

Phase-space open-system dynamics of second-order nonlinear interactions with pulsed quantum light

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The theoretical description of broadband, multimode quantum pulses undergoing a second-order $\chi^{(2)}$ -nonlinear interaction can be quite intricate due to the large dimensionality of the underlying phase space. However, in many cases only a few broadband (temporal) modes are relevant before and after the nonlinear interaction. Here we present an efficient framework to calculate the relation between the quantum states at the input and output of a nonlinear element in their respective relevant modes. Since the number of relevant input and output modes may differ, resulting in an open quantum system, we introduce the generalized Bloch-Messiah decomposition (GBMD), reducing the description to an equal number of input and output modes. The GBMD enables us to calculate the multimode Wigner function of the output state by convolving the rescaled Wigner function of the reduced input quantum pulse with a multivariate Gaussian phase-space function. We expand on this result by considering two examples: a parametric amplification of a broadband Fock state in the infrared frequencies and a THz-to-optical frequency up-conversion of a pulsed two-mode squeezed vacuum state. We investigate the effect, the convolution and thermalization due to entanglement breakage have on the output Wigner function by calculating the von Neumann entropy of the output Wigner function. The methods presented here can be used to optimize the amplification or frequency conversion of broadband quantum states, opening an avenue to the generation and characterization of optical quantum states on ultrafast timescales.

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I. INTRODUCTION

Nonlinear optics has been essential for the development of quantum optics. The generation of quantum pulses of light, such as the squeezed vacuum, enables a plethora of quantum information technologies spanning from quantum metrology (including the subshotnoise detection of gravitational waves [1,2] or precise spectroscopy [3–6]), quantum key distribution [7] and quantum networks [8], to the generation of even more resourceful states like photon subtracted squeezed states [9].

In recent years, the nonlinear interaction of light has enabled the generation of broadband quantum pulses [10–14] and the ultrafast detection of (quantum) light [15–19], a development which requires novel theoretical tools to describe the intricate mode structure and statistics of broadband quantum states. While the (second-order) nonlinear interaction of a few monochromatic modes of light can be described using symplectic matrices [20,21], it is not always possible to extend this formalism to pulsed quantum states. After parametric amplification, the phase-space dimension of the broadband

output pulses is in general twice the dimension of the input phase space [22], and can possibly be infinite, for example, if the input is in the ground state (vacuum) [22,23]. However, in many scenarios, only a few or even a single pulsed mode is relevant for further processing, e.g., if the quantum pulse is detected after the nonlinear interaction [24], which results in an open-system dynamics. Thus, for many applications, the input-output relation of the nonlinear interaction may be between a different number of relevant input and output modes, as schematically sketched in Fig. 1.

Here we present a method based on the symplectic singular value decomposition (SVD) introduced by Xu [32,33], which enables a reduction of the number of input or output modes to a set of effective modes with matching number of input-output modes. Since the method is based on a symplectic version of the SVD we refer to it as the generalized Bloch-Messiah decomposition (GBMD). While the usual Bloch-Messiah decomposition is a remarkable resourceful tool in nonlinear quantum optics, as long as the dynamics is unitary and closed [23,34–36], the generalized version presented here is applicable to arbitrary sets of input and output modes, which comprise the open systems. The effect of a mismatch between the transformed input modes and the relevant output modes on the quantum state is a nonunitary evolution during the effective nonlinear interaction. Thus, the output quantum state can be a mixed state, even if the quantum state at the input of the nonlinear interaction is pure. We describe this effect using Wigner function representation of the quantum state in phase space.

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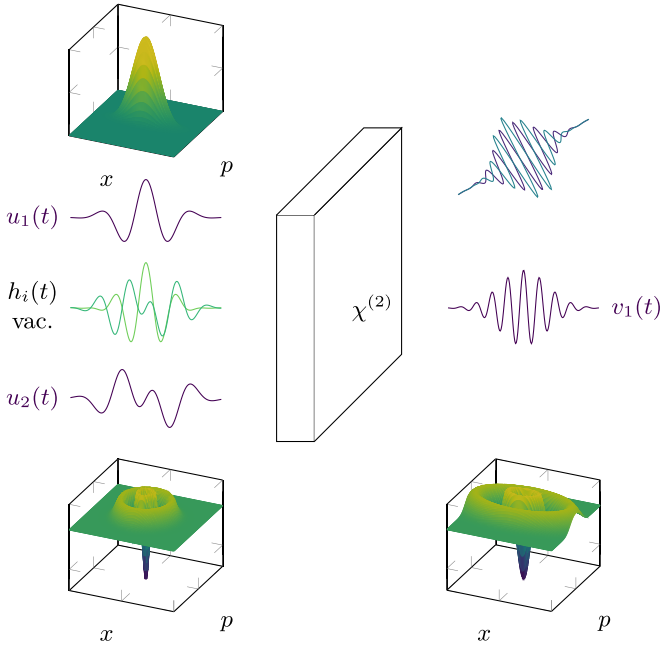


FIG. 1. A schematic representation of the scenario considered in this work. In the most general setting, a quantum pulse of light, composed of multiple temporal modes with profiles $u_i(t)$, traverses a nonlinear medium with a second-order nonlinear susceptibility $\chi^{(2)}$. After the nonlinear interaction some, possibly different, modes $v_i(t)$ are relevant, e.g., for detection. While we do not explicitly include the measurement in the model, this setting is inspired by nonlinear homodyne detection [25–27] and electro-optic sampling [28–31]. Since the set of input and output modes are not necessarily closed under the transformation by the nonlinear interaction, some additional modes $h_i(t)$ in the ground state (vacuum) are required for the description. The main achievement of this work is a relation between the input quantum pulse and the quantum state occupying the output modes based on the (multimode) Wigner function in Eq. (14). This relation only includes the relevant modes $u_i(t)$ and $v_i(t)$, but not $h_i(t)$, which we achieve by introducing the generalized Bloch-Messiah decomposition (GBMD) in Sec. IV.

The main result of this work is the phase-space input-output relation of the quantum states in Eq. (14) connecting the state prior and after the nonlinear interaction. The Wigner function representing the output quantum state is given by a convolution of the Wigner function corresponding to transformed and reduced input state with a Gaussian phase-space function and a subsequent rescaling.

As examples of the main result, we consider two scenarios: first, the parametric amplification of a broadband near-infrared Fock state, with a mismatch between the input and output mode, and, second, the up-conversion of a broadband two-mode squeezed state from the terahertz (THz) to the optical frequencies. The former scenario could prove useful to the description of nonlinear homodyne detection [25–27], which utilizes a parametric amplification of the quantum signal prior to a homodyne detection. The latter scenario is motivated by electro-optic sampling, which comprises an up-conversion process and a field-quadrature measurement to enable the measurement of THz (quantum) states of the electromagnetic field with a detection in the optical frequencies

[28–31]. In both scenarios, the output mode is determined by the broadband mode of the classical pulse (local oscillator). Since the input quantum state may occupy several pulsed modes, this necessarily leads to a mismatch between the dimension of the phase space the input state resides in, and the output mode relevant after the nonlinear interaction. Here we are not concerned with a comprehensive description of nonlinear homodyne detection or electro-optic sampling, since it merely serves as a motivation for the formalism presented here. Thus, we do not explicitly model the detection of the light. Yet the measurement statistics could be obtained from the Wigner function of the output state [37]. The formalism developed here may be applicable to a much broader set of scenarios. While we choose examples based on broadband modes, the formalism could also benefit the description of the nonlinear interaction between modes with a spacial mismatch. For example, if the output of an optical parametric oscillator is coupled into an optical fiber [38].

Here we describe the up-conversion using the formalism of the first-order unitary introduced in [39,40], which can be understood as a renormalized, first-order perturbation theory based on the Baker-Campbell-Hausdorff formula. In the first scenario we consider, the parametric amplification of a broadband Fock state, the mismatch between input and output mode leads to an attenuation of the Wigner negativity. We show that the parametric amplification can make the negativity more resilient to the mode mismatch. In the second example, the up-conversion of a broadband two-mode squeezed state, we apply the generalized Bloch-Messiah decomposition to reduce the phase-space dimension of the input state which can lead to thermalization originating from the breakage of entanglement either present in the two-mode squeezed vacuum or generated in the nonlinear interaction. We quantify the thermalization using the von Neumann entropy and show how it depends on the central frequency of the output mode. We show theoretically that the up-conversion exhibits two different regimes, which we call the squeezing and beam-splitting regime, determined by the central frequency of the output mode. In the example considered here, the squeezing regime exhibits excess thermalization from the breakage of state-independent entanglement generated in the up-conversion. In the beam-splitter regime, the entanglement present in the quantum state prior to the up-conversion becomes relevant, which could enable the direct measurement of entanglement in a two-mode squeezed THz pulse.

This work is organized as follows. In Sec. II we give a short overview of the formalism used in this work, including broadband modes and the multimode Wigner function. In Sec. III the input-output relation of the quadrature operators is presented. The generalized Bloch-Messiah decomposition is introduced in Sec. IV, which is then utilized in Sec. V to derive the main result of this work, which is the phase-space input-output relation of the pulsed quantum state. In Sec. VI we present an exemplary nonlinear interaction, which uses a first-order unitary approach to derive the input-output relation of parametric amplification and THz-to-optical frequency up-conversion. We use this in Sec. VII to demonstrate the application of the formalism in two examples. In the first example, we consider a parametric amplification of a broadband Fock state in the infrared-frequency range. In the second

example, we describe the up-conversion of a two-mode squeezed vacuum in two broadband THz modes to a single broadband mode in the optical frequencies. The work is summarized and concluded in Sec. VIII with an outlook on further research.

II. A BRIEF INTRODUCTION TO BROADBAND MODES

Quantum optics is based on the assumption that light of a single color with (angular) frequency ω consists of quanta of a well-defined energy, which are annihilated or created by the operators $\hat{a}(\omega)$ or $\hat{a}^\dagger(\omega)$ (as shorthand $\hat{a}_\omega$ or $\hat{a}_\omega^\dagger$). This single frequency basis may be good for some setups, but since nonlinear optics enables the interaction of light in an ultrabroad band of frequencies, the description of all individual monochromatic modes can be computationally exhaustive. In this case, it can be instructive to define broadband (temporal) modes which comprise a continuum of frequencies instead of a monochromatic wave [41,42]. The quanta of a broadband mode, with frequency distribution $u(\omega)$, are annihilated by $\hat{a}_u = \int_0^\infty u^*(\omega)\hat{a}_\omega d\omega$. The extension to N broadband modes $u_j(\omega)$, requires the mode functions to be orthogonal with respect to the Hilbert-Schmidt inner product $\langle u_i, u_j \rangle = \int_0^\infty u_i^*(\omega)u_j(\omega)d\omega$, to ensure the canonical commutation relations are fulfilled, $[\hat{a}_{u_i}, \hat{a}_{u_j}^\dagger] = \delta_{ij}$. In this case the modes are said to be orthogonal.

The mode operators can be used to define quadratures $\hat{x}_{u_j} = \frac{1}{\sqrt{2}}(\hat{a}_{u_j} + \hat{a}_{u_j}^\dagger)$ and $\hat{p}_{u_j} = \frac{1}{\sqrt{2i}}(\hat{a}_{u_j} - \hat{a}_{u_j}^\dagger)$, as well as a quantum phase space $\hat{\Gamma} = (\hat{x}_{u_1}, \dots, \hat{x}_{u_N}, \hat{p}_{u_1}, \dots, \hat{p}_{u_N})^T$, where the transpose of an operator valued vector $\hat{v}$, denoted by $\hat{v}^T$, is only applied to the vector not the elements, i.e., the quadrature operators. A quantum phase space is distinguished by the canonical commutation relation between the quadratures, $[\hat{\Gamma}_i, \hat{\Gamma}_j] = i(\Omega_N)_{ij}$, with the matrix

$$\Omega_N = \begin{pmatrix} 0 & \mathbb{I}_N \\ -\mathbb{I}_N & 0 \end{pmatrix}, \quad (1)$$

defining the symplectic structure of the phase space and $\mathbb{I}_N$ being the $N \times N$ unit matrix. In this work we will represent the multimode quantum state in phase space using the Wigner function $W(\vec{\gamma})$. Starting from the density operator $\hat{\rho}$ of a quantum state, we can introduce the characteristic function $\chi(\vec{\beta}) = \text{tr}[\hat{\rho}\hat{D}(\vec{\beta})]$ as the expectation value of the multimode displacement operator $\hat{D}(\vec{\beta}) = \exp(-i\hat{\Gamma}^T\Omega_N\vec{\beta})$ with respect to the quantum state. The multimode Wigner function is the (symplectic) Fourier transform of the characteristic function [20],

$$W(\vec{\beta}) = \frac{1}{(2\pi)^{2N}} \int_{\mathbb{R}^{2N}} \exp(i\vec{\gamma}^T\Omega_N\vec{\beta})\chi(\vec{\beta})d^{2N}\beta. \quad (2)$$

The Wigner function can take on negative values and thus cannot be interpreted as a probability distribution. The phase-space representation of a multimode quantum state proves to be very resourceful, when dealing with a unitary time evolution based on a Hamiltonian quadratic in the quadrature operators of $\hat{\Gamma}$, as is the case for a (driven) $\chi^{(2)}$ -nonlinear interaction. Assuming a quadratic Hamiltonian and a closed system, the time evolved quadratures $\hat{\Gamma}_{\text{out}} = M\hat{\Gamma}_{\text{in}}$ are related

to the input quadratures, $\hat{\Gamma}_{\text{in}}$, by a symplectic matrix M . The symplectic matrices are defined by the property $M^T\Omega_N M = \Omega_N$. Thus, the Wigner function of the output quantum state W_{out} , after time evolution under a quadratic Hamiltonian, is related to the input quantum state Wigner function, W_{in} , by $W_{\text{out}}(\vec{\gamma}) = W_{\text{in}}(M^{-1}\vec{\gamma})$, if the system is closed. The symplectic matrices allow for a special singular value decomposition, $M = OZO'$, with O, O' being orthogonal symplectic matrices and Z being a diagonal symplectic matrix which squeezes one quadrature by a factor and the corresponding orthogonal quadrature by the inverse factor. This decomposition is called the Bloch-Messiah decomposition (BMD). The matrices O, O' correspond to the passive, multiport beam-splitting action, while Z corresponds to the active single-mode squeezing action of the time evolution mediated by M [20,21,34,43,44]. In the next section we will show that a second-order nonlinear interaction of a pulsed open quantum system cannot always be described by a symplectic transformation and we thus need to generalize the formalism presented in this section.

III. OPERATOR INPUT-OUTPUT RELATION

The general setting we consider consist of a pulsed quantum state of light as the input to an optical element with a second-order nonlinearity, which transforms the statistics and mode function of the input into an output quantum state residing in potentially different mode functions. Our goal is to find the output quantum state in a predetermined set of broadband output modes, given an input quantum pulse and the input-output relation connecting the quadrature operators before and after the nonlinear interaction. We denote the operators annihilating a photon of continuous (angular) frequency ω in the Fock space of the optical system at the input and output of the nonlinear optical element with $\hat{a}_{\text{in}}(\omega)$ and $\hat{b}_{\text{out}}(\omega)$ respectively. Furthermore, we assume the input and output mode operators are related by the Bogoliubov transformation,

$$\begin{aligned} \hat{b}_{\text{out}}(\omega) = & \int_0^\infty F(\omega, \Omega)\hat{a}_{\text{in}}(\Omega)d\Omega \\ & + \int_0^\infty G^*(\omega, \Omega)\hat{a}_{\text{in}}^\dagger(\Omega)d\Omega, \end{aligned} \quad (3)$$

linear in the creation and annihilation operators [22,23,35]. The integration kernels F and G conserve the canonical commutation relation $\delta(\omega - \omega') = [\hat{b}_{\text{out}}(\omega), \hat{b}_{\text{out}}^\dagger(\omega')]$. The Bogoliubov transformation in Eq. (3) can result from a Hamiltonian quadratic in the creation and annihilation operators as shown in [11,23,35] and Appendix B. Defining quantum states in the Hilbert space of continuous frequency can be intricate, e.g., leading to divergences. For this reason, it can be advantageous to discretize the input-output relation in Eq. (3) using broadband modes (also known as temporal modes). One such broadband mode basis can be generated using the Bloch-Messiah decomposition (BMD), reducing Eq. (3) to the squeezing of individual broadband modes [23,34,44,45]. For some cases it might be sufficient to only consider a few of these most significant broadband eigenmodes of the BMD. However, in general, the basis given by the BMD will not be the most efficient one, when describing the transformation of the quantum state in an arbitrary input mode $u(\omega)$, and we

must resort to a description of only the relevant (and possibly different) input and output modes [22]. For example, the first Bloch-Messiah mode of the nonlinear interaction in a thin nonlinear crystal, as discussed in Sec. VI, is centered at half the central frequency of the classical pulse driving the interaction [46]. However, in the up-conversion of a THz quantum pulse to the detectable optical frequencies, the involved broadband modes do not correspond to the Bloch-Messiah modes of the nonlinear interaction. Thus, a BMD does not result in a resourceful description.

In the scenario we consider, the input quantum state occupies N broadband modes with operators $\hat{a}_{u_i} = \int d\omega u_i^*(\omega) \hat{a}_{\text{in}}(\omega)$, while all other orthogonal modes are in the vacuum, and the M relevant output modes are described by $\hat{b}_{v_i} = \int d\omega v_i^*(\omega) \hat{b}_{\text{out}}(\omega)$. At this point the output modes are completely arbitrary. However, they could be determined by further processing, e.g., by the local oscillator pulse of a homodyne detection, or be obtained using the formalism presented in [22]. By inserting the input-output relation of Eq. (3) into the definition of the output modes, we can relate

$$\hat{b}_{v_i} = \zeta_i \hat{a}_{f_i} + \xi_i \hat{a}_{g_i}^\dagger \quad (4)$$

to the (possibly overlapping) mode functions

$$f_i(\Omega) = \frac{1}{\zeta_i} \int F^*(\omega, \Omega) v_i(\omega) d\omega, \quad (5)$$

$$g_i(\Omega) = \frac{1}{\xi_i} \int G^*(\omega, \Omega) v_i^*(\omega) d\omega, \quad (6)$$

normalized through ζ_i and ξ_i . Since the modes f_i and g_i are possibly overlapping, they are not orthogonal with respect to the Hilbert-Schmidt inner product and in turn, do not obey the canonical commutation relation. Yet, to write the input-output relation in terms of the occupied input modes and some vacuum modes, we can utilize the Gram-Schmidt procedure to transform the linearly independent modes $\{u_1, \dots, u_N, g_1, \dots, g_M, f_1, \dots, f_M\}$ into a set $\mathcal{B} = \{u_1, \dots, u_N, h_1, \dots, h_{2M}\}$ of orthogonal modes. The linear independency is assumed here, but holds true for most practical examples. In some cases it is possible to use a smaller set of modes than $\mathcal{B}$ [22], however here we focus on the most general case. Inverting the Gram-Schmidt basis allows us to express the output modes, $v_i(\omega)$, with elements of the Gram-Schmidt basis $\mathcal{B}$. To facilitate an input-output relation of the quantum state using the Wigner function, we introduce the quadrature operators $\hat{x}_f = \frac{1}{\sqrt{2}}(\hat{a}_f + \hat{a}_f^\dagger)$ and $\hat{p}_f = \frac{1}{\sqrt{2i}}(\hat{a}_f - \hat{a}_f^\dagger)$. Organizing the quadratures of the output, input, and orthogonal (vacuum) modes into the vectors

$$\hat{\Gamma}_{\text{out}} = (\hat{x}_{v_1}, \dots, \hat{x}_{v_M}, \hat{p}_{v_1}, \dots, \hat{p}_{v_M})^T, \quad (7)$$

$$\hat{\Gamma}_{\text{in}} = (\hat{x}_{u_1}, \dots, \hat{x}_{u_N}, \hat{p}_{u_1}, \dots, \hat{p}_{u_N})^T, \quad (8)$$

$$\hat{\Gamma}_{\perp} = (\hat{x}_{h_1}, \dots, \hat{x}_{h_{2M}}, \hat{p}_{h_1}, \dots, \hat{p}_{h_{2M}})^T, \quad (9)$$

allows recasting the input-output relation in terms of phase-space operators,

$$\hat{\Gamma}_{\text{out}} = A \hat{\Gamma}_{\text{in}} + B \hat{\Gamma}_{\perp}. \quad (10)$$

The matrix

$$A = \begin{pmatrix} \text{Re}(F_{\text{in}} + G_{\text{in}}) & \text{Im}(-F_{\text{in}} + G_{\text{in}}) \\ \text{Im}(F_{\text{in}} + G_{\text{in}}) & \text{Re}(F_{\text{in}} - G_{\text{in}}) \end{pmatrix} \quad (11)$$

is determined by the matrix elements $[F_{\text{in}}]_{ji} = \langle u_i, f_j \rangle^* = \iint v_j^*(\omega) F(\omega, \Omega) u_i(\Omega) d\omega d\Omega$ and $[G_{\text{in}}]_{ji} = \langle u_i, g_j \rangle = \iint v_j^*(\omega) G^*(\omega, \Omega) u_i^*(\Omega) d\omega d\Omega$, while B depends on the matrix elements of F and G with respect to the modes v_j and h_i (see Appendix A 1 for details). If no vacuum modes are involved, $B = 0$, and the input-output relation of the quadrature operators conserves the canonical commutation relation, the matrix A in Eq. (10) becomes symplectic. In general, however, A might not even be a square matrix. To resolve this issue we introduce the generalized Bloch-Messiah decomposition in the following section.

IV. THE GENERALIZED BLOCH MESSIAH DECOMPOSITION

If the number of input modes N differs from the number of output modes M , the matrix A in Eq. (10) is not a square matrix (in general $A \in \mathbb{R}^{2M \times 2N}$). Yet, the rectangular shape of A suggest a more efficient mode basis for which a square matrix exists, only involving $\min\{N, M\}$ input and $\min\{N, M\}$ output modes. For this reason we introduce the *generalized Bloch-Messiah decomposition* (GBMD). As we show in Appendix C, we can decompose

$$A = \begin{cases} S_L^T P_L^T \mathcal{A} & M > N \\ \mathcal{A} & M = N \\ \mathcal{A} P_R S_R & M < N \end{cases} \quad (12)$$

with $S_L \in \mathbb{R}^{2M \times 2M}$, $S_R \in \mathbb{R}^{2N \times 2N}$ being symplectic, while $P_L \in \mathbb{R}^{2N \times 2M}$, $P_R \in \mathbb{R}^{2M \times 2N}$ reduce the dimensionality and $\mathcal{A}$ being a real square matrix with dimension $2 \min\{M, N\}$. The index L indicate the left-sided decomposition and R the right-sided one.

We call Eq. (12) the generalized Bloch-Messiah decomposition, since it is based on the symplectic version of the singular value decomposition of a rectangular matrix A introduced by Xu [32,33]. Different to the usual Bloch-Messiah decomposition, A does not need to be symplectic. However, we assume $A \Omega A^T$ is nonsingular. This is usually the case as long as all output modes overlap with at least one input mode after it is transformed by the nonlinear interaction. There exists an algorithm to compute the GBMD derived by Xu [33] based on symplectic Housholder and Givens rotation. We present an example of the algorithm in Appendix C 1, and the corresponding Python implementation can be found in [47]. The decomposition in Eq. (12) has great potential for many settings in quantum optics. The GBMD could potentially be used to describe the single-mode measurement of a multimode quantum state after transformation by a nonlinear element [27,38,40] or to orthogonalize overlapping mode functions as proposed for the correlation tomography in [46]. The simplified input-output relation resulting from the GBMD allows us to formulate an input-output relation of the quantum states. However, since the GBMD reduces the dimension of either the input or output phase space, the quantum-state input-output

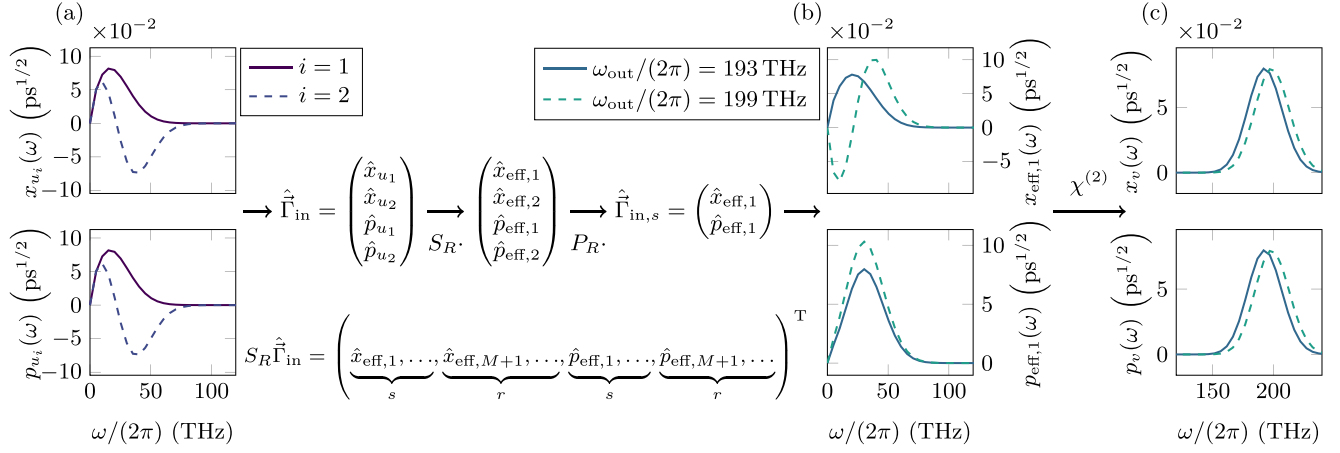


FIG. 2. (a) The input modes of $\hat{\Gamma}_{in}$ in the frequency domain. (b) The effective input modes, which are obtained from the generalized Bloch-Messiah decomposition in Eq. (12), by first transforming the input modes using the symplectic matrix S_R and afterwards reducing the number of effective modes using P_R to match the number, M , of output modes v_i (in this case we consider one output mode, $M = 1$, centered at ω_{out}). We call the modes which are kept system modes (labeled by s) and label the residual modes by r . The effective modes are up-converted to the output modes in $\hat{\Gamma}_{out}$, shown in (c), by the $\chi^{(2)}$ -nonlinear interaction resulting in Eq. (10). Here we used the nonlinear interaction introduced in Sec. VI as an example.

relation, derived in the following section, necessarily needs to be an open-system description.

V. THE INPUT-OUTPUT RELATION IN PHASE SPACE

In this section we derive the phase-space input-output relation based on the multimode Wigner function of the input and output state. The generalized Bloch-Messiah decomposition in Eq. (12) allows us to find a set of $\min\{N, M\}$ effective modes, describing the transformation of the quantum state due to the nonlinear interaction. The output Wigner function presents the main result of this work. Since only one of the three cases $M > N$, $M = N$ or $M < N$ can be true at the same time, we will derive the phase-space input-output relation for all three cases separately.

A. Input mode surplus ($M < N$)

In the case of an input-mode surplus ($M < N$), the generalized Bloch-Messiah decomposition is $A = \mathcal{A}P_R S_R$. Thus, we can collect the effective system modes, i.e., the modes transformed into the relevant output modes by the nonlinear interaction, in $\hat{\Gamma}_{in,s} = P_R S_R \hat{\Gamma}_{in}$. While S_R transforms the phase space of the input modes, the matrix P_R reduces the $2N$ components of the vector $S_R \hat{\Gamma}_{in}$ to the $2M$ components of the output phase space, as visualized in Fig. 2. We define the reduced input state of the system modes, $\hat{\rho}_{in,s} = \text{tr}_r(\hat{\rho}_{in})$, by tracing over the remaining $N - M$ modes not contained in $\hat{\Gamma}_{in,s}$. The modes traced out are marked by r in Fig. 2 and will be called residual modes, as opposed to the remaining system modes marked by s . Since only the system modes are relevant to the output quantum state, we introduce the Wigner function, $W_{in,s}(\vec{\gamma})$, of the reduced input quantum state, $\hat{\rho}_{in,s}$, of the system modes. Furthermore, by defining the covariance matrix of the orthogonal (initial vacuum) modes, $\text{cov}_B = BB^T$, we can express the output Wigner function as a convolution of

the multivariate Gaussian phase-space function,

$$G(\vec{\gamma}) = \frac{2^M (2\pi)^M}{\sqrt{\det(\text{cov}_B)}} \exp(-\vec{\gamma}^T \mathcal{A} \text{cov}_B^{-1} \mathcal{A}^T \vec{\gamma}), \quad (13)$$

and the input Wigner function, resulting in

$$W_{out}(\vec{\gamma}) = (W_{in,s} * G)(\mathcal{A}^{-1} \vec{\gamma}), \quad (14)$$

with $*$ denoting the convolution. Thus, the reduced input Wigner function is smoothed out by the convolution with G and transformed by $\mathcal{A}$. A detailed derivation of Eq. (14) can be found in Appendix D. As a remark, it should be mentioned that for multiple nonlinear interactions in a setup, all Gaussian phase-space functions involved in the convolution can be collected into one Gaussian. An example of Eq. (14) for a two-mode squeezed input pulse and a single (broadband) output mode is discussed in Sec. VII B.

To better understand Eq. (14), it is instructive to elaborate on a special case of the phase-space input-output relation. Let us consider an input quantum pulse described by a Gaussian Wigner function,

$$W_{in}(\vec{\gamma}) = \frac{1}{\pi^M \sqrt{\det(\text{cov}_{in})}} \exp(-\vec{\gamma}^T \text{cov}_{in}^{-1} \vec{\gamma}), \quad (15)$$

with covariance matrix cov_{in} and no displacement in phase space. Since the nonlinear interaction is described by a quadratic Hamiltonian, the output state is still Gaussian with a covariance matrix

$$\text{cov}_{out} = \text{cov}_B + \mathcal{A} \text{cov}_{in,s} \mathcal{A}^T, \quad (16)$$

and the covariance matrix $\text{cov}_{in,s} = P_R S_R \text{cov}_{in} S_R^T P_R^T$ corresponding to the covariance matrix of the reduced input state of the system modes.

B. Output mode surplus ($M > N$)

In the case of an output mode surplus ($M > N$), only a fraction of the output modes are necessary to represent

the transformed input quantum state. Utilizing the GBMD $A = S_L^T P_L^T \mathcal{A}$, we can define the phase space of the reduced output modes, i.e., the system modes, as $\vec{\gamma}_s = P_L S_L^{-1} \vec{\gamma}$ and the phase space of the environment $\vec{\gamma}_e$, which together comprise the phase space of the full output Wigner function, $\vec{\gamma}$. Furthermore, by separating the covariance matrix of the orthogonal modes into system, $\text{cov}_{B,ss} = P_L S_L \text{cov}_B S_L^T P_L^T$, and an environment part, $\text{cov}_{B,ee}$, with the latter corresponding to the remaining blocks on the diagonal of $S_L \text{cov}_B S_L^T$ besides $\text{cov}_{B,ss}$ (see Appendix D for more details). If we transform to the basis $T \hat{\Gamma}_\perp = (\hat{x}_{h_1}, \hat{p}_{h_1}, \hat{x}_{h_2}, \hat{p}_{h_2}, \dots, \hat{x}_{h_{2M}}, \hat{p}_{h_{2M}})^T$, the transformed covariance matrix disintegrates into the four blocks,

$$T S_L \text{cov}_B S_L^T T = \begin{pmatrix} \text{cov}_{B,ss} & \text{cov}_{B,se} \\ \text{cov}_{B,es} & \text{cov}_{B,ee} \end{pmatrix}, \quad (17)$$

with the system and environment blocks on the diagonal. By introducing the Schur complement, $\text{cov}_{\text{schur}} = \text{cov}_{B,ss} - \text{cov}_{B,se} \text{cov}_{B,ee}^{-1} \text{cov}_{B,es}$, together with $\vec{\gamma}_{\text{schur}} = \text{cov}_{B,es} \text{cov}_{B,ee}^{-1} \vec{\gamma}_e$, connecting the system and environment by the off-diagonal blocks $\text{cov}_{B,se} = \text{cov}_{B,es}^T$, we can again express the output Wigner function by a convolution. Compared to the case in the previous section, the multivariate Gaussian phase-space function,

$$G(\vec{\gamma}_s) = \frac{2^M}{(2\pi)^{M-2L}} [\det(\text{cov}_{\text{schur}}) \det(\text{cov}_{B,ee})]^{-\frac{1}{2}} \times \exp(-\vec{\gamma}_s^T \mathcal{A} \text{cov}_{\text{schur}}^{-1} \mathcal{A}^T \vec{\gamma}_s), \quad (18)$$

is determined by Schur complement $\text{cov}_{\text{schur}}$ of the covariance matrix cov_B with respect to the system modes. The output Wigner function,

$$W_{\text{out}}(\vec{\gamma}) = (W_{\text{in}} * G)(\mathcal{A}^{-1}[\vec{\gamma}_s + \vec{\gamma}_{\text{schur}}]), \quad (19)$$

is again the input Wigner function, but smoothed out by the convolution with G and rescaled by $\mathcal{A}$, however, additionally displaced by $\vec{\gamma}_{\text{schur}}$.

C. Equal mode number ($M = N$)

If the number of input modes equals the number of output modes, we cannot reduce the dimension of the input state nor the output state. However, this is not necessary since the matrix A , describing the nonlinear interaction between in- and output modes, is a square matrix. Thus, the GBMD is not necessary. The transformation A given by the nonlinear process is still not symplectic, as long as the vacuum modes $h_i(\omega)$ are involved. Yet, we can describe the phase-space input-output relation using the Gaussian phase-space function

$$G(\vec{\gamma}) = \frac{2^{2M} \pi^M}{\sqrt{\det(\text{cov}_B)}} \exp(-\vec{\gamma}^T \mathcal{A} \text{cov}_B^{-1} \mathcal{A}^T \vec{\gamma}) \quad (20)$$

by

$$W_{\text{out}}(\vec{\gamma}) = (W_{\text{in}} * G)(\mathcal{A}^{-1} \vec{\gamma}). \quad (21)$$

This case will be further discussed in Sec. VII A for a pulse of a single (broadband) mode Fock state ($N = M = 1$).

VI. EXAMPLE OF THE SECOND-ORDER NONLINEAR INTERACTION

To illustrate the input-output theory derived above, we consider the frequency conversion of THz radiation to optical frequencies as well as the parametric amplification of infrared radiation using a zinc-telluride nonlinear crystal in free space [48,49]. With a crystal orientation as in [16], the effect of the second-order nonlinear susceptibility $\chi^{(2)}$ is described by a unitary time-evolution operator,

$$\hat{U}_{\text{NL}} = \exp \left(\frac{1}{2} \iint_0^\infty J(\Omega, \omega) \hat{a}_\omega \hat{a}_\Omega d\omega d\Omega - \text{H.c.} + \iint_0^\infty K(\Omega, \omega) \hat{a}_\omega^\dagger \hat{a}_\Omega d\omega d\Omega \right), \quad (22)$$

quadratic in the operators acting on the modes polarized orthogonal to the strong classical pulse driving the nonlinear interaction. The kernels $J(\Omega, \omega)$ and $K(\Omega, \omega)$, accounting for a squeezing and beam-splitting interaction respectively, are given in Eqs. (B2) and (B3). The squeezing interaction simultaneously creates or annihilates a photon at frequency ω and Ω , while the beam-splitting interaction creates a photon at frequency ω and annihilating a photon at frequency Ω . Both kernels scale linearly with the coherent amplitude α of the classical pulse driving the nonlinear interaction. Here we neglect any time-ordering effects, which is valid as long as the nonlinear interaction is weak. In general, time ordering becomes relevant if higher orders in a perturbative series contribute to the time evolution, since the higher orders account for time ordering. Yet the quadratic form of Eq. (22) is unaffected by time-ordering effects, only the kernels $J(\Omega, \omega)$ and $K(\Omega, \omega)$ may change. There are theoretical tools to include time ordering [12,35,50,51]; however, we will resort to the simplified kernels in Eqs. (B2) and (B3), since this is sufficient to demonstrate the general features of the formalism derived in the previous section. An example of the kernels for a thin crystal of length $L = 20 \mu\text{m}$ and a probe pulse centered at 200 THz with a bandwidth of 118 THz is shown in Fig. 3.

Using the first-order unitary introduced in [39,40] and defining $f_K(\Omega) = -\int K^*(\Omega, \omega) v(\omega) d\omega$, $f_J(\Omega) = \int J^*(\Omega, \omega) v^*(\omega) d\omega$ we can calculate the parameters

$$\theta_K = \left(\iint_0^\infty |f_K(\Omega)|^2 d\Omega \right)^{1/2}, \quad (23)$$

$$\theta_J = \left(\iint_0^\infty |f_J(\Omega)|^2 d\Omega \right)^{1/2}, \quad (24)$$

determining the amount of beam splitting and squeezing involved in the nonlinear interaction which results in a given output mode $v(\omega)$. Depending on whether beam splitting or squeezing dominates, the parameter $\theta = \sqrt{|\theta_K^2 - \theta_J^2|}$ defines a beam-splitting or squeezing relation,

$$(\mu, v) = \begin{cases} (\cos[\theta], \sin[\theta]) & \theta_K > \theta_J \\ (\cosh[\theta], \sinh[\theta]) & \theta_K < \theta_J \end{cases}, \quad (25)$$

resulting in $F(\omega, \Omega) = \mu \delta(\omega - \Omega) - v K(\Omega, \omega)/\theta$ and $G(\omega, \Omega) = v J(\Omega, \omega)/\theta$. The first-order unitary can be understood as a first-order perturbation theory (using

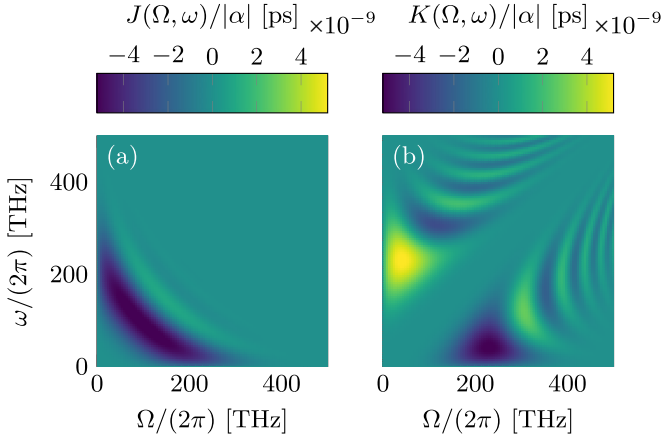


FIG. 3. An example of the kernels $J(\Omega, \omega)$ and $K(\Omega, \omega)$ describing the nonlinear interaction mediating between the input and output modes, which is defined in Eq. (22). The kernel $J(\Omega, \omega)$ describes the strength of a squeezing interaction between the continuous (angular) frequency mode at Ω and the mode at ω , simultaneously creating or annihilating a photon in each mode. The kernel $K(\Omega, \omega)$ corresponds to a beam-splitter interaction between the continuous frequency modes and creates a photon in the mode ω , while annihilating one at Ω . The nonlinear crystal is assumed to be zinc telluride with a refractive index $n(\omega)$ [52], length $L = 20 \mu\text{m}$ and an electro-optic coefficient $r_{41} = 4 \times 10^{-12} \text{ mV}^{-1}$ [53]. The classical coherent pulse driving the nonlinear interaction is assumed to be centered at $\omega_p/(2\pi) = 200 \text{ THz}$, to have a bandwidth of $\Delta\omega_p/(2\pi) = 118 \text{ THz}$ and a coherent amplitude of α . We assume the nonlinear crystal is in free space, the coherent pulse is in the fundamental paraxial mode with beam waist area $A = 28 \mu\text{m}^2$ and polarized orthogonal to the mode of the quantum interaction. Note that for one frequency, Ω or ω , fixed to ω_p , most of the contribution from the squeezing interaction is below the central frequency of the coherent pulse, but above for the beam-splitting contribution.

the Baker-Campbell-Hausdorff formula) renormalized to fulfill the bosonic commutation relation. We will refer to the parameter range with $\theta_K > \theta_J$ as the beam-splitting regime, while $\theta_K < \theta_J$ is called the squeezing regime. A simplified version of single (monochromatic) mode frequency conversion in the beam-splitter regime can be found in [54] and for a broadband but mode selective case in [55]. The mode function $v(\omega)$ could be determined by using the formalism introduced by Tziperman *et al.* [22]. However, here we follow an example suited for a subsequent detection of the output mode, e.g., using homodyne detection with the local oscillator defining the broadband output mode. Since the local oscillator mode and the broadband modes of the quantum state do not necessarily correspond to the Bloch-Messiah modes of the nonlinear interaction, the relation between the input and output quantum pulse is (in the most general case) not described by a symplectic transformation.

VII. EXAMPLES

In these sections we will investigate two examples of the phase-space input-output relation. First, we consider a broadband Fock state in the infrared-frequency range parametrically amplified by the nonlinear interaction introduced in the

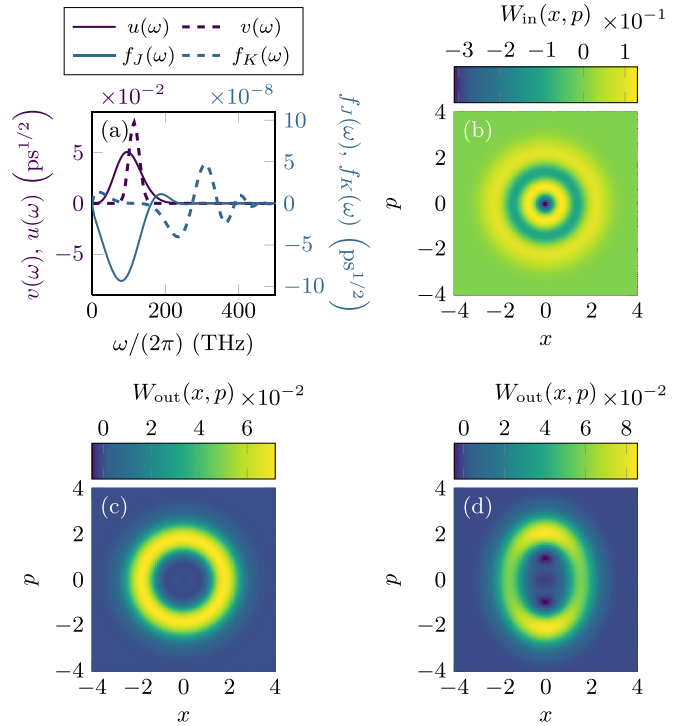


FIG. 4. (a) The input and output mode functions $u(\omega)$ and $v(\omega)$ as well as the mode functions $f_K(\omega)$ and $f_J(\omega)$, defined in Sec. VI, in the frequency domain. (b) The input Wigner function of a pulsed infrared three-photon Fock state in the mode $u(\omega)$. (c) The output Wigner function in the mode $v(\omega)$ without a nonlinear interaction between the input and output ($\alpha = 0$). (d) The output Wigner function in the same mode $v(\omega)$ but including a nonlinear interaction ($\alpha = 7 \times 10^4$). Since the output mode transformed by the nonlinear interaction includes some overlap between the input mode and the mode $f_J(\omega)$ in addition to the overlap between $u(\omega)$ and $v(\omega)$, more of the input states Wigner negativity is visible in the output state as compared to (c).

previous section. The output mode is assumed to have a much narrower bandwidth, which results in an attenuation of the negativity in the Wigner function of the output state. The parametric amplification can recover some of the negativity. Second, we investigate the up-conversion of a broadband two-mode squeezed state in the THz range to one optical broadband mode. In both cases we stay in the perturbative regime of the nonlinear interaction with $\alpha = 7 \times 10^4$, as numerically verified in [39]. Higher coherent amplitudes α of the classical pulse may require to take time ordering into account [35,50,51].

A. Single-mode Fock state

The scenario we consider in this section consists of one broadband mode $u(\omega)$ in the infrared-frequency regime, which is occupied by a n -photon Fock state, while the output mode $v(\omega)$ is narrower in the frequency [cf. Fig. 4(a)]. The broadband mode of the Fock state is centered at $\omega_{in}/(2\pi) = 100 \text{ THz}$ and has a frequency spread of 59 THz , while the output mode $v(\omega)$ has a smaller bandwidth of $\Delta\omega/(2\pi) = 24 \text{ THz}$ and a central frequency of $\omega_{out}/(2\pi) = 115 \text{ THz}$.

Since we only have one input and one output mode, we can use the expression in Sec. VC. Using a singular value decomposition, the matrix $A^{-1}\text{cov}_B[A^{-1}]^T = \sigma_x \vec{e}_x \vec{e}_x^T + \sigma_p \vec{e}_p \vec{e}_p^T$ is diagonal in the phase space basis $\vec{e}_x, \vec{e}_p$ with singular values σ_x, σ_p . By defining $A^{-1}\vec{y} = y\vec{e}_x + z\vec{e}_p$ we can express the Wigner function of the output state for an n -photon input state as

$$W_{\text{out}}(\vec{y}) = \frac{\exp\left(-\frac{z^2}{\sigma_x+1} - \frac{y^2}{\sigma_p+1}\right)}{\pi |\det(A)| \sqrt{(\sigma_x+1)(\sigma_p+1)}} \times \sum_{m=1}^n \left(\frac{\sigma_x-1}{\sigma_x+1}\right)^m L_m^{(-\frac{1}{2})}\left(\frac{-2z^2}{\sigma_x^2-1}\right) \left(\frac{\sigma_p-1}{\sigma_p+1}\right)^{n-m} \times L_{n-m}^{(-\frac{1}{2})}\left(\frac{-2y^2}{\sigma_p^2-1}\right). \quad (26)$$

Examples of Eq. (26) for a three-photon input state can be seen in Fig. 4 using the nonlinear process presented in Sec. VI. The three-photon input state ($n=3$) is shown in Fig. 4(b), while the output state in the mode $v(\omega)$, calculated according to Eq. (26), is visualized in Fig. 4(c) excluding the nonlinear interaction ($\alpha=0$) and in Fig. 4(d) including the interaction ($\alpha=7 \times 10^4$).

It is important to note here that in general the time evolution connecting the input and output Wigner function is not unitary and the system of input and output modes is not closed since the vacuum modes in Eq. (10) are traced out during the calculation of the output quantum state. Tracing over the additional modes will admix some of the (transformed) vacuum modes to the output quantum state which is quantified by the singular values σ_x, σ_p . Due to the additional contributions from the traced out vacuum modes, A does not need to be symplectic to ensure the conservation of the canonical commutation relation. Thus, the output Wigner function can be thermal, even if the input state was pure. The thermalization is determined by σ_x and σ_p , since the matrix A only rescales the arguments of the Wigner function.

The thermalization of the output state can be seen in Fig. 4(c), since the output Wigner function is non-Gaussian, but the negativity is reduced. The thermalization originates from the mismatch between the input and output mode, which requires introducing two modes $h_1(\omega)$ and $h_2(\omega)$ to describe the operator input-output relation (see Sec. III). The nonlinear interaction, introduced in Sec. VI can be used to increase the mode overlap again, since the transformation of the output mode operator $\hat{b}_v$ with the unitary in Eq. (22) of the nonlinear interaction will result in two (overlapping) modes $f(\Omega) = \frac{1}{\xi}[\mu v(\Omega) + \frac{\nu}{\theta} f_K(\Omega)]$ and $g(\Omega) = \frac{\nu}{\theta \xi} f_J(\Omega)$ (see Appendix B). Because $f_J(\Omega)$ overlaps well with the input mode, the nonlinear interaction reduces the contribution of the modes $h_1(\omega)$ and $h_2(\omega)$ to the input-output relation and thus decreases σ_x and σ_p . For the example in Fig. 4, the minimum of the output Wigner function is decreased from -7.11×10^{-4} without the nonlinear interaction to -4.66×10^{-3} with the nonlinear interaction. The Wigner negativity is not necessarily improved due to the nonlinear interaction, since it can introduce some squeezing which can lead to some additional thermalization, as we will discuss in the next section.

Our results open a path to a possible optimization in the detection of pulsed quantum states exhibiting Wigner negativity. By combining the input quantum state in the mode $u(\Omega)$ with a squeezed vacuum pulse in the mode $h_1(\Omega)$ orthogonal to $u(\Omega)$, the smoothing parameter in one direction, e.g., σ_p , could be reduced by the expense of enlarging the other. The combined effect still could increase the total Wigner negativity in the output state. We elaborate on the extension to multiple input modes using a two-mode squeezed state in the following section.

B. Two-mode squeezed vacuum

In the example considered in this section, two broadband modes in the THz regime, containing a two-mode squeezed vacuum state, are up-converted to the optical frequency range, in which only one mode is of further interest, as depicted in Fig. 2. Multimode squeezed states possess a Gaussian Wigner function, thus we can utilize Eq. (16) to describe the output Wigner function.

Since the number of output modes is smaller than the number of input modes, $M < N$ we need to find the GBMD, introduced in Sec. IV, $A = \mathcal{A}P_R S_R$. A Python implementation of the algorithm, presented in [33] and described in Appendix C 1, for the specific setting considered in this section, can be found in [47]. We use the same parameters for the nonlinear crystal as in the previous example (and introduced in Sec. VI). In the case of a real drive amplitude $\alpha \in \mathbb{R}$, the mode functions are real as well, $v(\omega), u(\omega) \in \mathbb{R}$ for all ω , and the matrices $\mathcal{A}, \text{cov}_B$ become diagonal. Therefore, we can represent the dependence of the matrix elements $\mathcal{A}_{11}, \mathcal{A}_{22}$ and $[\text{cov}_B]_{11}, [\text{cov}_B]_{22}$ on $\omega_{\text{out}}, \alpha$ in Fig. 5. A prominent feature of Fig. 5 is the different dependency on α in the beam-splitting and squeezing regime, introduced in Sec. VI. While the central frequency of the output mode determines the regime, with a transition from squeezing to beam-splitting at $\omega_{\text{out}}/(2\pi) = 186$ THz, the parameter α affects the squeezing and beam-splitting interaction the same and thus is proportional to θ . Therefore, the quantities determining the input-output relation of the quadratures, shown in Fig. 5, exhibits a periodic dependence on the coherent amplitude α in the beam-splitting regime and exponential dependence on α in the squeezing regime. Increasing the central frequency ω_{out} of the output mode increases the overlap of the mode function $f(\Omega)$ in Eq. (5) with the input mode function $u(\Omega)$, while decreasing the overlap between the mode function $g(\Omega)$ in Eq. (6) and $u(\Omega)$. The change of the overlap between the respective modes results in a sign change of $\mathcal{A}_{11}$, as visible in Fig. 5(a), and thus leads to a zero crossing of $\mathcal{A}_{11}$. The attenuation of the x quadrature close to the transition from beam-splitting to squeezing regime is observed in e.g. classical electro-optic sampling [56]. It is an open question, if the transition from beam-splitting to squeezing regime is a phase transition, which will not be addressed in this work.

The two parameter sets exemplified in this section are marked by (i) for $\omega_{\text{out}} = 193$ THz and $\alpha = 7 \times 10^4$ or (ii) for $\omega_{\text{out}} = 199$ THz and $\alpha = 7 \times 10^4$.

The most significant difference to the previous example in Sec. VII A, is the reduction of the input modes to one effective input mode with quadrature operators $\hat{\Gamma}_{\text{in},s} = P_R S_R$

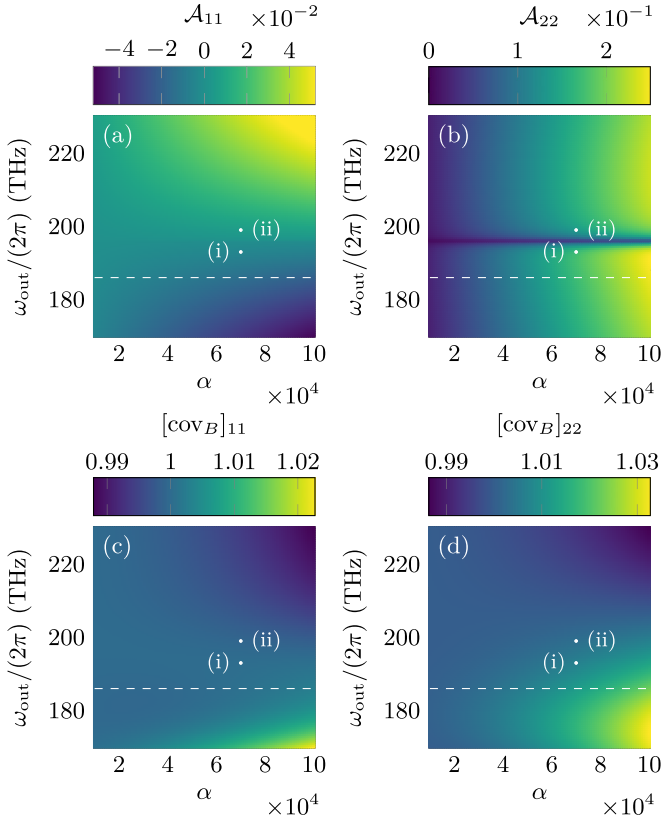


FIG. 5. The four parameters, independent of the input quantum state, determining the Wigner function of the output quantum state as a function of the central frequency ω_{out} of the output mode and the amplitude α of the classical pulse driving the nonlinear interaction. (a, b) The nonzero elements $\mathcal{A}_{11}$, $\mathcal{A}_{22}$ of the matrix $\mathcal{A}$. The matrix $\mathcal{A}$ is calculated using the generalized Bloch-Messiah decomposition as described in Sec. VII B. The matrix $\mathcal{A}$ scales the phase space of the output Wigner functions. (c, d) The two nonzero entries of the covariance matrix cov_B in Eq. (16), independent of the input states covariance matrix, as a function of the same parameters. The horizontally dashed lines mark the transition from the squeezing to the beam-splitting regime, introduced in Sec. VI, as ω_{out} is increased.

$\hat{\Gamma}_{\text{in}} = (\hat{x}_{\text{eff}}, \hat{p}_{\text{eff}})^T$. The effective input modes corresponding to the two different output modes are shown in Fig. 2(b). While the p -quadrature is mostly independent of ω_{out} , the x -quadrature changes drastically. This is visible in Fig. 6(a), which shows how much the modes $u_1(\Omega)$ and $u_2(\Omega)$ contribute to the effective input quadratures as a function of the output modes central frequency. The contribution of each input mode is determined by S_R since the effective input mode corresponds to the first mode in $S_R \hat{\Gamma}_{\text{in}}$. While at $\omega_{\text{out}} = 193$ THz, the effective input mode mostly consists of contribution from the first input mode $u_1(\Omega)$, going to $\omega_{\text{out}} = 199$ THz exchanges the roles of $\hat{x}_{u_1}$ and $\hat{x}_{u_2}$ while the p -quadrature stays almost unaffected.

The impact of the different effective input modes on the output Wigner function is best exemplified by a two-mode squeezed input state with covariance matrix

$$\text{cov}_{\text{in}} = \text{diag}(e^{-2r}, e^{2r}, e^{2r}, e^{-2r}), \quad (27)$$

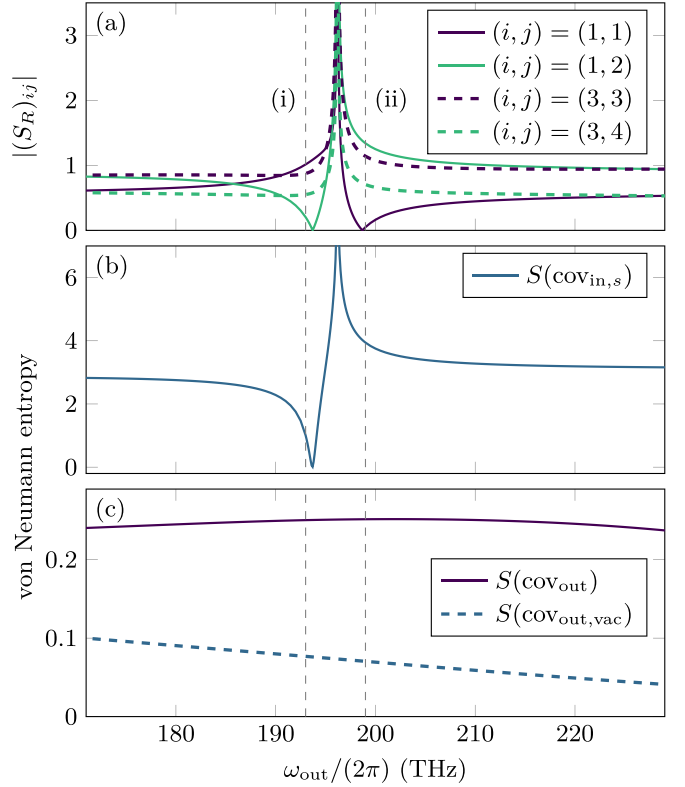


FIG. 6. (a) The nonzero elements of the symplectic matrix S_R transforming the quadratures of the input quantum state as a function of the central frequency ω_{out} of the coherent pulse driving the nonlinear interaction. The entries $(S_R)_{11}$ and $(S_R)_{12}$ determine the amount $\hat{x}_{u_1}$ and $\hat{x}_{u_2}$ contribute to the effective input quadrature $\hat{x}_{\text{eff}}$ shown in Fig. 2, while $(S_R)_{33}$ and $(S_R)_{34}$ determine the amount $\hat{p}_{u_1}$ and $\hat{p}_{u_2}$ contribute $\hat{p}_{\text{eff}}$. (b) The von Neumann entropy of transformed input state covariance matrix $\text{cov}_{\text{in},s}$ as a function of the central frequency ω_{out} of the output mode, calculated according to [20,21,57]. (c) The von Neumann entropy of the output state with covariance matrix cov_{out} , if the two-mode squeezed state with covariance matrix in Eq. (27) is used, and $\text{cov}_{\text{out},\text{vac}}$, if the input state is the vacuum.

expressed in the basis of $\hat{\Gamma}_{\text{in}}$. An example of the Wigner function with the covariance matrix of Eq. (27) and $r = 1$ can be seen in Fig. 7(a) for the reduced state of the phase space $(\hat{x}_{u_1}, \hat{p}_{u_1})$ and in Fig. 7(b) reduced to $(\hat{x}_{u_2}, \hat{p}_{u_2})$.

The effect of the exchange of the quadratures $\hat{x}_{u_1}$ and $\hat{x}_{u_2}$ on the squeezed state as the output frequency goes from $\omega_{\text{out}}/(2\pi) = 193$ THz to $\omega_{\text{out}}/(2\pi) = 199$ THz is shown in Figs. 7(c) and 7(d), respectively. At (i) the Wigner function with the covariance matrix, $\text{cov}_{\text{in},s}$, of the transformed input state used in Eq. (16) follows the reduced state of the first mode in Fig. 7(a). For (ii) the transformed state inherits the antisqueezing of the first mode in the p quadrature and of the second mode in the x quadrature, resulting in a thermal state, i.e. the Wigner function is spread over a larger phase-space volume. The thermalization can be quantified with the von Neumann entropy shown in Fig. 6(b) as a function of the central frequency of the output mode ω_{out} . For a Gaussian quantum state, the von Neumann entropy can be calculated according to [20,21,57]. Since the transformation with S_R is symplectic and the initial state is pure, the thermalization

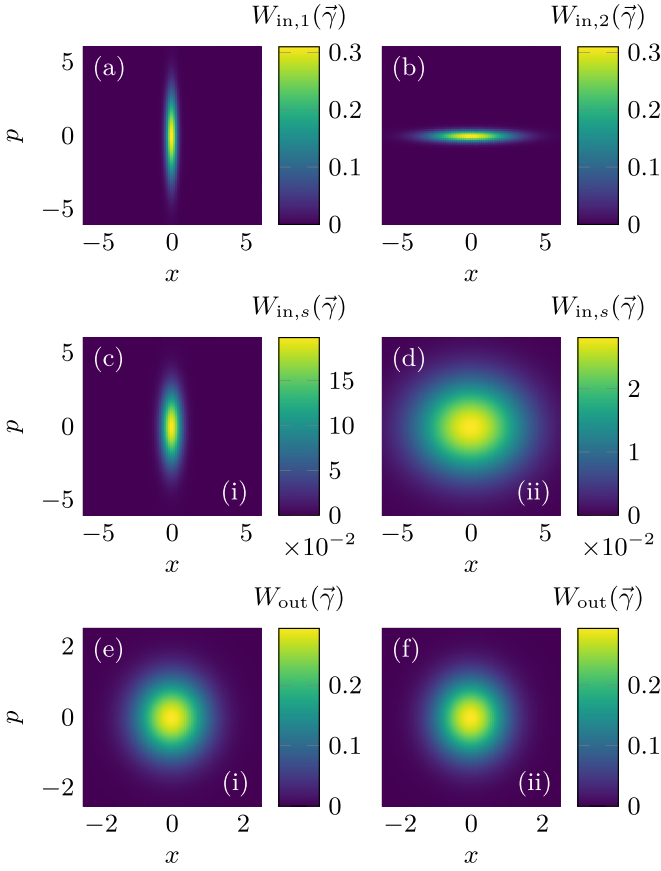


FIG. 7. (a, b) The two-mode squeezed input Wigner function of the first mode $u_1(\Omega)$ in (a) with a squeezing parameter of $r_1 = -1$ and of the second mode $u_2(\Omega)$ in (b) with a squeezing parameter of $r_2 = 1$. The covariance matrix of the input state is defined in Eq. (27). (c, d) The Wigner function of the transformed and reduced input state with covariance matrix $\text{cov}_{\text{in},s}$ for an output mode centered at $\omega_{\text{out}}/(2\pi) = 193$ THz or $\omega_{\text{out}}/(2\pi) = 199$ THz, respectively. (e, f) The Wigner function of the output state with the covariance matrix of Eq. (16) for the same output modes as in (c) and (d). The amplitude of the coherent pulse driving the nonlinear interaction is $\alpha = 7 \times 10^4$.

has to originate from entanglement breakage during the mode reduction and thus the von Neumann entropy measures the entanglement between the modes in $S_R \hat{\Gamma}$.

The Wigner function of the output state with covariance matrix according to Eq. (16) for the parameters (i) and (ii) are shown in Figs. 7(e) and 7(f) respectively. Both distributions are thermal, as can be seen from the von Neumann entropy as a function of the central frequency of the output mode in Fig. 6(c). The von Neumann entropy of the output state does not show the increase which the transformed input state exhibits, as the central frequency of the output mode is shifted from the squeezing to the beam-splitting regime. However, the difference in the von Neumann entropy of the up-converted two-mode squeezed state to the von Neumann entropy of the output state with covariance matrix $\text{cov}_{\text{out},\text{vac}} = \mathcal{A}\mathcal{A}^T + \text{cov}_B$, resulting from the input state being the vacuum, shows a clear increase. This is because some of the thermalization generated in the up-conversion is state independent and captured by

cov_B . We can utilize this fact to calculate

$$\text{cov}_{\text{diff}} = \text{cov}_{\text{out}} - \text{cov}_{\text{out},\text{vac}} = \mathcal{A}(\text{cov}_{\text{in},s} - \mathbb{I})\mathcal{A}^T. \quad (28)$$

Since for a real drive amplitude $\alpha \in \mathbb{R}$ all matrices are diagonal, it is sufficient to consider the variances. If one of the variances in cov_{diff} is negative, the transformed input state exhibits some squeezing, since the only negativity can originate from $\text{cov}_{\text{in},s} - \mathbb{I}$. While we are not including the measurement here, the variances cov_{diff} can potentially be accessed using, e.g., the electro-optic sampling setup from [10]. For the example in Fig. 7(e), the difference between the covariance matrices is $\text{cov}_{\text{diff}} = \text{diag}(-1.16 \times 10^{-5}, 0.14)$ and thus hints at some squeezing in the x quadrature of the transformed input state below the vacuum fluctuations. For the example in Fig. 7(f), the difference covariance matrix is $\text{cov}_{\text{diff}} = \text{diag}(3.30 \times 10^{-4}, 0.15)$ and therefore the transformed input state shows excess fluctuations in both quadratures compared to the vacuum.

Thus, the generalized Bloch-Messiah decomposition allows us to find an input-output relation between a different number of input and output modes. By doing so, we can investigate the emergence of thermalization in a parametric up-conversion and find suitable modes to achieve the conversion of desired states.

VIII. CONCLUSIONS

Nonlinear optics is key for the development of multi-mode (ultrafast) quantum optics. In this work, we present the phase-space input-output relation of multimode Wigner functions representing pulsed quantum states undergoing a $\chi^{(2)}$ -nonlinear interaction. While the input modes are determined by the pulsed quantum state, the output modes are arbitrary and could, for example, be determined by a homodyne detection after the nonlinear interaction. As the number of broadband input modes might not match the number of output modes, we introduce the generalized Bloch-Messiah decomposition (GBMD), based on a symplectic version of the singular value decomposition, to reduce the number of modes such that they coincide. With the GBMD we show that the output Wigner function is related to the Wigner function of the reduced input state, convolved with a Gaussian phase-space function and rescaled by a nonsymplectic matrix.

We elaborate on this result using two examples. First, we consider a Fock state in a single infrared-broadband mode parametrically amplified and spectrally filtered to a narrow bandwidth output mode. We present an example where the Wigner negativity of the input state is more resilient to the mode mismatch if the nonlinear interaction is included. Second, we describe the up-conversion of a squeezed pulse in two THz-broadband modes to a single optical pulse and show that the reduction of the input modes due to the GBMD can lead to entanglement breakage in the two-mode squeezed input state, resulting in thermalization of the output state. We identify two regimes of the up-conversion determined by the central frequency of the output mode, which we call beam-splitter and squeezing regime, and characterize the effect of the different regimes on the output Wigner function. We quan-

tify the thermalization using the von Neumann entropy and identify a state independent thermalization due to the breakage of entanglement generated in the nonlinear interaction itself, which leads to excess thermalization in the squeezing regime.

With the tools presented in this work, we are able to efficiently describe the nonlinear interactions of pulsed quantum states, necessary to advance quantum optics to the ultrafast regime, with possible applications spanning from optical quantum information technologies to efficient fast photo detection.

ACKNOWLEDGMENTS

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DATA AVAILABILITY

The data that support the findings of this article are openly available [47].

APPENDIX A: INPUT-OUTPUT RELATION OF THE QUADRATURE OPERATORS

In this Appendix we derive the quadrature input-output relation of Eq. (10).

1. Orthogonalization procedure

Applying the Gram-Schmidt orthogonalization procedure to $\{u_1, \dots, u_N, g_1, \dots, g_M, f_1, \dots, f_M\}$ results in a set $\{u_1, \dots, u_N, h_1, \dots, h_{2M}\}$ of orthogonal mode functions with

$$h_i(\Omega) = \begin{cases} \frac{1}{x_i} [g_i(\Omega) - \sum_{j=1}^N \langle u_j, g_i \rangle u_j(\Omega) - \sum_{j=1}^{i-1} \langle h_j, g_i \rangle h_j(\Omega)] & \text{for } 1 \leq i \leq M \\ \frac{1}{z_{i-M}} [f_{i-M}(\Omega) - \sum_{j=1}^N \langle u_j, f_{i-M} \rangle u_j(\Omega) - \sum_{j=1}^{i-1} \langle h_j, f_{i-M} \rangle h_j(\Omega)] & \text{for } M < i \leq 2M \end{cases} \quad (\text{A1})$$

If we solve the above equations for $g_i(\omega)$ and $f_i(\omega)$, we obtain

$$g_i(\Omega) = x_i h_i(\Omega) + \sum_{j=1}^N \langle u_j, g_i \rangle u_j(\Omega) + \sum_{j=1}^{i-1} \langle h_j, g_i \rangle h_j(\Omega), \quad (\text{A2})$$

$$f_i(\Omega) = z_i h_{i+M}(\Omega) + \sum_{j=1}^N \langle u_j, f_i \rangle u_j(\Omega) + \sum_{j=1}^{i+M-1} \langle h_j, f_i \rangle h_j(\Omega), \quad (\text{A3})$$

which we can insert into the output operators of Eq. (4),

$$\hat{b}_{v_i} = \hat{a}_{f_i} + \hat{a}_{g_i}^\dagger = z_i \hat{a}_{h_{i+M}} + \sum_{j=1}^N \langle u_j, f_i \rangle^* \hat{a}_{u_j} + \sum_{j=1}^{i+M-1} \langle h_j, f_i \rangle^* \hat{a}_{h_j} + x_i \hat{a}_{h_i}^\dagger + \sum_{j=1}^N \langle u_j, g_i \rangle \hat{a}_{u_j}^\dagger + \sum_{j=1}^{i-1} \langle h_j, g_i \rangle \hat{a}_{h_j}^\dagger. \quad (\text{A4})$$

By collecting the input, output, and Gram-Schmidt operators into the vectors

$$\hat{b}_{\text{out}} = (\hat{b}_{v_1}, \dots, \hat{b}_{v_M}, \hat{b}_{v_1}^\dagger, \dots, \hat{b}_{v_M}^\dagger)^\text{T}, \quad \hat{a}_{\text{in}} = (\hat{a}_{u_1}, \dots, \hat{a}_{u_N}, \hat{a}_{u_1}^\dagger, \dots, \hat{a}_{u_N}^\dagger)^\text{T}, \quad \hat{a}_\perp = (\hat{a}_{h_1}, \dots, \hat{a}_{h_{2M}}, \hat{a}_{h_1}^\dagger, \dots, \hat{a}_{h_{2M}}^\dagger)^\text{T},$$

and defining the matrix elements $[F_{\text{in}}]_{ji} = \langle u_i, f_j \rangle^* = \iint v_j^*(\omega) F(\omega, \Omega) u_i(\Omega) d\omega d\Omega$ and $[G_{\text{in}}]_{ji} = \langle u_i, g_j \rangle = \iint v_j^*(\omega) G^*(\omega, \Omega) u_i^*(\Omega) d\omega d\Omega$, which correspond to the matrices

$$F_{\text{in}} = \begin{pmatrix} \langle u_1, f_1 \rangle^* & \dots & \langle u_N, f_1 \rangle^* \\ \vdots & & \vdots \\ \langle u_1, f_M \rangle^* & \dots & \langle u_N, f_M \rangle^* \end{pmatrix}, \quad G_{\text{in}} = \begin{pmatrix} \langle u_1, g_1 \rangle & \dots & \langle u_N, g_1 \rangle \\ \vdots & & \vdots \\ \langle u_1, g_M \rangle & \dots & \langle u_N, g_M \rangle \end{pmatrix}, \quad (\text{A5})$$

as well as defining the matrices

$$F_\perp = \begin{pmatrix} \langle h_1, f_1 \rangle^* & \dots & \langle h_M, f_1 \rangle^* & z_1 & 0 & 0 & \dots & 0 \\ \langle h_1, f_2 \rangle^* & \dots & \langle h_M, f_2 \rangle^* & \langle h_{M+1}, f_2 \rangle^* & z_2 & 0 & \dots & 0 \\ \vdots & \ddots & & \vdots & & & & \\ \langle h_1, f_M \rangle^* & \dots & \langle h_M, f_M \rangle^* & \dots & & \langle h_{2M-1}, f_M \rangle^* & z_M \end{pmatrix}, \quad (\text{A6})$$

$$G_\perp = \begin{pmatrix} x_1 & 0 & 0 & \dots & 0 & \dots \\ \langle h_1, g_2 \rangle & x_2 & 0 & \dots & 0 & \dots \\ \vdots & \ddots & \vdots & & & \\ \langle h_1, g_M \rangle & \dots & \langle h_{M-1}, g_M \rangle & x_M & 0 & \dots \end{pmatrix}, \quad (\text{A7})$$

we can utilize

$$A_c = \begin{pmatrix} F_{\text{in}} & G_{\text{in}} \\ G_{\text{in}}^* & F_{\text{in}}^* \end{pmatrix}, \quad B_c = \begin{pmatrix} F_{\perp} & G_{\perp} \\ G_{\perp}^* & F_{\perp}^* \end{pmatrix} \quad (\text{A8})$$

to recast Eq. (A4) into $\hat{b}_{\text{out}} = A_c \hat{a}_{\text{in}} + B_c \hat{a}_{\perp}$. To go from the complex mode space to the real phase space, we can use the matrix

$$T_c = \frac{1}{\sqrt{2}} \begin{pmatrix} \mathbb{I} & i\mathbb{I} \\ \mathbb{I} & -i\mathbb{I} \end{pmatrix} \quad (\text{A9})$$

to transform the vectors $\hat{\Gamma}_{\text{out}} = T_c^\dagger \hat{b}_{\text{out}}$, $\hat{\Gamma}_{\text{in}} = T_c^\dagger \hat{a}_{\text{in}}$, $\hat{\Gamma}_{\perp} = T_c^\dagger \hat{a}_{\perp}$ and the matrices

$$A = T_c^\dagger A_c T_c = \begin{pmatrix} \text{Re}(F_{\text{in}} + G_{\text{in}}) & \text{Im}(-F_{\text{in}} + G_{\text{in}}) \\ \text{Im}(F_{\text{in}} + G_{\text{in}}) & \text{Re}(F_{\text{in}} - G_{\text{in}}) \end{pmatrix} = \begin{pmatrix} \text{Re}(F_{\text{in}}) & \text{Im}(-F_{\text{in}}) \\ \text{Im}(F_{\text{in}}) & \text{Re}(F_{\text{in}}) \end{pmatrix} + \begin{pmatrix} \text{Re}(G_{\text{in}}) & \text{Im}(G_{\text{in}}) \\ \text{Im}(G_{\text{in}}) & \text{Re}(-G_{\text{in}}) \end{pmatrix}, \quad (\text{A10})$$

$$B = T_c^\dagger B_c T_c = \begin{pmatrix} \text{Re}(F_{\perp} + G_{\perp}) & \text{Im}(-F_{\perp} + G_{\perp}) \\ \text{Im}(F_{\perp} + G_{\perp}) & \text{Re}(F_{\perp} - G_{\perp}) \end{pmatrix} = \begin{pmatrix} \text{Re}(F_{\perp}) & \text{Im}(-F_{\perp}) \\ \text{Im}(F_{\perp}) & \text{Re}(F_{\perp}) \end{pmatrix} + \begin{pmatrix} \text{Re}(G_{\perp}) & \text{Im}(G_{\perp}) \\ \text{Im}(G_{\perp}) & \text{Re}(-G_{\perp}) \end{pmatrix}, \quad (\text{A11})$$

accordingly. Thus, multiplying the equation for $\hat{b}_{\text{out}}$ by T_c and using $T_c T_c^\dagger = \mathbb{I}$, we arrive at the input-output relation of Eq. (10).

2. Example $N = M = 1$

Let us start with the simplest example used in the main text: one input mode $u(\omega)$ and one output mode $v(\omega)$ or $M = N = 1$. In this case the matrix connecting the input and the output quadratures is

$$A = \begin{pmatrix} \text{Re}(\langle u, f + g \rangle) & \text{Im}(\langle u, f + g \rangle) \\ \text{Im}(\langle u, -f + g \rangle) & \text{Re}(\langle u, f - g \rangle) \end{pmatrix}. \quad (\text{A12})$$

The matrix A is symplectic up to a scalar since $A^T \Omega_2 A = \Omega_2 (|\langle u, f \rangle|^2 - |\langle u, g \rangle|^2)$. With the matrix

$$B = \begin{pmatrix} \text{Re}(\langle h_1, f \rangle) + x & z & -\text{Im}(\langle h_1, f \rangle) & 0 \\ \text{Im}(\langle h_1, f \rangle) & 0 & \text{Re}(\langle h_1, f \rangle) - x & z \end{pmatrix}, \quad (\text{A13})$$

the contribution due to the initially unoccupied modes is

$$\text{cov}_B = \begin{pmatrix} (\text{Re}(\langle h_1, f \rangle) + x)^2 & 0 \\ 0 & (\text{Re}(\langle h_1, f \rangle) - x)^2 \end{pmatrix} + [\text{Im}(\langle h_1, f \rangle)^2 + z^2] \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \quad (\text{A14})$$

APPENDIX B: THE FIRST-ORDER UNITARY FOR THZ-TO-OPTICAL FREQUENCY CONVERSION

In this Appendix we will derive the input-output relation of the quadrature operators used in the example of THz-to-optical frequency conversion as described in Sec. VI. We start from the time evolution operator,

$$\hat{U}_{\text{NL}} \approx \exp \left(\frac{1}{2} \iint_0^\infty J[\Omega, \omega] \hat{a}_\omega \hat{a}_\Omega d\omega d\Omega - \text{H.c.} + \iint_0^\infty K[\Omega, \omega] \hat{a}_\omega^\dagger \hat{a}_\Omega d\omega d\Omega \right), \quad (\text{B1})$$

which is expressed in terms of a squeezing interaction given by the kernel $J(\Omega, \omega)$ and beam-splitting interaction described by $K(\Omega, \omega)$. The unitary time-evolution operator is derived in [46] and is based on a first-order Magnus expansion [58]. For the geometric arrangement of the classical pulse and nonlinear crystal as in [16], the kernels are given by [46]

$$J(\Omega, \omega) = 2 \frac{1}{\hbar} (2\pi)^{3/2} \left(\frac{\hbar}{4\pi \varepsilon_0 c A} \right)^{3/2} \sqrt{\frac{\omega + \Omega}{n(\omega + \Omega)}} \alpha(\omega + \Omega) \sqrt{\frac{\omega \Omega}{n(\omega) n(\Omega)}} \hat{\lambda}[\Delta k(\Omega, \omega)], \quad (\text{B2})$$

$$K(\Omega, \omega) = 2 \frac{1}{\hbar} (2\pi)^{3/2} \left(\frac{\hbar}{4\pi \varepsilon_0 c A} \right)^{3/2} \sqrt{\frac{|\omega - \Omega|}{n(\omega - \Omega)}} [\alpha(\Omega - \omega) + \alpha^*(\omega - \Omega)] \sqrt{\frac{\omega \Omega}{n(\omega) n(\Omega)}} \hat{\lambda}[\Delta k(-\Omega, \omega)], \quad (\text{B3})$$

with $\hat{\lambda}(k) = \lambda \frac{L}{\sqrt{2\pi}} \text{sinc}(kL/2)$ being the Fourier transform of the transversal profile of the crystal, the wave-vector mismatch $\Delta k(\Omega, \omega) = k_{\Omega+\omega} - k_\Omega - k_\omega$ and the coherent amplitude $\alpha(\Omega)$ of the classical pulse, which vanishes for negative Ω , the refractive index $n(\omega)$ of zinc telluride [52], and the beam-waist area $A = \pi (3 \mu\text{m})^2$ as well as $\hbar$, c and ε_0 being the reduced Planck constant, the speed of light in vacuum and the permittivity, respectively.

The first-order unitary approach developed in [39,40] allows to calculate the output operator $\hat{a}_v$ in terms of input operators. As a first step, we use the first order of the Baker-Campbell-Hausdorff-formula together with the definition

$f_K(\Omega) = -\int K^*(\Omega, \omega)v(\omega)d\omega$, $f_J(\Omega) = \int J^*(\Omega, \omega)v^*(\omega)d\omega$ to find

$$\hat{U}_{\text{NL}}^\dagger \hat{a}_v \hat{U}_{\text{NL}} = \hat{a}_v + \int_0^\infty f_K^*(\Omega) \hat{a}_\Omega d\Omega + \int_0^\infty f_J(\Omega) \hat{a}_\Omega^\dagger d\Omega = \hat{a}_v + \theta_K \hat{a}_K + \theta_J \hat{a}_J^\dagger. \quad (\text{B4})$$

We normalize the operators $\hat{a}_K$ and $\hat{a}_J$, defined by $f_K(\Omega)$ and $f_J(\Omega)$, using

$$\theta_K = \left(\int |f_K(\Omega)|^2 d\Omega \right)^{1/2}, \quad \theta_J = \left(\int |f_J(\Omega)|^2 d\Omega \right)^{1/2}. \quad (\text{B5})$$

By defining $\theta = \sqrt{|\theta_K^2 - \theta_J^2|}$, $\vartheta_J = \theta_J/\theta$ and $\vartheta_K = \theta_K/\theta$, we can find a unitary operator, which transforms the output mode operator as

$$\hat{U}_{\text{NL},1} \hat{a}_v \hat{U}_{\text{NL},1}^\dagger = \begin{cases} \cos(\theta) \hat{a}_v + \sin(\theta) [\vartheta_K \hat{a}_K + \vartheta_J \hat{a}_J^\dagger] & \theta_K > \theta_J \\ \cosh(\theta) \hat{a}_v + \sinh(\theta) [\vartheta_K \hat{a}_K + \vartheta_J \hat{a}_J^\dagger] & \theta_K < \theta_J \end{cases}. \quad (\text{B6})$$

Since the commutator

$$[\vartheta_K \hat{a}_K + \vartheta_J \hat{a}_J^\dagger, \vartheta_K \hat{a}_K^\dagger + \vartheta_J \hat{a}_J] = \begin{cases} 1 & \theta_K > \theta_J \\ -1 & \theta_K < \theta_J \end{cases} \quad (\text{B7})$$

corresponds to $\vartheta_K \hat{a}_K + \vartheta_J \hat{a}_J^\dagger$ being either a creation or annihilation operator, the unitary evolution in Eq. (B6) has to correspond to either a beam-splitting or squeezing interaction. In both cases the first order of a Taylor expansion around $\theta = 0$ agrees with the result obtained from the Baker-Campbell-Hausdorff-formula in Eq. (B4). Although the above argument only guarantees the validity of the first-order unitary for small θ , calculations involving the second-order unitary suggest a large parameter range, the first-order unitary holds valid [39].

By using μ and ν as in Eq. (25) and comparing the input-output relation from Eqs. (B6) and (10), we find $F(\omega, \Omega) = \mu \delta(\omega - \Omega) - \frac{\nu}{\theta} K(\Omega, \omega)$ and $G(\Omega, \omega) = \frac{\nu}{\theta} J(\Omega, \omega)$, resulting in

$$f(\Omega) = \frac{1}{\zeta} \int F^*(\omega, \Omega) v(\omega) d\omega = \frac{1}{\zeta} \left[\mu v(\Omega) + \frac{\nu}{\theta} f_K(\Omega) \right], \quad (\text{B8})$$

$$\zeta^2 = \langle f, f \rangle = \mu^2 \langle v, v \rangle + \frac{2\nu\mu}{\theta} \text{Re}(\langle v, f_K \rangle) + \frac{\nu^2}{\theta^2} \langle f_K, f_K \rangle = \mu^2 + \frac{2\nu\mu}{\theta} \text{Re}(\langle v, f_K \rangle) + \nu^2 \vartheta_K, \quad (\text{B9})$$

$$g(\Omega) = \frac{1}{\xi} \int d\omega G^*(\omega, \Omega) v^*(\omega) = \frac{\nu}{\theta\xi} f_J(\Omega), \quad (\text{B10})$$

$$\xi^2 = \langle g, g \rangle = \frac{\nu^2}{\theta^2} \langle f_J, f_J \rangle = \nu^2 \vartheta_J^2. \quad (\text{B11})$$

Therefore, the matrix elements connecting the input and output modes are

$$[F_{\text{in}}]_{1i} = \langle u_i, f \rangle^* = \frac{1}{\zeta} \left[\mu \langle u_i, v \rangle^* + \frac{\nu}{\theta} \langle u_i, f_K \rangle^* \right], \quad (\text{B12})$$

$$[G_{\text{in}}]_{1i} = \langle u_i, g \rangle = \frac{\nu}{\theta\xi} \langle u_i, f_J \rangle. \quad (\text{B13})$$

APPENDIX C: GENERALIZED BLOCH-MESSIAH DECOMPOSITION

In general it is not possible to decompose A using a Bloch-Messiah decomposition, since the matrix is not symplectic. For this reason we introduce the generalized Bloch-Messiah decomposition (GBMD) based on the symplectic singular value-like decomposition (SSVD) introduced by Xu [32,33]. In principle the matrix A could not be full rank, if for example the overlap between one input mode and all modes $f_i(\omega)$ and $g_i(\omega)$ is zero or the other way around. However, this transformation is trivial and will be excluded here. We therefore assume $A\Omega A^T$ is nonsingular, i.e., $\text{rank}(A\Omega A^T) = \min(2M, 2N)$.

First, we consider the case with more input modes than output modes $N > M$. The SSVD then states that there exist a decomposition

$$A = O_R \begin{pmatrix} \Sigma_R & 0 & 0 & 0 \\ 0 & 0 & \Sigma_R & 0 \end{pmatrix} S_R = O_R \begin{pmatrix} \Sigma_R & 0 \\ 0 & \Sigma_R \end{pmatrix} \begin{pmatrix} \mathbb{I}_M & 0 & 0 & 0 \\ 0 & 0 & \mathbb{I}_M & 0 \end{pmatrix} S = \mathcal{A} P_R S_R \quad (\text{C1})$$

with $O_R \in \mathbb{R}^{2M \times 2M}$ orthogonal, $\mathcal{A} = O_R \text{diag}(\Sigma_R, \Sigma_R) \in \mathbb{R}^{2M \times 2M}$ regular, Σ_R positive diagonal, $S_R \in \mathbb{R}^{2N \times 2N}$ symplectic and P_R projecting from $2N$ dimensions to $2M$ dimensions. By defining $\hat{\Gamma}_{\text{in},s} = P_R S_R \hat{\Gamma}_{\text{in}} = (\hat{x}_{\text{eff},1}, \dots, \hat{x}_{\text{eff},M}, \hat{p}_{\text{eff},1}, \dots, \hat{p}_{\text{eff},M})^T$ the

input-output relation in Eq. (10) can be recast into

$$\hat{\Gamma}_{\text{out}} = \mathcal{A} \hat{\Gamma}_{\text{in},s} + B \vec{\Gamma}_{\perp}, \quad (\text{C2})$$

which now only includes the first $2M$ quadrature operators and the orthogonal modes in $\vec{\Gamma}_{\perp}$.

Second, we assume more output modes than input modes, i.e., $M > N$, and decompose

$$A^T = O_L \begin{pmatrix} \Sigma_L & 0 & 0 & 0 \\ 0 & 0 & \Sigma_L & 0 \end{pmatrix} S_L = O_L \begin{pmatrix} \Sigma_L & 0 \\ 0 & \Sigma_L \end{pmatrix} \begin{pmatrix} \mathbb{I}_N & 0 & 0 & 0 \\ 0 & 0 & \mathbb{I}_N & 0 \end{pmatrix} S_L = \mathcal{A}^T P_L S_L \quad (\text{C3})$$

with $O_L \in \mathbb{R}^{2N \times 2N}$ orthogonal, $\mathcal{A} = O_L \text{diag}(\Sigma_L, \Sigma_L) \in \mathbb{R}^{2N \times 2N}$ regular, Σ_L positive diagonal, $S_L \in \mathbb{R}^{2M \times 2M}$ symplectic and P_L projecting from $2M$ dimensions to $2N$ dimensions.

Together we can write

$$A = S_L^T P_L^T \mathcal{A} P_R S_R \quad (\text{C4})$$

with either $S_L = P_L = \mathbb{I}$ for $N > M$, $S_R = P_R = \mathbb{I}$ for $M > N$ or $S_L = P_L = S_R = P_R = \mathbb{I}$ if $N = M$. In all three cases, $\mathcal{A} \in \mathbb{R}^{2k \times 2k}$ with $k = \min(M, N)$ is regular and we call Eqs. (C1) and (C3) the generalized Bloch-Messiah decompositions. An algorithm to compute the SSVD can be found in [33], and in the next section we will present an example of the GBMD for one output mode and N input modes. In the version we present here, Σ_R might not be positive definite, which does not matter for our purposes here, since Σ will be absorbed into $\mathcal{A}$ anyways.

1. Example of the generalized Bloch-Messiah decomposition with one output mode and N input modes

If we assume one output mode, $M = 1$, and multiple input modes, $N > 1$, the matrix A generally takes the form

$$A = \left(\begin{array}{cccc|cccc} \bullet & \bullet & \bullet & \cdots & \bullet & \bullet & \bullet & \cdots \\ \bullet & \bullet & \bullet & \cdots & \bullet & \bullet & \bullet & \cdots \end{array} \right), \quad (\text{C5})$$

with $\bullet$ marking the (possibly nonzero) entries of the matrix. Following Xu [33], we can use the symplectic Housholder transformations $U_1, U_2, U = U_2 U_1$ acting on the second row of A , and symplectic Givens rotation O_R (in the order U_1, O_R, U_2) to transform the matrix A . First, U_1^T eliminates A_{22} to A_{2N} , then O_R^T eliminates A_{21} and lastly U_2^T can be used to eliminate $A_{2(N+1)}$ to $A_{2(2N)}$, since A_{21} to A_{2N} is already zero, resulting in

$$O_R^T A U = \left(\begin{array}{cccc|cccc} \bullet & \bullet & \bullet & \cdots & \bullet & \bullet & \bullet & \cdots \\ 0 & 0 & 0 & \cdots & \bullet & 0 & 0 & \cdots \end{array} \right) = \left(\begin{array}{cc|cc} R_{11} & R_{12} & R_{13} & R_{14} \\ 0 & 0 & R_{23} & 0 \end{array} \right). \quad (\text{C6})$$

Since $R_{23}R_{11}$ might be negative, one can follow [33] or in this case we can use $s = \text{sign}(R_{23}R_{11})$ and multiply Eq. (C6) from the left with $\begin{pmatrix} -s & 0 \\ 0 & 1 \end{pmatrix}$, which will later be absorbed into $\mathcal{A}$. With $R_{23}R_{11}$ being positive, we can define the symplectic matrix

$$S_R^{-1} = U \begin{pmatrix} \sqrt{R_{23}/R_{11}} & -R_{12}/R_{11} & -R_{13}^T/\sqrt{R_{11}R_{23}} & -R_{14}/R_{11} \\ 0 & \mathbb{I} & -R_{14}^T/\sqrt{R_{11}R_{23}} & 0 \\ 0 & 0 & \sqrt{R_{11}/R_{23}} & 0 \\ 0 & 0 & -R_{12}^T/\sqrt{R_{11}R_{23}} & \mathbb{I} \end{pmatrix}. \quad (\text{C7})$$

Using $\Sigma_R = \sqrt{R_{23}R_{11}}$ we obtain the GBMD

$$A = O_R \left(\begin{array}{cccc|cccc} s\Sigma_R & 0 & \cdots & 0 & 0 & 0 & \cdots \\ 0 & 0 & \cdots & \Sigma_R & 0 & 0 & \cdots \end{array} \right) S_R = \mathcal{A} P_R S_R, \quad (\text{C8})$$

with $\mathcal{A} = O_R \begin{pmatrix} s\Sigma_R & 0 \\ 0 & \Sigma_R \end{pmatrix}$ and $P_R = \begin{pmatrix} -1 & 0 & \cdots & 0 & 0 & \cdots \\ 0 & 0 & \cdots & 1 & 0 & \cdots \end{pmatrix}$. It should be mentioned, that the phase-space basis obtained from $P_R S_R \hat{\Gamma}_{\text{in}}$, might be squeezed. To undo the squeezing, we can calculate the effective squeezing parameter using $r_{\text{eff}} = \ln(\sqrt{\sum_{i=1}^N [S_R]_{1j}} / \sqrt{\sum_{i=1}^N [S_R]_{3j}})$ and multiply S_R from the left and $\mathcal{A}$ from the left by the respective squeezing or anti squeezing of the first mode.

APPENDIX D: QUANTUM STATE INPUT-OUTPUT RELATION IN PHASE SPACE

To calculate the Wigner function of the output state we use the characteristic function of the initial quantum state $\hat{\rho}$ with some modes occupied by the quantum state and the rest in a vacuum $\hat{\rho} = \hat{\rho}_{\text{in}} \otimes |0\rangle_{\perp} \langle 0|$. Using Eqs. (10) and (C4) the characteristic

function can be rewritten as

$$\begin{aligned}\chi_{\text{out}}(\vec{\beta}) &= \text{tr}[\hat{\rho} \exp(-i\hat{\Gamma}_{\text{out}}^T \Omega_M \vec{\beta})] = \text{tr}[\hat{\rho} \exp(-i[A\hat{\Gamma}_{\text{in}} + B\hat{\Gamma}_{\perp}]^T \Omega_M \vec{\beta})] \\ &= \text{tr}[\hat{\rho}_{\text{in}} \exp(-i[A\hat{\Gamma}_{\text{in}}]^T \Omega_M \vec{\beta})] \text{tr}[|0\rangle_{\perp} \langle 0| \exp(-i[B\hat{\Gamma}_{\perp}]^T \Omega_M \vec{\beta})] \\ &= \text{tr}[\hat{\rho}_{\text{in}} \exp(-i[P_R S_R \hat{\Gamma}_{\text{in}}]^T \mathcal{A}^T \Omega_M P_L S_L^{-1} \vec{\beta})] \text{tr}[|0\rangle_{\perp} \langle 0| \exp(-i\hat{\Gamma}_{\perp}^T B^T \Omega_M \vec{\beta})].\end{aligned}\quad (\text{D1})$$

Therefore, the generalized Bloch Messiah decomposition in Eq. (12) allows us to find a mode basis for which we can use the reduced state $\hat{\rho}_{\text{in},s} = \text{tr}_r(\hat{\rho}_{\text{in}})$ of the $k = \min\{N, M\}$ modes and the trace over the last $N - k$ modes in $S_R \hat{\Gamma}_{\text{in}}$. Thus, only the modes in $\hat{\Gamma}_{\text{in},s} = P_R S_R \hat{\Gamma}_{\text{in}}$ are relevant to the output quantum state. Furthermore, we define $\vec{\beta}_s = P_L S_L^{-1} \vec{\beta}$ and $\vec{\beta}_e$ containing the $M - k$ modes in $S_L^{-1} \vec{\beta}$, but not in $\vec{\beta}_s$. Thus, we have

$$\chi_{\text{out}}(\vec{\beta}) = \text{tr}[\hat{\rho}_{\text{in},s} \exp(-i\hat{\Gamma}_{\text{in},s}^T \mathcal{A}^T \Omega_L \vec{\beta}_s)] \text{tr}[|0\rangle_{\perp} \langle 0| \exp(-i\hat{\Gamma}_{\perp}^T B^T \Omega_M \vec{\beta})]. \quad (\text{D2})$$

From here on we will drop the index of Ω , the dimension of which is implied by the context. By defining $\tilde{\mathcal{A}} = \Omega^T \mathcal{A}^T \Omega$ ($\tilde{\mathcal{A}} = \mathcal{A}^{-1}$ if $\mathcal{A}$ is symplectic) and $\tilde{B} = \Omega^T B^T \Omega$ we can write

$$\begin{aligned}\chi_{\text{out}}(\vec{\beta}) &= \text{tr}[\hat{\rho}_{\text{in},s} \exp(-i\hat{\Gamma}_{\text{in},s}^T \Omega \tilde{\mathcal{A}} \vec{\beta}_s)] \text{tr}[|0\rangle_{\perp} \langle 0| \exp(\hat{\Gamma}_{\perp}^T \Omega \tilde{B} \vec{\beta})] \\ &= \chi_{\text{in},s}(\tilde{\mathcal{A}} \vec{\beta}_s) \chi_{\text{vac}}(\tilde{B} \vec{\beta}),\end{aligned}\quad (\text{D3})$$

with

$$\chi_{\text{vac}}(\tilde{B} \vec{\beta}) = \exp\left(-\frac{1}{4}[\tilde{B} \vec{\beta}]^T \tilde{B} \vec{\beta}\right) = \exp\left(-\frac{1}{4}\vec{\beta}^T \Omega^T B \Omega \Omega^T B^T \Omega \vec{\beta}\right) = \exp\left(-\frac{1}{4}\vec{\beta}^T \Omega \text{cov}_B \Omega^T \vec{\beta}\right) = \chi_B(\vec{\beta}) \quad (\text{D4})$$

and $\text{cov}_B = BB^T$. We decompose $\vec{\gamma}^T \Omega \vec{\beta} = \vec{\gamma}_e^T \Omega \vec{\beta}_e + \vec{\gamma}_s^T \Omega \vec{\beta}_s$ and $\vec{\beta}^T \Omega \text{cov}_B \Omega^T \vec{\beta} = \vec{\beta}_s^T \Omega \text{cov}_{\text{th},ss} \Omega^T \vec{\beta}_s + \vec{\beta}_e^T \Omega \text{cov}_{\text{th},es} \Omega^T \vec{\beta}_s + \vec{\beta}_s^T \Omega \text{cov}_{\text{th},se} \Omega^T \vec{\beta}_e + \vec{\beta}_e^T \Omega \text{cov}_{\text{th},ee} \Omega^T \vec{\beta}_e$ and using that cov_B is symmetric we can solve

$$\begin{aligned}&\int_{\mathbb{R}^{2(M-L)}} \exp(i\vec{\gamma}_e^T \Omega \vec{\beta}_e - \frac{1}{4}[\vec{\beta}_e^T \Omega \text{cov}_{\text{th},es} \Omega^T \vec{\beta}_s + \vec{\beta}_s^T \Omega \text{cov}_{\text{th},se} \Omega^T \vec{\beta}_e + \vec{\beta}_e^T \Omega \text{cov}_{\text{th},ee} \Omega^T \vec{\beta}_e]) d^{2(M-L)} \beta_e \\ &= \sqrt{\frac{(4\pi)^{2(M-L)}}{\det(\text{cov}_{\text{th},ee})}} \exp\left(\frac{1}{4}\vec{\beta}_s^T \Omega \text{cov}_{\text{th},se} \text{cov}_{\text{th},ee}^{-1} \text{cov}_{\text{th},es} \Omega^T \vec{\beta}_s + i\vec{\gamma}_e^T \text{cov}_{\text{th},ee}^{-1} \text{cov}_{\text{th},es} \Omega^T \vec{\beta}_s - \vec{\gamma}_e^T \Omega \text{cov}_{\text{th},ee}^{-1} \vec{\gamma}_e\right).\end{aligned}\quad (\text{D5})$$

This motivates the definition $\text{cov}_{\text{schur}} = \text{cov}_{\text{th},ss} - \text{cov}_{\text{th},se} \text{cov}_{\text{th},ee}^{-1} \text{cov}_{\text{th},es}$ and $\vec{\gamma}_{\text{schur}} = \text{cov}_{\text{th},es} \text{cov}_{\text{th},ee}^{-1} \vec{\gamma}_e$. Since $\tilde{\mathcal{A}}$ is regular and invertible, such that $\tilde{\mathcal{A}}^{-1} = \Omega^T (\mathcal{A}^{-1})^T \Omega$, we can calculate the Wigner function of the amplified state,

$W_{\text{out}}(\vec{\gamma})$

$$\begin{aligned}&= \frac{1}{(2\pi)^{2M}} \int_{\mathbb{R}^{2M}} \exp(i\vec{\gamma}^T \Omega \vec{\beta}) \chi_{\text{out}}(\vec{\beta}) d^{2M} \beta = \frac{1}{(2\pi)^{2M}} \int_{\mathbb{R}^{2M}} \exp\left(i\vec{\gamma}^T \Omega \vec{\beta} - \frac{1}{4}\vec{\beta}^T \Omega \text{cov}_B \Omega^T \vec{\beta}\right) \chi_{\text{in},s}(\tilde{\mathcal{A}} \vec{\beta}) d^{2M} \beta \\ &= [(2\pi)^{2(M+L)} \det(\text{cov}_{\text{th},ee}/2)]^{-\frac{1}{2}} \int_{\mathbb{R}^{2L}} \exp\left(i[\vec{\gamma}_s + \vec{\gamma}_{\text{schur}}]^T \Omega \vec{\beta}_s - \frac{1}{4}\vec{\beta}_s^T \Omega \text{cov}_{\text{schur}} \Omega^T \vec{\beta}_s\right) \chi_{\text{in},s}(\tilde{\mathcal{A}} \vec{\beta}_s) d^{2L} \beta_s \\ &= [(2\pi)^{2(M+L)} \det(\text{cov}_{\text{th},ee}/2)]^{-\frac{1}{2}} \int_{\mathbb{R}^{2L}} \exp\left(i[\vec{\gamma}_s + \vec{\gamma}_{\text{schur}}]^T \Omega \tilde{\mathcal{A}}^{-1} \tilde{\mathcal{A}} \vec{\beta}_s - \frac{1}{4}[\tilde{\mathcal{A}}^{-1} \tilde{\mathcal{A}} \vec{\beta}_s]^T \Omega \text{cov}_{\text{schur}} \Omega^T \tilde{\mathcal{A}}^{-1} \tilde{\mathcal{A}} \vec{\beta}_s\right) \chi_{\text{in},s}(\tilde{\mathcal{A}} \vec{\beta}_s) d^{2L} \beta_s \\ &= \frac{1}{|\det(\mathcal{A})|} [(2\pi)^{2(M+L)} \det(\text{cov}_{\text{th},ee}/2)]^{-\frac{1}{2}} \int_{\mathbb{R}^{2L}} \exp\left(i[\vec{\gamma}_s + \vec{\gamma}_{\text{schur}}]^T \Omega \tilde{\mathcal{A}}^{-1} \vec{\beta}_s - \frac{1}{4}\vec{\beta}_s^T [\tilde{\mathcal{A}}^{-1}]^T \Omega \text{cov}_{\text{schur}} \Omega^T \tilde{\mathcal{A}}^{-1} \vec{\beta}_s\right) \chi_{\text{in},s}(\vec{\beta}_s) d^{2L} \beta_s \\ &= [(2\pi)^{2(M+L)} \det(\text{cov}_{\text{th},ee}/2) |\det(\mathcal{A})|^2]^{-\frac{1}{2}} \int_{\mathbb{R}^{2L}} \exp\left(i[\mathcal{A}^{-1} \{\vec{\gamma}_s + \vec{\gamma}_{\text{schur}}\}]^T \Omega \vec{\beta}_s - \frac{1}{4}\vec{\beta}_s^T \Omega \mathcal{A}^{-1} \text{cov}_{\text{schur}} [\mathcal{A}^{-1}]^T \Omega^T \vec{\beta}_s\right) \chi_{\text{in},s}(\vec{\beta}_s) d^{2L} \beta_s.\end{aligned}\quad (\text{D6})$$

By defining

$$\begin{aligned}G(\vec{\gamma}_s) &= [(2\pi)^{2(M-L)} \det(\text{cov}_{\text{th},ee}/2) |\det(\mathcal{A})|^2]^{-\frac{1}{2}} \int_{\mathbb{R}^{2L}} \exp\left(i\vec{\gamma}_s^T \Omega \vec{\beta}_s - \frac{1}{4}\vec{\beta}_s^T \Omega \mathcal{A}^{-1} \text{cov}_{\text{schur}} [\mathcal{A}^{-1}]^T \Omega^T \vec{\beta}_s\right) d^{2L} \beta_s \\ &= 2^M [(2\pi)^{2(M-2L)} \det(\text{cov}_{\text{schur}}) \det(\text{cov}_{\text{th},ee})]^{-\frac{1}{2}} \exp(-\vec{\gamma}_s^T \mathcal{A} \text{cov}_{\text{schur}}^{-1} \mathcal{A}^T \vec{\gamma}_s)\end{aligned}\quad (\text{D7})$$

and

$$W_{\text{in},s}(\vec{\gamma}_s) = \frac{1}{(2\pi)^{2L}} \int_{\mathbb{R}^{2N}} \exp(i\vec{\gamma}_s^T \Omega \vec{\beta}_s) \chi_{\text{in},s}(\vec{\beta}_s) d^{2L} \beta_s, \quad (\text{D8})$$

the output Wigner function is a smoothed out and transformed version of the input Wigner function,

$$W_{\text{out}}(\vec{\gamma}) = (W_{\text{in},s} * G)(\mathcal{A}^{-1}[\vec{\gamma}_s + \vec{\gamma}_{\text{schur}}]). \quad (\text{D9})$$

1. Fock states

Let us assume an n -photon Fock state in the first mode, then the characteristic function is

$$\begin{aligned} \exp(-i\vec{\Gamma}_{\text{in}}\Omega_2\vec{\beta}) &= \exp(-i\beta_x\hat{p} + i\beta_p\hat{x}) = \exp\left[\frac{1}{\sqrt{2}}\beta_x(\hat{a}^\dagger - \hat{a}) + \frac{i}{\sqrt{2}}\beta_p(\hat{a}^\dagger + \hat{a})\right] \\ &= \exp\left[\frac{1}{\sqrt{2}}(\beta_x + i\beta_p)\hat{a}^\dagger - \frac{1}{\sqrt{2}}(\beta_x - i\beta_p)\hat{a}\right], \end{aligned} \quad (\text{D10})$$

i.e., $\alpha = \frac{1}{\sqrt{2}}(\beta_x + i\beta_p)$. This allows us to apply the known formula for the characteristic function of the Fock state [59],

$$\chi_{\text{in}}(\vec{\beta}) = \langle n | \exp(\alpha\hat{a}^\dagger - \alpha^*\hat{a}) | n \rangle = L_n(|\alpha|^2)e^{-|\alpha|^2/2} = L_n(\|\vec{\beta}\|^2/2)e^{-\|\vec{\beta}\|^2/4}, \quad (\text{D11})$$

and by inserting the charactersitic function of the Fock state into Eq. (D9), we obtain the output Wigner function

$$W_{\text{out}}(\vec{\gamma}) = \frac{1}{(2\pi)^2|\det(A)|} \int_{-\infty}^{\infty} \exp\left(-\frac{1}{4}\vec{\beta}^T\Omega_2A^{-1}\text{cov}_B[A^{-1}]^T\Omega_2^T\vec{\beta} + i[A^{-1}\vec{\gamma}]^T\Omega_2\vec{\beta}\right) \chi_{\text{in}}(\vec{\beta}) d^4\beta. \quad (\text{D12})$$

Using the SVD $A^{-1}\text{cov}_B[A^{-1}]^T = \sigma_x\vec{e}_x\vec{e}_x^T + \sigma_p\vec{e}_p\vec{e}_p^T$, defining $A^{-1}\vec{\gamma} = y\vec{e}_x + z\vec{e}_p$ and integrate over $\vec{\beta} = x\vec{e}_x + p\vec{e}_p$, we find the output Wigner function,

$$\begin{aligned} W_{\text{out}}(\vec{\gamma}) &= \frac{1}{(2\pi)^2|\det(A)|} \iint_{-\infty}^{\infty} \exp\left(-\frac{1}{4}(\sigma_x + 1)x^2 - \frac{1}{4}(\sigma_p + 1)p^2 + iyp - izx\right) L_n\left(\frac{x^2 + p^2}{2}\right) dx dp \\ &= \frac{\exp\left(-\frac{z^2}{\sigma_x + 1} - \frac{y^2}{\sigma_p + 1}\right)}{\pi|\det(A)|\sqrt{(\sigma_x + 1)(\sigma_p + 1)}} \sum_{m=1}^n \left(\frac{\sigma_x - 1}{\sigma_x + 1}\right)^m L_m^{(-\frac{1}{2})}\left(\frac{-2z^2}{\sigma_x^2 - 1}\right) \left(\frac{\sigma_p - 1}{\sigma_p + 1}\right)^{n-m} L_{n-m}^{(-\frac{1}{2})}\left(\frac{-2y^2}{\sigma_p^2 - 1}\right). \end{aligned} \quad (\text{D13})$$

If $\sigma_x = \sigma_p =: \sigma$ the output Wigner function corresponds to a ensemble of Fock states with different photon occupations,

$$\begin{aligned} W_{\text{out}}(\vec{\gamma}) &= \frac{4\pi}{(\sigma + 1)} e^{-\frac{z^2 + y^2}{\sigma + 1}} \frac{1}{(2\pi)^2|\det(A)|} \left(\frac{\sigma - 1}{\sigma + 1}\right)^n \sum_{m=1}^n L_m^{(-\frac{1}{2})}\left(\frac{-2z^2}{\sigma^2 - 1}\right) L_{n-m}^{(-\frac{1}{2})}\left(\frac{-2y^2}{\sigma^2 - 1}\right) \\ &= \frac{1}{\pi(\sigma + 1)|\det(A)|} \left(\frac{\sigma - 1}{\sigma + 1}\right)^n L_n\left(-2\frac{\|A^{-1}\vec{\gamma}\|^2}{\sigma^2 - 1}\right) \exp\left(-\frac{\|A^{-1}\vec{\gamma}\|^2}{\sigma}\right). \end{aligned} \quad (\text{D14})$$

2. Gaussian states

For the case of Gaussian input state, the convolution in Eq. (D9) can be solved, since the convolution of two Gaussians is again a Gaussian. Starting from the characteristic funtion of the reduced input state,

$$\chi_{\text{in},s}(\vec{\beta}) = \exp\left(-\frac{1}{4}\vec{\beta}^T\Omega\text{cov}_{\text{in},s}\Omega^T\vec{\beta}\right), \quad (\text{D15})$$

we find the output Wigner function,

$$\begin{aligned} W_{\text{out}}(\vec{\gamma}) &= [(2\pi)^{2(M+L)} \det(\text{cov}_{\text{th},ee}/2) |\det(\mathcal{A})|^2]^{-\frac{1}{2}} \int_{\mathbb{R}^{2L}} \exp\left\{i[\mathcal{A}^{-1}(\vec{\gamma}_s + \vec{\gamma}_{\text{schur}})]^T\Omega\vec{\beta}_s\right. \\ &\quad \left.- \frac{1}{4}\vec{\beta}_s^T\Omega(\mathcal{A}^{-1}\text{cov}_{\text{schur}}[\mathcal{A}^{-1}]^T + \text{cov}_{\text{in},s})\Omega^T\vec{\beta}_s\right\} d^{2L}\beta_s \\ &= [\pi^{2M} \det(\text{cov}_{\text{th},ee}) |\det(\mathcal{A})|^2 \det(\mathcal{A}^{-1}\text{cov}_{\text{schur}}[\mathcal{A}^{-1}]^T + \text{cov}_{\text{in},s})]^{-\frac{1}{2}} \\ &\quad \times \exp\{-[\mathcal{A}^{-1}(\vec{\gamma}_s + \vec{\gamma}_{\text{schur}})]^T(\mathcal{A}^{-1}\text{cov}_{\text{schur}}[\mathcal{A}^{-1}]^T + \text{cov}_{\text{in},s})^{-1}\mathcal{A}^{-1}(\vec{\gamma}_s + \vec{\gamma}_{\text{schur}})\} \\ &= [\pi^{2M} \det(\text{cov}_{\text{th},ee}) \det(\text{cov}_{\text{schur}} + \mathcal{A}\text{cov}_{\text{in},s}\mathcal{A}^T)]^{-\frac{1}{2}} \\ &\quad \times \exp(-[\vec{\gamma}_s + \vec{\gamma}_{\text{schur}}]^T\{\text{cov}_{\text{schur}} + \mathcal{A}\text{cov}_{\text{in},s}\mathcal{A}^T\}^{-1}[\vec{\gamma}_s + \vec{\gamma}_{\text{schur}}]), \end{aligned} \quad (\text{D16})$$

with a covariance matrix $\text{cov}_{\text{out}} = \text{cov}_{\text{schur}} + \mathcal{A}\text{cov}_{\text{in},s}\mathcal{A}^T$ and a displacement in phase space of $\langle\hat{\vec{\Gamma}}_{\text{out}}\rangle = -\vec{\gamma}_{\text{schur}}$. If the number of input modes exceeds the number of output modes, i.e., $N > M$, the output Wigner function simplifies to

$$W_{\text{out}}(\vec{\gamma}) = [\pi^{2M} \det(\text{cov}_B + \mathcal{A}\text{cov}_{\text{in},s}\mathcal{A}^T)]^{-\frac{1}{2}} \exp[-\vec{\gamma}^T(\text{cov}_B + \mathcal{A}\text{cov}_{\text{in},s}\mathcal{A}^T)^{-1}\vec{\gamma}]. \quad (\text{D17})$$

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