

Beyond mean-field effects in dynamics of Bose-Einstein condensate in two-core trap

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The nonlinear dynamics of quantum droplets in a quasi-one-dimensional BEC loaded in a two-core trap potential are investigated. The cases of standing and moving quantum droplets are considered. The frequencies of the Josephson oscillations and the condition for self-trapping are found. An analytical description is based on the variational approach to a system of coupled extended Gross-Pitaevskii (EGP) equations, including Lee-Huang-Yang correction terms. In the case of moving droplets, the system of equations describing the coupling of the relative motion of droplets in two cores with the dynamics of the relative phase and population imbalance (Josephson currents) is derived. Different regimes of droplet dynamics are found, including attraction and repulsion in dependence on the relative phase evolution. These phenomena are the realization of the Andreev-Bashkin effect for superfluids in a two-core trap potential. Analytical predictions of the model are confirmed by full numerical simulations of the coupled extended Gross-Pitaevskii equations.

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I. INTRODUCTION

Recently, much attention has been paid to the problem of beyond mean-field (BMF) effects in the dynamics of Bose-Einstein condensates. First, such corrections to the energy appearing from the zero-point energy of Bogolyubov excitations have been calculated by Lee, Huang, and Yang [1]. The corrections from quantum fluctuations are small, so they influence the dynamics of a single-component Bose condensate. However, in the case of unstable collapsing Bose-Bose mixtures with a residual weak attraction, the repulsive interaction induced by quantum fluctuations (QFs) can play a stabilizing role and lead to the generation of quantum droplets (QDs) [2,3]. Later, the existence of quantum droplets has been confirmed in the experiment [4,5]. Now QD is found in many systems, such as dipolar BEC [6], Bose-Fermi mixtures [7], etc.

Other phenomena, when the BMF effects are considered, have been studied recently. Between them are the influence of QFs on the modulational instability of matter waves [8], on the miscible-immiscible transition in two-component BEC [9], on the dynamics of the Townes soliton [10], on the spectrum of impurities embedded into quantum droplets [11,12], on the dynamics in atomic Josephson junctions [13,14], etc.

An interesting problem is the dynamics of two interacting quantum droplets. In distinction from the interaction of classical fluid droplets, the collision of QD shows new phenomena such as the Josephson oscillations of the population between droplets. In a three-dimensional (3D) Bose-Bose mixture, this system has been investigated in Ref. [15]. It was shown,

using the phase-dependent potential interaction picture, that the relative motion of the droplets is influenced by the relative phase and population imbalance dynamics. The occurrence of the Andreev-Bashkin effect was shown.

The direct overlap of 3D droplets during the collision process makes the investigation of the problem complicated and is based mainly on intensive numerical simulations of the interaction of 3D localized structures in multidimensional extended Gross-Pitaevskii (EGP) equations. Also, the three-body losses shortened the time for experimental observations of the beyond mean-field effects.

Another system exists where the influence of QFs on the interaction of droplets and so on the Josephson oscillations and droplet motion can be investigated. It is the BEC loaded in a cigar-type dual-core trap. The advantage of this configuration is that it admits an analytical description. Also, the three-body losses in comparison with the 3D case are suppressed in such a system. The stationary states and the spontaneous symmetry breaking of quantum droplets in a system of binary bosonic gas trapped in a two-core cigar-shaped trap, have been investigated in Ref. [16]. Exact front solutions for this coupler system have been found.

In this work, we will study nonstationary phenomena such as macroscopic quantum tunneling and localization phenomena in this system. In particular, we will study the Josephson oscillation phenomena between quantum droplets, posed in different cores of the dual-core trap. The cases of standing and moving quantum droplets will be considered.

The structure of the paper is as follows. In Sec. II, we describe the model of the BEC in a two-core trap potential, including the effects of quantum fluctuations, by the Lee-Huang-Yang correction terms. Next, in Sec. III, we consider two standing QDs including the case of Lee-Huang-Yang (LHY) fluids, when the residual mean-field interaction is equal to zero. The variational approach is applied to derive

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the effective two-mode model, describing the Josephson oscillations between the droplets and localization processes. The results are compared with the full numerical simulations of the system of coupled EGP equations. In Sec. IV, the interaction of two QDs moving in a two-core trap potential is investigated. Using the variational approach, the system of coupled equations describing the relative motion of droplets and the relative phases and populations is derived. Analytical predictions of this model are compared with the full numerical simulations of EGP equations. In conclusion, the obtained results are reviewed and discussed.

II. MODEL

We consider the dynamics of a two-component BEC loaded in the dual-core cigar-type trap, when the beyond mean-field effects, describing quantum fluctuations, are taken into account. The case when the wave functions of each component inside each core are equal is considered. In this case, the system is described by two coupled modified Gross-Pitaevskii equations (GPEs) of the form [2,3,16,17]

$$i\hbar\Psi_{1,T} = -\frac{\hbar^2}{2m}\Psi_{1,XX} + V(X)\Psi_1 + \delta g|\Psi_1|^2\Psi_1 - \gamma|\Psi_1|\Psi_1 - K_0\Psi_2, \quad (1)$$

$$i\hbar\Psi_{2,T} = -\frac{\hbar^2}{2m}\Psi_{2,XX} + V(X)\Psi_2 + \delta g|\Psi_2|^2\Psi_2 - \gamma|\Psi_2|\Psi_2 - K_0\Psi_1, \quad (2)$$

where $\delta g = 2\hbar\omega_\perp\delta a_s$, $\delta a_s = a_{s,12} + a_s$, $a_{s,11} = a_{s,22} = a_s$, $\omega_\perp$ is the transverse atomic scattering length, and $a_{s,ij}$ are the intra- and interspecies atomic scattering lengths. The strength of the quantum fluctuation $\gamma = \sqrt{2mg^3/2}/(\pi\hbar)$, and K_0 is the parameter of the tunnel coupling.

The coupled-mode system (1) and (2) is derived for GPEs with beyond mean-field terms in the trap potential, having a double-well form in the transverse direction (see, for example, Refs. [17,18]). The tight-binding approximation is applied for the weak (linear) coupling K between modes. The condition $\hbar\omega_\perp \gg gn$ corresponding to the quasi-one-dimensional description is used. In general, there should be four coupled equations for the modes. In the case of the symmetric configuration considered here [2], the system is reduced to two coupled GP equations. A similar system appears in the problem of Rabi-coupled condensates. In this case, the beyond mean-field corrections to the energy have the form of effective two- and three-body interactions [19,20]. The *internal* Josephson oscillations and related quantum droplet dynamics in this system require a separate investigation.

In dimensionless form, with parameters

$$\begin{aligned} t &= \frac{T}{t_0}, \quad x = \frac{X}{x_0}, \quad u(v) = \frac{\Psi}{\psi_0}, \quad x_0 = \frac{\pi\hbar^2\sqrt{\delta g_0}}{\sqrt{2mg^{3/2}}}, \\ t_0 &= \frac{\pi^2\hbar^3\delta g_0}{2mg^3}, \quad \psi_0 = \frac{\sqrt{2m}}{\pi\hbar\delta g_0g^{3/2}}, \\ K &= \frac{K_0}{E_0}, \quad E_0 = \frac{\hbar^2}{2mx_0^2}, \end{aligned} \quad (3)$$

the system (1) and (2) has the form [21]

$$\begin{aligned} iu_t + u_{xx} + \delta|u|^2u + |u|u + Ku &= 0, \\ iv_t + v_{xx} + \delta|v|^2v + |v|v + Kv &= 0, \end{aligned} \quad (4)$$

$\delta = \delta g/\delta g_0$, and δg_0 is the characteristic scale for the residual atomic scattering length δa_s , which is typically of order of a few a_0 , where a_0 is the Bohr radius. Note that this equation is valid when the condition $\xi = gn/\hbar\omega_\perp = na_s l_\perp^2 \ll 1$ is satisfied, $l_\perp = \sqrt{\hbar/(m\omega_\perp)}$. The number of atoms N_{ph} is related with $N_1 = \int_{-\infty}^{+\infty} |u|^2 dx$ and $N_2 = \int_{-\infty}^{+\infty} |v|^2 dx$ as

$$N_{\text{ph}} = N_j \psi_0^2 x_0 = N_j \frac{2}{\pi} \left(\frac{g}{\delta g_0} \right)^{3/2} \quad (j = 1, 2). \quad (5)$$

The important limiting case is when the intra- and interspecies strengths are balanced, i.e., $\delta g = 0$, the so-called LHY fluid [22,23]. Thus the effective nonlinearity is produced solely by the quantum fluctuations.

The values of the parameters for the experiment are $(l_\perp/l_x)^2 \sim 10^{-2}$, $\gamma \gtrsim 0.1$, and $N_j \sim 10$ for $N_{\text{ph}} \sim 10^3$, $a_s \sim 0.3l_\perp$.

III. JOSEPHSON OSCILLATIONS OF POPULATION BETWEEN TWO DROPLETS

Let us consider the dynamics of two quantum droplets in each core of the trap. The unperturbed solution of the system (2) ($K = \delta g = 0$) is

$$u(x, t) = A \text{sech}^2(ax) e^{-i\mu t}. \quad (6)$$

For an analysis of the dynamics of two interacting droplets, it is useful to apply the variational approach [24,25].

We will look for the solutions of the form

$$\begin{aligned} u &= A_1 \text{sech}^2(ax) e^{i\theta_1}, \\ v &= A_2 \text{sech}^2(ax) e^{i\theta_2} \end{aligned} \quad (7)$$

where $A_j, \theta_j, j = (1, 2)$ are functions of time, and the width a is assumed to be the same for both components and constant [25]. Note that using the Gaussian ansatz gives small deviating results. In the variational approach, we should calculate the averaged Lagrangian $\bar{L}$,

$$\bar{L} = \int_{-\infty}^{\infty} L(u, v) dx, \quad (8)$$

where L is the Lagrangian of the system (4). The total number of atoms is equal to

$$N = \int_{-\infty}^{\infty} (|u|^2 + |v|^2) dx = \frac{4}{3a} (A_1^2 + A_2^2). \quad (9)$$

Substituting the ansatz (7) into the expression for the averaged Lagrangian (8), we obtain the following:

$$\begin{aligned} \bar{L} &= \sum_{j=1}^2 \left[\frac{4}{3a} A_j^2 \theta_{jt} - \frac{32}{45a} A_j^3 + \frac{16}{15} a A_j^2 - \frac{16\delta}{35a} A_j^4 \right] \\ &\quad - \frac{8KA_1A_2}{3a} \cos(\theta_1 - \theta_2). \end{aligned} \quad (10)$$

Using this $\bar{L}$ and the corresponding Euler-Lagrange equations we obtain the system of equations for the droplet

parameters,

$$\begin{aligned} \frac{1}{2}(A_1^2 - A_2^2)_t &= 2KA_1A_2 \sin(\theta_1 - \theta_2), \\ \theta_t &= \frac{4}{5}(A_1 - A_2) + \frac{24\delta}{35}(A_1^2 - A_2^2) \\ &\quad + K\left(\frac{A_2}{A_1} - \frac{A_1}{A_2}\right) \cos(\theta_1 - \theta_2), \end{aligned} \quad (11)$$

where $\theta = \theta_1 - \theta_2$ is the relative phase. Introducing the relative imbalance

$$Z = \frac{N_1 - N_2}{N}, \quad N = N_1 + N_2,$$

we can rewrite the system in the form

$$\begin{aligned} Z_t &= 2K\sqrt{1 - Z^2} \sin \theta, \\ \theta_t &= -\frac{2KZ}{\sqrt{1 - Z^2}} \cos \theta + \frac{18}{35}\delta NZ \\ &\quad + \frac{\sqrt{6aN}}{5}(\sqrt{1 + Z} - \sqrt{1 - Z}). \end{aligned} \quad (12)$$

We can write the Hamiltonian function for this system of equations

$$\begin{aligned} H &= 2K\sqrt{1 - Z^2} \cos \theta + \frac{9}{35}\delta NZ^2 \\ &\quad + \frac{2\sqrt{6}}{15}\sqrt{aN}[(1 + Z)^{3/2} + (1 - Z)^{3/2}]. \end{aligned} \quad (13)$$

The Hamiltonian equations for the canonical variables Z, θ are

$$Z_t = -\frac{\partial H}{\partial \theta}, \quad \theta_t = \frac{\partial H}{\partial Z}.$$

In that form, the system is similar to that considered in Refs. [26,27]. Here, two modes are possible: the zero-phase mode ($\theta = 0$) and the π -phase mode ($\theta = \pi$).

Linearizing the system around the points $Z = 0, \theta = 0$ (the zero-phase mode), we obtain for the Josephson oscillations frequencies

$$\omega_J = 2K\sqrt{1 - \frac{9\delta N}{35K} - \frac{\sqrt{6aN}}{10K}}. \quad (14)$$

In the π -phase mode we have the Josephson frequency as

$$\omega_J = \omega_J = 2K\sqrt{1 + \frac{9\delta N}{35K} + \frac{\sqrt{6aN}}{10K}}. \quad (15)$$

A. Self-trapping mode

A self-trapping mode means that the atoms are localized in each well, i.e., $\langle Z \rangle \neq 0$. This regime usually occurs in the large-amplitude oscillation of the atomic population imbalance $Z(t)$. Using the Hamiltonian function, the conditions for the appearance of the localization mode can be studied,

$$H(Z(0), \theta(0)) > 2K + \frac{4\sqrt{6}}{15}\sqrt{aN}.$$

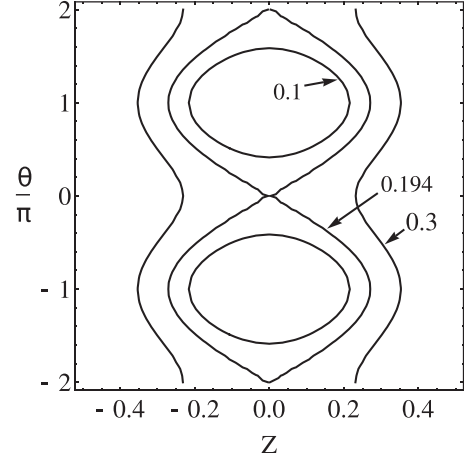


FIG. 1. This phase portrait is presented for the parameters $K = 0.01$, $\delta = 0$ in different initial values of $Z(0)$. Here, $a = 0.47$, $N = 10$, $\theta(0) = \pi/2$.

Then we can obtain the critical value of K for the symmetry breaking in the LHY droplet case ($\delta = 0$):

$$K > K_c = \frac{\sqrt{6aN}\{2 - [1 + Z(0)]^{3/2} - [1 - Z(0)]^{3/2}\}}{15[\sqrt{1 - Z(0)^2} \cos \theta(0) - 1]}. \quad (16)$$

We can obtain a phase portrait using the Hamiltonian function and draw general conclusions about the self-trapping regime. It can be seen in Fig. 1 that Josephson oscillations ($\langle Z \rangle = 0$) can be seen when $Z(0) = 0.1$, and $Z(0)_c = 0.194$ is the critical point of transition from Josephson oscillations to the self-trapping mode. When $Z(0) > Z(0)_c$, the self-trapping mode starts. These results prove that Eq. (16) is also correct, because if we put $Z(0) = 0.194$ into Eq. (16), we get $K \approx 0.01$.

B. Numerical simulations

We look for the solution of Eq. (4) in the following form:

$$\begin{aligned} u(x, t) &= \bar{u}e^{-i\mu t}, \\ v(x, t) &= \bar{v}e^{-i\mu t}. \end{aligned} \quad (17)$$

By putting Eq. (17) into Eq. (4), the following system of equations can be obtained:

$$\begin{aligned} -\mu \bar{u}_t + \bar{u}_{xx} + \delta |\bar{u}|^2 \bar{u} + |\bar{u}| \bar{u} + K \bar{v} &= 0, \\ -\mu \bar{v}_t + \bar{v}_{xx} + \delta |\bar{v}|^2 \bar{v} + |\bar{v}| \bar{v} + K \bar{u} &= 0. \end{aligned} \quad (18)$$

If $\delta = 0$ and $u \approx v$ are taken, the system of Eq. (18) can be analytically solved,

$$\bar{u} = 1.5(\mu - K) \text{sech}^2(0.5\sqrt{\mu - K}x). \quad (19)$$

Comparing Eq. (19) with Eq. (6) and considering Eq. (9), the following results are obtained:

$$\begin{aligned} a &= 0.5\sqrt{\mu - K}, \quad A = 1.5(\mu - K), \\ N_1 &= N_2 = 6(\mu - K)^{3/2} = 48a^3. \end{aligned}$$

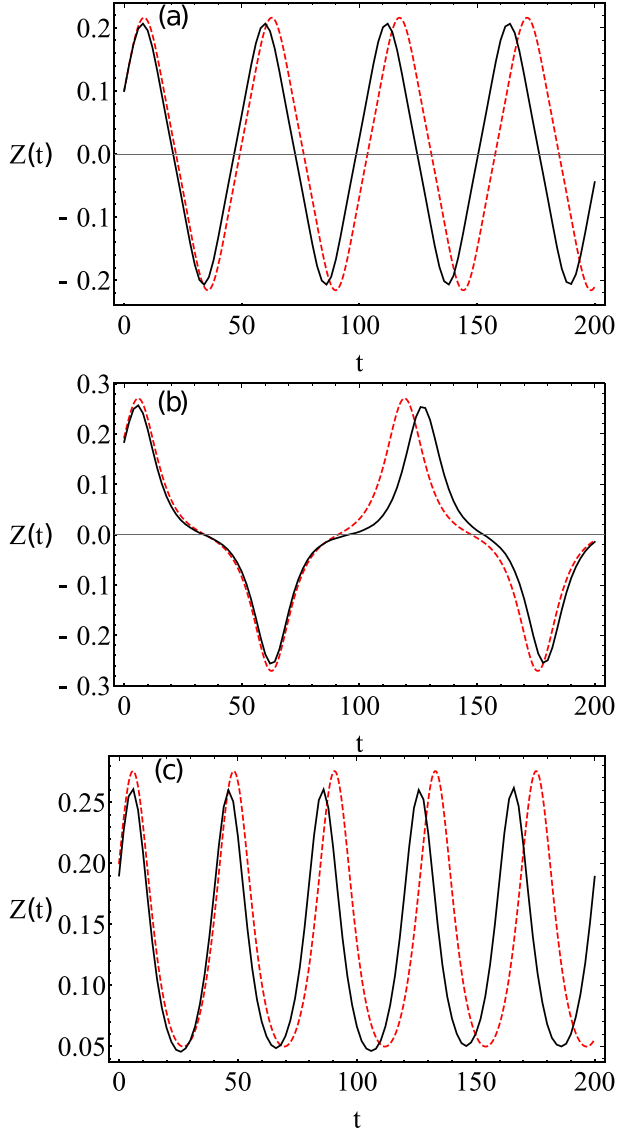


FIG. 2. The dynamics of $Z(t)$ are presented. The solid and dashed lines correspond to the numerical and variational results, respectively. The initial values of $Z(t)$ are (a) $Z(0) = 0.1$ (for both lines), (b) $Z(0)_c = 0.184$ (for the solid line), $Z(0)_c = 0.193$ (for the dashed line), (c) $Z(0) = 0.19$ (for the solid line), $Z(0) = 0.2$ (for the dashed line), and $\theta = 0.5\pi$ for all cases. In all cases, the parameters are $K = 0.01$, $\delta = 0$, $a = 0.47$, $N_1 + N_2 = 10$.

Let us compare the numerical and variational results. In Fig. 2(a), the Josephson oscillations are generated when $Z(0) = 0.1$. In this case, it can be seen that the numerical and variational results are in good agreement. Figure 2(b) depicts the transition from Josephson oscillations to the self-trapping regime. Here, it can be seen that the variational and numerical critical values differ by about 5%. In Fig. 2(c), the self-trapping regimes are presented.

Figure 3 shows the Josephson frequency for the LHY fluid ($\delta = 0$). In this case, the results of Eq. (15) and numerical calculations are compared in the π -phase mode. In the zero-phase mode, there is a localization mode for all initial imbalances $Z(0)$. To observe Josephson oscillations in the zero-phase mode, the condition $aN < 100K^2/6$ must be

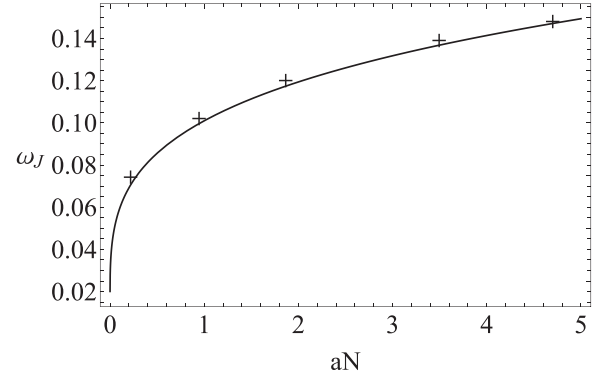


FIG. 3. A graph showing the dependence of the Josephson frequency on aN is presented for the LHY fluid ($\delta = 0$). The solid line is based on Eq. (15) when $\delta = 0$, and the “+” points are based on numerical simulations. The initial values are $Z(0) = 0.1$, $\theta(0) = \pi$.

satisfied. Given that $K = 0.01$, it follows that $aN < 0.0017$. It is only when this condition is met can Josephson oscillations exist.

IV. QUANTUM DROPLETS MOVING IN TWO CHANNELS

In this section we will study the general case when quantum droplets are moving in two channels, when $\delta g \neq 0$. For an analytical description of this problem we will apply the variational approach (VA). Since $\delta g \neq 0$, we will use another ansatz for the trial functions, namely the Gaussian one:

$$u(x, t) = A_1 \exp\left(\frac{(x - \xi_1)^2}{2a^2} + ik_1(x - \xi) + i\theta_1\right), \quad (20)$$

$$v(x, t) = A_2 \exp\left(\frac{(x - \xi_2)^2}{2a^2} + ik_2(x - \xi) + i\theta_2\right). \quad (21)$$

As was shown in Ref. [21], this ansatz works well for $N \ll 1$. To describe the flat-top droplets, the super-Gaussian trial functions give a good description of the dynamics [28]. But in this case, the simplicity of VA is lost and the numerical simulations of the original system are preferable. We substitute these ansatz functions into the set of equations (4) and obtain the averaged Lagrangian function,

$$\begin{aligned} \bar{L}_{\text{tot}} = \sum_{j=1}^2 \left[A_j^2 (\theta_{jt} - k_j \xi_{jt}) \sqrt{\pi} a - \left(\frac{2}{3}\right)^{3/2} \sqrt{\pi} A_j^3 a \right. \\ \left. - \frac{\sqrt{\pi}}{2\sqrt{2}} \delta A_j^4 a + \frac{\sqrt{\pi} A_j^2}{2a} + \sqrt{\pi} A_j^2 k_j^2 a \right] \\ - 2\sqrt{\pi} K A_1 A_2 a \cos(\theta) \exp\left(-\frac{\xi^2}{4a^2} - \frac{k^2 a^2}{4}\right), \quad (22) \end{aligned}$$

where $\xi = \xi_1 - \xi_2$ is the relative center of mass and $k_1 = -k/2$, $k_2 = k/2$. Using this Lagrangian function and the Euler-Lagrange equations, we obtain a set of equations for the relative variables:

$$\begin{aligned} Z_t &= 2K\sqrt{1-Z^2} \sin(\theta) \exp\left(-\frac{\xi^2}{4a^2} - \frac{k^2 a^2}{4}\right), \\ \theta_t &= -\frac{KZ}{\sqrt{1-Z^2}} (k^2 a^2 + 2) \cos(\theta) \exp\left(-\frac{\xi^2}{4a^2} - \frac{k^2 a^2}{4}\right) \end{aligned}$$

$$\begin{aligned}
& + \sqrt{\frac{N}{3\sqrt{\pi}a}}(\sqrt{1+Z} - \sqrt{1-Z}) + \frac{\delta N}{\sqrt{2\pi}a}Z, \\
\xi_t &= -2k - \frac{2Kka^2}{\sqrt{1-Z^2}} \cos(\theta) \exp\left(-\frac{\xi^2}{4a^2} - \frac{k^2a^2}{4}\right), \\
k_t &= \frac{kZZ_t}{\sqrt{1-Z^2}} + \frac{2K\xi \cos(\theta)}{a^2\sqrt{1-Z^2}} \exp\left(-\frac{\xi^2}{4a^2} - \frac{k^2a^2}{4}\right). \quad (23)
\end{aligned}$$

The effective potential, describing the interaction of two QDs, can be obtained from the equation

$$\xi_{tt} = -\frac{\partial U}{\partial \xi}, \quad (24)$$

and is (for $ka \ll 1$, $kK \ll 1$)

$$U = -\frac{8K}{\sqrt{1-Z^2}} \exp\left(-\frac{\xi^2}{4a^2}\right) \cos(\theta). \quad (25)$$

It is seen that the interaction potential of QD depends on the relative distance and the relative phase difference between the droplets. See the analogous result for solitons in the nonlinear directional optical couplers [29,30]. Thus, in dependence on θ , the interaction potential can be attractive or repulsive. Since the internal motion of the droplet depends on the relative phase, the evolution of the latter can lead to the change of the character of interaction. In particular, the growth of the relative phase with time can lead to the change of the attraction between droplets to the repulsion and vice versa. Coupling between Josephson currents and the relative motion of the superfluids in a two-core trap system is an example of the Andreev-Bashkin effect [15,31,32].

Numerical simulations

Let us compare the analytical results taken from Eq. (23) with numerical simulations. We can obtain $N = \frac{1024}{9\pi^{3/2}a^3}$ using the equality of the amplitudes and the number of particles of the trial functions (19) and (20). Of course, this is true for $\delta = 0$ and $u = v$, but if δ is close to zero and the functions u and v are not very different from each other, this equality can be considered sufficiently accurate.

In Fig. 4, the dynamics of the center of mass, $\xi(t)$, and the relative imbalance, $Z(t)$, are presented for different initial values of $Z(t)$. As evident from the graphs, the results of the variational approach and numerical calculations align qualitatively well. Figure 4 shows graphs for only two initial phases, $\theta(0) = 0$ and $\theta(0) = \pi$, while similar behavior for $\theta(0) = \pi$ is also observed in the remaining phases. The main changes are observed when varying the value of $Z(0)$; therefore, we focused more attention on this case. Figures 4(a) and 4(b) show the dynamics for the zero-phase and π -phase modes when $Z(0) = 0.1$. In the zero-phase mode, the QDs interact with each other for some time, oscillating around the zero point, but eventually move away from each other. The reason for this is that, as seen from Eq. (25), the interaction potential between the QDs depends on the phase $\theta(t)$, which increases with time [see the blue line in Fig. 4(a)]. Phase growth transforms an initially attractive potential into a repulsive potential, or vice versa, QDs naturally move away from each other in a repulsive potential. In the π -phase mode, the QDs move

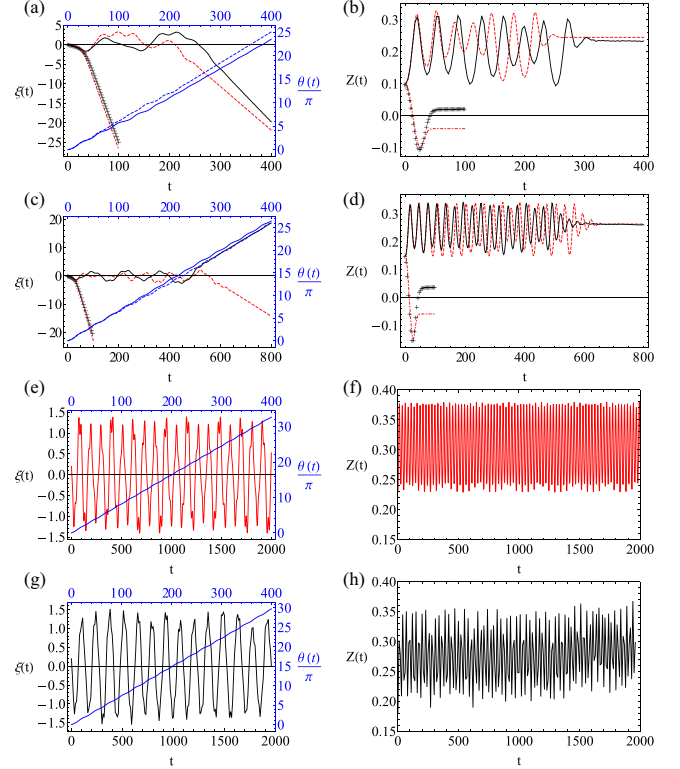


FIG. 4. The dynamics of the relative imbalance $Z(t)$, the relative center of mass $\xi(t)$, and the relative phase $\theta(t)$ (blue line) are presented for zero-phase and π -phase modes. The solid line and “+” points correspond to the numerical results for zero-phase and π -phase modes, respectively. The dashed and dotted-dashed lines correspond to the variational approach, respectively. The initial values are (a), (b) $Z(0) = 0.1$, (c), (d) $Z(0) = 0.15$, (e), (f) (only the variational approach) $Z(0) = 0.23$, (g), (h) (only numerical) $Z(0) = 0.18$, $k(0) = 0.02$, $\xi(0) = 0.2$, and the parameters are $N_1 + N_2 = 10$, $a = 1.6$, $\delta = -0.1$, $K = 0.01$.

away from each other with almost no oscillation because, in this case, the interaction potential is repulsive from the beginning.

As shown in Figs. 4(c) and 4(d), an increase in the value of $Z(0)$ leads to a longer oscillation time for the QDs. However, this behavior is observed only in the zero-phase regime, with almost no change occurring in the π -phase regime. If the values of $Z(0)$ are further increased, the oscillation time of QDs increases suddenly [see Figs. 4(e)–4(h)]. This situation can be explained as follows: When the value of $Z(0)$ increases, the phase grows faster, and the repulsive potential transforms into an attractive potential before the QDs have time to move away from each other. As a result, QDs can continue their oscillations around the zero point for a long time. This sharp increase in oscillation time was observed at $Z(0) = 0.18$ in the numerical calculation and $Z(0) = 0.23$ in the variational approach. Naturally, as $Z(0)$ increases, the error also increases, and as the parameter a is chosen, the functions $u(x)$ and $v(x)$ are taken to be approximately equal. There are only self-trapping regimes in all graphs in Fig. 4.

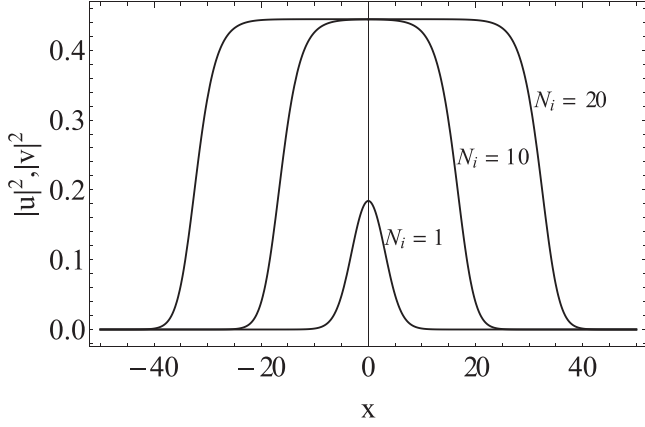


FIG. 5. The form of solution (26) is presented for different normalizations $N_i = 1, 10, 20$.

If $u \approx v$ is taken, Eq. (18) has an exact flat-top solution, as shown in Ref. [28],

$$\bar{u}, \bar{v} = -\frac{3(\mu + K)}{1 + \sqrt{1 - 9(\mu + K)\delta/2} \cosh[\sqrt{-(\mu + K)x}]}, \quad (26)$$

and we can also find the relation between the number of atoms N and the chemical potential μ ,

$$N_i = \frac{4}{3} \left(\sqrt{-\frac{2}{\delta^3}} \ln \frac{\sqrt{4.5(\mu + K)\delta} + 1}{\sqrt{1 - 4.5(\mu + K)\delta}} - \sqrt{-\frac{9\mu}{\delta^2}} \right). \quad (27)$$

If $\delta = -1$ is chosen, it can be seen from Eq. (27) that $N_i \rightarrow \infty$ ($i = 1, 2$) when $\mu + K \rightarrow -2/9$. The form of the solution given by Eq. (26) is illustrated in Fig. 5 for different normalizations N . As the value of N increases, the solution adopts a flat-top shape.

Due to the fact that it is difficult to obtain equations similar to the system of equations (23) from a solution of this form, the solutions were presented only based on numerical simulations (see Fig. 6). The dynamics of the relative imbalance $Z(t)$, the relative center of mass $\xi(t)$, and the relative phase $\theta(t)$ [the insets of Figs. 6(a), 6(c) and 6(e)] are presented for different phase modes. As seen in Fig. 6(a), in this type of solution, the QDs oscillate around the zero point and do not move away from each other. The interaction potential between them remains attractive because the relative phase changes very little near the zero point [see the inset of Fig. 6(a)]. In this case, Josephson oscillations can also be observed [see Fig. 6(b)]. Similarly, the situations in Figs. 6(c) and 6(d) are almost the same as before. Here too the interaction potential remains attractive, the QDs oscillate around the zero point, and Josephson oscillations occur between them as a result of the tunneling effect. However, it can be seen from the inset of Fig. 6(e) that when the initial phase is π , the phase oscillates around the point π and the interaction potential is repulsive. As a result, QDs move away from each other [see Fig. 6(e)].

The Andreev-Bashkin effect is shown in Fig. 7 for different initial relative imbalances $Z(0)$, where the flow of one quantum fluid causes another to flow in the same direction without viscosity. In Figs. 7(a) and 7(b), as a result of phase growth, the interaction potential changes from attractive to

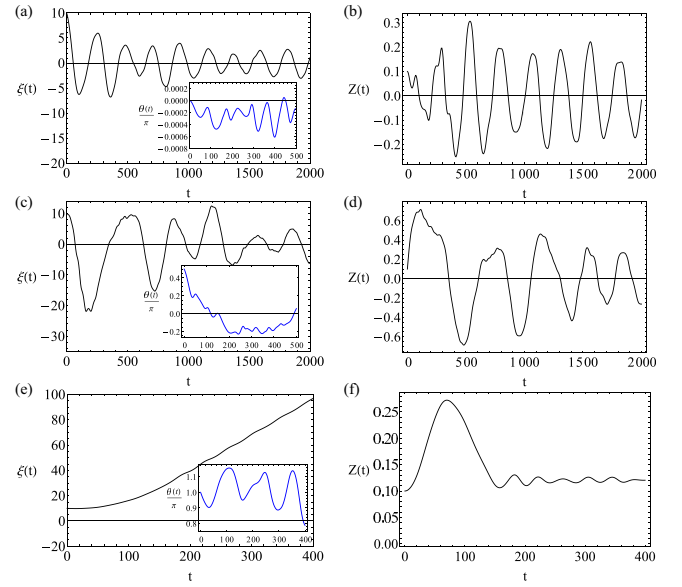


FIG. 6. The dynamics of the relative imbalance $Z(t)$, the relative center of mass $\xi(t)$, and the relative phase $\theta(t)$ [inset of (a), (c), (e)] are presented for different initial phases. The initial values are (a), (b) $\theta(0) = 0$, (c), (d) $\theta(0) = 0.5\pi$, (e), (f) $\theta(0) = \pi$, $Z(0) = 0.1$, $k(0) = 0.02$, $\xi(0) = 10$, and the parameters are $N_1 + N_2 = 20$, $K = 0.01$, $\delta = -1$.

repulsive and vice versa. In this case, the second QD moves away after following the first one for some time. Figures 7(c) and 7(d) show that the QDs flow without moving away from each other. If the value of $Z(0)$ is increased, the phase increases rapidly and turns into an attractive potential before moving away from each other due to the mutual repulsion potential, and the Andreev-Bashkin effect can be seen to exist.

V. CONCLUSION

We have investigated the nonlinear dynamics of quantum droplets in a quasi-one-dimensional BEC loaded in a two-core

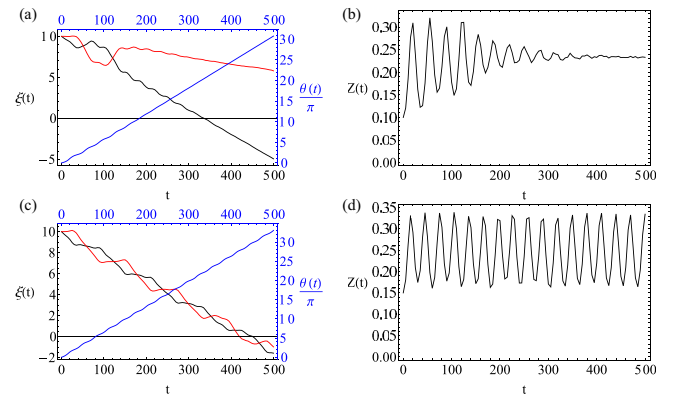


FIG. 7. The dynamics of the relative imbalance $Z(t)$, the relative center of mass $\xi(t)$, and the relative phase $\theta(t)$ (blue line) are presented for different initial relative imbalances $Z(0)$. The initial values are (a), (b) $Z(0) = 0.1$, (c), (d) $Z(0) = 0.15$, $\theta(0) = 0$, $k_1(0) = 0.02$, $k_2(0) = 0$, $\xi_1(0) = \xi_2(0) = 10$, and the parameters are $N_1 + N_2 = 10$, $\delta = -0.1$, $K = 0.01$, $a = 1.6$.

trap potential. The cases of standing and moving quantum droplets were considered. The frequencies of the Josephson oscillations and the condition for the self-trapping were found. An analytical description was based on the variational approach to the system of coupled extended Gross-Pitaevskii (EGP) equations, including Lee-Huang-Yang correction terms. It was found that for the case of the LHY fluid (when effects of the QF dominate over the mean-field contribution), in the zero-phase mode BEC is localized in one well for all initial population imbalances $Z(0)$. In the case of moving droplets, the system of equations describing the coupling of the relative motion of droplets in two cores with the dynamics of the relative phase and population imbalance (Josephson currents) was derived. Different regimes of droplet dynamics including attraction, and repulsion in dependence on the relative phase evolution were found. These phenomena are the realization of the Andreev-Bashkin effect for superfluids in a two-core trap potential. Analytical predictions of the

model were confirmed by full numerical simulations of the coupled extended Gross-Pitaevskii equations.

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DATA AVAILABILITY

The data that support the findings of this article are not publicly available. The data are available from the authors upon reasonable request.

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