

Holonomic multicontrolled gates for single-photon states

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Controlled and multicontrolled quantum gates whose action on a target qubit depends on the state of multiple control qubits, represent a fundamental logical building block for complex quantum algorithms. We propose a scheme for realizing this class of gates based on non-abelian holonomies in modulated photonic waveguide networks. Our approach relies on linear photonic circuits formed by two star networks coupled via a two-path circuit. A star network with M peripheral waveguides coupled to a single central site, or M pod, naturally generalizes the tripod structure used in non-abelian Thouless pumping and stimulated Raman adiabatic passage. In the present work, we first analyze the minimal case of two connected tripods and design adiabatic driving loops that implement single-qubit, CNOT, and SWAP gates. We then show how extending the approach to larger M -pod structures enables the realization of multiply controlled operations, which we exemplify by designing TOFFOLI and the OR gate on two coupled pentapods. Finally, we demonstrate that networks of connected tripods can implement the Deutsch quantum query algorithm.

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I. INTRODUCTION

Non-abelian holonomies [1,2] represent the natural $U(N)$ generalization of Berry's geometric phase [3]. They were originally introduced by Wilczek and Zee in their seminal work [1], which elucidated the relation between non-abelian gauge theories and the geometric structure of Hilbert space emerging in the adiabatic evolution of quantum systems with degenerate energy levels. Since then various experiments generated and detected non-abelian holonomies in photonic platforms [4,5], acoustic waveguides [6], quantum optical setups [7,8], cold atoms [9], and superconducting devices [10]. At the same time, theoretical works highlighted their fundamental role in modeling quantum transport [11–14] and describing topological phenomena [15–17].

The basic relevance of non-abelian holonomies is closely intertwined with their practical interest in quantum control [18–20] and quantum computing [21,22]. In particular, as first demonstrated by Zanardi and Rasetti, holonomies can be exploited to implement a universal set of *holonomic quantum gates* (HQGs) [21]. This idea, introduced in Ref. [21], is based on encoding qubits in the eigenstates of a degenerate Hamiltonian H and performing unitary operations through the Wilczek-Zee holonomy by adiabatically modulating a set of classical control parameters defining H . A key advantage

of HQGs lies in their robustness against certain kinds of noise and decoherence [23–26]. Since the pioneering work of Ref. [21], HQGs have been both theoretically proposed and experimentally realized across various physical platforms [27–32]. Furthermore, the potentially crucial role of HQGs in enabling more robust quantum computing architectures [33–37] has motivated various works aimed at realizing holonomic gates beyond the adiabatic regime. Nonadiabatic holonomic gates are quantum logic gates based on geometric phases, but implemented without requiring slow (adiabatic) evolution. They thus combine the robustness of geometric gates with the speed of dynamical ones. This could potentially reduce the exposure to decoherence, leading to enhanced performance in physical devices [38–44].

In both photonic and atomic implementations of holonomic systems, a crucial role is played by Hamiltonians with a tripod structure, such as those illustrated in Fig. 1(a). This type of Hamiltonian exhibits two degenerate energy levels for any value of the coupling parameters, and the holonomy is typically generated by adiabatically controlling these couplings. Hamiltonians of this kind lie at the heart of non-abelian Thouless pumping [13] and stimulated Raman adiabatic passage (STIRAP) protocols [8,18,22,44–47].

In this work, we go beyond standard single-qubit holonomic protocols, by focusing on the design of multicontrolled holonomic gates with m control qubits and one target qubit within an adiabatically modulated array of $2(2^m + 2)$ optical waveguides. This array consists of two coupled M pods, each having $M = 2^m + 1$ legs connected to a central waveguide. The system proposed in the present work encodes information in the state of a single photon in a multimode setup and uses holonomies to manipulate this information implementing an holonomic version of linear optic quantum computing

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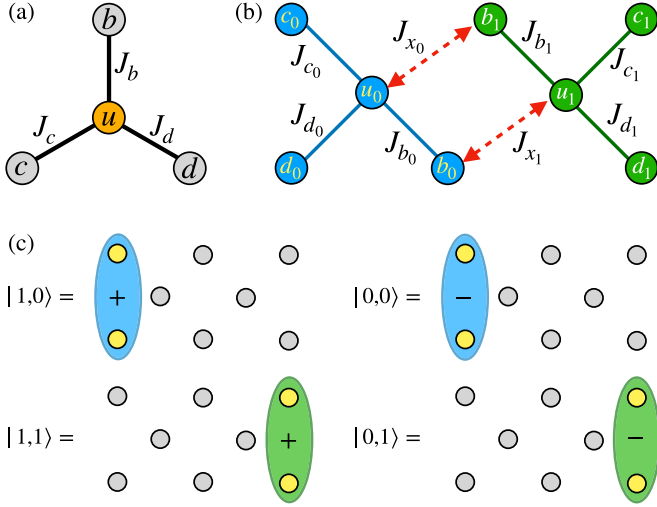


FIG. 1. (a) Scheme describing the structure of the Hamiltonian of a single tripod. (b) Schematic of the array composed of two tripods respectively colored in blue and green. The two tripods are connected via the hopping J_{x_0} and J_{x_1} , with respect to panel (a) the geometrical structure of each tripod is rotated. (c) Two qubits encoding states $|s, p\rangle$ for the target qubit $p = 0$ (upper row) and $p = 1$ (lower row). At the initial parameters $J_{\ell_p} = 0$ for $\ell = c, d, x$ and $J_{b_p} = J$ for $p = 0, 1$ of the driving cycle, the states correspond to the four zero-energy degenerate eigenstates.

protocols [48–50]. In the simplest case, the array structure is based on two tripods, ($m = 1$), connected via hoppings that preserve the four-fold degeneracy of the single-photon space for any coupling strength as shown in Fig. 1(b). By properly choosing the qubit encoding scheme and the driving cycles, we engineer a universal set of holonomic transformations [51–54]. In particular, in addition to single-qubit operations, such as Hadamard and phase gates, we design controlled two-qubit gates of the form $C_U = |0\rangle\langle 0| \otimes \mathbb{I} + |1\rangle\langle 1| \otimes U$ and the SWAP gate [55].

Multicontrolled quantum gates based on two coupled M pods extend the CNOT by applying a gate U to a target qubit p conditioned on the state of m control qubits. These gates represent a basic logical building block in the design of complex quantum algorithms, see, e.g., Refs. [56–59]. Here we encode the qubits in single-photon states on two coupled M pods and we engineer driving protocols to control holonomically the evolution of the qubit states. As specific examples we propose the realization of the TOFFOLI gate [60] and the quantum OR gate on a structure formed by two coupled 5-pods which we call pentapods. Eventually we show that similar structures can be used to implement the Deutsch quantum query algorithm [61,62].

II. PHOTONIC LATTICE AND QUBIT ENCODING

We consider a chiral symmetric photonic lattice of eight single-mode waveguides coupled through evanescent hoppings J_{i_p} with $i = b, c, d, x$, and $p = 0, 1$ shown in Fig. 1(b). The lattice consists of two tripods, colored in blue and green, connected via the hoppings J_{x_0} and J_{x_1} . The different coupling strengths J_{i_p} with $i = b, c, d, x$ and $p = 0, 1$ are given by the spatial distance between the neighboring guides.

The dynamics of photon propagation is described by a Schrödinger equation $i\partial_z|\psi(z)\rangle = H(z)|\psi(z)\rangle$ with Hamiltonian

$$H = H_0 + H_1 + H_X, \quad (1)$$

where the 0,1 tripod Hamiltonian are given by $H_p = (J_{b_p}b_p^\dagger u_p + J_{c_p}c_p^\dagger u_p + J_{d_p}d_p^\dagger u_p + \text{H.c.})$ with $p = 0, 1$, while H_X denotes the coupling between the two tripods $H_X = (J_{x_0}b_1^\dagger u_0 + J_{x_1}b_0^\dagger u_1 + \text{H.c.})$. Chirality ensures that, for any choice of the hopping parameters, the spectrum of H consists of four degenerate modes $\mathcal{B} = \{|\psi_1\rangle, \dots, |\psi_4\rangle\}$ at energy $E_0 = 0$ and four nondegenerate nonzero modes. Chirality furthermore enforces that the zero modes lie only on the majority sublattice $\mathcal{M} = \{b_p, c_p, d_p\}_{p=0,1}$, i.e., they are zero on the minority sublattice $m = \{u_p\}_{p=0,1}$ [63].

We focus on holonomic gates obtained by periodic and adiabatic modulation of the couplings along z , i.e., the propagation distance along the waveguides, which plays the role of time in the evolution process. At $z = 0$ only J_{b_0} and J_{b_1} are different from zero while all other couplings vanish. In this initial configuration, the four orthonormal zero modes shown in Fig. 1(c), are

$$\begin{aligned} |\psi_1(0)\rangle &= \frac{|c_0\rangle - |d_0\rangle}{\sqrt{2}}, & |\psi_2(0)\rangle &= \frac{|c_1\rangle - |d_1\rangle}{\sqrt{2}}, \\ |\psi_3(0)\rangle &= \frac{|c_0\rangle + |d_0\rangle}{\sqrt{2}}, & |\psi_4(0)\rangle &= \frac{|c_1\rangle + |d_1\rangle}{\sqrt{2}}, \end{aligned} \quad (2)$$

yielding a natural encoding for two qubits in a single-photon excitation. We can, therefore, recast the logic states in Eq. (2) as

$$\frac{|c_p\rangle + (2s-1)|d_p\rangle}{\sqrt{2}} \equiv |s, p\rangle, \quad (3)$$

with $p = 0, 1$ denoting the position and $s = 0, 1$ the symmetry. For the realization of controlled gates the position p is the *target* qubit, while s is the *control* qubit. The notation in Eq. (3) for two-qubit states therefore refers to a single-photon state implementing both qubits at once via spatial encoding in single-mode waveguides.

III. SINGLE- AND TWO-QUBIT GATES

We focus on the implementation of geometric gates by initializing the system in the degenerate subspace of zero modes $|\psi(0)\rangle = \sum_v c_v |\psi_v(0)\rangle$ defined in Eq. (2) and by manipulating adiabatically the coupling of the Hamiltonian H along the z coordinate following a cycle γ . To this end, we consider systems of single-mode optical waveguides coupled through evanescent interactions, where the separation between neighboring guides controls the hopping strength and thus directly defines the coupling structure. Over the cycle γ of duration λ , the input state $|\psi(0)\rangle$ transforms as

$$|\psi(\lambda)\rangle = W_\gamma(\lambda, 0)|\psi(0)\rangle, \quad (4)$$

where $W_\gamma(\lambda, 0) = P \exp[i \oint_\gamma \Gamma_z dz]$ indicates the $U(4)$ holonomy, P denotes the path ordering, and $[\Gamma_z]_{vv'} = \langle \psi_v | i\partial_z | \psi_{v'} \rangle$ is the Wilczek-Zee connection.

As first highlighted by Zanardi and Rasetti [21], non-abelian holonomies can be exploited to implement universal

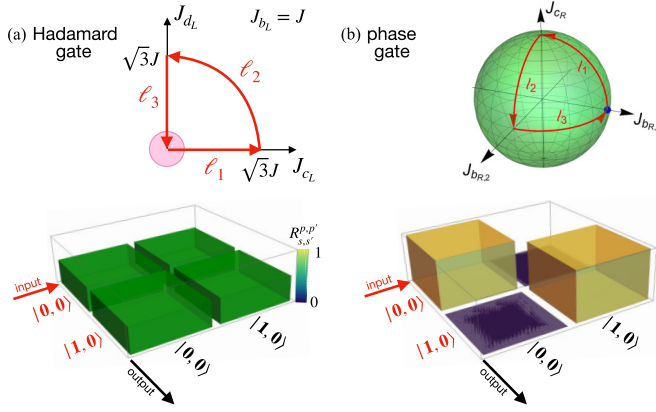


FIG. 2. (a) Driving cycle in the parameter space (the circle indicates the initial point) and quantum state detection probability for the single qubit Hadamard gate. (b) Same as (a) for the single-qubit phase gate.

set gates. In particular, within the setup of Fig. 1(b), they allow for the implementation of rotations and phase gates acting on the symmetry quantum number s , respectively given by $V_\xi = e^{i\xi\sigma_y} \otimes \sigma_0$ and $V_\varphi = e^{i\frac{\varphi}{2}(\sigma_0 - \sigma_z)} \otimes \sigma_0$.

Both transformations can be generated holonomically through adiabatic manipulations acting on both tripods simultaneously and setting $H_T = 0$ following the route presented in Refs. [13,46,47] for single tripod systems. For example, modulating each tripod with the driving cycles shown in Fig. 2, one can realize the Hadamard gate and the phase gate, respectively, see Appendix B for more details. See Figs. 2(a) and 2(b) for the corresponding final state detection probability, extracted from the numerical data as $R_{s,s'}^{p,p'} = |\langle s', p' | \psi_{s,p}(\lambda) \rangle|^2$.

For the four states system considered in this work, the holonomic evolution along suitably defined cycles also offers a natural playground to design two-qubit gates. We start by focusing on the generation of conditional gates of the form

$$C_\eta = |0\rangle\langle 0| \otimes \mathbb{I} + |1\rangle\langle 1| \otimes U_\eta, \quad (5)$$

where $U_\eta = e^{i\eta\sigma_y}$ is a rotation around the y axis of the Bloch sphere and the angle η depends only on the geometry of the cycle and the structure of the Hilbert space, see Appendix C for more details. An example is illustrated in Fig. 3 where we demonstrate a CNOT gate, i.e., a gate that SWAPS the target qubit p if and only if the control qubit s is $|1\rangle$. The latter corresponds to setting $\eta = \pi/2$, hence, $U_\eta = i\sigma_y$ in Eq. (5).

In our photonic array, this gate is achieved holonomically by setting $J_{x_0} = J_{x_1} \equiv J_x$, $J_{b_0} = J_{b_1} \equiv J_b$, and $J_{c_p} = J_{d_p} \equiv J_c$ for both $p = 0, 1$. The symmetry relations for $J_{c_p} = J_{d_p}$ imply that $|0, 0\rangle$ and $|0, 1\rangle$ are exact eigenstates of the array along the entire cycle γ while the states $|1, 0\rangle$ and $|1, 1\rangle$ evolve adiabatically according to Eq. (4) leading to $W_\gamma(\lambda, 0) = C_{\pi/2}$, see Appendix C for more details. Consequently, the geometrical meaning of the couplings arrangement is twofold: (i) it defines the physical configuration of the waveguide network and (ii) it controls the evolution on the manifold of degenerate states.

In Figs. 3(c) and 3(d) we show the evolution of the states $|1, 0\rangle$ and $|1, 1\rangle$ obtained by numerically solving the Schrödinger equation $i\partial_z |\psi(z)\rangle = H(z) |\psi(z)\rangle$. As predicted, the transformation flips the target qubit p while preserving

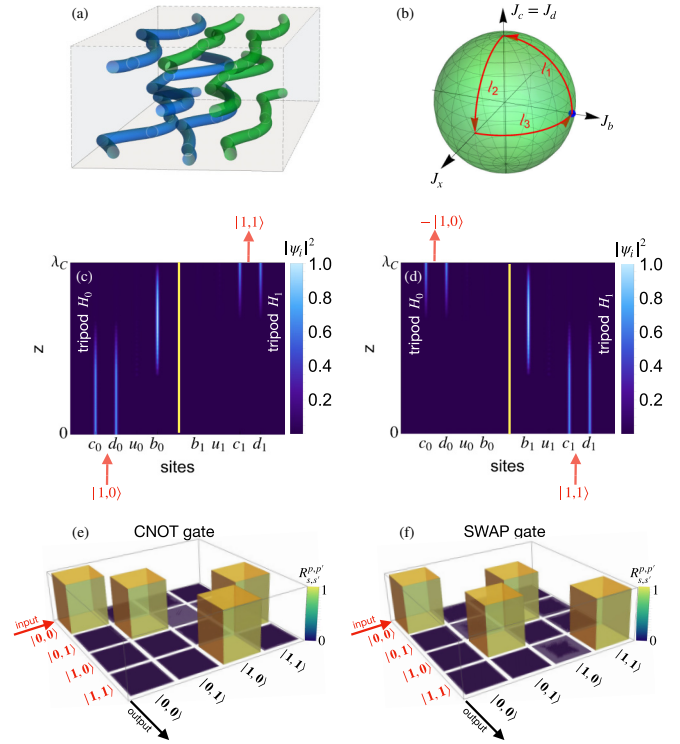


FIG. 3. (a) Illustration of the lattice pumped to implement a CNOT gate. The left tripod $p = 0$ is colored in green while the right tripod $p = 1$ is colored in blue. (b) Driving cycle in the parameter space. The circle indicates the initial point. (c), (d) Propagation of $|1, 0\rangle$ and $|1, 1\rangle$ states, respectively, over one cycle period. The yellow lines separate the 0 and the 1 tripods. (e), (f) Quantum state detection probability for the CNOT gate and the SWAP gate, respectively.

their control qubit $s = 1$. In Fig. 3(e) we show the numerical quantum state detection probability matrix R . However, by setting $J_{c_1} = -J_{d_1}$ while keeping $J_{c_0} = J_{d_0}$ in the cycle in Fig. 3(b), the CNOT gate can be turned into a SWAP gate, see Appendix C for more details. This gate indeed, SWAPS the two qubits $|0, 1\rangle$ and $|1, 0\rangle$ while keeping $|0, 0\rangle$ and $|1, 1\rangle$ unchanged. The numerically computed quantum state detection probability is shown in Fig. 3(f). This implementation of the SWAP gate requires changing the sign of the hopping J_{d_1} from positive to negative, which can be done experimentally by introducing an auxiliary waveguide between u_1 and d_1 [64,65], through a resistor applied to array [66], or by replacing certain waveguide with multiorbital ones [67,68].

IV. MULTICONTROLLED GATES

The array shown in Fig. 1(b) can be modified to host multiccontrolled gates by replacing the two tripods with two M pods. An M pod is a system of $M + 1$ waveguides with a star topology, where the M ones are connected to a central guide u , and it has $M - 1$ zero-energy states [46,47,69]. The Hamiltonians of the p th M pod are

$$H_p^{(M)} = \sum_{\ell \in [1, M]} J_{\ell, p} (a_{\ell, p}^\dagger u_p + \text{H.c.}) \quad p = 0, 1, \quad (6)$$

with $J_{\ell, p}$ denoting the coupling strengths of the ℓ th peripheral waveguide $a_{\ell, p}$ to the central central one u_p . For $M = 2^m + 1$,

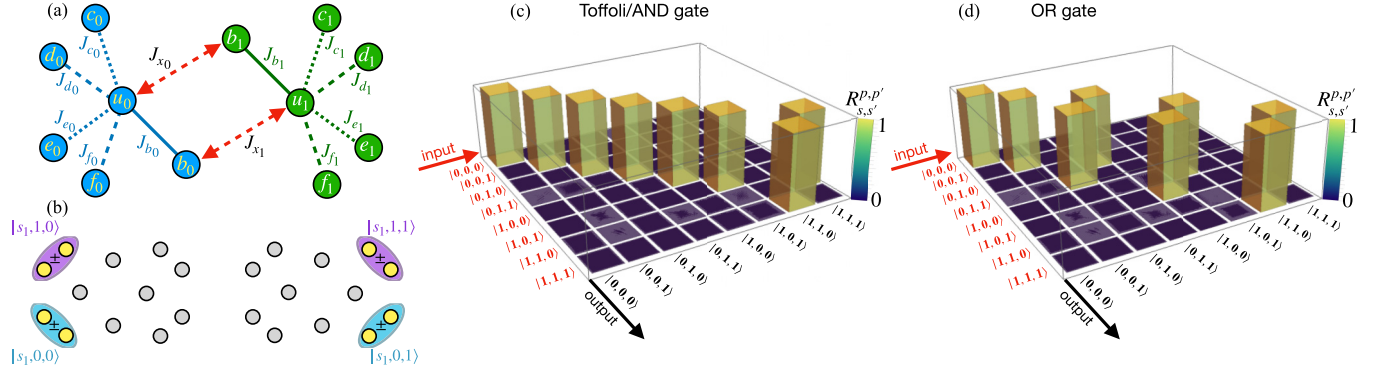


FIG. 4. (a) Schematic of the lattice composed of left (0) and right (1) pentapods respectively colored in blue and green. The two tripods are connected via the hopping J_{xL} and J_{xR} . (b) Three qubits encoding states $|b(s_1, s_2, p)\rangle$ with two control qubits $s_1, s_2 = \pm$ and a target $p = 0, 1$ qubits. At the initial parameters $J_{\ell p} = 0$ for $\ell = c, d, e, f, x$ and $J_{bp} = J$ for $p = L, R$ of the driving cycle, the states correspond to the eight zero-energy degenerate eigenstates. (c) Quantum state detection probability for the TOFFOLI gate. The driving cycle is shown in the top-right corner. (d) Same as (c) for a quantum version of the OR gate.

each M pod has 2^m zero-energy states which can be used to encode the m qubit.

The array with two coupled M pods has the following Hamiltonian:

$$H^{(M)} = H_0^{(M)} + H_1^{(M)} + H_X, \quad (7)$$

where the coupling Hamiltonian between the M pods is $H_X = (J_{x0}a_{1,1}^\dagger u_0 + J_{x1}a_{1,0}^\dagger u_1 + \text{H.c.})$. The 2×2^m zero-energy states of the Hamiltonian $H^{(M)}$ yield the encoding of m control qubits and one target qubit [70].

Let us unfold this extension on the TOFFOLI gate. This gate is a doubly controlled not, i.e., it applies a NOT gate conditioned on $m = 2$ control qubits, namely, $\text{CCNOT} = |0\rangle\langle 0| \otimes \mathbb{I}_4 + |1\rangle\langle 1| \otimes \text{CNOT}$, which is an important component in quantum search algorithms [71–73]. We realize the TOFFOLI gate by replacing each tripod in Fig. 1(b) with 5-pods, as shown in Fig. 4(a). The Hamiltonian of each 5-pods $H_p^{(5)}$, which we dub *pentapods*, is $H_p = (J_{bp}b_p^\dagger + J_{cp}c_p^\dagger + J_{dp}d_p^\dagger + J_{ep}e_p^\dagger + J_{fp}f_p^\dagger)u_p + \text{H.c.}$. The spectrum of the resulting array in Fig. 4(a) consists of eight degenerate modes $\{|\psi_j\rangle\}_{j=1}^8$ at energy $E_0 = 0$ and four nondegenerate modes, see Appendix A and Ref. [69] for more details.

For $J_{b0} = J_{b1} = J$ and all other couplings' vanishing, the orthonormal zero modes are supported exclusively on the disconnected external sites of the two pentapods, namely, they are linear combinations of the states $|c_0\rangle, |d_0\rangle, |e_0\rangle$, and $|f_0\rangle$ and $|c_1\rangle, |d_1\rangle, |e_1\rangle$, and $|f_1\rangle$. Choosing this initial configuration and limiting ourselves to the single-excitation space, we can encode the three qubits as follows:

$$|s_1, s_2, p\rangle = \frac{s_2}{\sqrt{2}}(|c_p\rangle + (2s_1 - 1)|d_p\rangle) + \frac{1 - s_2}{\sqrt{2}}(|e_p\rangle + (2s_1 - 1)|f_p\rangle), \quad (8)$$

where the *control* qubits are identified by the quantum numbers s_1 and s_2 while the *target* qubit is identified by p . The waveguides b_0 and b_1 are instead used only to couple the two pentapods.

By modulating the hopping terms J_{bp}, J_{cp} , and J_{xp} according to the driving cycle shown in Fig. 3(b), while setting all other

couplings to zero ($J_{\ell p} = 0$ for $\ell = e, f$ and $p = 0, 1$), a TOFFOLI gate is implemented. The corresponding quantum state detection probability R for this gate is shown in Fig. 4(c). This gate is particularly notable because it realizes a reversible AND operation [74]. During the implementation of the TOFFOLI gate, the waveguides e and f remain decoupled from the rest of the circuit; however, they are activated to implement more complex logical operations.

As an example, Fig. 4(d) shows the quantum state detection probability of a reversible OR gate [74]. This gate is obtained by composing three driving cycles of the form shown in Fig. 3(b), each time setting a different group of hopping amplitudes to zero. Specifically, the OR gate is implemented by applying the cycle in Fig. 3(b) three times with the following configurations:

- (i) modulating J_{bp}, J_{cp} , and J_{xp} for $p = 0, 1$, while zeroing all other couplings;
- (ii) modulating J_{b0}, J_{c0} , and J_{x0} on the $p = 0$ tripod and J_{e1}, J_{f1} , and J_{x1} on the $p = 1$ tripod, while zeroing the others;
- (iii) modulating J_{e0}, J_{f0} , and J_{x0} on the $p = 0$ tripod and J_{b1}, J_{c1} , and J_{x1} on the $p = 1$ tripod, while zeroing the others.

These two gates shown in Figs. 4(c) and 4(d) are conditional gates of the form

$$C = \sum_{s_1, s_2} |s_1, s_2\rangle\langle s_1, s_2| \otimes V^{s_1, s_2}, \quad (9)$$

where V^{s_1, s_2} is a 2×2 unitary transformation which depend on s_1 and s_2 , hence, extending the conditional gate C_η in Eq. (5) from a single to a doubly controlled gate. The unitaries V^{s_1, s_2} can be implemented by either composing driving cycles that selectively involve a few waveguides only, as in our examples above, or driving cycles that involve all guides simultaneously.

V. IMPLEMENTATION OF A QUERY PROBLEM

We now show that the array in Fig. 1(b), together with the driving cycle in Figs. 2 and 3, can be used to implement quantum computing gate-models. A quantum gate-model consists of (i) a set of qubits, (ii) a sequence of quantum gates acting on them, and (iii) measurements performed on selected qubits.

TABLE I. Table of the four Boolean functions.

s	$f_1(s)$	$f_2(s)$	$f_3(s)$	$f_4(s)$
0	0	0	1	1
1	0	1	0	1

A specific class of gate models is that of query models, which involve an oracle U_f that encodes an unknown black-box function f to be investigated. Deutsch's algorithm is a paradigmatic example of this class [62]. It determines whether a single-bit Boolean function $f : \{0, 1\} \rightarrow \{0, 1\}$ is constant, $f(0) = f(1)$, or balanced, $f(0) \neq f(1)$, using a single oracle query, whereas any classical strategy requires two. The four possible Boolean functions are summarized in Table I, with f_1 and f_4 constant and f_2 and f_3 balanced.

As presented in Ref. [62], the oracle U_f which implement the four Boolean functions is an operator acting on the two qubit states $|s, p\rangle$ defined as $U_f(|s, p\rangle) = |s, p \oplus f(s)\rangle$ where $\oplus$ is the XOR operation (i.e., the sum Modulo 2). The query of the Deutsch's algorithm, as schematically recapped in Fig. 5(a), consists in considering a superposition state $|+, -\rangle$

obtained by applying the Hadamard gates on both control and target qubits on an initial input $|0, 1\rangle$, as explicitly detailed in Appendix D. The oracle U_f discerns the functions in the two classes as

$$U_f(|+, -\rangle) = \begin{cases} (-1)^{f(0)}|+, -\rangle & \text{constant,} \\ (-1)^{f(0)}|-, -\rangle & \text{balanced.} \end{cases} \quad (10)$$

The four operators U_f can be obtained via four different combinations of the CNOT gate and single-qubit NOT gate [62]. In our photonic array Fig. 1(b), these four oracle operators are implemented through proper combination of the driving cycles in Figs. 2(a) and 2(b) and Fig. 3(b). However, an alternative and less cumbersome way to implement the oracles comes from exploiting the symmetries of the array Fig. 1(b). Let us impose $J_{d_p} = \alpha_p J_{c_p}$ for $\alpha_p = \pm 1$ in each tripod that, i.e., $H_p = (J_{b_p} b_p^\dagger + J_{c_p} c_p^\dagger + \alpha_p J_{c_p} d_p^\dagger) u_p + \text{H.c.}$ for $p = 0, 1$. Consequently, the driving cycle in Fig. 3(b) yields four holonomies W_{α_0, α_1} , among which the holonomy corresponding to the CNOT can be obtained for $\alpha_0 = 1$ and $\alpha_1 = 1$ and the holonomy corresponding to the SWAP can be obtained for $\alpha_0 = 1$ and $\alpha_1 = -1$. In our photonic array in Fig. 1(b), the superposition states read $|-, -\rangle = \frac{1}{\sqrt{2}}(-|d_0\rangle + |d_1\rangle)$ and $|+, -\rangle = \frac{1}{\sqrt{2}}(|c_0\rangle - |c_1\rangle)$, respectively shown in Figs. 5(b) and 5(c). In Figs. 5(d) and 5(g) we show the single queries which distinguish the identity functions f_1, f_4 [shown in Figs. 5(d) and 5(e)] from the balanced functions f_2, f_3 [shown in Figs. 5(f) and 5(g)].

Let us conclude by conjecturing that the proposed extension of the array from interconnected tripods to interconnected M pods may pave the way to generalize this procedure. Indeed, setting $M = 2^m + 1$ could allow the photonic implementation of the Deutsch-Jozsa quantum algorithm via non-abelian holonomies. This is an algorithm that efficiently distinguishes m -bits binary functions $f(s) : \{0, 1\}^m \rightarrow \{0, 1\}$ between constant (the function returns the same value for all 2^m inputs) and balanced (the function returns 0 for half $2^m - 1$ inputs and 1 for the remaining half).

VI. CONCLUSION

Recent work on non-abelian Thouless pumping [4, 13, 16], non-abelian braiding [5, 12, 75, 76], and quantum walks [20] has demonstrated that holonomies in photonic waveguide arrays can be used to control light propagation, enabling the engineering of topological effects.

In the present work, we explore the possibility of encoding the state of single and multiple qubits in the position of a single photon within a waveguide array, and we investigate how non-abelian holonomies can be used to implement different quantum gates realizing a holonomic version of linear optic quantum computing protocols [50].

Our control strategy is based on the paradigm of *adiabatic population transfer*, similar to that employed in STIRAP atomic protocols. Accordingly, we focus on lattices with the structure of single or coupled M pods, or star graphs, where a single site is coupled to M peripheral sites. Within these structures, we show that it is possible to engineer manipulations in which the structure of the single-photon wave function on a single M pod with $M = 2^m + 1$ acts as m

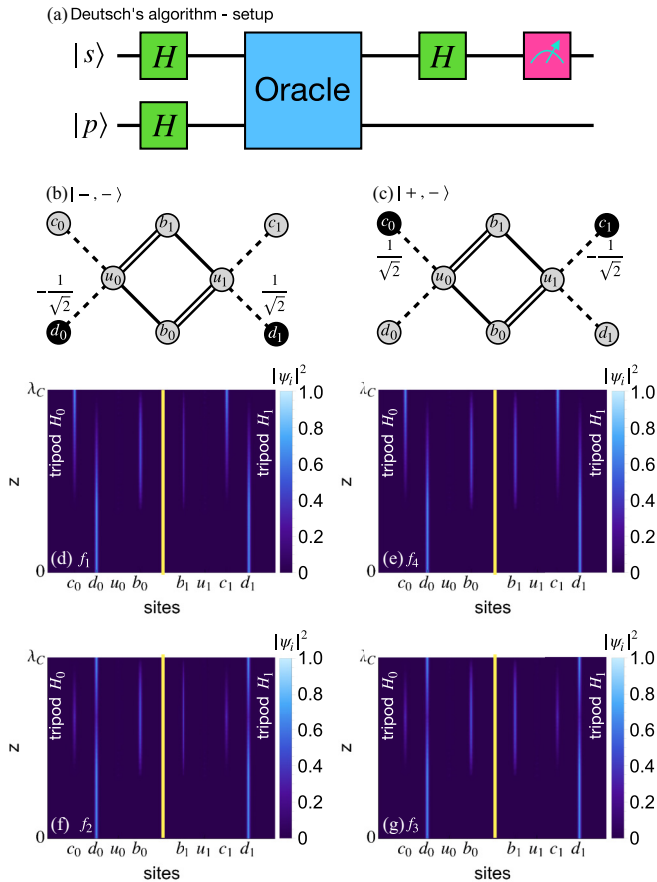


FIG. 5. (a) Illustration of the Deutsch's algorithm. (b), (c) Superposition states $|+, -\rangle$ and $|-, -\rangle$. (d), (e) Propagation of $|-, -\rangle$ over one cycle period for $\alpha_0 = \pm 1$ and $\alpha_1 = \pm 1$ which respectively implement the constant functions f_1 and f_4 . (f), (g) Same as (d), (e) for $\alpha_0 = \pm 1$ and $\alpha_1 = \mp 1$ which respectively implement the balanced functions f_2 and f_3 . The yellow lines separate the 0 and the 1 tripods.

control qubits, while the placement on one or another M pods serves as the target. Within this holonomic framework of driven waveguide networks, we discuss the implementation of (i) single-qubit gates, such as the Hadamard and phase gates (shown in Fig. 2), and (ii) multi-qubit gates, such as the CNOT (shown in Fig. 3) and the TOFFOLI gate (shown in Fig. 4). This pair of multicontrolled gates, CNOT and TOFFOLI, supplemented with single-qubit Hadamard and phase gates, constitute a universal set since they approximate any unitary circuit [51–54]. Within our approach, the network topology (which guides are connected and with what strength) governs how the zero modes mix by implementing different controlled gates. Moreover, under the approximation of adiabatic evolution, the evolution of the system is completely determined by Wilczek-Zee connection. Geometry thus enters with two distinct meanings: it refers both to the physical geometry of the waveguide network and to the geometric evolution within the space of zero-energy states.

We further show that the proposed photonic array can realize notable quantum query algorithms, such as the Deutsch algorithm, where the adiabatic evolution operator associated with a holonomy acts as the oracle. This implementation can, in principle, be extended to the Deutsch-Jozsa quantum algorithm. Such an approach, on the one hand, does not require many-body interaction, which may degrade the fidelity of the gates due to noise. On the other hand, realizing M pods with a large number of waveguides may be challenging due to unwanted couplings. This technical issue may be tamed by switching to multimode waveguides. This holonomic implementation of quantum query problems reframes both the oracle and queries in terms of geometric holonomies. Our approach thus opens an alternative direction for leveraging non-abelian holonomies in quantum algorithms and quantum computing.

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DATA AVAILABILITY

The data that support the findings of this article are not publicly available. The data are available from the authors upon reasonable request.

APPENDIX A: EIGHT-SITES ARRAY AND HOLONOMIES

The Hamiltonian of the photonic array written explicitly reads

$$H = H_0 + H_1 + H_T$$

$$= (J_{b_0}b_0^\dagger + J_{c_0}c_0^\dagger + J_{d_0}d_0^\dagger)u_0 + (J_{b_1}b_1^\dagger + J_{c_1}c_1^\dagger + J_{d_1}d_1^\dagger)u_1$$

$$+ J_{x_0}b_1^\dagger u_0 + J_{x_1}b_0^\dagger u_1 + \text{H.c.}, \quad (\text{A1})$$

which has the matrix

$$H = \begin{pmatrix} 0 & J_{b_0} & J_{c_0} & J_{d_0} & 0 & J_{x_0} & 0 & 0 \\ J_{b_0} & 0 & 0 & 0 & J_{x_1} & 0 & 0 & 0 \\ J_{c_0} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ J_{d_0} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & J_{x_1} & 0 & 0 & 0 & J_{b_1} & J_{c_1} & J_{d_1} \\ J_{x_0} & 0 & 0 & 0 & J_{b_1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & J_{c_1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & J_{d_1} & 0 & 0 & 0 \end{pmatrix}. \quad (\text{A2})$$

Due to chiral symmetry, this array has four degenerate eigenenergies at $E_0 = 0$ for any value of the parameters. The other four nondegenerate eigenenergies are

$$E = \pm \frac{1}{\sqrt{2}} \{ J_{b_0}^2 + J_{b_1}^2 + J_{c_0}^2 + J_{c_1}^2 + J_{d_0}^2 + J_{d_1}^2 + J_{x_0}^2 + J_{x_1}^2$$

$$\pm [(J_{b_0}^2 + J_{b_1}^2 + J_{c_0}^2 + J_{c_1}^2 + J_{d_0}^2 + J_{d_1}^2 + J_{x_0}^2 + J_{x_1}^2)^2$$

$$+ 4(2J_{b_0}J_{b_1}J_{x_0}J_{x_1} - (J_{c_0}^2 + J_{d_0}^2 + J_{x_0}^2)(J_{c_1}^2 + J_{d_1}^2 + J_{x_1}^2)$$

$$- J_{b_1}^2(J_{c_0}^2 + J_{d_0}^2) - J_{b_0}^2(J_{c_1}^2 + J_{d_1}^2)]^{\frac{1}{2}} \}^{\frac{1}{2}}. \quad (\text{A3})$$

Chirality constricts the four eigenstates onto the majority subarray $\mathcal{M} = \{b_0, c_0, d_0, b_1, c_1, d_1\}$. Degeneracy implies that there exist arbitrary choices of the orthonormal basis $\mathcal{B} = \{|\psi_1\rangle, \dots, |\psi_4\rangle\}$ of the $E_0 = 0$ degenerate subspace. In the following computation of the Wilczek-Zee connection:

$$[\Gamma_z]_{\nu\nu'} = \langle \psi_\nu | i\partial_z | \psi_{\nu'} \rangle \quad (\text{A4})$$

is used to obtain the holonomy

$$W(z_0, z_1) = e^{i\theta_a} P \exp \left[i \int_{z_0}^{z_1} \Gamma_z dz \right], \quad (\text{A5})$$

we specify that the basis $\mathcal{B}$ is used to compute Γ_z for each gate. This does not affect the calculation nor loose generality, but avoids us having to report four cumbersome eigenstates.

APPENDIX B: SINGLE-QUBIT GATES

To implement single-qubit Pauli gates represented by σ_y ad σ_z , we set $J_{x_0} = J_{x_1} \equiv 0$ for all $z \in \mathbb{R}$. In this setup, the four nonzero modes in Eq. (A3) reduce to $E = \{\pm\sqrt{J_{c_0}^2 + J_{d_0}^2 + J_{b_0}^2}, \pm\sqrt{J_{c_1}^2 + J_{d_1}^2 + J_{b_1}^2}\}$. An orthonormal basis of zero modes for this specific configuration is

$$|\psi_1^y(z)\rangle = \frac{J_{d_0}|c_0\rangle - J_{c_0}|d_0\rangle}{\delta_0},$$

$$|\psi_2^y(z)\rangle = \frac{J_{d_1}|c_1\rangle - J_{c_1}|d_1\rangle}{\delta_1},$$

$$|\psi_3^y(z)\rangle = \frac{J_{b_0}(J_{c_0}|c_0\rangle + J_{d_0}|d_0\rangle) - \delta_0^2|b_0\rangle}{\delta_0\Delta_0},$$

$$|\psi_4^y(z)\rangle = \frac{J_{b_1}(J_{c_1}|c_1\rangle + J_{d_1}|d_1\rangle) - \delta_1^2|b_1\rangle}{\delta_1\Delta_1}, \quad (\text{B1})$$

where $\delta_p = \sqrt{J_{c_p}^2 + J_{d_p}^2}$ and $\Delta_p = \sqrt{J_{c_p}^2 + J_{d_p}^2 + J_{b_p}^2}$ for $p = 0, 1$. At the starting point of the evolution where the parameters are $J_{c_p} = J_{d_p} = 0$, $J_{b_p} = J$, these eigenstates reduce to the encoding states $|s, p\rangle$ for $p = 0, 1$ and $s = 0, 1$ as in Eq. (C5). Let us compute the auxiliary terms $A_{\nu\nu'}$ useful to obtain the

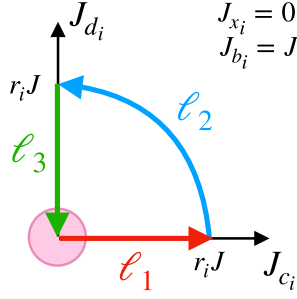


FIG. 6. Driving cycle γ_{σ_y} split in three branches ℓ_1 (red), ℓ_2 (blue), and ℓ_3 (green).

connection Γ_z in Eq. (C5), here written as

$$A_{vv'} = i \{ \langle \psi_v | \partial_{J_{c_0}} | \psi_{v'} \rangle, \langle \psi_v | \partial_{J_{d_0}} | \psi_{v'} \rangle, \langle \psi_v | \partial_{J_{b_0}} | \psi_{v'} \rangle, \langle \psi_v | \partial_{J_{c_1}} | \psi_{v'} \rangle, \langle \psi_v | \partial_{J_{d_1}} | \psi_{v'} \rangle, \langle \psi_v | \partial_{J_{b_1}} | \psi_{v'} \rangle \}. \quad (\text{B2})$$

The only relevant terms are A_{13} and A_{24} , which read

$$A_{13} = \frac{iJ_{b_0}}{\delta_0^2 \Delta_0} \{ J_{c_0}, -J_{d_0}, 0, 0, 0, 0 \}, \quad (\text{B3})$$

$$A_{24} = \frac{iJ_{b_1}}{\delta_1^2 \Delta_1} \{ 0, 0, 0, J_{c_1}, -J_{d_1}, 0 \},$$

with $A_{31} = -A_{13}$ and $A_{42} = -A_{24}$. The Wilczek-Zee connection then reads

$$\Gamma_z = \frac{J_{b_0}(J_{d_0}J_{c_0} - J_{c_0}J_{d_0})}{\delta_0^2 \Delta_0} \sigma_y \otimes \frac{\sigma_0 + \sigma_z}{2} + \frac{J_{b_1}(J_{d_1}J_{c_1} - J_{c_1}J_{d_1})}{\delta_1^2 \Delta_1} \sigma_y \otimes \frac{\sigma_0 - \sigma_z}{2}. \quad (\text{B4})$$

For closed paths γ_p in the parameter space, the path integrals yield $\xi_0 = \oint_{\gamma_0} [\Gamma_z]_{13} d\gamma$ and $\xi_1 = \oint_{\gamma_1} [\Gamma_z]_{24} d\gamma$. In Eq. (A5), this results in

$$W(\gamma) = \begin{pmatrix} \cos \xi_0 & 0 & \sin \xi_0 & 0 \\ 0 & \cos \xi_1 & 0 & \sin \xi_1 \\ -\sin \xi_0 & 0 & \cos \xi_0 & 0 \\ 0 & -\sin \xi_1 & 0 & \cos \xi_1 \end{pmatrix}$$

$$= [\cos \xi_0 \sigma_0 + i \sin \xi_0 \sigma_y] \otimes \frac{\sigma_0 + \sigma_z}{2} + [\cos \xi_1 \sigma_0 + i \sin \xi_1 \sigma_y] \otimes \frac{\sigma_0 - \sigma_z}{2}. \quad (\text{B5})$$

These are two independent σ_y rotations for single qubits in each tripod. Loops of the kind shown in Fig. 6, split in three separate branches ℓ_1 , ℓ_2 , and ℓ_3 , respectively, colored in red, blue, and green, allow implementing, among others, the Hadamard and the NOT gates.

The first and third branches ℓ_1 and ℓ_3 yield zeros in Eq. (B4) as along them J_{c_p} and J_{d_p} are respectively zero. Along ℓ_2 , $J_{b_p} = J$ while $J_{c_p} = r_p J \cos \theta$ and $J_{d_p} = r_p J \sin \theta$ for $\theta \in [0, \frac{\pi}{2}]$. The prefactors of the Pauli matrix tensor products in Eq. (B4) then reduce to $\frac{1}{\sqrt{r_p^2 + 1}}$. Hence, in Eq. (B5) the angles are $\eta_p = \frac{\pi}{2\sqrt{r_p^2 + 1}}$. The Hadamard gate is achieved by choosing $r_p = \sqrt{3}$, which implies $\eta_p = \frac{\pi}{4}$. A NOT gate obtained for $\eta_p = \frac{\pi}{2}$ in Eq. (B5) can be achieved by either extending

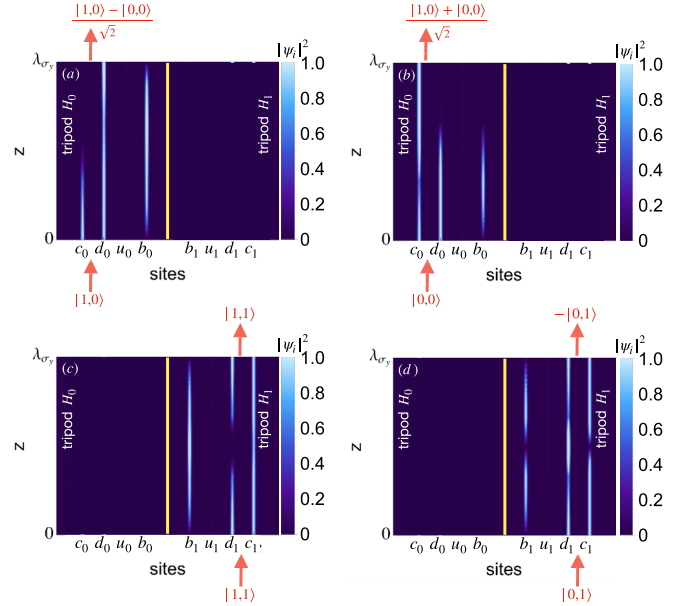


FIG. 7. (a), (b) Evolution of the $|1, 0\rangle$ and $|0, 0\rangle$, respectively, via cycle in Fig. 6 with $\xi_0 = \frac{\pi}{4}$ in the tripod H_0 . (c), (d) Evolution of the $|1, 1\rangle$ and $|0, 1\rangle$, respectively, via cycle in Fig. 8 in the tripod H_1 . The yellow lines separate the 0 and the 1 tripods.

the quarter circle to a half circle (i.e., for $\theta \in [0, \pi]$) or by simply repeating the loop γ_{σ_y} in Fig. 6 twice. The holonomy in Eq. (B5) for $\xi_0 = \frac{\pi}{4}$ and $\xi_1 = \frac{\pi}{2}$ hence reduces to

$$W = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & \sqrt{2} \\ -1 & 0 & 1 & 0 \\ 0 & -\sqrt{2} & 0 & 0 \end{pmatrix}$$

$$= \frac{\sigma_0 + i\sigma_y}{\sqrt{2}} \otimes \frac{\sigma_0 + \sigma_z}{2} + i\sigma_y \otimes \frac{\sigma_0 - \sigma_z}{2}. \quad (\text{B6})$$

In Fig. 7, we show the evolution of the states $|1, 0\rangle$ [Fig. 7(a)] and $|0, 0\rangle$ [Fig. 7(b)] in the tripod H_0 via the cycle in Fig. 6 defined for $\xi_0 = \frac{\pi}{4}$. The two tripods are disjoint as the connecting Hamiltonian $H_T = 0$. Both panels show that, for $\xi_0 = \frac{\pi}{4}$, the initial states end respectively in a superposition $\frac{|1,0\rangle - |0,0\rangle}{\sqrt{2}}$ and $\frac{|1,0\rangle + |0,0\rangle}{\sqrt{2}}$, which correspond to a $\frac{\pi}{4}$ rotation.

To perform a phase qubit σ_z , we assume $J_{c_p} = J_{d_p}$ in both tripods $p = 0, 1$ and turn the hoppings J_{b_p} complex, i.e., $J_{b_p} = J_{b_{i,1}} + e^{i\phi_p} J_{b_{i,2}}$ for phases ϕ_p . An orthonormal basis of zero modes, for this specific configuration, reads

$$|\psi_1^z(z)\rangle = \frac{|c_0\rangle - |d_0\rangle}{\sqrt{2}},$$

$$|\psi_2^z(z)\rangle = \frac{|c_1\rangle - |d_1\rangle}{\sqrt{2}},$$

$$|\psi_3^z(z)\rangle = \frac{J_{b_0}(|c_0\rangle + |d_0\rangle) - J_{c_0}|b_1\rangle}{\epsilon_0},$$

$$|\psi_4^z(z)\rangle = \frac{J_{b_1}(|c_1\rangle + |d_1\rangle) - J_{c_1}|b_0\rangle}{\epsilon_1}, \quad (\text{B7})$$

with $\epsilon_p = \sqrt{2(J_{c_p}^2 + |J_{b_p}|^2)}$. At the initial point of the cycle where the parameters are $J_{c_p} = 0$, $J_{b_{i,1}} = J$, and $J_{b_{i,2}} = 0$,

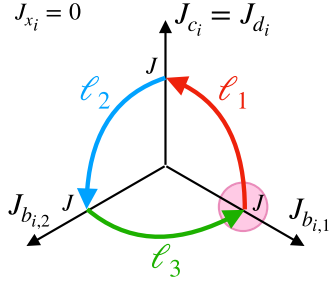


FIG. 8. Driving cycle γ_{σ_z} split in three branches ℓ_1 (red), ℓ_2 (blue), and ℓ_3 (green).

these eigenstates reduce to the encoding states $|s, p\rangle$ as in Eq. (3). The auxiliary terms $A_{vv'}$ useful in obtaining the connection Γ_z in Eq. (C5), here are

$$A_{vv'} = i\{\langle\psi_v|\partial_{J_{c_0}}|\psi_{v'}\rangle, \langle\psi_v|\partial_{J_{b_{0,1}}}|\psi_{v'}\rangle, \langle\psi_v|\partial_{J_{b_{0,2}}}|\psi_{v'}\rangle, \langle\psi_v|\partial_{J_{c_1}}|\psi_{v'}\rangle, \langle\psi_v|\partial_{J_{b_{1,1}}}|\psi_{v'}\rangle, \langle\psi_v|\partial_{J_{b_{1,2}}}|\psi_{v'}\rangle\}. \quad (\text{B8})$$

The only relevant terms are A_{33} and A_{44} , which read

$$A_{33} = \frac{\sin\phi_0}{\epsilon_0^2} \{0, J_{b_{0,2}}, -J_{b_{0,1}}, 0, 0, 0, 0\},$$

$$A_{44} = \frac{\sin\phi_1}{\epsilon_1^2} \{0, 0, 0, 0, J_{b_{1,1}}, -J_{b_{1,2}}\}. \quad (\text{B9})$$

The Wilczek-Zee connection then reads

$$\Gamma_z = \frac{\sin\phi_0(J_{b_{0,2}}\dot{J}_{b_{0,1}} - J_{b_{0,1}}\dot{J}_{b_{0,2}})}{\epsilon_0^2} \frac{\sigma_0 - \sigma_z}{2} \otimes \frac{\sigma_0 + \sigma_z}{2} + \frac{\sin\phi_1(J_{b_{1,2}}\dot{J}_{b_{1,1}} - J_{b_{1,1}}\dot{J}_{b_{1,2}})}{\epsilon_1^2} \frac{\sigma_0 - \sigma_z}{2} \otimes \frac{\sigma_0 - \sigma_z}{2}. \quad (\text{B10})$$

For a closed path γ in the parameter space, the path integrals yield $\varphi_0 = \oint_{\gamma_0} [\Gamma_z]_{33} d\gamma$ and $\varphi_1 = \oint_{\gamma_1} [\Gamma_z]_{44} d\gamma$. In Eq. (A5), this results in

$$W(z_0, z_1) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & e^{i\varphi_0} & 0 \\ 0 & 0 & 0 & e^{i\varphi_1} \end{pmatrix} = \frac{\sigma_0 - \sigma_z}{2} \otimes \left[e^{i\varphi_0} \frac{\sigma_0 + \sigma_z}{2} + e^{i\varphi_1} \frac{\sigma_0 - \sigma_z}{2} \right] + \frac{\sigma_0 + \sigma_z}{2} \otimes \sigma_0. \quad (\text{B11})$$

A typical loop γ_{σ_z} that provides a phase shift gate is shown in Fig. 8, split in three branches ℓ_1 , ℓ_2 , and ℓ_3 colored in red, blue, and green, respectively.

The first and the third branches ℓ_1 and ℓ_3 yield zeros in Eq. (B10) as along them $J_{b_{1,1}}$ and $J_{b_{1,2}}$ are respectively zero. Along ℓ_2 the hopping $J_{c_p} = 0 = J_{d_p}$, which implies that the prefactors of the Pauli matrix tensor products in Eq. (B10) are the derivative in z of the complex parameters, i.e., $\partial_z J_{b_L} = [\Gamma_x]_{33}$ and $\partial_z J_{b_R} = [\Gamma_x]_{44}$. In this branch ℓ_2 , where $J_{b_{1,2}} = J \cos\theta$ and $J_{b_{1,1}} = J \sin\theta$ for $\theta \in [0, \frac{\pi}{2}]$, the path integrals reduce to $\varphi_0 = \oint_{\gamma} [\Gamma_z]_{33} d\gamma = \phi_L$ and $\varphi_1 = \oint_{\gamma} [\Gamma_z]_{44} d\gamma = \phi_R$. Hence, the holonomy in Eq. (B11) represent two relative

phase rotation gates, one in each tripod H_L and H_R . These rotations reduce to two σ_z gates, one per tripod, by choosing $\phi_p = \pi$ whose matrix is

$$W(z_0, z_1) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} = \sigma_z \otimes \sigma_0. \quad (\text{B12})$$

In Fig. 7, we show the evolution of the states $|1, 1\rangle$ [Fig. 7(c)] and $|0, 1\rangle$ [Fig. 7(d)] in the tripod H_1 via the cycle in Fig. 8. The two tripods are disjoint as the connecting Hamiltonian $H_T = 0$. These panels show that for $\xi_0 = \frac{\pi}{4}$, the initial state $|1, 1\rangle$ is mapped into itself, while $|0, 1\rangle$ flips sign.

APPENDIX C: TWO QUBITS GATES

Let us discuss the CNOT and the SWAP gates. To realize the former one, we consider a fully symmetric structure where $J_{x_0} = J_{x_1} \equiv J_x$, $J_{b_0} = J_{b_1} \equiv J_b$, and $J_{c_0} = J_{c_1} = J_{d_0} = J_{d_1} \equiv J_c$ for which the Hamiltonian reduces to

$$H(z) = J_b(b_0^\dagger u_0 + b_1^\dagger u_1) + J_x(b_1^\dagger u_0 + b_0^\dagger u_1) + J_c[(c_0^\dagger + d_0^\dagger)u_0 + (c_1^\dagger + d_1^\dagger)u_1] + \text{H.c.} \quad (\text{C1})$$

In this setup, the four nonzero modes in Eq. (A3) reduce to $E = \pm\sqrt{2J_c^2 + (J_x \pm J_b)^2}$. We introduce an orthonormal basis of zero modes for this specific driving cycle, hence, we denote the basis with an apex c

$$|\psi_1^c(z)\rangle = \frac{|c_0\rangle - |d_0\rangle}{\sqrt{2}}, \quad (\text{C2})$$

$$|\psi_2^c(z)\rangle = \frac{|c_1\rangle - |d_1\rangle}{\sqrt{2}},$$

$$|\psi_3^c(z)\rangle = \frac{J_b(2J_c^2 - J_x^2 + J_b^2)|s_0\rangle + J_x(2J_c^2 + J_x^2 - J_b^2)|s_1\rangle}{R_3} + \frac{\sqrt{2}J_c(J_b^2 + 2J_c^2 + J_x^2)|b_0\rangle - 2\sqrt{2}J_x J_b J_c |b_1\rangle}{R_3},$$

$$|\psi_4^c(z)\rangle = \frac{J_x|s_0\rangle + J_b|s_1\rangle - \sqrt{2}J_c|b_1\rangle}{R_4}, \quad (\text{C3})$$

where $|s_0\rangle = \frac{|c_0\rangle + |d_0\rangle}{\sqrt{2}}$, $|s_1\rangle = \frac{|c_1\rangle + |d_1\rangle}{\sqrt{2}}$, $N_{\pm} = \sqrt{2J_c^2 + (J_x \pm J_b)^2}$, $R_4 = \sqrt{2J_c^2 + J_x^2 + J_b^2}$, and $R_3 = R_4 N_+ N_-$. At the initial parameters considered for any cycle $J_x = J_c = 0$, $J_b = J$, these eigenstates in Eq. (C5) reduce to the two qubits encoding states $|s, p\rangle$ for $p = 0, 1$ and $s = 0, 1$

$$|\psi_1^c(0)\rangle = \frac{|c_0\rangle - |d_0\rangle}{\sqrt{2}} \equiv |0, 0\rangle,$$

$$|\psi_2^c(0)\rangle = \frac{|c_1\rangle - |d_1\rangle}{\sqrt{2}} \equiv |0, 1\rangle,$$

$$|\psi_3^c(0)\rangle = \frac{|c_0\rangle + |d_0\rangle}{\sqrt{2}} \equiv |1, 0\rangle,$$

$$|\psi_4^c(0)\rangle = \frac{|c_1\rangle + |d_1\rangle}{\sqrt{2}} \equiv |1, 1\rangle. \quad (\text{C4})$$

We calculate each element of Γ_z in Eq. (A4)

$$[\Gamma_z]_{vv'} = \mathbf{A}_{vv'} \mathbf{V}, \quad (\text{C5})$$

where $\mathbf{V} = (J_b, J_c, J_x)$ and $\mathbf{A}_{vv'}$ are given by

$$A_{vv'} = i\{\langle\psi_v|\partial_{J_b}|\psi_{v'}\rangle, \langle\psi_v|\partial_{J_c}|\psi_{v'}\rangle, \langle\psi_v|\partial_{J_x}|\psi_{v'}\rangle\}. \quad (\text{C6})$$

Since all eigenvectors are real, then $\langle\psi_v|\partial_{J_\ell}|\psi_v\rangle = 0$ for all $v = 1, \dots, 4$ and for each $\ell = b, c, x$. Hence, the diagonal elements of the connection Γ_z are zero. Furthermore, since $|\psi_1\rangle$ and $|\psi_2\rangle$ are independent on the parameter J_ℓ , the only nonzero terms in Γ_z are those induced by $|\psi_3\rangle$ and $|\psi_4\rangle$. It then follows

$$A_{34} = \frac{i}{R_3^2 N_+ N_-} \{J_x(J_b^2 - 2J_c^2 - J_x^2), 4J_b J_c J_x, J_b(J_x^2 - J_b^2 - 2J_c^2)\}, \quad (\text{C7})$$

with $A_{43} = -A_{34}$. The Wilczek-Zee connection then reads

$$\begin{aligned} \Gamma_z &= \frac{2J_x J_c^2 J_b - 4J_b J_c J_x J_c + 2J_b J_c^2 J_x}{R_3^2 N_+ N_-} \\ &\quad - \frac{J_b J_x (J_b J_c + J_x J_c) + J_b^3 J_c + J_b^3 J_x}{R_3^2 N_+ N_-} \frac{\sigma_0 - \sigma_z}{2} \otimes \sigma_y \\ &= f_{\text{CN}}(J_c, J_b, J_x)(\sigma_0 - \sigma_z) \otimes \sigma_y \end{aligned} \quad (\text{C8})$$

for the Pauli matrices σ_p with $i = x, y, z$ and the identity σ_0 . For a closed path γ in the parameter space $\{J_b, J_c, J_x\}$, let us define the result of the path integral yields $\eta = \oint_\gamma [\Gamma_z]_{34} d\gamma$. In Eq. (A5), this results

$$\begin{aligned} W(z_0, z_1) &= \frac{\sigma_0 - \sigma_z}{2} \otimes [\cos \eta \sigma_0 + i \sin \eta \sigma_y] + \frac{\sigma_0 + \sigma_z}{2} \otimes \sigma_0 \\ &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos \eta & \sin \eta \\ 0 & 0 & -\sin \eta & \cos \eta \end{pmatrix}. \end{aligned} \quad (\text{C9})$$

Let us consider the driving cycle γ_C for the CNOT, and split it in three separate branches ℓ_1 , ℓ_2 , and ℓ_3 respectively colored in red, blue, and green. In each branch ℓ_1 , ℓ_2 , and ℓ_3 , only pairs of parameters evolve while the third is set to zero. This immediately sets the first three terms in Eq. (C8) irrelevant as they require the simultaneous variations of all three parameters. The four remaining terms require the simultaneous variation of J_b and J_x , hence, the branches ℓ_1 and ℓ_2 immediately yield zero. Branch ℓ_3 instead reads set $J_b = \cos \theta$ and $J_x = \sin \theta$, with $J_c \equiv 0$. From Eq. (C8)

$$\begin{aligned} [\Gamma_z]_{34} &= \frac{J_x^3 J_b + J_b^3 J_x - J_b J_x (J_b J_b + J_x J_x)}{(J_x^2 + J_b^2)(J_x^2 - J_b^2)} \\ &= \frac{\cos^4 \theta - \sin^4 \theta}{\sin^2 \theta - \cos^2 \theta} = 1. \end{aligned} \quad (\text{C10})$$

Integrated from 0 to $\frac{\pi}{2}$, it follows that $\eta = \int_0^{\frac{\pi}{2}} [\Gamma_z]_{34} dz = \frac{\pi}{2}$ which, via Eq. (C9), implies that the holonomy W is

$$\begin{aligned} W_{\text{CN}} &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \\ &= \frac{1}{2}[(\sigma_0 + \sigma_z) \otimes \sigma_0 + i(\sigma_0 - \sigma_z) \otimes \sigma_y]. \end{aligned} \quad (\text{C11})$$

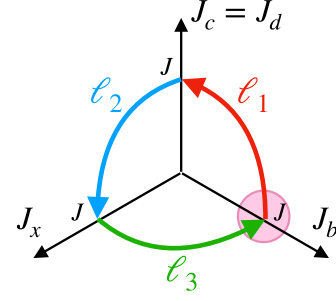


FIG. 9. Driving cycle γ_C split in three branches ℓ_1 (red), ℓ_2 (blue), and ℓ_3 (green).

For the SWAP, we keep the symmetry conditions $J_{x_L} = J_{x_R} \equiv J_x$, $J_{b_L} = J_{b_R} \equiv J_b$, and $J_{c_L} = J_{d_L} \equiv J_c$ set for the CNOT, and impose an antisymmetry condition $J_{c_R} = -J_{d_R} \equiv J_c$. The orthonormal basis of zero modes for this specific driving cycle, hence, we denote the basis with an apex s

$$\begin{aligned} |\psi_1^s(z)\rangle &= \frac{|c_0\rangle - |d_0\rangle}{\sqrt{2}}, \\ |\psi_2^s(z)\rangle &= \frac{J_x |s_0\rangle + J_b |a_1\rangle - \sqrt{2} J_c |b_1\rangle}{R_3}, \\ |\psi_3^s(z)\rangle &= \frac{J_b (2J_c^2 - J_x^2 + J_b^2) |s_0\rangle + J_x (2J_c^2 + J_x^2 - J_b^2) |a_1\rangle}{R_4} \\ &\quad + \frac{\sqrt{2} J_c (J_b^2 + 2J_c^2 + J_x^2) |b_0\rangle - 2\sqrt{2} J_x J_b J_c |b_1\rangle}{R_3}, \\ |\psi_4^s(z)\rangle &= \frac{|c_1\rangle + |d_1\rangle}{\sqrt{2}}, \end{aligned} \quad (\text{C12})$$

where $|a_1\rangle = \frac{|c_1\rangle - |d_1\rangle}{\sqrt{2}}$. Similarly, for the CNOT, at $J_x = J_c = 0$, $J_b = J$ these eigenstates in Eq. (C13) reduce to the two qubits' encoding states $|s, p\rangle$ for $p = 0, 1$ and $s = 0, 1$ as in Eq. (C5). For the same driving cycle in Fig. 9, the holonomy results in

$$\begin{aligned} W_{\text{SW}} &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\ &= \frac{1}{2}(\sigma_0 \otimes \sigma_0 - i\sigma_x \otimes \sigma_y + i\sigma_y \otimes \sigma_x + \sigma_z \otimes \sigma_z). \end{aligned} \quad (\text{C13})$$

Let us obtain the SWAP from the CNOT. A holonomic NOT gate acting on the control qubit on the right tripod H_R can be obtained in Eq. (B5) by setting $\eta_L = 0$ and $\eta_R = \frac{\pi}{2}$. These two angles are obtained by (i) in the left tripod H_L keep all the hopping to zero $J_{\ell_L} = 0$ for $\ell = c, d, b$, while (ii) running two times the driving cycle in Fig. 6 in the right tripod H_R . Equation (B5) then reduces to

$$\begin{aligned} W_{\text{nR}} &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \\ &= \sigma_0 \otimes \frac{\sigma_0 + \sigma_z}{2} + i\sigma_y \otimes \frac{\sigma_0 - \sigma_z}{2}. \end{aligned} \quad (\text{C14})$$

The product

$$W_{\text{nR}}^\dagger W_{\text{CN}} W_{\text{nR}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = W_{\text{SW}} \quad (\text{C15})$$

corresponds to the holonomy W_{SW} of the SWAP gate in Eq. (C14).

APPENDIX D: DEUTSCH'S ALGORITHM

The Deutsch's algorithm leverages on quantum superposition to distinguish binary functions f in Table I between constant and balanced with a single measurement. Formally, for a binary function f and a state $|s, p\rangle$ with control qubit s and target qubit p , the oracle U_f of Deutsch's algorithm acts as $U_f(|s, p\rangle) = |s, p \oplus f(s)\rangle$ the XOR $a \oplus b = [a + b]_2$ (sum modulo 2).

Applied on a superposition

$$|s, -\rangle = \frac{1}{\sqrt{2}}(|s, 0\rangle - |s, 1\rangle), \quad (\text{D1})$$

the oracle U_f yields the *phase kickback* property $U_f|s, -\rangle = (-1)^{f(s)}|s, -\rangle$. For a superposition state

$$\begin{aligned} |+, -\rangle &= \frac{1}{\sqrt{2}}(|0, -\rangle + |1, -\rangle) \\ &= \frac{1}{2}(|0, 0\rangle - |0, 1\rangle + |1, 0\rangle - |1, 1\rangle), \end{aligned} \quad (\text{D2})$$

obtained by applying the Hadamard gates on both control and target on an initial input $|0, 1\rangle$, it then follows that

$$\begin{aligned} U_f|+, -\rangle &= |-\rangle \otimes \frac{(-1)^{f(0)}|0\rangle + (-1)^{f(1)}|1\rangle}{\sqrt{2}} \\ &= |-\rangle \otimes (-1)^{f(0)} \frac{|0\rangle + (-1)^{f(0) \oplus f(1)}|1\rangle}{\sqrt{2}} \\ &= \begin{cases} (-1)^{f(0)}|+, -\rangle & f(0) \oplus f(1) = 0, \\ (-1)^{f(0)}|-, -\rangle & f(0) \oplus f(1) = 1. \end{cases} \end{aligned} \quad (\text{D3})$$

A final Hadamard gate on the control qubit s allows then to distinguish constant functions from balanced ones.

We now explicitly show that the photonic array proposed in Fig. 1(a) and the driving cycle in Fig. 3(b) with proper signs of the hopping terms implement the oracle identity in Eq. (D3). Consider the Hamiltonian $H = H_0 + H_1 + H_T$, where $H_T = J_{x_0} b_1^\dagger u_0 + J_{x_1} b_0^\dagger u_1 + \text{H.c.}$ and for $p = 0, 1$ we have that $H_p = (J_{b_p} b_p^\dagger + J_{c_p} c_p^\dagger + \alpha_p J_{c_p} d_p^\dagger) u_p + \text{H.c.}$ with $J_{d_p} = \alpha_p J_{c_p}$ for $\alpha_p = \pm 1$. The driving cycle in Fig. 3(b) yields four holonomies W_{α_0, α_1}

$$\begin{aligned} W_{1,1} &= \frac{1}{2}[(\sigma_0 + \sigma_z) \otimes \sigma_0 + i(\sigma_0 - \sigma_z) \otimes \sigma_y], \\ W_{1,-1} &= \frac{1}{2}[\sigma_0 \otimes \sigma_0 + \sigma_z \otimes \sigma_z + i(\sigma_y \otimes \sigma_x - \sigma_x \otimes \sigma_y)], \\ W_{-1,1} &= \frac{1}{2}[\sigma_0 \otimes \sigma_0 - \sigma_z \otimes \sigma_z + i(\sigma_y \otimes \sigma_x + \sigma_x \otimes \sigma_y)], \\ W_{-1,-1} &= \frac{1}{2}[(\sigma_0 - \sigma_z) \otimes \sigma_0 + i(\sigma_0 + \sigma_z) \otimes \sigma_y]. \end{aligned} \quad (\text{D4})$$

Straightening the minus signs in the holonomies via the phase gate, we obtain the four transformations

$$\begin{aligned} V_{f_1} &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, & V_{f_4} &= \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \\ V_{f_2} &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, & V_{f_3} &= \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}. \end{aligned} \quad (\text{D5})$$

We define the functions $f_1, \dots, f_4$ in Table I via the transformations $V_{f_1}, \dots, V_{f_4}$ in Eq. (D5) by first setting the input $x \in \{0, 1\}$ in the two-qubits states as

$$\text{input } \mathcal{I} : \begin{cases} x = 0 \mapsto |0, 0\rangle, \\ x = 1 \mapsto |1, 1\rangle. \end{cases} \quad (\text{D6})$$

Then we compose $V_{f_1}, \dots, V_{f_4}$ with a measurement done via scalar product $P_1 = \langle 1, 0| + \langle 1, 1|$

$$f_\ell(x) = |P_1 \circ V_{f_\ell} |x, x\rangle|^2. \quad (\text{D7})$$

Indeed,

$$f_1(x) = \begin{cases} 0 \xrightarrow{\mathcal{I}} |0, 0\rangle \xrightarrow{V_{f_1}} |0, 0\rangle \xrightarrow{P_1} 0 \\ 1 \xrightarrow{\mathcal{I}} |1, 1\rangle \xrightarrow{V_{f_1}} |1, 0\rangle \xrightarrow{P_1} 0, \end{cases} \quad (\text{D8})$$

$$f_2(x) = \begin{cases} 0 \xrightarrow{\mathcal{I}} |0, 0\rangle \xrightarrow{V_{f_2}} |0, 0\rangle \xrightarrow{P_1} 0 \\ 1 \xrightarrow{\mathcal{I}} |1, 1\rangle \xrightarrow{V_{f_2}} |1, 1\rangle \xrightarrow{P_1} 1, \end{cases} \quad (\text{D9})$$

$$f_3(x) = \begin{cases} 0 \xrightarrow{\mathcal{I}} |0, 0\rangle \xrightarrow{V_{f_3}} |0, 1\rangle \xrightarrow{P_1} 1 \\ 1 \xrightarrow{\mathcal{I}} |1, 1\rangle \xrightarrow{V_{f_3}} |1, 0\rangle \xrightarrow{P_1} 0, \end{cases} \quad (\text{D10})$$

$$f_4(x) = \begin{cases} 0 \xrightarrow{\mathcal{I}} |0, 0\rangle \xrightarrow{V_{f_4}} |0, 1\rangle \xrightarrow{P_1} 1 \\ 1 \xrightarrow{\mathcal{I}} |1, 1\rangle \xrightarrow{V_{f_4}} |1, 1\rangle \xrightarrow{P_1} 1. \end{cases} \quad (\text{D11})$$

The superposition states where we perform the Deutsch's algorithm are

$$\begin{aligned} |+, -\rangle &= \frac{1}{2}(|0, 0\rangle - |0, 1\rangle + |1, 0\rangle - |1, 1\rangle), \\ |-, -\rangle &= \frac{1}{2}(|0, 0\rangle - |0, 1\rangle - |1, 0\rangle + |1, 1\rangle). \end{aligned} \quad (\text{D12})$$

Then

$$\begin{aligned} V_{f_1}(|-, -\rangle) &= \frac{1}{2}(|0, 0\rangle - |0, 1\rangle - |1, 1\rangle + |1, 0\rangle) = |+, -\rangle, \\ V_{f_2}(|-, -\rangle) &= \frac{1}{2}(|0, 0\rangle - |1, 0\rangle - |0, 1\rangle + |1, 1\rangle) = |-, -\rangle, \\ V_{f_3}(|-, -\rangle) &= \frac{1}{2}(|1, 1\rangle - |0, 1\rangle - |1, 0\rangle + |0, 0\rangle) = -|-, -\rangle, \\ V_{f_4}(|-, -\rangle) &= \frac{1}{2}(|0, 1\rangle - |0, 0\rangle - |1, 0\rangle + |1, 1\rangle) = -|+, -\rangle. \end{aligned} \quad (\text{D13})$$

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