^\perp}$.

Proof. The statements follow immediately by definition of P_η and Q_η :

- (i) Let $|\psi\rangle \in S_V$. Then $P_\eta|\psi\rangle = \sqrt{\eta}|\psi\rangle \in S_V^{\sqrt{\eta}}$. Moreover, $Q_\eta|\psi\rangle = \sqrt{1-\eta}|\psi\rangle \in S_V^{\sqrt{1-\eta}}$.
- (ii) Let $|\psi\rangle \in S_{V^\perp}$. Then $P_\eta|\psi\rangle = \sqrt{\eta}P|\psi\rangle = 0$. Moreover, $Q_\eta|\psi\rangle = Q|\psi\rangle = |\psi\rangle \in S_{V^\perp}$. ■

b. Weak recurrence

For shorter notation, we sometimes abbreviate $U := U(t)$ and $\tilde{U}_\eta := \tilde{U}_\eta(t)$. It is mentioned that any unitary operator U depends continuously on time $t \in \mathbb{R}$, i.e., $U \equiv U_t = U(t) = e^{-iHt}$.

Lemma A1. Let $|\psi\rangle \in S_V$ be an initial state and $U(t)$ be a unitary operator. Then, after measuring $N \in \mathbb{N}$ times, the η -dependent return probability to V , denoted by $R_N(\eta, t)$, and the expected first return time, denoted by $\tau_N(\eta, t)$, are

$$\begin{aligned} R_N(\eta, t) &= 1 - \|\tilde{U}_\eta(t)^N |\psi\rangle\|^2, \\ \tau_N(\eta, t) &= \frac{\sum_{k=0}^{N-1} \|\tilde{U}_\eta(t)^k |\psi\rangle\|^2 - N \|\tilde{U}_\eta(t)^N |\psi\rangle\|^2}{1 - \|\tilde{U}_\eta(t)^N |\psi\rangle\|^2}. \end{aligned}$$

Proof. We follow the procedure of Ref. [22] and compute an analogous decomposition of the form $\hat{a}_n + \tilde{U}^n P = U \tilde{U}^{n-1} P$. Naturally, the “pseudo-orthogonality” of (18) is inherited in such a decomposition for weak measurements. For $|\psi\rangle \in S_V$, Proposition A3 yields $P_\eta|\psi\rangle \in S_V^{\sqrt{\eta}}$. The η -dependent quantities become (compare with Definition A1)

$$P \mapsto P_\eta, \quad \tilde{U} \mapsto \tilde{U}_\eta := Q_\eta U, \quad \hat{a}_n \mapsto \hat{a}_n^\eta := P_\eta U \tilde{U}_\eta^{n-1} P.$$

Recall that the projector on the right-hand side controls that $|\psi\rangle \in \text{ran } P$. This is not a measurement after the zeroth step, so that there is no η term arising here. The modified decomposition reads

$$\hat{a}_n^\eta + \tilde{U}_\eta^n P = [\mathbb{1} + f(\eta)P] U \tilde{U}_\eta^{n-1} P. \quad (\text{A3})$$

Details are presented in Appendix B 1. To transfer the construction from Ref. [22], we have to show that the telescoping series for the return probability is preserved in the case of measuring weakly. Indeed, one computes (see Appendix B 1)

$$p_n^\eta(t) := \|\hat{a}_n^\eta(t)|\psi\rangle\|^2 = \|\tilde{U}_\eta(t)^{n-1} |\psi\rangle\|^2 - \|\tilde{U}_\eta(t)^n |\psi\rangle\|^2. \quad (\text{A4})$$

By definition of the overall V -return probability [22] and the above telescoping sum,

$$\begin{aligned} R_N(\eta, t) &:= \sum_{n=1}^N p_n^\eta(t) = \sum_{n=1}^N \|\hat{a}_n^\eta(t)|\psi\rangle\|^2 \\ &= 1 - \|\tilde{U}_\eta(t)^N |\psi\rangle\|^2, \end{aligned}$$

which shows (22). We insert the above expression for the expected first V -return time [15], i.e., $\tau_N = \sum_{n=1}^N n p_n / R_N$, so

that we have for $\tau_N(\eta, t)$:

$$\begin{aligned}\tau_N(\eta, t) &= \frac{\sum_{n=1}^N n \|\hat{a}_n^\eta(t)|\psi\rangle\|^2}{\sum_{n=1}^N \|\hat{a}_n^\eta(t)|\psi\rangle\|^2} \\ &= \frac{\sum_{k=0}^{N-1} \|\tilde{U}_\eta(t)^k|\psi\rangle\|^2 - N\|\tilde{U}_\eta(t)^N|\psi\rangle\|^2}{1 - \|\tilde{U}_\eta(t)^N|\psi\rangle\|^2}.\end{aligned}$$

Corollary A1. Let $|\psi\rangle \in S_V$ be V -recurrent, i.e., $R_\infty(t) = 1$. Then, $|\psi\rangle$ is also V -recurrent in the weak measurement protocol, i.e., $R_\infty(\eta, t) = 1$, for all $\eta \in (0, 1]$.

Proof. For $\eta = 1$, the statement is trivial. Let $\eta \in (0, 1)$. Then, we compute:

$$\begin{aligned}R_\infty(\eta, t) &= \lim_{N \rightarrow \infty} R_N(\eta, t) = \lim_{N \rightarrow \infty} (1 - \|\tilde{U}_\eta(t)^N|\psi\rangle\|^2) \\ &= 1 - \underbrace{\lim_{N \rightarrow \infty} \|\tilde{U}_\eta(t)^N|\psi\rangle\|^2}_{\eta > 0} = 1.\end{aligned}$$

The fact that $\lim_{N \rightarrow \infty} \|\tilde{U}_\eta(t)^N|\psi\rangle\|^2 = 0$ holds true is seen as follows: For $\eta \in (0, 1]$, we have for any unit vectors $|\psi\rangle$ and $|\psi'\rangle := U|\psi\rangle$,

$$\begin{aligned}0 \leq \|\tilde{U}_\eta|\psi\rangle\|^2 &= \langle\psi|\tilde{U}_\eta^\dagger\tilde{U}_\eta|\psi\rangle = \langle\psi'|\mathbb{1} - \eta P|\psi'\rangle \\ &= 1 - \eta\langle\psi'|PP|\psi'\rangle \leq 1,\end{aligned}\quad (\text{A5})$$

Proof. As usual, $\eta \in (0, 1]$. We distinguish between two cases. First, $\eta = 1$ and second, $0 < \eta < 1$. In the first case, we always have to recover the projection-valued quantities from [22]. Due to $\tilde{U}_\eta := Q_\eta U = (Q + \sqrt{1 - \eta}P)U$, we have $\tilde{U}_1 = QU \equiv \tilde{U}$. Thus, $R_N(t, 1) = 1 - \|\tilde{U}_1(t)^N|\psi\rangle\|^2 = 1 - \|\tilde{U}(t)^N|\psi\rangle\|^2 \equiv R_N(t)$. We take the limit $N \rightarrow \infty$ and have $R_\infty(t) = 1 - \lim_{N \rightarrow \infty} \|\tilde{U}(t)^N|\psi\rangle\|^2$.

Now, let $0 < \eta < 1$. For two positive (diagonal) operators Λ_1 and Λ_2 , $N \in \mathbb{N}$, it holds true that

$$\|((\Lambda_1 + \Lambda_2)U)^N|\psi\rangle\| \geq \|(\Lambda_1 U)^N|\psi\rangle\|,$$

$$|\psi\rangle \in S_V \implies \|\tilde{U}_\eta^N|\psi\rangle\|^2 = \|(Q + \sqrt{1 - \eta}P)U\|^N \geq \|(QU)^N|\psi\rangle\|^2 = \|\tilde{U}^N|\psi\rangle\|^2.$$

Consequently,

$$R_N(\eta, t) = 1 - \|\tilde{U}_\eta(t)^N|\psi\rangle\|^2 \leq 1 - \|\tilde{U}(t)^N|\psi\rangle\|^2 \equiv R_N(t).$$

The monotonicity of taking the limit proves the assertion. $\blacksquare$

Corollary A3. Let $\tau_N(\eta, t)$ denote the expected V first return time of an initial state $|\psi\rangle \in S_V$ after N weak measurements with coupling strength $\eta \in (0, 1]$. Then, $\tau_N(\eta, t) \geq \tau_N(1, t) \equiv \tau_N(t)$. Clearly, also $\tau_\infty(\eta, t) \geq \tau_\infty(1, t) \equiv \tau_\infty(t)$, where $\tau_\infty(\eta, t) := \lim_{N \rightarrow \infty} \tau_N(\eta, t)$.

Proof. We apply Corollary A2 and note that, for all $n \geq 1$,

$$\begin{aligned}\|\tilde{U}_1^n|\psi\rangle\|^2 &= \|\tilde{U}^n|\psi\rangle\|^2 = \|(QU)^n|\psi\rangle\|^2 \leq \|((Q + \sqrt{1 - \eta}P)U)^n|\psi\rangle\|^2 \leq 1, \\ &\implies 1 - \|\tilde{U}^n|\psi\rangle\|^2 \geq 1 - \|\tilde{U}_\eta^n|\psi\rangle\|^2.\end{aligned}$$

From this, and $\|\tilde{U}_\eta^k|\psi\rangle\| \geq \|\tilde{U}^k|\psi\rangle\|$ for all $k \geq 1$ (cf. Corollary A2), it follows that

$$\begin{aligned}\tau_N(\eta, t) &= \frac{\sum_{k=0}^{N-1} \|\tilde{U}_\eta(t)^k|\psi\rangle\|^2 - N\|\tilde{U}_\eta(t)^N|\psi\rangle\|^2}{1 - \|\tilde{U}_\eta(t)^N|\psi\rangle\|^2} \geq \frac{\sum_{k=0}^{N-1} \|\tilde{U}(t)^k|\psi\rangle\|^2 - N\|\tilde{U}(t)^N|\psi\rangle\|^2}{1 - \|\tilde{U}(t)^N|\psi\rangle\|^2} \\ &\geq \frac{\sum_{k=0}^{N-1} \|\tilde{U}(t)^k|\psi\rangle\|^2 - N\|\tilde{U}(t)^N|\psi\rangle\|^2}{1 - \|\tilde{U}_\eta(t)^N|\psi\rangle\|^2} \geq \frac{\sum_{k=0}^{N-1} \|\tilde{U}(t)^k|\psi\rangle\|^2 - N\|\tilde{U}(t)^N|\psi\rangle\|^2}{1 - \|\tilde{U}(t)^N|\psi\rangle\|^2} \stackrel{\text{def.}}{=} \tau_N(t).\end{aligned}$$

The monotonicity of taking the limit proves the assertion. $\blacksquare$

since P is a projector (i.e., $P = P^2$). Moreover, Eq. (A5) yields $\langle\psi|\tilde{U}_\eta^\dagger\tilde{U}_\eta|\psi\rangle = 1$, if and only if $P|\psi'\rangle = PU|\psi\rangle = 0$. Focusing on this case, we define $|\tilde{\psi}\rangle := \tilde{U}_\eta^{N-1}|\psi\rangle$ and assume that $\|\tilde{U}_\eta|\tilde{\psi}\rangle\|^2 = \langle\tilde{\psi}|\tilde{U}_\eta^\dagger\tilde{U}_\eta|\tilde{\psi}\rangle = 1$, for some initial state $|\psi\rangle \in S_V$. Then,

$$\begin{aligned}1 &= \langle\psi|\tilde{U}_\eta^\dagger\tilde{U}_\eta^N|\psi\rangle = \langle\psi|\tilde{U}_\eta^\dagger\tilde{U}_\eta^{N-1}\tilde{U}_\eta|\psi\rangle \\ &= \langle\tilde{\psi}|\tilde{U}_\eta^\dagger\tilde{U}_\eta|\tilde{\psi}\rangle = \langle\tilde{\psi}|U^\dagger(\mathbb{1} - \eta P)U|\tilde{\psi}\rangle \\ &= 1 - \eta\langle\tilde{\psi}|U^\dagger PU|\tilde{\psi}\rangle\end{aligned}$$

implies that $PU|\tilde{\psi}\rangle = 0$. Thus, when $|\tilde{\psi}\rangle$ is a unit vector and $|\psi\rangle \in S_V$ is an initial state,

$$|\langle\psi|U|\tilde{\psi}\rangle|^2 = |\langle\psi|PU|\tilde{\psi}\rangle|^2 = |\langle\psi|PU\tilde{U}_\eta^{N-1}|\psi\rangle|^2 = 0,$$

which implies that the return probability $|\langle\psi|\hat{a}_N^\eta|\psi\rangle|^2 := |\langle\psi|P_\eta U\tilde{U}_\eta^{N-1}|\psi\rangle|^2 = \eta|\langle\psi|PU\tilde{U}_\eta^{N-1}|\psi\rangle|^2 = 0$. Hence, we conclude that $\|\tilde{\psi}\rangle\| = \|\tilde{U}_\eta^{N-1}|\psi\rangle\| < 1$ for all initial states $|\psi\rangle \in S_V$ and $N \geq 2$, i.e., for $M := N - 1$, $\lim_{M \rightarrow \infty} \|\tilde{U}_\eta(t)^M|\psi\rangle\|^2 = 0$.

Note that, in the case of $\eta = 0$ (full decoupling), $\lim_{N \rightarrow \infty} \|\tilde{U}_\eta(t)^N|\psi\rangle\|^2 = \lim_{N \rightarrow \infty} \|U(t)^N|\psi\rangle\|^2 = 1$, i.e., $R_\infty(0, t) = 0$, coinciding with the intuition that no state is recurrent, when nothing is being measured. $\blacksquare$

Corollary A2. Let $R_N(\eta, t)$ denote the V -return probability of an initial state $|\psi\rangle \in S_V$ after N weak measurements with coupling strength $\eta \in (0, 1]$. Then, $R_N(\eta, t) \leq R_N(1, t) \equiv R_N(t)$. Clearly, also $R_\infty(\eta, t) \leq R_\infty(1, t) \equiv R_\infty(t)$, where $R_\infty(\eta, t) := \lim_{N \rightarrow \infty} R_N(\eta, t)$.

Lemma A2. Let $|\psi\rangle \in S_V$. For $t = 0$ and for all $\eta \in (0, 1]$, the expected return time $\tau_N(\eta, t = 0)$ is given by

$$\tau_N(\eta, 0) = \frac{1}{\eta} - \frac{N(1-\eta)^N}{1 - (1-\eta)^N}.$$

Especially, $\tau_\infty(\eta, 0) = 1/\eta$. The rate, of which $\tau_N(\eta, 0)$ converges point wise to $1/\eta$ is $c(\eta) = 1 - \eta$.

Proof. For $t = 0$, $U(0) = \mathbb{1}$. As mentioned in the text, if $t = 0$, clearly $\tau_N(1, 0) = 1$. So let $\eta \in (0, 1)$. Since there is no evolution at all, $\tilde{U}_\eta \equiv \tilde{\mathbb{1}}_\eta = [(Q + \sqrt{1-\eta}P)\mathbb{1}]$, which implies $\tilde{\mathbb{1}}_\eta^k = Q + (\sqrt{1-\eta})^k P$ due to P, Q being projection operators. Then, for all $|\psi\rangle \in S_V$, $\|\tilde{\mathbb{1}}_\eta^k |\psi\rangle\|^2 = (1-\eta)^k$. Inserting this in Eq. (25) gives

$$\tau_N(\eta, 0) = \frac{\sum_{k=0}^{N-1} (1-\eta)^k - N(1-\eta)^N}{1 - (1-\eta)^N}.$$

Since $|1-\eta| < 1$, we can simply use the geometric sum $\sum_{k=0}^{N-1} q^k = \frac{1-q^N}{1-q}$ ($|q| < 1$), from which we get

$$\tau_N(\eta, t = 0) = \frac{\frac{1-(1-\eta)^N}{1-(1-\eta)} - N(1-\eta)^N}{1 - (1-\eta)^N} = \frac{1}{\eta} - \frac{N(1-\eta)^N}{1 - (1-\eta)^N}.$$

Taking the limit $N \rightarrow \infty$ yields $\tau_\infty(\eta, 0) = 1/\eta$.

The rate of convergence is defined by $c(\eta) := \lim_{N \rightarrow \infty} \frac{|\tau_{N+1}(\eta, 0) - \tau_\infty(\eta, 0)|}{|\tau_N(\eta, 0) - \tau_\infty(\eta, 0)|}$. A simple computation yields

$$\begin{aligned} c(\eta) &= \lim_{N \rightarrow \infty} \frac{\left| \frac{(N+1)(1-\eta)^{N+1}}{1-(1-\eta)^{N+1}} \right|}{\left| \frac{N(1-\eta)^N}{1-(1-\eta)^N} \right|} \\ &= \lim_{N \rightarrow \infty} \frac{(N+1)(1-\eta)^{N+1}[1 - (1-\eta)^N]}{N(1-\eta)^N[1 - (1-\eta)^{N+1}]} \\ &= \lim_{N \rightarrow \infty} \frac{(1-\eta + \frac{1-\eta}{N})[1 - (1-\eta)^N]}{1 - (1-\eta)^{N+1}} = 1 - \eta. \end{aligned}$$

The subsequent Lemma shows that $\tau_\infty(\eta, t)$ is lower-bounded by $1/\eta$ for all $t \in \mathbb{R}$. ■

Lemma A3. Let $t \in \mathbb{R}$ and $\eta \in (0, 1]$. Then,

$$\frac{1}{\eta} \leq \tau_\infty(\eta, t). \quad (\text{A6})$$

Proof. Let $k \in \mathbb{N}$. We define for $|\psi_k(\eta, t)\rangle := (Q_\eta U_t)^k |\psi\rangle$, $\psi_k(\eta, t) := \|\psi_k(\eta, t)\|^2$ and $q_k(\eta, t) := \|PU_t |\psi_k(\eta, t)\rangle\|^2$. We show the claim in three steps:

(i) At first, we show that $q_k \leq \psi_k$.

Due to the orthogonal projections P and Q , any $v \in \mathcal{H}$, is uniquely decomposed as $v = Pv + Qv$, which gives $\|v\|^2 = \|Pv\|^2 + \|Qv\|^2$ and consequently $\|Pv\|^2 \leq \|v\|^2$. Take any unitary operator U on $\mathcal{H}$, so that we have $\|P(Uv)\|^2 \leq \|Uv\|^2 = \|v\|^2$. Therefore, $q_k(\eta, t) := \|P(U_t |\psi_k(\eta, t)\rangle)\|^2 \leq \|\psi_k(\eta, t)\|^2 = \psi_k(\eta, t)$. Abbreviating the η - and t -dependencies, this shows that $q_k \leq \psi_k$.

(ii) Second, $\psi_{k+1} = \psi_k - \eta q_k$.

The (pseudo-) orthogonal decomposition through Q_η also gives rise to the following: $\|\psi_{k+1}\|^2 = \|(Q_\eta U_t)^{k+1} |\psi\rangle\|^2 = \|(Q + \sqrt{1-\eta}P)U_t(Q_\eta U_t)^k |\psi\rangle\|^2 = \|(Q + \sqrt{1-\eta}P)U_t |\psi_k(\eta, t)\rangle\|^2 = \|QU_t |\psi_k(\eta, t)\rangle\|^2 + \|\sqrt{1-\eta}PU_t |\psi_k(\eta, t)\rangle\|^2 = \|(\mathbb{1} - P)U_t |\psi_k(\eta, t)\rangle\|^2 + \|\sqrt{1-\eta}PU_t |\psi_k(\eta, t)\rangle\|^2 = \|U_t |\psi_k(\eta, t)\rangle\|^2 - (1 - (\sqrt{1-\eta})^2) \|PU_t |\psi_k(\eta, t)\rangle\|^2 = \|\psi_k(\eta, t)\|^2 - \eta \|PU_t |\psi_k(\eta, t)\rangle\|^2 \stackrel{\text{def.}}{=} \psi_k - \eta q_k$.

(iii) $1/\eta \leq \tau_\infty(\eta, t)$.

Combining (i) and (ii), we find that $\psi_{k+1} = \psi_k - \eta q_k \geq \psi_k - \eta \psi_k = (1-\eta)\psi_k$. Iterating this over $k \in \mathbb{N}$, we get $\psi_k \geq (1-\eta)^k \psi_0 = (1-\eta)^k$, because of $\psi_0 = 1$ by construction.

Hence, $(1-\eta)^k \psi_0 \leq \psi_k$, so that we can lower bound, due to monotonicity and Lemma A2, the following partial sum: $s_N := \sum_{k=0}^N \psi_k \geq \sum_{k=0}^N \psi_0 (1-\eta)^k \stackrel{\psi_0=1}{=} \frac{1-(1-\eta)^{N+1}}{1-(1-\eta)}$. Since $\tau_\infty \equiv s_\infty = \lim_{N \rightarrow \infty} s_N = \lim_{N \rightarrow \infty} \sum_{k=0}^N \psi_k$ we conclude by monotonicity and writing the η - and t -dependencies: $\tau_\infty(\eta, t) \geq 1/\eta$. ■

It is worth mentioning, that the choice of a continuous-time unitary operator does not affect the generality of the proof. The reasoning works equivalently for all unitary operators on a finite-dimensional Hilbert space.

APPENDIX B: DETAILS OF PROOFS

1. Lemma A1

To show Eq. (A3), we start by computing the left side of (A3) and conclude the right.

$$\begin{aligned} \hat{a}_\eta^n + \tilde{U}_\eta^n P &= P_\eta U \tilde{U}_\eta^{n-1} P + \tilde{U}_\eta^n P = P_\eta U \tilde{U}_\eta^{n-1} P + \tilde{U}_\eta \tilde{U}_\eta^{n-1} P = (P_\eta U + \tilde{U}_\eta) \tilde{U}_\eta^{n-1} P \\ &= (P_\eta U + Q_\eta U) \tilde{U}_\eta^{n-1} P = (P_\eta + Q_\eta) U \tilde{U}_\eta^{n-1} P = (\sqrt{\eta}P + (\mathbb{1} - P) + \sqrt{1-\eta}P) U \tilde{U}_\eta^{n-1} P \\ &= (\mathbb{1} + f(\eta)P) U \tilde{U}_\eta^{n-1} P, \end{aligned}$$

where

$$f : (0, 1] \rightarrow [0, \sqrt{2} - 1], \quad f(\eta) := \sqrt{\eta} + \sqrt{1-\eta} - 1. \quad (\text{B1})$$

Moreover, Eq. (A4) is shown. From the above pseudo-orthogonal decomposition, one writes

$$\hat{a}_\eta^n = U \tilde{U}_\eta^{n-1} P - \tilde{U}_\eta^n P + f(\eta) P U \tilde{U}_\eta^{n-1} P.$$

For $|\psi\rangle \in S_V$, we have by construction $P|\psi\rangle = |\psi\rangle$. Thus,

$$\hat{a}_\eta^n |\psi\rangle = \underbrace{U\tilde{U}_\eta^{n-1}|\psi\rangle - \tilde{U}_\eta^n |\psi\rangle}_{=:a} + \underbrace{f(\eta)PU\tilde{U}_\eta^{n-1}|\psi\rangle}_{=:b},$$

so that $\|\hat{a}_\eta^n |\psi\rangle\|^2$ is of the form

$$\|\hat{a}_\eta^n |\psi\rangle\|^2 = \|a + b\|^2 = \|a\|^2 + \|b\|^2 + 2\text{Re}\langle a, b\rangle.$$

Here, we have defined $a := (U\tilde{U}_\eta^{n-1} - \tilde{U}_\eta^n)|\psi\rangle$ and $b := f(\eta)PU\tilde{U}_\eta^{n-1}|\psi\rangle$. Hence, we compute separately the subsequent three expressions: $\|a\|^2$, $\|b\|^2$, and $2\text{Re}\langle a, b\rangle$.

The results are

- (1) $\|a\|^2 = -\|\tilde{U}_\eta^{n-1}|\psi\rangle\|^2 + \|\tilde{U}_\eta^n |\psi\rangle\|^2 + 2\frac{g(\eta)}{\eta}\|\hat{a}_\eta^n |\psi\rangle\|^2$,
- (2) $\|b\|^2 = \frac{f^2(\eta)}{\eta}\|\hat{a}_\eta^n |\psi\rangle\|^2$,
- (3) $2\text{Re}\langle a, b\rangle = \frac{2g(\eta)f(\eta)}{\eta}\|\hat{a}_\eta^n |\psi\rangle\|^2$.

Details are presented in Sec. B 2. For simplicity, we define $h(\eta) := \frac{1}{\eta}[2g(\eta) + f^2(\eta) + 2g(\eta)f(\eta)] = 2$. Then, we find by combining all three expressions:

$$\begin{aligned} \|\hat{a}_\eta^n |\psi\rangle\|^2 &= -\|\tilde{U}_\eta^{n-1}|\psi\rangle\|^2 + \|\tilde{U}_\eta^n |\psi\rangle\|^2 + 2\frac{g(\eta)}{\eta}\|\hat{a}_\eta^n |\psi\rangle\|^2 + \frac{f^2(\eta)}{\eta}\|\hat{a}_\eta^n |\psi\rangle\|^2 + \frac{2g(\eta)f(\eta)}{\eta}\|\hat{a}_\eta^n |\psi\rangle\|^2 \\ &= -\|\tilde{U}_\eta^{n-1}|\psi\rangle\|^2 + \|\tilde{U}_\eta^n |\psi\rangle\|^2 + h(\eta)\|\hat{a}_\eta^n |\psi\rangle\|^2 = -\|\tilde{U}_\eta^{n-1}|\psi\rangle\|^2 + \|\tilde{U}_\eta^n |\psi\rangle\|^2 + 2\|\hat{a}_\eta^n |\psi\rangle\|^2, \\ \iff \|\hat{a}_\eta^n |\psi\rangle\|^2 &= \|\tilde{U}_\eta^{n-1}|\psi\rangle\|^2 - \|\tilde{U}_\eta^n |\psi\rangle\|^2. \end{aligned}$$

2. A few details

a. Details to B 1

(1) Such an expression was already computed in Ref. [22]. It corresponds to the $\eta = 1$ projection-valued measurement. However, we repeat the computation to follow closely what will change in the case of our weak measurement protocol. The result is

$$\begin{aligned} \|a\|^2 &= \|(U\tilde{U}_\eta^{n-1} - \tilde{U}_\eta^n)|\psi\rangle\|^2 = \|U\tilde{U}_\eta^{n-1}|\psi\rangle\|^2 + \|\tilde{U}_\eta^n |\psi\rangle\|^2 - 2\text{Re}\langle U\tilde{U}_\eta^{n-1}|\psi\rangle, \tilde{U}_\eta^n |\psi\rangle \\ &= \|U\|^2\|\tilde{U}_\eta^{n-1}|\psi\rangle\|^2 + \|\tilde{U}_\eta^n |\psi\rangle\|^2 - 2\text{Re}\langle U\tilde{U}_\eta^{n-1}|\psi\rangle, \tilde{U}_\eta^n |\psi\rangle \\ &= \|\tilde{U}_\eta^{n-1}|\psi\rangle\|^2 + \|\tilde{U}_\eta^n |\psi\rangle\|^2 - 2\text{Re}\langle U\psi', \tilde{U}_\eta^n \psi' \rangle, \end{aligned}$$

where we have defined $|\psi'\rangle := \tilde{U}_\eta^{n-1}|\psi\rangle$.

We now compute $\text{Re}\langle U\psi', \tilde{U}_\eta^n \psi' \rangle$. Recall that $Q_\eta = Q + \sqrt{1-\eta}P = \mathbb{1} - P + \sqrt{1-\eta}P = \mathbb{1} - (1 - \sqrt{1-\eta})P =: [\mathbb{1} - g(\eta)]P$ for $g : (0, 1] \rightarrow (0, 1]$, $g(\eta) := 1 - \sqrt{1-\eta}$.

$$\begin{aligned} \langle U\psi', \tilde{U}_\eta^n \psi' \rangle &= \langle U\psi', (Q_\eta U)\psi' \rangle = \langle \psi', U^*(Q_\eta U)\psi' \rangle \\ &= \langle \psi', U^*(\mathbb{1} - g(\eta)P)U\psi' \rangle = \langle \psi', (U^*U - U^*g(\eta)PU)\psi' \rangle = \langle \psi', (\mathbb{1} - U^*g(\eta)PU)\psi' \rangle \\ &= \langle \psi', \psi' \rangle - g(\eta)\langle \psi', U^*PU\psi' \rangle = \langle \psi', \psi' \rangle - g(\eta)\langle \psi', U^*P^*PU\psi' \rangle \\ &= \langle \psi', \psi' \rangle - g(\eta)\langle \psi', (PU)^*PU\psi' \rangle = \langle \psi', \psi' \rangle - g(\eta)\langle PU\psi', PU\psi' \rangle \\ &= \|\psi'\|^2 - g(\eta)\langle PU\psi', PU\psi' \rangle. \end{aligned}$$

Since $|\psi\rangle \in S_V$ and $\eta > 0$, $|\psi\rangle = \frac{1}{\sqrt{\eta}}(\sqrt{\eta}P|\psi\rangle)$. Using this identity, one computes

$$\begin{aligned} \langle PU\psi', PU\psi' \rangle &= \langle PU\tilde{U}_\eta^{n-1}\psi, PU\tilde{U}_\eta^{n-1}\psi \rangle \\ &= \left\langle \frac{1}{\sqrt{\eta}}\sqrt{\eta}PU\tilde{U}_\eta^{n-1}P\psi, \frac{1}{\sqrt{\eta}}\sqrt{\eta}PU\tilde{U}_\eta^{n-1}P\psi \right\rangle \\ &= \frac{1}{\eta}\langle P_\eta U\tilde{U}_\eta^{n-1}P\psi, P_\eta U\tilde{U}_\eta^{n-1}P\psi \rangle = \frac{1}{\eta}\|\hat{a}_\eta^n |\psi\rangle\|^2. \end{aligned}$$

From this, we summarize $\langle U\psi', \tilde{U}_\eta^n \psi' \rangle = \|\psi'\|^2 - g(\eta)\langle PU\psi', PU\psi' \rangle = \|\psi'\|^2 - \frac{g(\eta)}{\eta}\|\hat{a}_\eta^n |\psi\rangle\|^2$. Thus,

$$\text{Re}\langle U\psi', \tilde{U}_\eta^n \psi' \rangle = \text{Re}\left(\|\psi'\|^2 - \frac{g(\eta)}{\eta}\|\hat{a}_\eta^n |\psi\rangle\|^2\right) = \|\tilde{U}_\eta^{n-1}|\psi\rangle\|^2 - \frac{g(\eta)}{\eta}\|\hat{a}_\eta^n |\psi\rangle\|^2.$$

Altogether, we finally write

$$\begin{aligned} \|a\|^2 &= \|\tilde{U}_\eta^{n-1}|\psi\rangle\|^2 + \|\tilde{U}_\eta^n|\psi\rangle\|^2 - 2\operatorname{Re}\langle U\psi', \tilde{U}_\eta\psi'\rangle \\ &= \|\tilde{U}_\eta^{n-1}|\psi\rangle\|^2 + \|\tilde{U}_\eta^n|\psi\rangle\|^2 - 2\left(\|\tilde{U}_\eta^{n-1}|\psi\rangle\|^2 - \frac{g(\eta)}{\eta}\|\hat{a}_n^\eta|\psi\rangle\|^2\right), \\ \iff \|a\|^2 &= -\|\tilde{U}_\eta^{n-1}|\psi\rangle\|^2 + \|\tilde{U}_\eta^n|\psi\rangle\|^2 + 2\frac{g(\eta)}{\eta}\|\hat{a}_n^\eta|\psi\rangle\|^2. \end{aligned}$$

From the structure of $\|a\|^2$, we already see that the telescoping property is preserved. The expressions $\|b\|^2$ and $2\operatorname{Re}\langle a, b\rangle$ will erase the prefactors in η .

(2) Again, since $|\psi\rangle \in S_V$:

$$b = f(\eta)PU\tilde{U}_\eta^{n-1}|\psi\rangle = f(\eta)\frac{1}{\sqrt{\eta}}\sqrt{\eta}PU\tilde{U}_\eta^{n-1}P|\psi\rangle = \frac{f(\eta)}{\sqrt{\eta}}\underbrace{P_\eta U\tilde{U}_\eta^{n-1}P}_{=\hat{a}_n^\eta}|\psi\rangle = \frac{f(\eta)}{\sqrt{\eta}}\hat{a}_n^\eta|\psi\rangle.$$

Hence, we have $\|b\|^2 = \frac{f^2(\eta)}{\eta}\|\hat{a}_n^\eta|\psi\rangle\|^2$.

(3) The expression for b was computed in the above step 2. Compute a and combine:

$$\begin{aligned} a &= U\tilde{U}_\eta^{n-1}|\psi\rangle - \tilde{U}_\eta^n|\psi\rangle = (U - \tilde{U}_\eta)\tilde{U}_\eta^{n-1}|\psi\rangle = (\mathbb{1} - Q_\eta)U\tilde{U}_\eta^{n-1}|\psi\rangle = \underbrace{(\mathbb{1} - Q_\eta)U}_{=g(\eta)P}\tilde{U}_\eta^{n-1}P|\psi\rangle \\ &= g(\eta)PU\tilde{U}_\eta^{n-1}P|\psi\rangle = \frac{g(\eta)}{\sqrt{\eta}}\sqrt{\eta}PU\tilde{U}_\eta^{n-1}P|\psi\rangle = \frac{g(\eta)}{\sqrt{\eta}}\underbrace{P_\eta U\tilde{U}_\eta^{n-1}P}_{=\hat{a}_n^\eta}|\psi\rangle = \frac{g(\eta)}{\sqrt{\eta}}\hat{a}_n^\eta|\psi\rangle. \end{aligned}$$

Finally,

$$2\operatorname{Re}\langle a, b\rangle = 2\operatorname{Re}\left\langle \frac{g(\eta)}{\sqrt{\eta}}\hat{a}_n^\eta\psi, \frac{f(\eta)}{\sqrt{\eta}}\hat{a}_n^\eta\psi \right\rangle = \frac{2g(\eta)f(\eta)}{\eta}\operatorname{Re}\langle \hat{a}_n^\eta\psi, \hat{a}_n^\eta\psi \rangle = \frac{2g(\eta)f(\eta)}{\eta}\|\hat{a}_n^\eta|\psi\rangle\|^2.$$

It is mentioned that the telescoping sum in the numerator of (25) is of the form (compare with Lemma A.3 in Ref. [22])

$$\sum_{n=1}^N n(u_{n-1} - u_n) = u_0 - u_1 + 2u_1 - 2u_2 + 3u_2 - 3u_3 + \dots + N(u_{N-1} - u_N) = \sum_{k=0}^{N-1} u_k - Nu_N.$$

b. Details of the integral representation

The following statement follows immediately from the findings of Bourgain *et al.* [22] and were subsequently worked out in more generality by Grünbaum and Velazquez [23]. Expression (30) is derived in the following. A similar derivation of such a scalar-valued complex return integral was presented in Ref. [15] and can also be found, for instance, in Ref. [41]. We explicitly write out Eq. (30) and start by $(\hat{a}_n^\eta(t) \equiv \hat{a}_n^\eta)$

$$\tau_\infty(\eta, t) = \sum_{n=1}^{\infty} n \|\hat{a}_n^\eta(t)|\psi\rangle\|^2.$$

Note that, for $\alpha \in \mathbb{R}$, we can write $n = -i\frac{\partial}{\partial\alpha}e^{in\alpha}|_{\alpha=0}$ and similarly $ine^{in\theta} = \frac{\partial}{\partial\theta}e^{in\theta}$. Moreover, we use $\delta_{nm} = \frac{1}{2\pi}\int_0^{2\pi} d\theta e^{i\theta(n-m)}$ for $n, m \in \mathbb{N}$ and the generating function for the first return operator, i.e., $\hat{a}(\eta, z, t) := \sum_{k=1}^{\infty} \hat{a}_k^\eta(t)z^k$, which can be written for $|z| = 1$ as $\hat{a}(\eta, e^{i\theta}, t) := \sum_{k=1}^{\infty} \hat{a}_k^\eta(t)e^{ik\theta}$, for $z := e^{i\theta}$, $\theta \in [0, 2\pi]$. Then,

$$\|\hat{a}_n^\eta|\psi\rangle\|^2 = \langle \psi, \hat{a}_n^{\eta\dagger}\hat{a}_n^\eta\psi \rangle = \langle \psi, \delta_{nm}\hat{a}_n^{\eta\dagger}\hat{a}_m^\eta\psi \rangle = \left\langle \psi, \frac{1}{2\pi}\int_0^{2\pi} d\theta e^{i\theta(n-m)}\hat{a}_n^{\eta\dagger}\hat{a}_m^\eta\psi \right\rangle.$$

Combining all, we can write

$$\begin{aligned} \tau_\infty(\eta, t) &= \sum_{n=1}^{\infty} n \|\hat{a}_n^\eta(t)|\psi\rangle\|^2 = \sum_{n,m=1}^{\infty} \left(-i\frac{\partial}{\partial\alpha}e^{in\alpha}\Big|_{\alpha=0}\right) \left\langle \psi, \frac{1}{2\pi}\int_0^{2\pi} d\theta e^{i\theta(n-m)}\hat{a}_n^{\eta\dagger}(t)\hat{a}_m^\eta(t)\psi \right\rangle \\ &= \sum_{n,m=1}^{\infty} \frac{1}{2\pi i} \int_0^{2\pi} d\theta \left\langle \psi, e^{-im\theta}\hat{a}_m^{\eta\dagger}(t) \left(\frac{\partial}{\partial\alpha}e^{in\alpha}\Big|_{\alpha=0}\right) e^{in\theta}\hat{a}_n^\eta(t)\psi \right\rangle \end{aligned}$$

$$\begin{aligned}
&= \sum_{n=1}^{\infty} \frac{1}{2\pi i} \int_0^{2\pi} d\theta \left\langle \psi, \underbrace{\sum_{m=1}^{\infty} e^{-im\theta} \hat{a}_m^{\eta\dagger}(t)}_{=\hat{a}^\dagger(\eta, e^{i\theta}, t)} \left(\frac{\partial}{\partial \alpha} e^{in(\alpha+\theta)} \Big|_{\alpha=0} \right) \hat{a}_n^\eta(t) \psi \right\rangle. \\
&= \frac{1}{2\pi i} \int_0^{2\pi} d\theta \left\langle \psi, \hat{a}^\dagger(\eta, e^{i\theta}, t) \sum_{n=1}^{\infty} \left(\frac{\partial}{\partial \alpha} e^{in(\alpha+\theta)} \Big|_{\alpha=0} \right) \hat{a}_n^\eta(t) \psi \right\rangle.
\end{aligned}$$

Now, we use

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{\partial}{\partial \alpha} (e^{in(\alpha+\theta)} \hat{a}_n^\eta(t)) \Big|_{\alpha=0} &= \sum_{n=1}^{\infty} in e^{in(\alpha+\theta)} \hat{a}_n^\eta(t) \Big|_{\alpha=0} = \sum_{n=1}^{\infty} in e^{in\theta} \hat{a}_n^\eta(t) = \sum_{n=1}^{\infty} \left(\frac{\partial}{\partial \theta} e^{in\theta} \right) \hat{a}_n^\eta(t) \\
&= \frac{\partial}{\partial \theta} \sum_{n=1}^{\infty} e^{in\theta} \hat{a}_n^\eta(t) \stackrel{\text{def.}}{=} \frac{\partial}{\partial \theta} \hat{a}(\eta, e^{i\theta}, t),
\end{aligned}$$

which concludes

$$\tau_\infty(\eta, t) = \frac{1}{2\pi i} \int_0^{2\pi} d\theta \left\langle \psi, \hat{a}^\dagger(\eta, e^{i\theta}, t) \frac{\partial}{\partial \theta} \hat{a}(\eta, e^{i\theta}, t) \psi \right\rangle.$$

APPENDIX C: ANALYTICAL EXPRESSION OF THE TWO-LEVEL SYSTEM

Even for an example of minimal dimension, the involved quantities quickly become infeasible to control analytically. The involved operators for a complex contour integral are

$$\begin{aligned}
\tilde{U}_\eta(z, t) &= \frac{U_0(t, z, \eta)}{1 + z^2 \sqrt{1-\eta} - z(1 + \sqrt{1-\eta}) \cos t}, \\
U_0(t, z, \eta) &:= \begin{pmatrix} 1 - z \cos t & -iz\sqrt{1-\eta} \sin t \\ -iz \sin t & 1 - z\sqrt{1-\eta} \cos t \end{pmatrix},
\end{aligned}$$

and

$$\hat{a}(\eta, z, t) = \begin{pmatrix} \frac{\sqrt{\eta} z (\cos t - z)}{1 + z^2 \sqrt{1-\eta} - z(1 + \sqrt{1-\eta}) \cos t} & 0 \\ 0 & 0 \end{pmatrix}.$$

However, evaluating the integral representation of $\tau_\infty(\eta, t)$ symbolically does not work, so we take the vectorization approach as described in Refs. [23,24]. We define $g(t, x, \eta) := p(t, x, \eta)/q(t, x, \eta)$ with

$$\begin{aligned}
p(t, x, \eta) &= \eta(1 + z\{-2\sqrt{1-\eta} + z[1 + 2\sqrt{1-\eta} + 2z(-1 + \eta) - \eta]\} \\
&\quad + \{1 + z[-2 + z - 2\sqrt{1-\eta} + 2z\sqrt{1-\eta} - z\eta]\} \cos 2t) \\
q(t, x, \eta) &= 2\{1 + z[-\sqrt{1-\eta} - z^2(1 - \eta)^{\frac{3}{2}} - z\sqrt{1-\eta}(-2 + \eta) + z^3(-1 + \eta)^2] \\
&\quad - z[-1 + z^2(-1 + \eta)](-2 + \eta) \cos^2 t + z\{-\sqrt{1-\eta} + 2z[1 + \sqrt{1-\eta} - \eta] \\
&\quad + z\sqrt{1-\eta}[z(-1 + \eta) - \eta]\} \cos 2t\}.
\end{aligned}$$

APPENDIX D: PROOF OF LEMMA 1: THE MEAN RETURN TIME SCALES WITH $1/\eta$

With $x := \sqrt{1-\eta}$, $P := |\psi\rangle\langle\psi|$, and $Q := \mathbb{1} - P$ we redefine the survival operator $\tilde{U}_x := (Q + xP)U$ for $0 \leq x < 1$. Then, the return amplitude $\tilde{\psi}_x := (1 - x^2)^{1/2} \langle\psi|(z - \tilde{U}_x)^{-1}|\psi\rangle \tilde{\psi}_\eta = \sqrt{\eta} \langle\psi|(z - QU - xPU)^{-1}|\psi\rangle$ we can write for the resolvent

$$\begin{aligned}
&(z - QU - xPU)^{-1}P \\
&= (z - QU)^{-1}[\mathbb{1} - xPU(z - QU)^{-1}]^{-1}P,
\end{aligned}$$

which is expanded in powers of x as

$$\begin{aligned}
&= (z - QU)^{-1} \sum_{l \geq 0} x^l [PU(z - QU)^{-1}]^l P \\
&= (z - QU)^{-1} \sum_{l \geq 0} x^l [PU(z - QU)^{-1}P]^l P.
\end{aligned}$$

Since $PU(z - QU)^{-1}P = P\langle\psi|(z - QU)^{-1}|\psi\rangle P = P\tilde{\psi}_1$ we obtain for the right-hand side

$$= (z - QU)^{-1} \sum_{l \geq 0} x^l \tilde{\psi}_1^l P = (z - QU)^{-1} P \frac{1}{1 - x\tilde{\psi}_1},$$

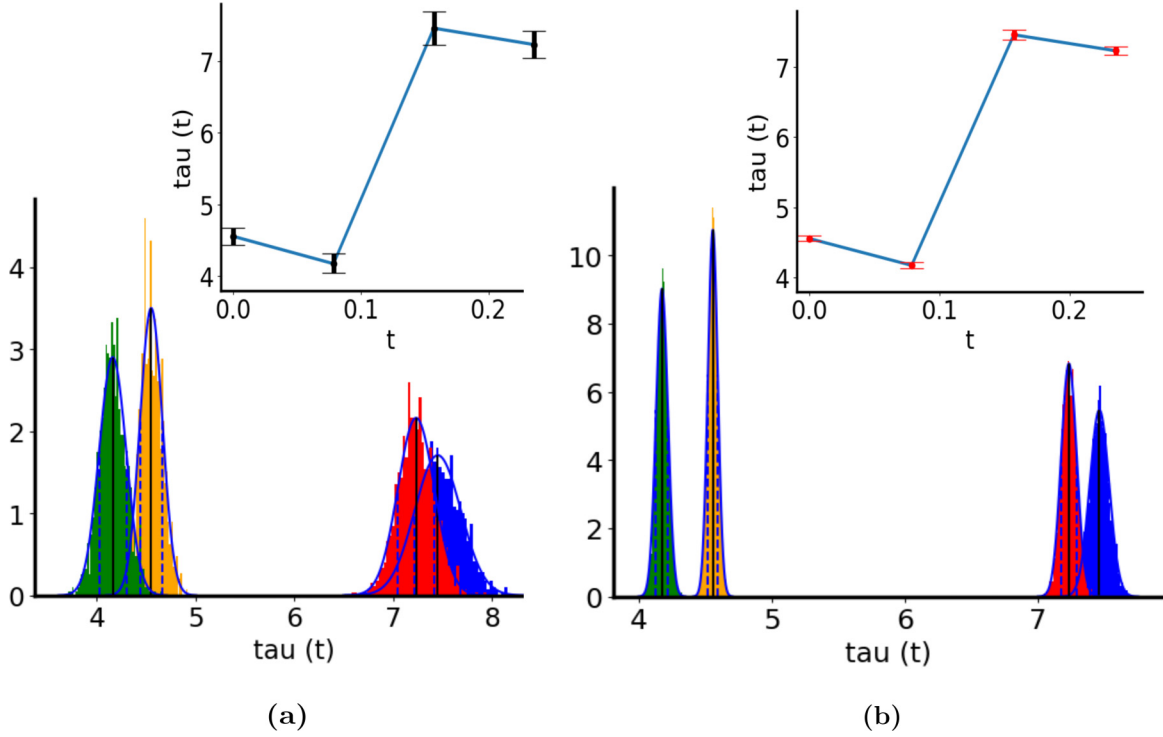


FIG. 11. Effect of the sampling size on the error bars through statistical bootstrapping, when drawing over 40 000 shots executed on IBM-Aachen. Exemplary distributions for $t_i = i\Delta t$, for $i = 1, 2, 3, 4$ and $\Delta t = \frac{2\pi}{80}$ (a) $R = 1000$ repetitions of $S = 1000$ samples. (b) $R = 5000$ repetitions of 10 000 samples. As the sample size and the number of repetitions increase, the clean normal distributions around the obtained first hitting time become visible. The standard deviation decreases, so the error bars become extremely small. Numerical values are presented in Table I.

such that the return amplitude $\tilde{\psi}_x$ reads

$$\tilde{\psi}_x = \sqrt{1-x^2} \frac{\tilde{\psi}_1}{1-x\tilde{\psi}_1}.$$

Then, in Ref. [57], it was shown that

$$\begin{aligned} \tilde{\psi}_1(e^{i\omega}) &= -\frac{\tilde{u}}{\tilde{u}^*} \equiv e^{i\varphi}, \\ \tilde{u}(e^{i\omega}) &= -\langle \Phi | (e^{iHt} - e^{i\omega})^{-1} | \Phi \rangle. \end{aligned}$$

This can be used to calculate the mean return time as

$$\begin{aligned} & \frac{-i}{2\pi} \int_0^{2\pi} \tilde{\psi}_x(e^{i\omega})^* \partial_\omega \tilde{\psi}_x(e^{i\omega}) d\omega \\ &= \frac{-i(1-x^2)}{2\pi} \int_0^{2\pi} \frac{e^{-i\varphi}}{1-xe^{-i\varphi}} \partial_\omega \frac{e^{i\varphi}}{1-xe^{i\varphi}} d\omega. \end{aligned}$$

From the expansion of the denominators in geometric series, we get

$$\begin{aligned} & \frac{1-x^2}{2\pi x^2} \int_0^{2\pi} \sum_{l_1 \geq 1} x^{l_1} e^{-i\varphi_{l_1}} \sum_{l_2 \geq 1} l_2 x^{l_2} e^{i\varphi_{l_2}} \frac{d\varphi}{d\omega} d\omega \\ &= \sum_{l_1, l_2 \geq 1} l_2 x^{l_1+l_2} \frac{1-x^2}{2\pi x^2} \int_0^{2\pi} e^{i\varphi(l_2-l_1)} \frac{d\varphi}{d\omega} d\omega. \end{aligned} \quad (\text{D1})$$

Then for $l_2 \neq l_1$ we have

$$\frac{1}{2\pi} \int_0^{2\pi} e^{i\varphi(l_2-l_1)} \frac{d\varphi}{d\omega} d\omega$$

$$= \frac{-i}{2\pi(l_2-l_1)} \int_0^{2\pi} \frac{de^{i\varphi(l_2-l_1)}}{d\omega} d\omega = 0$$

due to the periodicity of $e^{i\varphi(l_2-l_1)}$, and for $l_2 = l_1$

$$\frac{1}{2\pi} \int_0^{2\pi} \frac{d\varphi}{d\omega} d\omega = \tau_w \equiv \tau_\infty,$$

with the winding number τ_w . Note that this winding number is the same as τ_∞ . Thus, we obtain for the right-hand side of Eq. (D1)

$$\begin{aligned} (1-x^2)\tau_w \sum_{l \geq 1} l x^{2(l-1)} &= (1-x^2)\tau_w \frac{d}{d(x^2)} \sum_{l \geq 0} (x^2)^l \\ &= \frac{(1-x^2)\tau_w}{(1-x^2)^2} = \frac{\tau_w}{1-x^2} = \frac{\tau_\infty}{1-x^2} \\ &= \frac{\tau_\infty}{\eta}. \end{aligned} \quad (\text{D2})$$

This concludes the proof.

APPENDIX E: STATISTICAL BOOTSTRAPPING

Ideally, we would repeat the quantum-computational investigation multiple times. However, due to limited resources, one can obtain statistical deviations by statistical bootstrapping. We follow the construction presented in Appendix C in Ref. [58].

Given $\mathcal{N} = 40\,000$ trajectories for 80 different angles t and a first-return statistics, we can perform a sampling with

TABLE I. First four values of the first hitting time: theory, quantum computation, and statistical resampling.

Theory, quantum computing, and bootstrapping.			
Theory values, Eq. (24)	IBM-Aachen	$R = 1000, S = 2000$	$R = 5000, S = 10\,000$
$t_1 = 0, \quad \tau(t_1) = 4.767$	$\tau(t_1) = 4.556$	$\tau(t_1) = 4.551 \pm 0.119$	$\tau(t_1) = 4.556 \pm 0.037$
$t_2 = 0.079, \quad \tau(t_2) = 3.663$	$\tau(t_2) = 4.174$	$\tau(t_2) = 4.177 \pm 0.136$	$\tau(t_2) = 4.174 \pm 0.043$
$t_3 = 0.157, \quad \tau(t_3) = 7.328$	$\tau(t_3) = 7.461$	$\tau(t_3) = 7.460 \pm 0.237$	$\tau(t_3) = 7.463 \pm 0.072$
$t_4 = 0.236, \quad \tau(t_4) = 7.185$	$\tau(t_4) = 7.233$	$\tau(t_4) = 7.233 \pm 0.186$	$\tau(t_4) = 7.233 \pm 0.058$

replacement experiment to obtain an empirical mean value. Since such a randomized sampling procedure converges to a normal distribution by the central limit theorem, one can find a reliable estimate on the true mean first hitting time. Note that we can also compare the result to our theoretically obtained curve by Eq. (25).

The bootstrapping procedure for the depicted values from Table I is visualized in Fig. 11. For each angle t_i repeat the following: For R repetitions and S samples, draw one out of $\mathcal{N}$ bit strings and compute its first hitting event. Example: 010 would be a yes measurement after the second measurement (first 1 appears at the second position, i.e., $n = 2$). In practice,

the bit strings have length $N = 20$ (recall that N denotes the circuit depth). Then, sort the bit strings according to their first hitting appearance and compute the statistical probability $p_n = (\text{number of strings with first hitting event } n)/\mathcal{N}$. Compute $\tau = \sum_{n=1}^N n p_n / \sum_{n=1}^N p_n$ according to Eq. (24). This gives a list $(\tau_1, \dots, \tau_R)$, from which we compute the empirical mean $\mu = \frac{1}{R} \sum_{m=1}^R \tau_m$. Similarly, the standard deviation is computed by $\sigma = [\frac{1}{R} \sum_{m=1}^R (\tau_m - \mu)^2]^{1/2}$. In our case, $\mathcal{N} = 40\,000$ already provides a large sample size. The error bars presented in Fig. 4 correspond to $\mu \pm \sigma$, which are computed for $R = 1000$ and $S = 2000$.

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