

Characterizing stationary optomechanical entanglement in the presence of non-Markovian noise

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We study an optomechanical system, where a mechanical oscillator interacts with a Gaussian input optical field. In the linearized picture, we analytically prove that if the input light field is the vacuum state, or is frequency-independently squeezed, the stationary entanglement between the oscillator and the output optical field is independent of the coherent coupling between them, which we refer to as the universality of entanglement. Furthermore, we demonstrate that entanglement cannot be generated by performing arbitrary frequency-dependent squeezing on the input optical field. Our results hold in the presence of general, Gaussian environmental noise sources, including non-Markovian noise.

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I. INTRODUCTION

The transition from quantum to classical behavior has been an intriguing question in past decades. Quantum systems interact with the environment and develop correlations with environmental degrees of freedom, which in turn prohibit observing the quantum nature of the system when viewed by an independent external observer, a phenomenon that is referred to as *decoherence* [1,2]. Above some rate of decoherence, it is expected that a system starts displaying classical behavior [3].

Optomechanical systems are examples of quantum systems, where a mechanical oscillator interacts with an optical field in the simplest case. In such systems, the momentum exchange between the mechanical and electromagnetic degrees of freedom, a phenomenon that is referred to as radiation pressure [4,5], gives rise to the entanglement between these two components. Optomechanical entanglement has applications in quantum sensing [6], quantum control [7–9], testing fundamental physics [10–12], and new quantum technologies [13,14].

Optomechanical systems suffer from decoherence due to environmental noise sources, which can destroy or diminish the amount of generated optomechanical entanglement [15–17]. Examples of such noises are Brownian thermal noise [18,19], suspension thermal noise [20,21], and other technical noises. In this paper, we classify environmental noise sources

based on their effect on the mechanical oscillator: we refer to noise sources that get transduced by the mechanical oscillator before being read out by the probe (i.e., the light field) as force noise, and noise sources affecting the probe only as sensing noise, respectively.

In [17], the behavior of optomechanical entanglement was studied assuming a linearized interaction and Gaussian dynamics between an oscillator and the vacuum fluctuations in the input light field [5]. Given this system, it has been observed numerically that optomechanical entanglement vanishes above a certain threshold concerning the environmental noise sources only, and cannot be recovered by increasing the interaction strength between the oscillator and the input light field. In other words, the entangling-disentangling transition is *universal* with respect to the interaction strength. Furthermore, this phenomenon has been observed for environmental noises that are not necessarily Markovian, due to the presence of, for example, structural damping [22].

Given this observation, in this paper, we analytically prove the universality of optomechanical entanglement in the presence of general, not necessarily Markovian, environmental noise sources, when the input field is the vacuum state (or is frequency-independently squeezed). Furthermore, we show that if the system is separable when the input field is the vacuum state, entanglement cannot be achieved with arbitrary frequency-dependent squeezing in the input light field. The universality of the optomechanical entanglement can be interpreted as a quantum-to-classical transition, where the system appears to be “classical” due to the environment, i.e. it cannot get entangled with the light field since it is strongly correlated with the environment. Calculating the corresponding threshold for environmental noise sources is crucial for upcoming quantum technologies with optomechanical devices, and observing quantum behavior in the macroscopic regime. We also discuss the results in the companion Letter [23].

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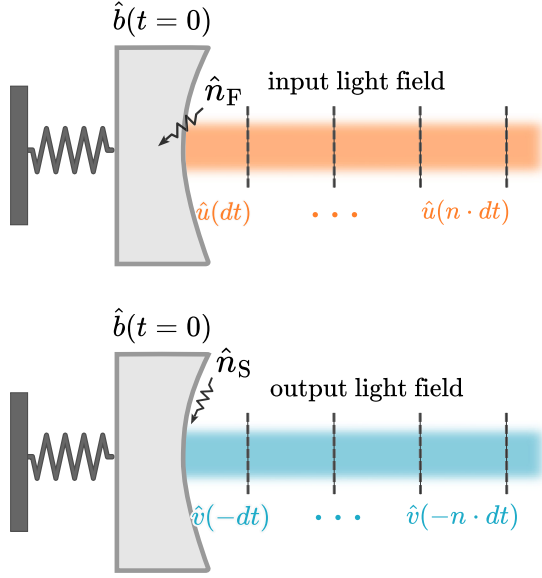


FIG. 1. Schematic of the optomechanical system. The mechanical mode $\hat{b}$ interacts with a light field and reaches a steady state. We refer to the modes of the input light field as $\hat{u}$, and the reflected (output) light field as $\hat{v}$, respectively. The schematic shows the equivalence between the spatial and the temporal modes of the light field: for example, the temporal modes of the input field located at the light-mechanics interface for $t > 0$ can be thought of as the spatial modes at $t = 0$ that have yet to interact with the mirror. Similarly, the temporal modes of the output field for $t < 0$ can be thought of as the spatial modes at $t = 0$ that have already reflected off the mirror. We label the duration of the temporal modes as dt in the schematic; however, we are interested in the continuum limit, where $dt \rightarrow 0$. Lastly, the mechanical mode is subjected to a force noise $\hat{n}_F$, and the output field is subjected to a sensing noise $\hat{n}_S$.

This paper is organized as follows. We state the system dynamics, the covariance matrix, and the entanglement criterion in Sec. II. Then, we demonstrate the universality of the entangling-disentangling transition for arbitrary environmental noise sources in Sec. III, assuming that the input light field is in the vacuum state. We relax this assumption and consider arbitrary frequency-dependent squeezing in the input light field in Sec. IV.

II. SYSTEM DYNAMICS

The system consists of a mechanical oscillator driven by a coherent, input optical field. For a schematic, see Fig. 1. We work in a suitably displaced frame where the first moments of the dynamical variables are zero. We denote the position fluctuations of the center of mass of the mechanical oscillator with $\hat{b}_1$, its associated momentum fluctuations with $\hat{b}_2$, and the quantum fluctuations in the amplitude (phase) quadratures of the input (output) optical fields with $\hat{u}_1(t)$, $\hat{u}_2(t)$ ($\hat{v}_1(t)$, and $\hat{v}_2(t)$), respectively. The dimensionless, linearized equations of motion in Heisenberg picture are [24]

$$\dot{\hat{v}}_1(t) = \hat{u}_1(t), \quad (1a)$$

$$\dot{\hat{v}}_2(t) = \hat{u}_2(t) + \frac{\Omega_q}{\sqrt{\omega_m}} [\hat{b}_1(t) + \hat{n}_S(t)], \quad (1b)$$

TABLE I. Classification of the noise sources affecting the dynamics, given in Eqs. (1). The noise sources fall under two orthogonal categories, demonstrated by the x and y axes of the Table. The first category (x axis) arises due to the nature of the noise: we refer to the fundamental noise that occurs due to the continuous monitoring of the oscillator as quantum noise. Conversely, any noise arising due to the environment (e.g., seismic noise, coating thermal noise, etc.) is referred to as environmental noise. The second category (y axis) describes how the noise source affects the dynamics: a force noise introduces stochastic momentum kicks on the oscillator, whereas a sensing noise is sensed with the position of the oscillator and does not affect its dynamics.

	Force noise	Sensing noise
Quantum (measurement) noise	$\hat{u}_1$	$\hat{u}_2$
Environmental noise	$\hat{n}_F$	$\hat{n}_S$

$$\dot{\hat{b}}_2(t) = -\gamma_m \hat{b}_2(t) - \omega_m \hat{b}_1(t) + \frac{\Omega_q}{\sqrt{\omega_m}} \hat{u}_1(t) + \hat{n}_F(t), \quad (1c)$$

$$\dot{\hat{b}}_1(t) = \omega_m \hat{b}_2(t), \quad (1d)$$

where Ω_q is the characteristic light-mechanics interaction strength, or the coherent coupling [25], ω_m is the mechanical resonance frequency, γ_m is the damping rate of the oscillator, and $\hat{n}_F(t)$ and $\hat{n}_S(t)$ are the force and sensing noises, respectively. The commutation relation for the same-time quadrature operators is $[\hat{b}_1(t), \hat{b}_2(t)] = 2i$ [26].

First, we assume that the driving light (in the displaced frame) is in the vacuum state, with temporally uncorrelated fluctuations. This assumption will be relaxed at the end of the paper, where an arbitrarily frequency-dependent squeezed input light field will be considered. The commutation relations for the quadratures of the input light field are given as $[\hat{u}_1(t), \hat{u}_2(t')] = 2i\delta(t - t')$, $[\hat{u}_i(t), \hat{u}_i(t')] = 0$, $i = 1, 2$. The same relations hold for the output light field.

As described in the companion Letter [23], the noise sources driving the mechanical oscillator, $\hat{u}_{1,2}$, $\hat{n}_F$, $\hat{n}_S$, can be categorized in two manners, summarized in Table I. First, we refer to the fundamental measurement noise occurring due to the continuous monitoring of the oscillator position as *quantum noise*. This noise is sourced by $\hat{u}_{1,2}$ in Eqs. (1). Conversely, any additional noise in the system that can in principle be eliminated is referred to as *environmental noise*. This noise is sourced by $\hat{n}_F$ and $\hat{n}_S$ in Eqs. (1). An orthogonal characterization of the noise sources can be obtained by considering their effect on the system. We refer to the noise sources that produce stochastic momentum kicks to the oscillator as *force noise*, sourced by $\hat{u}_1$ and $\hat{n}_F$. Lastly, the noises that do not affect the dynamics of the oscillator, but are sensed simultaneously with the mechanical position are referred to as *sensing noise*. They are sourced by $\hat{u}_2$ and $\hat{n}_S$.

Examples to $\hat{n}_F$ are seismic noise [27,28], suspension thermal noise [29,30], and structural damping [22,31,32] in systems with suspended oscillators such as gravitational wave detectors. Further examples include thermal excitations of a substrate in contact with the mechanical oscillator [33], and collision with gas molecules or photon scattering in levitated nanoparticles [34–36]. Additionally, examples to $\hat{n}_S$ are

thermal coating noise for coated mirrors [37,38], and additional mechanical modes contributing to the outgoing light in membranes or levitated systems [39]. Note that the sensing noise here couples to the detection channel with a rate proportional to the interaction strength Ω_q [cf. Eqs. (1)]. There can be other types of sensing noises that do not couple with this rate, e.g., passive or photon losses and detector dark noise [40]. We only consider the former type of sensing noise in $\hat{n}_S$, as in the presence of the latter, the universality of optomechanical entanglement does not hold (see Sec. III C).

Equations (1) describe a system where the mechanical oscillator is driven by the input light field and a force noise, represented by the terms $\frac{\Omega_q}{\sqrt{\omega_m}}\hat{u}_1(t)$ and $\hat{n}_F(t)$ in Eq. (1c), respectively. The system is unconditionally stable and reaches a steady state. Taking the Fourier transform [41] of Eqs. (1c) and (1d), the steady-state solution for the mechanical oscillator is written as $\hat{b}_1(\Omega) = \chi(\Omega)F_{\text{ext}}(\Omega)$, where $F_{\text{ext}}(\Omega) = \Omega_q\sqrt{\omega_m}\hat{u}_1(\Omega) + \omega_m\hat{n}_F(\Omega)$ is the Fourier transform of the sum of external forces driving the oscillator, and $\chi(\Omega) = (-\Omega^2 - i\gamma_m\Omega + \omega_m^2)^{-1}$ is the mechanical susceptibility. We refer to the inverse Fourier transform of the mechanical susceptibility as $\chi(t)$, which is a real, causal function of time.

A. Structure of the covariance matrix

We want to investigate the entanglement between the mechanical oscillator at $t = 0$, and the output light field that was reflected from the mirror during $t < 0$. For this purpose, we first define the covariance matrix $\mathbf{V}$, which can be written in the following block matrix form:

$$\mathbf{V} = \begin{bmatrix} \mathbf{V}^{bb} & \mathbf{V}^{bv} \\ \mathbf{V}^{vb} & \mathbf{V}^{vv} \end{bmatrix}, \quad \mathbf{V}^{mn} = \begin{bmatrix} V_{11}^{mn} & V_{12}^{mn} \\ V_{21}^{mn} & V_{22}^{mn} \end{bmatrix}, \quad (2)$$

where $m, n \in \{v, b\}$, $\mathbf{V}^{bb}$ and $\mathbf{V}^{vv}$ are the covariance matrices of the mechanical mode and the light field, respectively, and $\mathbf{V}^{bv}, \mathbf{V}^{vb}$ contain the cross-correlations between the mechanical mode and the modes of the optical field, with $\mathbf{V}^{vb} = (\mathbf{V}^{bv})^T$ (T denotes the transpose). $\mathbf{V}^{bb}$ is a 2×2 matrix of real numbers, as the mechanical mode is only relevant at the instant of $t = 0$. On the other hand, we are interested in the output light field for $t < 0$, where the continuous index t labels this infinite set of modes. $\mathbf{V}^{vv}$ is thus a 2-by-2 matrix of bounded operators on $\mathcal{L}^2(0, \infty)$, the Hilbert space of square-integrable real functions on the half real line. Similarly, the elements of the 2-by-2 matrix $\mathbf{V}^{vb}$ ($\mathbf{V}^{bv}$) are functions in $\mathcal{L}^2(0, \infty)$. For example, $V_{12}^{bv}(t)$ is a function denoting the correlations between $\hat{b}_1(t=0)$ and $\hat{v}_2(t)$ for $t < 0$. To compute the elements of $\mathbf{V}$, we take the inverse Fourier transform of cross spectra between quadratures, as

$$\begin{aligned} V_{kl}^{mn} &= \frac{1}{2} \langle \{\hat{m}_k(t_1), \hat{n}_l(t_2)\} \rangle \\ &= \int_{-\infty}^{+\infty} \frac{d\Omega}{2\pi} S_{m_k n_l}(\Omega) e^{-i\Omega(t_1 - t_2)}, \end{aligned} \quad (3)$$

where $m, n \in \{v, b\}$, $k, l \in \{1, 2\}$, $t_{1(2)} = 0$ if $m(n) = b$, $t_{1(2)} < 0$ otherwise, and $S_{m_k n_l}$ is the double-sided cross

spectrum between $\hat{m}_k$ and $\hat{n}_l$, given by

$$S_{m_k n_l}(\Omega) \delta(\Omega - \Omega') = \frac{1}{4\pi} \langle \hat{m}_k(\Omega) \hat{n}_l^\dagger(\Omega') + \hat{n}_l^\dagger(\Omega') \hat{m}_k(\Omega) \rangle. \quad (4)$$

Note that since the input light field is the vacuum, $S_{u_1 u_1} = S_{u_2 u_2} = 1$ and $S_{u_1 u_2} = 0$. Furthermore, we denote the spectra of $\hat{n}_F(t)$ and $\hat{n}_S(t)$ with $S_{n_F}(\Omega)$ and $S_{n_S}(\Omega)$, respectively.

First, let us compute $\mathbf{V}^{vv}$. The operators in $\mathbf{V}^{vv}$ are indexed by $\tau \in (-\infty, \infty)$, where τ increases (decreases) with increasing columns (rows) of the matrix of the operator. $\mathbf{V}^{vv}$ is given as

$$\mathbf{V}^{vv} = \begin{bmatrix} \delta(\tau) & \Omega_q^2 \chi(-\tau) \\ \Omega_q^2 \chi(\tau) & \delta(\tau) + \Omega_q^2 \chi_{\text{EN}}(\tau) + \Omega_q^4 \Theta(\tau) \end{bmatrix}, \quad (5)$$

where $\Theta(\Omega) = |\chi(\Omega)|^2$ and we define a “force noise susceptibility” $\chi_F(\tau)$, with $\chi_F(\Omega) = \omega_m |\chi(\Omega)|^2 S_{n_F}(\Omega)$, and a “sensing noise susceptibility” $\chi_S(\tau)$, with $\chi_S(\Omega) = S_{n_S}(\Omega)/\omega_m$ so that $\chi_{\text{EN}}(\tau) := \chi_F(\tau) + \chi_S(\tau)$ summarizes the susceptibility of the environmental noises. The matrix representations of V_{ij}^{vv} , $i, j = 1, 2$ are in the Toeplitz form, as their elements depend only on the difference between their respective row and column indices.

Next, let us compute $\mathbf{V}^{bv}$, which will be a matrix of functions indexed by $t, t > 0$. It contains the correlations between the mechanical oscillator and the light field:

$$\mathbf{V}^{bv} = \sqrt{\omega_m} \Omega_q \begin{bmatrix} \chi(t) & \chi_F(t) + \Omega_q^2 \Theta(t) \\ \frac{1}{\omega_m} \dot{\chi}(t) & \frac{1}{\omega_m} \dot{\chi}_F(t) + \frac{\Omega_q^2}{\omega_m} \dot{\Theta}(t) \end{bmatrix}, \quad (6)$$

where t increases with increasing columns. Since $\hat{b}_2(t) = \dot{\hat{b}}_1(t)/\omega_m$ from Eq. (1d), the cross-correlations involving $\hat{b}_2(t)$ are derivatives of the cross-correlations involving $\hat{b}_1(t)$, normalized by ω_m . The terms containing Θ in the second columns correspond to the amplitude fluctuations of the input light, transduced twice by the mechanical oscillator (recall that Θ is $|\chi|^2$ in the frequency domain). These fluctuations emerge as noise in the phase quadrature of the output light field. This phenomenon is referred to as *back action*.

Lastly, let us compute $\mathbf{V}^{bb}$, which is a matrix of real numbers:

$$\mathbf{V}^{bb} = \begin{bmatrix} V_{11}^{bb} & 0 \\ 0 & V_{22}^{bb} \end{bmatrix}, \quad (7a)$$

$$V_{11}^{bb} = \omega_m (\chi_F(t=0) + \Omega_q^2 \Theta(t=0)), \quad (7b)$$

$$V_{22}^{bb} = \frac{-1}{\omega_m} (\ddot{\chi}_F(t=0) + \Omega_q^2 \ddot{\Theta}(t=0)). \quad (7c)$$

This matrix is diagonal due to vanishing cross-correlations between $\hat{b}_1(t)$ and $\hat{b}_2(t)$ at equal times.

B. Entanglement criterion

We are interested in the bipartite entanglement between the mechanical oscillator and the continuum of temporal modes of the output light. Since we assume Gaussian dynamics and noise sources, the state of the mechanical mode and the temporal modes of the light are Gaussian. In this configuration,

the positivity of the partial transpose (PPT) criterion is necessary and sufficient for verifying the presence of bipartite entanglement [42–45]. For continuous variables, the criterion asserts that a state is separable iff $\mathbf{V}_{\text{pt}} + i\mathbf{K} \geq 0$; $\mathbf{V}_{\text{pt}}$ denotes the partial transpose of $\mathbf{V}$ with respect to one of the parties [43,46], and $\mathbf{K}$ is the commutator matrix, defined as

$$\mathbf{K} = \begin{bmatrix} \mathbf{K}^b & \\ & \mathbf{K}^v \end{bmatrix}, \quad (8)$$

with

$$\mathbf{K}^b = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad \mathbf{K}^v = \begin{bmatrix} 0 & \delta(t) \\ -\delta(t) & 0 \end{bmatrix}. \quad (9)$$

In our configuration, the partial transpose operation amounts to reversing the sign of the momentum of the mechanical mode: $\hat{b}_2(t) \rightarrow -\hat{b}_2(t)$, which also puts a minus sign on the rows and columns of $\mathbf{V}$ associated with $\hat{b}_2(t)$. For example,

$$\mathbf{V}_{\text{pt}}^{bv} = \begin{bmatrix} V_{11}^{bv}(t) & V_{12}^{bv}(t) \\ -V_{21}^{bv}(t) & -V_{22}^{bv}(t) \end{bmatrix}. \quad (10)$$

Since we are looking for entanglement of one of the modes against the rest, $\mathbf{V}_{\text{pt}} + i\mathbf{K}$ can have at most one negative eigenvalue [46,47]. Therefore, the optomechanical entanglement exists if the following condition holds:

$$\det[\mathbf{V}^{bb} + i\mathbf{K}^b - \mathbf{V}_{\text{pt}}^{bv} \cdot (\mathbf{V}^{vv} + i\mathbf{K}^v)^{-1} \cdot \mathbf{V}_{\text{pt}}^{vb}] < 0. \quad (11)$$

Note that the dot product $\cdot$ is defined as, for an n -by- n matrix of operators $\mathbf{A}$, an n -by- m (m -by- n) matrix of functions $\mathbf{f}$ ($\mathbf{g}$),

$$(\mathbf{g} \cdot \mathbf{A} \cdot \mathbf{f})_{kl} = \sum_{i,j} \int_0^\infty \int_0^\infty dt dt' g_{ki}(t') A_{ij}(t - t') f_{jl}(t). \quad (12)$$

From Eq. (11), the existence of optomechanical entanglement depends on a complex trade-off between the strength of the force noise, the sensing noise, and the quantum noise due to vacuum fluctuations. We explore this relation in depth in the following sections.

Lastly, entanglement is quantified with the *logarithmic negativity*, defined as [48,49]

$$E_{\mathcal{N}} = \sum_j \max\{0, -\log_{10}(\lambda_j)\}, \quad (13)$$

where the sum is over the symplectic eigenvalues λ_j of the partially transposed covariance matrix, and a nonzero logarithmic negativity indicates the presence of optomechanical entanglement.

III. UNIVERSAL ENTANGLEMENT

We first simplify the condition to observe entanglement in Eq. (11) in Sec. III A. Subsequently, we demonstrate the universality of the entangling-disentangling transition in Sec. III B. Lastly, we give an example as to how this transition constrains the amount of environmental noise allowable in order to observe optomechanical entanglement in Sec. III C.

A. Entanglement indicator

To simplify Eq. (11), we first evaluate $(\mathbf{V}_{\text{pt}}^{vv} + i\mathbf{K}^v)^{-1}$. Using standard properties of the 2-by-2 block-matrices (see Appendix B), we write

$$(\mathbf{V}_{\text{pt}}^{vv} + i\mathbf{K}^v)^{-1} = \mathbf{\Sigma} \cdot \mathbf{D}_M \cdot \mathbf{\Sigma}^\dagger + \mathbf{\Delta}, \quad (14a)$$

$$\mathbf{\Delta} = \begin{bmatrix} V_{11}^{vv}(\tau)^{-1} & 0 \\ 0 & 0 \end{bmatrix}, \quad (14b)$$

$$\mathbf{\Sigma} = \begin{bmatrix} V_{11}^{vv}(\tau)^{-1} & 0 \\ 0 & \delta(\tau) \end{bmatrix} \cdot \begin{bmatrix} V_{12}^{vv}(\tau) + i\delta(\tau) & 0 \\ -\delta(\tau) & 0 \end{bmatrix}, \quad (14c)$$

$$\mathbf{D}_M = \begin{bmatrix} M^{-1}(\tau) & 0 \\ 0 & M^{-1}(\tau) \end{bmatrix}, \quad (14d)$$

with [50]

$$M(\tau) = V_{22}^{vv}(\tau) - [-i\delta(\tau) + V_{21}^{vv}(\tau)] \cdot V_{11}^{vv}(\tau)^{-1} \times [i\delta(\tau) + V_{12}^{vv}(\tau)] \quad (15)$$

and $M^{-1}(\tau)$ is an operator calculated with the Wiener-Hopf method (see Appendix C for a summary). Plugging in $V_{ij}^{vv}(t)$, $i, j = 1, 2$ from Eq. (5), and using the dot product convention in Eq. (12), we obtain

$$(\mathbf{V}_{\text{pt}}^{vv} + i\mathbf{K}^v)^{-1} = \mathbf{\Sigma} \cdot \mathbf{D}_M \cdot \mathbf{\Sigma}^\dagger + \begin{bmatrix} \delta(\tau) & 0 \\ 0 & 0 \end{bmatrix}, \quad (16a)$$

$$\mathbf{\Sigma} = \begin{bmatrix} \Omega_q^2 \chi(-\tau) + i\delta(\tau) & 0 \\ -\delta(\tau) & 0 \end{bmatrix}, \quad (16b)$$

$$M(\tau) = -i\Omega_q^2[\chi(\tau) - \chi(-\tau)] + \Omega_q^2 \chi_{\text{EN}}(\tau). \quad (16c)$$

When the second operator in Eq. (16a) operates on $\mathbf{V}_{\text{pt}}^{bv}$ and $\mathbf{V}_{\text{pt}}^{vb}$ in Eq. (11), we obtain

$$\int_0^\infty dt \begin{bmatrix} V_{11}^{bv}(t)^2 & -V_{11}^{bv}(t)V_{21}^{bv}(t) \\ -V_{21}^{bv}(t)V_{11}^{bv}(t) & V_{21}^{bv}(t)^2 \end{bmatrix} \quad (17a)$$

$$= \int_0^\infty dt \Omega_q^2 \begin{bmatrix} \omega_m \chi(t)^2 & -\chi(t)\dot{\chi}(t) \\ -\chi(t)\dot{\chi}(t) & \frac{1}{\omega_m} \dot{\chi}(t)^2 \end{bmatrix} \quad (17b)$$

$$= \int_{-\infty}^\infty dt \Omega_q^2 \begin{bmatrix} \omega_m \chi(t)^2 & -\chi(t)\dot{\chi}(t) \\ -\chi(t)\dot{\chi}(t) & \frac{1}{\omega_m} \dot{\chi}(t)^2 \end{bmatrix} \quad (17c)$$

$$= \int_{-\infty}^\infty d\Omega \Omega_q^2 \begin{bmatrix} \omega_m |\chi(\Omega)|^2 & 0 \\ 0 & \frac{\Omega^2}{\omega_m} |\chi(\Omega)|^2 \end{bmatrix} \quad (17d)$$

$$= \Omega_q^2 \begin{bmatrix} \omega_m \Theta(0) & 0 \\ 0 & \frac{-1}{\omega_m} \ddot{\Theta}(0) \end{bmatrix}, \quad (17e)$$

since $\int_0^\infty dt \chi(t)\dot{\chi}(t) = -\chi(0)^2/2 = 0$. We were able to extend the lower integration limit to $-\infty$ in Eq. (17c) due to the causality of $\chi(t)$. Then, we used the Plancherel theorem to go from the time domain to the frequency domain to calculate the diagonal elements in Eq. (17d). Lastly, we expressed the diagonal elements as inverse Fourier transforms at time $t = 0$ in Eq. (17e). We notice that this expression is contained in $\mathbf{V}^{bb}$ [Eq. (7)], therefore it cancels out the determinant in Eq. (11),

which becomes

$$\det[\tilde{\mathbf{V}}^{bb} + i\mathbf{K}^b - \mathbf{V}_{\text{pt}}^{bv} \cdot \boldsymbol{\Sigma} \cdot \mathbf{D}_M \cdot \boldsymbol{\Sigma}^\dagger \cdot (\mathbf{V}_{\text{pt}}^{bv})^T] < 0, \quad (18)$$

$$\tilde{\mathbf{V}}^{bb} := \begin{bmatrix} \omega_m \chi_F(0) & 0 \\ 0 & -\frac{1}{\omega_m} \ddot{\chi}_F(0) \end{bmatrix},$$

where we rewrote $\mathbf{V}_{\text{pt}}^{bv}$ as $(\mathbf{V}_{\text{pt}}^{bv})^T$. We refer to this simplification as the *vacuum cancellation*, since the variances of the quadratures of the mechanical oscillator due to the vacuum fluctuations factor out of the inequality in Eq. (11).

To calculate $\mathbf{V}_{\text{pt}}^{bv} \cdot \boldsymbol{\Sigma} \cdot \mathbf{D}_M \cdot \boldsymbol{\Sigma}^\dagger \cdot (\mathbf{V}_{\text{pt}}^{bv})^T$, we first operate with $\boldsymbol{\Sigma}$ on $\mathbf{V}_{\text{pt}}^{bv}$ from the right,

$$\begin{bmatrix} V_{11}^{bv}(t) & V_{12}^{bv}(t) \\ -V_{21}^{bv}(t) & -V_{22}^{bv}(t) \end{bmatrix} \cdot \begin{bmatrix} \Omega_q^2 \chi(-\tau) + i\delta(\tau) & 0 \\ -\delta(\tau) & 0 \end{bmatrix}. \quad (19)$$

Here, we use the index $t > 0$ to indicate functions, and $\tau \in (-\infty, \infty)$ to indicate operators. We use this convention throughout the rest of the paper. Performing the matrix product, we obtain

$$\begin{bmatrix} iV_{11}^{bv}(t) + \Omega_q^2 V_{11}^{bv}(t) \cdot \chi(-\tau) - V_{12}^{bv}(t) & 0 \\ -iV_{21}^{bv}(t) - \Omega_q^2 V_{21}^{bv}(t) \cdot \chi(-\tau) + V_{22}^{bv}(t) & 0 \end{bmatrix}, \quad (20)$$

where we used the fact that taking the dot product with the identity operator $\delta(\tau)$ does not modify the functions. Taking the dot product between $V_{11}^{bv}(t)$ and $\chi(-\tau)$ and going to the frequency domain, we have

$$\begin{aligned} (V_{11}^{bv}(t) \cdot \chi(-\tau))(\Omega) &= \mathcal{F} \left\{ \int_0^\infty dt' \chi(t' - t) V_{11}^{bv}(t') \right\} \\ &= \Omega_q \sqrt{\omega_m} \mathcal{F} \left\{ \int_{-\infty}^\infty dt' \chi(t' - t) \chi(t') \right\} \\ &= \Omega_q \sqrt{\omega_m} \chi(\Omega)^\dagger \chi(\Omega) \\ &= \Omega_q \sqrt{\omega_m} |\chi(\Omega)|^2, \end{aligned} \quad (21)$$

where we used the definition of V_{11}^{bv} given in Eq. (6), and we extended the integration boundary using that $\chi(t) = 0$ for $t < 0$. Then, we find that

$$\begin{aligned} V_{11}^{bv}(t) \cdot \chi(-\tau) &= \sqrt{\omega_m} \Omega_q \mathcal{F}^{-1} \{ |\chi(\Omega)|^2 \} \\ &= \sqrt{\omega_m} \Omega_q \Theta(t). \end{aligned} \quad (22)$$

Similarly, we can also write $V_{21}^{bv}(t) \cdot \chi(-\tau) = \sqrt{\omega_m} \Omega_q \dot{\Theta}(t)$. Substituting these expressions back in Eq. (20), we obtain

$$\begin{bmatrix} iV_{11}^{bv} - \tilde{V}_{12}^{bv} & 0 \\ -iV_{21}^{bv} + \tilde{V}_{22}^{bv} & 0 \end{bmatrix}, \quad (23)$$

where we have defined

$$\tilde{V}_{12}^{bv}(t) = \sqrt{\omega_m} \Omega_q \chi_F(t), \quad \tilde{V}_{22}^{bv}(t) = \frac{\Omega_q}{\sqrt{\omega_m}} \dot{\chi}_F(t), \quad (24)$$

i.e., we redefine $\tilde{V}_{i2}^{bv}(t)$, $i = 1, 2$ such that they only contain the correlations due to the force noise, and not the back action.

Similar calculations allow one to reexpress $\mathbf{V}_{\text{pt}}^{bv} \cdot \boldsymbol{\Sigma} \cdot \mathbf{D}_M \cdot \boldsymbol{\Sigma}^\dagger \cdot (\mathbf{V}_{\text{pt}}^{bv})^T$ as

$$\tilde{\mathbf{V}}_{\text{pt}}^{bv}(t') \cdot \mathbf{V}_M(\tau = t - t') \cdot [\tilde{\mathbf{V}}_{\text{pt}}^{bv}(t)]^T \quad (25)$$

with

$$\tilde{\mathbf{V}}_{\text{pt}}^{bv} := \begin{bmatrix} V_{11}^{bv} & \tilde{V}_{12}^{bv} \\ -V_{21}^{bv} & -\tilde{V}_{22}^{bv} \end{bmatrix}, \quad \mathbf{V}_M := \begin{bmatrix} M^{-1} & -iM^{-1} \\ iM^{-1} & M^{-1} \end{bmatrix}. \quad (26)$$

Therefore, we can rewrite Eq. (18) and define the following *entanglement indicator*:

$$\begin{aligned} \det[\tilde{\mathbf{V}}^{bb} + i\mathbf{K}^b - \tilde{\mathbf{V}}_{\text{pt}}^{bv} \cdot \mathbf{V}_M \cdot (\tilde{\mathbf{V}}_{\text{pt}}^{bv})^T] &< 0 \\ \iff \text{optomechanical entanglement.} \end{aligned} \quad (27)$$

This inequality is our starting point for the subsequent considerations.

B. Demonstrating universality

Given the entanglement indicator in Eq. (27), we explore the existence of entanglement for different configurations of environmental noise sources. For example, we first assume a very large sensing noise, i.e., the limit of $S_{n_s} \rightarrow \infty$. In this limit, $\chi_S(t) \rightarrow \infty$, therefore $M \rightarrow \infty$, and consequently $M^{-1} \rightarrow 0$. However, the elements of $\mathbf{V}_{\text{pt}}^{bv}$ are finite, since they do not contain the correlations due to the sensing noise. Then, the rightmost term in Eq. (27) goes to zero, and the entanglement indicator is modified to

$$\det[\tilde{\mathbf{V}}^{bb} + i\mathbf{K}^b] < 0 \iff \text{entanglement.} \quad (28)$$

$\tilde{\mathbf{V}}^{bb}$ contains the fluctuations in the quadratures of the mechanical oscillator due to the force noise only. Therefore, this matrix can be thought as the covariance matrix of an oscillator in the absence of any optomechanical interaction, subjected to force noise. Since this is a physical covariance matrix, $\tilde{\mathbf{V}}^{bb} + i\mathbf{K}^b \geq 0$ from the Heisenberg uncertainty principle. Therefore, Eq. (28) cannot be realized for any physical force noise spectrum, and there can be no optomechanical entanglement in this limit.

Next, we investigate entanglement in the absence of sensing noise. The condition in Eq. (27) reduces to the following expression for a high-Q oscillator ($\omega_m \gg \gamma_m$)

$$S_{n_F}(\omega_m) \geq 2\gamma_m \iff \text{entanglement.} \quad (29)$$

We derive this inequality in Appendix D, which holds for all force noise spectra by the fluctuation-dissipation theorem [51,52]. Therefore, the mechanical oscillator is entangled with the output light field for any force noise spectrum in the absence of sensing noise.

Summarizing the two scenarios discussed above, for a given force noise spectrum, the existence of entanglement depends on the amount of sensing noise present in the system, signifying that there exists an entangling-disentangling transition at which optomechanical entanglement emerges. Furthermore, this transition is unique for a given force noise and sensing noise spectrum, as demonstrated in Appendix E.

Furthermore, from Eqs. (16c), (24), (26), and (27), $M(t) \propto \Omega_q^2$ and $\tilde{\mathbf{V}}_{\text{pt}}^{bv} \propto \Omega_q$. Therefore, the term $\tilde{\mathbf{V}}_{\text{pt}}^{bv} \cdot \mathbf{V}_M \cdot (\tilde{\mathbf{V}}_{\text{pt}}^{bv})^T$ in Eq. (27) is independent of Ω_q . We also have that $\tilde{\mathbf{V}}^{bb}$ and $\mathbf{K}^b$ are already independent of Ω_q . Therefore, Ω_q factors out of Eq. (27), making the entangling-disentangling transition independent of the interaction strength between the mechanical

oscillator and the output light field. Therefore, this transition is universal with respect to the interaction strength. To restate our definition of universality, we call a transition universal if it (i) exists for a finite level of sensing noise, (ii) is unique with respect to the amount of sensing noise, and (iii) is independent of the coherent coupling Ω_q .

After the transition (i.e., for an entangled state), we saw numerically that the negativity increases with Ω_q , which supports our intuition that the optomechanical entanglement is strengthened with an increased interaction strength. However, most interestingly, the universality implies that, for a noise configuration before the transition (i.e., for a separable state), one cannot make the state entangled by increasing the interaction strength.

In Appendix F, we demonstrate another way to reach Eq. (27), which provides more physical intuition. That derivation suggests that the entanglement condition in Eq. (27) reduces to testing whether the conditional quantum state of a mechanical oscillator driven solely by a force noise $\hat{n}_F$ satisfies the Heisenberg uncertainty principle. The state is conditioned on the output light field, which senses the oscillator with a sensing noise $\hat{n}_S$. If the uncertainty principle is violated, optomechanical entanglement will be observed. We notice that the mechanical oscillator in this picture in the alternative derivation is not driven by vacuum fluctuations, which is due to the vacuum cancellation shown in Sec. III B.

C. Example case

To derive a simple analytical bound on the maximum allowable environmental noise in order to observe optomechanical entanglement, we assume white noise spectra parametrized with Ω_F and Ω_S for the force and the sensing noise, respectively (cf. [17,53]). More specifically, we assume that their spectra are in the form of

$$S_{n_F}(\Omega) = \frac{2\Omega_F^2}{\omega_m}, \quad S_{n_S}(\Omega) = \frac{2\omega_m}{\Omega_S^2}. \quad (30)$$

Consequently, $\chi_F(\Omega) = 2\Omega_F^2|\chi(\Omega)|^2$ and $\chi_S(\Omega) = 2/\Omega_S^2$.

Furthermore, we assume the *free-mass limit*, i.e., the mechanical resonance frequency ω_m of the oscillator is much smaller than the other frequencies in the system. In this limit, $\Omega_F, \Omega_S \gg \omega_m, \gamma_m$. Lastly, we assume the presence of passive losses occurring due to, e.g., a photodetector with nonunity efficiency, whose effect on the dynamics is described in Appendix J. Even though these losses act effectively as a sensing noise on the system, they do not couple to the detection channel with a rate of Ω_q , different from the sensing noise $\hat{n}_S$ in Eqs. (1). Hence, in the presence of such losses, optomechanical entanglement is no longer universal with respect to the coherent coupling Ω_q .

For a white force and sensing noise, and a photodetector efficiency of $\eta < 1$, the spectrum of M is given by [Eq. (16c)]

$$\frac{M(\Omega)}{2\eta\Omega_q^2} = \frac{1}{\Omega_S^2} + |\chi(\Omega)|^2 \left[\Omega_F^2 + \gamma_m\Omega + \frac{\Omega_q^2}{2}(1-\eta) \right]. \quad (31)$$

In order to apply the Wiener-Hopf method, we need to write $M(\Omega)$ as a product of its causal and anticausal poles and zeros, i.e., $M(\Omega) = M(\Omega)_+ M(\Omega)_-$, where $M(\Omega)_+ = M(\Omega)_-^\dagger$, and

$M(\Omega)_+ [M(\Omega)_-]$ contains the causal (anticausal) poles and zeros. In the free-mass limit, $M(\Omega)_+$ is given up to first order in ω_m, γ_m as

$$M(\Omega)_+ \approx \frac{\sqrt{2\eta}\Omega_q}{\Omega_S} \frac{(\Omega - \Omega_{z,+})(\Omega - \Omega_{z,-})}{\Omega^2 + i\gamma_m\Omega - \omega_m^2}, \quad (32)$$

where

$$\Omega_{z,\pm} = \frac{-i \pm 1}{\sqrt{2}} \sqrt{\Omega_S\Omega_N} \mp \frac{i\Omega_S\gamma_m}{4\Omega_N}, \quad (33)$$

with $\Omega_N = \sqrt{\Omega_F^2 + (1-\eta)\Omega_q^2/2}$. To evaluate the entanglement indicator in Eq. (27), we need to obtain the expressions for $\tilde{\mathbf{V}}^{bb}$ and $\tilde{\mathbf{V}}_{pt}^{bv}$. From Eqs. (18) and (24), we calculate

$$\tilde{\mathbf{V}}^{bb} = \frac{\Omega_N^2}{\gamma_m\omega_m} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \tilde{\mathbf{V}}_{pt}^{bv}(t) = \sqrt{\omega_m\eta}\Omega_q \begin{bmatrix} \chi(t) & 2\Omega_N^2\Theta(t) \\ \frac{-1}{\omega_m}\dot{\chi}(t) & \frac{-1}{\omega_m}2\Omega_N^2\dot{\Theta}(t) \end{bmatrix}, \quad (34)$$

where $\Theta(\Omega) = |\chi(\Omega)|^2$. Proceeding with the Wiener-Hopf method, we obtain up to the terms that contribute to constant order in ω_m, γ_m

$$\tilde{\mathbf{V}}_{pt}^{bv} \cdot \mathbf{V}_M \cdot (\tilde{\mathbf{V}}_{pt}^{bv})^T \approx \tilde{\mathbf{V}}^{bb} + \begin{bmatrix} 2^{3/2}\omega_m\sqrt{\frac{\Omega_N}{\Omega_S^3}} & -i + \frac{2\Omega_N}{\Omega_S} \\ i + \frac{2\Omega_N}{\Omega_S} & -\frac{2^{3/2}}{\omega_m}\sqrt{\frac{\Omega_N}{\Omega_S^3}} \end{bmatrix}. \quad (35)$$

Therefore, Eq. (27) reduces to the following condition in the limit of $\gamma_m \rightarrow 0$:

$$\Omega_N < \Omega_S \iff \text{optomechanical entanglement.} \quad (36)$$

When $\eta = 1$, i.e., there are no passive losses, this relation gets modified as $\Omega_F < \Omega_S \iff \text{entanglement}$, which signifies that the presence of optomechanical entanglement depends on a simple trade-off between the relative strengths of the environmental noise sources. This bound has been observed numerically in [17]. For $\eta < 1$, $\Omega_N > \Omega_F$, therefore it becomes harder to achieve optomechanical entanglement for nonunity photodetector efficiency, and it is more advantageous to have a small coherent coupling Ω_q for this purpose.

We compute the logarithmic negativity numerically, assuming that the input light field is the vacuum state, and the environmental noise sources are white, characterized by Ω_F and Ω_S in Eq. (30). The results are given in Fig. 2. In order to work in the free-mass limit, we fix $\omega_m/(2\pi) = 1$ Hz, $\gamma_m/(2\pi) = 0.01$ Hz, $\Omega_F/(2\pi) = 10$ Hz, and vary Ω_S in the range of $\Omega_S/(2\pi) \in [1, 500]$ Hz, such that $\Omega_F, \Omega_S \gg \omega_m, \gamma_m$. First, we assume a perfect photodetector with unity efficiency ($\eta = 1$). This case is plotted with dashed and plain lines in blue, where the interaction strength is taken to be $\Omega_q/(2\pi) = 10$ Hz and $\Omega_q/(2\pi) = 20$ Hz, respectively. The entangling threshold occurs at $\Omega_F \approx \Omega_S$, as observed in Fig. 2. Then, we assume a photodetector with nonunity efficiency ($\eta = 0.9$), which introduces passive losses to the system. We plot this case with dashed and plain lines in yellow, for $\Omega_q/(2\pi) = 10$ Hz, and $\Omega_q/(2\pi) = 20$ Hz, respectively. From Eq. (36), we compute the entangling threshold as $\Omega_S/\Omega_F = 1.025$ ($\Omega_S/\Omega_F = 1.095$) for $\Omega_q/(2\pi) = 10$ Hz [$\Omega_q/(2\pi) = 20$ Hz], which is drawn to

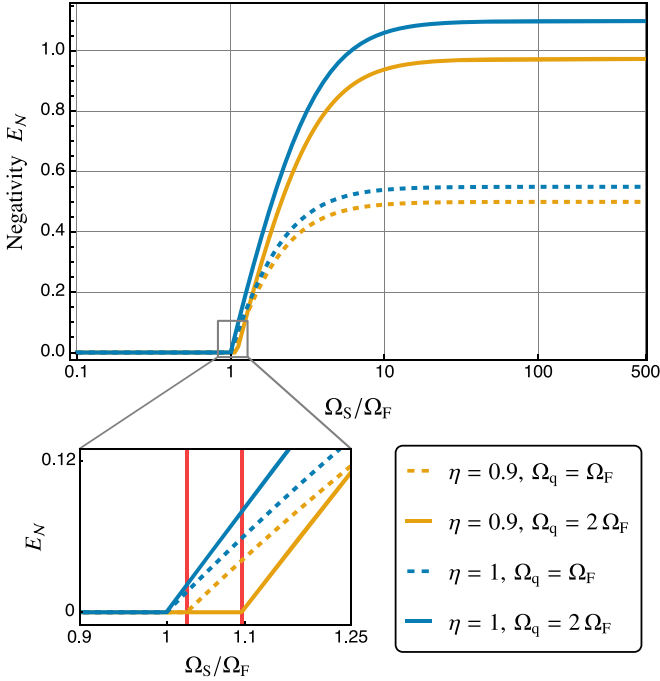


FIG. 2. Logarithmic negativity for the case where the input light field is the vacuum state, for a photodetector efficiency of $\eta = 0.9$ (plotted in yellow), $\eta = 1$ (plotted in blue), and white for environmental noise sources. We fix the force noise level with $\Omega_F/(2\pi) = 10$ Hz, and vary the sensing noise level in the range of $\Omega_S/(2\pi) \in [1, 500]$ Hz. The parameters of the mechanical oscillator are chosen as $\omega_m/(2\pi) = 1$ Hz and $\gamma_m/(2\pi) = 0.01$ Hz. We observe that for a unity photodetector efficiency ($\eta = 1$), the entangling transition is characterized by $\Omega_F \approx \Omega_S$, as observed numerically in Ref. [17]. For a nonunity photodetector efficiency ($\eta < 1$), the entangling transition is not universal with respect to the interaction strength Ω_q . For this case, the transition is characterized by $\sqrt{\Omega_F^2 + (1 - \eta)\Omega_q^2/2} \approx \Omega_S$, drawn by the red vertical lines in the inset.

the inset of Fig. 2 with a red, vertical curve. As observed analytically, passive losses impede achieving optomechanical entanglement. Furthermore, it is more advantageous to have a small interaction strength to obtain entanglement (i.e., reach the entangling threshold). However, once the threshold is reached, a larger interaction strength provides more entanglement, quantified by a larger logarithmic negativity.

IV. GENERALIZATION TO ARBITRARY SQUEEZING

Until this part of the paper, we have assumed that the input light field consisted of vacuum fluctuations. Now, let us assume that arbitrary squeezing operations have been performed on the input field before interacting with the mechanical oscillator.

First, we show in Appendix G that if the input field has been subjected to phase shifts or squeezing by a constant factor, the determinant condition in Eq. (27) is unchanged. Hence, such operations do not affect the entangling-disentangling transition. Therefore, we must apply a more general, potentially frequency-dependent transformation to

the input light field in order to act on the entangling-disentangling transition.

A general, unitary transformation performed on the input field must be causal and symplectic. The causality condition is because we have defined the input field to be the temporal modes of the light during $t > 0$. Then, any noncausal transformation mixes the input and the output temporal modes, which conflicts with the causality of the dynamics.

The most general causal symplectic transformation can be written as (see Appendix H)

$$\begin{bmatrix} \hat{u}_1(\Omega) \\ \hat{u}_2(\Omega) \end{bmatrix} \rightarrow e^{i\phi(\Omega)} \begin{bmatrix} A(\Omega) & B(\Omega) \\ C(\Omega) & D(\Omega) \end{bmatrix} \begin{bmatrix} \hat{u}_1(\Omega) \\ \hat{u}_2(\Omega) \end{bmatrix}, \quad (37)$$

where $A(\Omega)$, $B(\Omega)$, $C(\Omega)$, and $D(\Omega)$ are real functions of Ω , and $A(\Omega)D(\Omega) - B(\Omega)C(\Omega) = 1 \forall \Omega$. The phase factor $e^{i\phi(\Omega)}$ ensures the causality of the transformation.

We proceed with writing down the covariance matrix of the system, given this transformation. First, we calculate the correlations between the quadratures of the output light field,

$$V_{11}^{vv}(\Omega) = T_1(\Omega), \quad (38a)$$

$$V_{12}^{vv}(\Omega) = T_{12}(\Omega) + \Omega_q^2 T_1(\Omega) \chi(-\Omega), \quad (38b)$$

$$V_{22}^{vv}(\Omega) = T_2(\Omega) + \Omega_q^4 T_1(\Omega) |\chi(\Omega)|^2 + \Omega_q^2 \{\chi_{\text{EN}}(\Omega) + T_{12}(\Omega) [\chi(\Omega) + \chi(-\Omega)]\}, \quad (38c)$$

where we have defined $T_1(\Omega) = A(\Omega)^2 + B(\Omega)^2$, $T_2(\Omega) = C(\Omega)^2 + D(\Omega)^2$, and $T_{12}(\Omega) = A(\Omega)C(\Omega) + B(\Omega)D(\Omega)$. Next, the covariances of the quadratures of the mechanical oscillator are given by

$$V_{11}^{bb}(\Omega) = \omega_m \chi_F(\Omega) + \omega_m \Omega_q^2 |\chi(\Omega)|^2 T_1(\Omega), \quad (39)$$

and $V_{12}^{bb}(\Omega) = i\Omega V_{11}^{bb}(\Omega)/\omega_m$, $V_{22}^{bb}(\Omega) = \Omega^2 V_{11}^{bb}(\Omega)/\omega_m$. We see that the variance associated with the force noise is unchanged compared to Eq. (7), and the variance due to the optomechanical interaction is scaled by the frequency-dependent factor of $T_1(\Omega)$ in the frequency domain. Finally, we compute the cross-spectra of the mechanical oscillator and the output light field:

$$V_{11}^{bv}(\Omega) = \Omega_q \sqrt{\omega_m} \chi(\Omega) T_1(\Omega), \quad (40a)$$

$$V_{12}^{bv}(\Omega) = \Omega_q \sqrt{\omega_m} T_{12}(\Omega) \chi(\Omega) + \Omega_q \sqrt{\omega_m} \chi_F(\Omega) + \Omega_q^3 \sqrt{\omega_m} |\chi(\Omega)|^2 T_1(\Omega) \quad (40b)$$

and $V_{2i}^{bv}(\Omega) = -i\Omega V_{1i}^{bv}(\Omega)/\omega_m$, $i = 1, 2$. To continue with the formalism of Sec. III, let us recall the general form of the entanglement criterion. From Eqs. (11) and (14a), we have

$$\det[\mathbf{V}^{bb} + i\mathbf{K}^b - \mathbf{V}_{\text{pt}}^{bv} \cdot (\boldsymbol{\Sigma} \cdot \mathbf{D}_M \cdot \boldsymbol{\Sigma}^\dagger + \boldsymbol{\Delta}) \cdot (\mathbf{V}_{\text{pt}}^{bv})^T] < 0 \iff \text{entanglement}, \quad (41)$$

where $\boldsymbol{\Delta}$, $\boldsymbol{\Sigma}$, and $\mathbf{D}_M$ are given as [see Eq. (14)]

$$\boldsymbol{\Delta} = \begin{bmatrix} V_{11}^{vv}(\tau)^{-1} & 0 \\ 0 & 0 \end{bmatrix}, \quad (42a)$$

$$\boldsymbol{\Sigma} = \begin{bmatrix} V_{11}^{vv}(\tau)^{-1} \cdot V_{12}^{vv}(\tau) + iV_{11}^{vv}(\tau)^{-1} & 0 \\ -\delta(\tau) & 0 \end{bmatrix}, \quad (42b)$$

$$\mathbf{D}_M = \begin{bmatrix} M^{-1}(\tau) & 0 \\ 0 & M^{-1}(\tau) \end{bmatrix}, \quad (42c)$$

and $\mathbf{V}_{\text{pt}}^{bv}$ is defined in Eq. (10). To compute Eq. (41), we need to invert V_{11}^{vv} using the Wiener-Hopf method, therefore, we write it down in the Fourier domain as a product of its causal and anticausal poles and zeros,

$$V_{11}^{vv}(\Omega) = T_1(\Omega) = T_1(\Omega)_+ T_1(\Omega)_-, \quad (43)$$

where $T_1(\Omega)_+^\dagger = T_1(\Omega)_-$, and $T_1(\Omega)_+ [T_1(\Omega)_-]$ contains the causal (anticausal) poles and zeros of $V_{11}^{vv}(\Omega)$. In the absence of any squeezing in the input light field, we had found a simplification in the entanglement criterion, which we referred to as vacuum cancellation in Sec. III A. This simplification canceled out some of the terms in the entanglement criterion related to the mechanical oscillator [cf. Eq. (18)]. Specifically, we observed that the remaining terms were the variances of $\hat{b}_1$ and $\hat{b}_2$ generated due to the force noise only. Here, in the presence of frequency-dependent squeezing, we observe that the vacuum cancellation persists. To see this, we operate on $\mathbf{V}_{\text{pt}}^{bv}$ with Δ from the right,

$$(\mathbf{V}_{\text{pt}}^{bv} \cdot \Delta)(\Omega) = \begin{bmatrix} V_{11}^{bv} \cdot (V_{11}^{vv})^{-1} & 0 \\ -V_{21}^{bv} \cdot (V_{11}^{vv})^{-1} & 0 \end{bmatrix}. \quad (44)$$

Using Eqs. (40) and (43), we can simplify this expression to obtain

$$\begin{aligned} & \sqrt{\omega_m} \Omega_q \frac{1}{T_1(\Omega)_+} \left[\frac{T_1(\Omega)_+ T_1(\Omega)_- \chi(\Omega)}{T_1(\Omega)_-} \right]_+ \begin{bmatrix} 1 & 0 \\ -\frac{i\Omega}{\omega_m} & 0 \end{bmatrix} \\ &= \sqrt{\omega_m} \Omega_q \chi(\Omega) \begin{bmatrix} 1 & 0 \\ -\frac{i\Omega}{\omega_m} & 0 \end{bmatrix}. \end{aligned} \quad (45)$$

Then, in the frequency domain, we have

$$(\mathbf{V}_{\text{pt}}^{bv} \cdot \Delta \cdot (\mathbf{V}_{\text{pt}}^{bv})^T) = \omega_m \Omega_q^2 |\chi(\Omega)|^2 T_1(\Omega) \begin{bmatrix} 1 & \frac{i\Omega}{\omega_m} \\ -\frac{i\Omega}{\omega_m} & \frac{\Omega^2}{\omega_m^2} \end{bmatrix}. \quad (46)$$

This term is equal (in the frequency domain) to the variances of the quadratures of the mechanical oscillator arising from the optomechanical interaction [cf. the second term in Eq. (39)]. Therefore, we see that the vacuum cancellation is realized, and we can use Eq. (18) to test for entanglement, given that Σ and $\mathbf{D}_M$ are calculated accordingly.

Continuing with the calculation, we want to compute $\mathbf{V}_{\text{pt}}^{bv} \cdot \Sigma \cdot \mathbf{D}_M \cdot \Sigma^\dagger \cdot (\mathbf{V}_{\text{pt}}^{bv})^T$. For this purpose, we operate on $\mathbf{V}_{\text{pt}}^{bv}$ from the right with Σ . Using Eqs. (10) and (42b), we obtain

$$\begin{bmatrix} V_{11}^{bv}(t) \cdot (V_{11}^{vv})^{-1} \cdot [i\delta(\tau) + V_{12}^{vv}(\tau)] - V_{12}^{bv}(t) & 0 \\ -V_{21}^{bv}(t) \cdot (V_{11}^{vv})^{-1} \cdot [i\delta(\tau) + V_{12}^{vv}(\tau)] + V_{22}^{bv}(t) & 0 \end{bmatrix}. \quad (47)$$

We can simplify this expression by plugging in what we have found for $V_{i1}^{bv}(t) \cdot (V_{11}^{vv})^{-1}$, $i = 1, 2$, from Eq. (45), which results in

$$\begin{bmatrix} \sqrt{\omega_m} \Omega_q \chi(t) \cdot [i\delta(\tau) + V_{12}^{vv}(\tau)] - V_{12}^{bv}(t) & 0 \\ -\frac{\Omega_q}{\sqrt{\omega_m}} \dot{\chi}(t) \cdot [i\delta(\tau) + V_{12}^{vv}(\tau)] + V_{22}^{bv}(t) & 0 \end{bmatrix}. \quad (48)$$

To further simplify this expression, we want to compute $\chi(t) \cdot V_{12}^{vv}(\tau)$. Then, in the frequency domain, we have

$$\begin{aligned} & \Omega_q \sqrt{\omega_m} \mathcal{F} \left\{ \int_0^\infty dt' V_{12}^{vv}(t-t') \chi(t') \right\} \\ &= \Omega_q \sqrt{\omega_m} \mathcal{F} \left\{ \int_{-\infty}^\infty dt' V_{12}^{vv}(t-t') \chi(t') \right\} \\ &= \Omega_q \sqrt{\omega_m} T_{12}(\Omega) \chi(\Omega) + \Omega_q^3 \sqrt{\omega_m} |\chi(\Omega)|^2 T_1(\Omega) \\ &= V_{12}^{bv} \Big|_{\chi_F=0}, \end{aligned} \quad (49)$$

where V_{12}^{bv} is given in Eq. (40b). Similarly,

$$\frac{\Omega_q}{\sqrt{\omega_m}} \mathcal{F} \left\{ \int_0^\infty dt' V_{12}^{vv}(t-t') \dot{\chi}(t') \right\} = V_{22}^{bv} \Big|_{\chi_F=0}. \quad (50)$$

Then, similar to the case in Sec. III A, the terms related to the optomechanical interaction in V_{12}^{bv} and V_{22}^{bv} cancel out of Eq. (48).

We realize that thanks to these simplifications, the entanglement criterion in Eq. (41) is equivalent to the entanglement indicator defined in Eq. (27). However, while using Eq. (27), the operator M needs to be recomputed using Eq. (15). The expressions for the rest of the terms in Eq. (27), meaning $\tilde{\mathbf{V}}_{\text{pt}}^{bb}$, $\tilde{\mathbf{V}}_{\text{pt}}^{bv}$, and $\mathbf{V}_M$, are unchanged compared to the case where the input field is the vacuum state.

Now, let us calculate $M(t, t')$, whose definition is given in Eq. (15). Different from Sec. III, this operator has two time indices in the presence of arbitrary frequency-dependent squeezing. From the Wiener-Hopf method, we can first compute the columns of the matrix $V_{11}^{vv}(t, t')^{-1} \cdot [i\delta(t, t') + V_{12}^{vv}(t, t')]$. These columns will be functions of $t > 0$. Let us refer to them as $\{c_{t'}(t)\}$, where $t' > 0$ represents the column index, and t represents the row index, respectively. From Eqs. (38a) and (38b), we have in the frequency domain

$$\begin{aligned} c_{t'}(\Omega) &= \frac{1}{T_1(\Omega)_+} \left[\mathcal{F} \{ i\delta(t) + V_{12}^{vv}(-t) \} \frac{e^{i\Omega t'}}{T_1(\Omega)_-} \right]_+ \\ &= \frac{1}{T_1(\Omega)_+} \left[\frac{e^{i\Omega t'}}{T_1(\Omega)_-} [i + T_{12}(\Omega)] \right]_+ \\ &\quad + \Omega_q^2 e^{i\Omega t'} \chi(\Omega). \end{aligned} \quad (51)$$

Then, the elements of the matrix $[-i\delta(t, t') + V_{21}^{vv}(t, t')] \cdot V_{11}^{vv}(t, t')^{-1} \cdot [i\delta(t, t') + V_{12}^{vv}(t, t')]$ are given by

$$\int_0^\infty dx [-i\delta(t-x) + V_{21}^{vv}(x-t)] c_{t'}(x), \quad (52a)$$

$$\int_{-\infty}^\infty dx [-i\delta(t-x) + V_{21}^{vv}(x-t)] c_{t'}(x), \quad (52b)$$

$$\int_{-\infty}^\infty \frac{d\Omega}{2\pi} \mathcal{F} \{ -i\delta(t-x) + V_{21}^{vv}(x-t) \} c_{t'}(\Omega). \quad (52c)$$

We were able to extend the lower integration limit to $-\infty$ in Eq. (52b) since the function $c_{t'}(t)$ is causal. Then, we

used the Plancherel theorem to go from the time domain to the frequency domain in Eq. (52c). We can write Eq. (52c) explicitly as

$$\begin{aligned} & \int_{-\infty}^{\infty} \frac{d\Omega}{2\pi} \left\{ \frac{-i + T_{12}(\Omega)}{T_1(\Omega)_+} e^{-i\Omega t} \left[\frac{e^{i\Omega t'}}{T_1(\Omega)_-} [i + T_{12}(\Omega)] \right]_+ \right. \\ & \quad + \Omega_q^2 T_{12}(\Omega) e^{i\Omega(t'-t)} [\chi(\Omega) + \chi(-\Omega)] - i\Omega_q^2 e^{i\Omega(t'-t)} \\ & \quad \left. [\chi(\Omega) - \chi(-\Omega)] + \Omega_q^4 e^{i\Omega(t'-t)} |\chi(\Omega)|^2 T_1(\Omega) \right\}. \quad (53) \end{aligned}$$

This expression describes the elements of a matrix indexed by $t, t' > 0$, where t and t' are the row and the column indices, respectively. We observe that this matrix is Hermitian, but not necessarily Toeplitz. The Hermitian property is seen from the fact that interchanging t and t' corresponds to the conjugation of this expression. However, this matrix can be written as a sum of a Hermitian matrix and a Hermitian Toeplitz matrix, where the Toeplitz part is the sum of the terms in Eq. (53) that are a function of $t - t'$ (i.e., the terms in the second and third rows).

Therefore, M will also be the sum of a Hermitian matrix and a Hermitian Toeplitz matrix. More specifically, using Eqs. (15), (38c), and (53), we find that M can be written as

$$M(t, t') = M_{\text{vac}}(t - t') + \Lambda(t, t'), \quad (54)$$

where $M_{\text{vac}}(t - t')$ is what we have found for M when the input light field is the vacuum state [given in Eq. (16c)], and $\Lambda(t, t')$ is some Hermitian operator independent of Ω_q , whose entries depend on the squeezing in the input light field. The full expression for $\Lambda(t, t')$ is written as

$$\begin{aligned} \Lambda(t, t') &= T_2(t, t') - \int_{-\infty}^{\infty} \frac{d\Omega}{2\pi} \left(\frac{-i + T_{12}(\Omega)}{T_1(\Omega)_+} \right) e^{-i\Omega t} \\ & \quad \times \left[\frac{e^{i\Omega t'}}{T_1(\Omega)_-} [i + T_{12}(\Omega)] \right]_+. \quad (55) \end{aligned}$$

We observe that when the input light field is the vacuum state, $T_2(t, t') = \delta(t - t')$, $T_{12}(\Omega) = 0$, and $T_1(\Omega)_{\pm} = 1$. Then, using Eq. (55), we find that $\Lambda = \mathbf{0}$ for this case, and $M(t, t') = M_{\text{vac}}(t - t')$.

We want to investigate whether optomechanical entanglement is achievable with arbitrary, frequency-dependent squeezing in the input light field. For this purpose, we first observe that when $\Omega_q \rightarrow \infty$, $M \rightarrow M_{\text{vac}}$, and the determinant condition in Eq. (27) is equivalent to its form when the input light field is the vacuum state for a given force and sensing noise. We refer to this configuration as the “vacuum case.”

Now, we want to show that if the system is entangled for some Ω_q , it is bound to be entangled for $\Omega'_q > \Omega_q$. To show this, let us assume that the system is entangled for some Ω_q . We notice that the first two terms in Eq. (27) will be independent of Ω_q . The Ω_q dependency comes from the third term, where $\tilde{\mathbf{V}}_{\text{pt}}^{bv} \propto \Omega_q$, and $M(t, t') = M_{\text{vac}}(t - t') + \Lambda(t, t')$, $M_{\text{vac}} \propto \Omega_q^2$. Then, the third term can be written in the

form of

$$\begin{aligned} & \mathbf{C} \cdot \begin{bmatrix} (\frac{1}{\Omega_q^2} \Lambda + D)^{-1} & 0 \\ 0 & (\frac{1}{\Omega_q^2} \Lambda + D)^{-1} \end{bmatrix} \cdot \mathbf{C}^\dagger, \\ & \mathbf{C} = \frac{1}{\Omega_q} \tilde{\mathbf{V}}_{\text{pt}}^{bv} \cdot \begin{bmatrix} -i\delta(\tau) & 0 \\ \delta(\tau) & 0 \end{bmatrix}, \\ & D = \frac{1}{\Omega_q^2} M_{\text{vac}}, \end{aligned} \quad (56)$$

where $\mathbf{C}$, D , and Λ are independent of Ω_q . Furthermore, notice that M is the Schur complement of the block V_{11}^{vv} of the matrix $\mathbf{V}^{vv} + i\mathbf{K}^v$. Since V_{11}^{vv} is positive definite, and $\mathbf{V}^{vv} + i\mathbf{K}^v$ is positive semidefinite (from the Heisenberg uncertainty principle), M is positive semidefinite for all Ω_q . Consequently, Λ and M_{vac} are also positive semidefinite, since $M = \Lambda$ when $\Omega_q = 0$, and $M \rightarrow M_{\text{vac}}$ in the limit of $\Omega_q \rightarrow \infty$. Also, recall that both D and Λ are Hermitian. As a result, we can write for $\Omega'_q > \Omega_q$

$$\begin{aligned} & \frac{1}{\Omega_q'^2} \Lambda < \frac{1}{\Omega_q^2} \Lambda, \\ & \frac{1}{\Omega_q'^2} \Lambda + D < \frac{1}{\Omega_q^2} \Lambda + D, \\ & \left(\frac{1}{\Omega_q'^2} \Lambda + D \right)^{-1} > \left(\frac{1}{\Omega_q^2} \Lambda + D \right)^{-1}. \end{aligned} \quad (57)$$

Therefore, if we refer to the matrix in the determinant condition in Eq. (27) as $\mathbf{V}_{\text{det}}(\Omega_q)$, we can write $\mathbf{V}_{\text{det}}(\Omega'_q) < \mathbf{V}_{\text{det}}(\Omega_q)$ for $\Omega'_q > \Omega_q$. Since the system is entangled for Ω_q , the determinant of $\mathbf{V}_{\text{det}}(\Omega_q)$ is negative. Furthermore, since we are looking for the entanglement between one mode and the rest, $\mathbf{V}_{\text{det}}(\Omega_q)$ can have at most one negative eigenvalue (for this case, it has exactly one negative eigenvalue). Then, $\mathbf{V}_{\text{det}}(\Omega'_q)$ will also have a negative eigenvalue, and the system will be entangled. Hence, we proved that if optomechanical entanglement exists for Ω_q , it also exists for $\Omega'_q > \Omega_q$.

Consequently, we can make the following statements about the nature of optomechanical entanglement when the input light field is squeezed in a frequency-dependent fashion. First, we observe that universality with respect to the interaction strength, Ω_q , does not hold, since the entanglement indicator in Eq. (27) depends on Ω_q . Hence, increasing Ω_q may provide optomechanical entanglement to form. However, the transition is still unique with respect to the amount of environmental noise. Second, if there exists optomechanical entanglement for any frequency-dependent squeezing configuration, it will also exist when the input light field is in the vacuum state. Therefore, it is “easier” to achieve entanglement for a given force and sensing noise without frequency-dependently squeezing the input light field. In other words, the hyperspace of force and sensing noise spectra for which there exists optomechanical entanglement for the vacuum case encapsulates the space when the input light field is squeezed arbitrarily [54]. We observed numerically that after optomechanical entanglement is achieved, the amount of entanglement, quantified by the logarithmic negativity, can be greater or less than the case where the input light field

is the vacuum state, depending on the frequency-dependent squeezing transformation.

V. DISCUSSION FOR FREQUENCY-DEPENDENT SQUEEZING

In Sec. IV, it was proven that when the input light field is squeezed in a frequency-dependent fashion, it is harder to achieve optomechanical entanglement compared to the vacuum input case, i.e., there exists a region of the parameter space where optomechanical entanglement is observed only for the latter configuration. Intuitively, this result can be understood in the following way: when the ingoing light field is the vacuum state, the modes corresponding to different frequencies (or arrival times with respect to the surface of the oscillator) are not correlated. Conversely, when the ingoing light field is frequency-dependently squeezed, there is a nonzero correlation between these modes. Then, when we consider the outgoing light field that has reflected from the mechanical oscillator during time $t < 0$, it will be entangled with the ingoing light field that has yet to interact with the oscillator during this time (note that this field will be reflected from the oscillator during time $t > 0$). Hence, while computing the entanglement between this field and the mechanical oscillator at time $t = 0$, the correlation will induce an additional noise to the system when the ingoing light modes are traced out. Therefore, we can expect that it is harder to get entangled for this case, as it will appear to be more noisy than the vacuum input case.

This extra noise can be overcome by computing the entanglement between the mechanical oscillator at time $t = 0$, and the joint field combining the outgoing field that has left the oscillator during $t < 0$, and the ingoing field that has yet to interact with the oscillator. Since the joint field contains information about the light field during time $t \in (-\infty, \infty)$, the frequency-dependent squeezing can be “reversed” [55] by whitening the joint field. This would not be possible if we only had access to the outgoing light field, as the whitening transformation will be noncausal in the presence of frequency-dependent squeezing.

This situation is illustrated in Fig. 3. We fix the parameters of the mechanical oscillator as $\omega_m/(2\pi) = 1$ Hz, $\gamma_m/(2\pi) = 0.01$ Hz, and assume white environmental noise sources parametrized by $\Omega_F/(2\pi) = 10$ Hz and $\Omega_S/(2\pi) \in [1, 500]$ Hz. We first plot the logarithmic negativity for the vacuum input with red curves, for interaction strengths of $\Omega_q/(2\pi) = 5$ Hz and $\Omega_q/(2\pi) = 10$ Hz, with dashed and plain curves, respectively. We repeat this calculation for a frequency-dependently squeezed ingoing field (plotted in yellow). The squeezing transformation is such that the total quantum noise is squeezed with

$$S(\Omega) \rightarrow e^{-2r} S(\Omega), \quad (58)$$

where $S(\Omega)$ is the spectrum of the quantum noise and r is a squeezing parameter. For more details about the realization, see Appendix I. We fix $r = 1$, and observe that the entangling transition is not universal with respect to the interaction strength Ω_q in the presence of frequency-dependent squeezing, as was shown in Sec. IV. Finally, we repeat the calculation for the same squeezed input, but we compute the

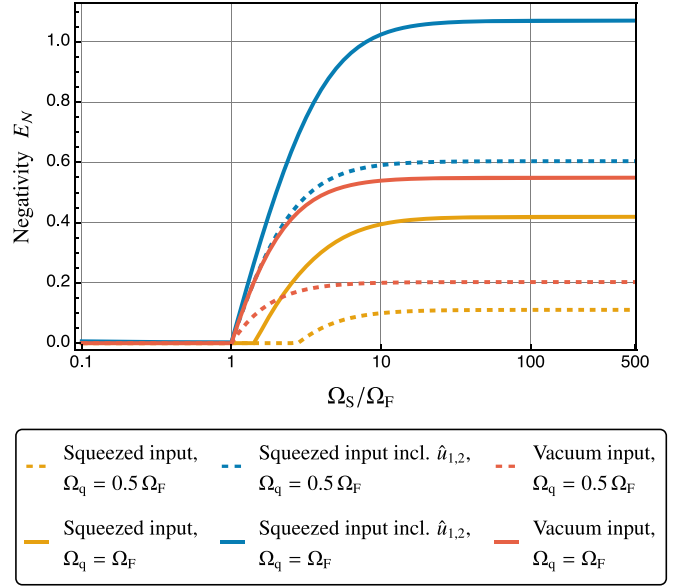


FIG. 3. Logarithmic negativity for the case where the input light field is frequency-dependently squeezed with $r = 1$; cf. Eq. (58)], and for the vacuum input. We further consider the cases where we compute the entanglement between the mechanical oscillator at time $t = 0$ and the light field, where the field is considered to be (i) the outgoing field that has left the oscillator during $t < 0$, (ii) the joint field combining the outgoing field that has left the oscillator during $t < 0$, and the ingoing field that has yet to interact with the oscillator, but will be reflected from the oscillator during $t > 0$. These cases (for a squeezed input) are plotted in yellow and blue curves, respectively. For the vacuum input plotted in red, we only consider case (i). We observe that for a squeezed input, entanglement is harder to achieve for case (i), and the entangling transition is not universal with respect to the interaction strength Ω_q . However, for case (ii), universality is restored.

entanglement between the oscillator and the joint field combining the outgoing and the ingoing light field during $t < 0$ (plotted in blue). We observe that the universality is restored, and the entangling transition occurs at $\Omega_F \approx \Omega_S$, as found in Sec. III C. Note that the amount of entanglement is the largest for this case.

VI. CONCLUSIONS

In this paper, we proved the existence of an entangling-disentangling transition in an optomechanical system consisting of a mechanical oscillator interacting with a light field in a pure state. We assumed arbitrary, possibly non-Markovian, environmental noise sources. Assuming Gaussian dynamics, we were able to compute necessary and sufficient conditions for entanglement using the PPT criterion.

First, assuming that the input light field is the vacuum state, we showed that the entangling-disentangling transition is independent of the interaction strength between the oscillator and the light field, hence, it is universal. Therefore, if the environmental noises are such that the system is not in the regime where optomechanical entanglement can occur, one cannot achieve it by increasing the interaction strength between the oscillator and the light field.

In the second part of the paper, we assumed arbitrary Gaussian and causal unitary transformations applied to the light field, including frequency-dependent squeezing. We demonstrated that the entangling-disentangling transition is not affected by frequency-independent squeezing and phase shifts, i.e., it is equivalent to the vacuum input case. However, when the light field is squeezed in a frequency-dependent manner, we showed that entanglement is harder to achieve. More specifically, we proved that if there is no optomechanical entanglement when the input light field is the vacuum state, it is impossible to generate it by squeezing the input light field arbitrarily.

We expand on the physical interpretation and consequences of these results in a companion Letter [23].

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DATA AVAILABILITY

The data that support the findings of this article are openly available [56].

APPENDIX A: EQUATIONS OF MOTION WITH PHYSICAL DIMENSIONS

Here, we present the equations of motion with physical dimensions. We denote the position (momentum) of the center of mass of the mechanical oscillator with $\hat{x}$ ($\hat{p}$), and the amplitude and phase quadratures of the input (output) light field with $\hat{u}_1(t)$ and $\hat{u}_2(t)$ [$\hat{v}_1(t)$ and $\hat{v}_2(t)$], respectively. The equations read

$$\hat{v}_1(t) = \hat{u}_1(t), \quad (\text{A1a})$$

$$\hat{v}_2(t) = \hat{u}_2(t) + \alpha[\hat{x}(t) + \hat{N}_S(t)], \quad (\text{A1b})$$

$$\frac{d\hat{p}(t)}{dt} = -\gamma_m \hat{p}(t) - m\omega_m^2 \hat{x}(t) + \hbar\alpha \hat{u}_1(t) + \hat{N}_F(t), \quad (\text{A1c})$$

$$\frac{d\hat{x}(t)}{dt} = \hat{p}(t)/m, \quad (\text{A1d})$$

where $\alpha = \Omega_q \sqrt{m/\hbar}$, Ω_q is the characteristic interaction strength as in the main text, γ_m is the damping rate of the oscillator, m is the mass of the oscillator, and $\hat{N}_F(t)$ and $\hat{N}_S(t)$ are the force and the sensing noise, respectively. The quadratures of the mechanical oscillator in the main text, $\hat{b}_1(t)$ and $\hat{b}_2(t)$, are related to $\hat{x}(t)$ and $\hat{p}(t)$ with

$$\hat{b}_1(t) = \sqrt{\frac{2m\omega_m}{\hbar}} \hat{x}(t), \quad \hat{b}_2(t) = \sqrt{\frac{2}{\hbar m\omega_m}} \hat{p}(t) \quad (\text{A2})$$

so that $[\hat{x}(t), \hat{p}(t)] = i\hbar$, $[\hat{b}_1(t), \hat{b}_2(t)] = 2i$. Note that here, the commutators are for same-time operators: $[\hat{x}(t), \hat{p}(t')] = 0$ for $t \neq t'$. Furthermore, $\hat{N}_F(t)$ and $\hat{N}_S(t)$ are related to $\hat{n}_F(t)$ and $\hat{n}_S(t)$ in the main text with

$$\hat{N}_S(t) = \sqrt{\frac{\hbar}{2m\omega_m}} \hat{n}_S(t), \quad \hat{N}_F(t) = \sqrt{\frac{\hbar m\omega_m}{2}} \hat{n}_F(t) \quad (\text{A3})$$

so that $\hat{n}_S$ is dimensionless and $\hat{n}_F$ has dimensions of Hertz.

APPENDIX B: MATRIX INVERSION USING SCHUR COMPLEMENT

In the main text, we encounter block matrices in the form of

$$\mathbf{L} = \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix}, \quad (\text{B1})$$

where $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, and $\mathbf{D}$ are square matrices. If $\mathbf{A}$ is invertible, we can write

$$\mathbf{L}^{-1} = \begin{bmatrix} \mathbf{A}^{-1} + \mathbf{A}^{-1}\mathbf{B}(\mathbf{L}/\mathbf{A})^{-1}\mathbf{C}\mathbf{A}^{-1} & -\mathbf{A}^{-1}\mathbf{B}(\mathbf{L}/\mathbf{A})^{-1} \\ -(\mathbf{L}/\mathbf{A})^{-1}\mathbf{C}\mathbf{A}^{-1} & (\mathbf{L}/\mathbf{A})^{-1} \end{bmatrix}, \quad (\text{B2})$$

where $\mathbf{L}/\mathbf{A} := \mathbf{D} - \mathbf{C}\mathbf{A}^{-1}\mathbf{B}$ is the Schur complement of block $\mathbf{A}$ of matrix $\mathbf{L}$. We can restate this formula as

$$\begin{aligned} \mathbf{L}^{-1} &= \begin{bmatrix} \mathbf{A}^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} \end{bmatrix} \begin{bmatrix} \mathbf{B}(\mathbf{L}/\mathbf{A})^{-1}\mathbf{C} & -\mathbf{B}(\mathbf{L}/\mathbf{A})^{-1} \\ -(\mathbf{L}/\mathbf{A})^{-1}\mathbf{C} & (\mathbf{L}/\mathbf{A})^{-1} \end{bmatrix} \begin{bmatrix} \mathbf{A}^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} \end{bmatrix} \\ &+ \begin{bmatrix} \mathbf{A}^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \\ &= \begin{bmatrix} \mathbf{A}^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} \end{bmatrix} \begin{bmatrix} \mathbf{B} & \mathbf{0} \\ -\mathbf{1} & \mathbf{0} \end{bmatrix} \begin{bmatrix} (\mathbf{L}/\mathbf{A})^{-1} & \mathbf{0} \\ \mathbf{0} & (\mathbf{L}/\mathbf{A})^{-1} \end{bmatrix} \\ &\begin{bmatrix} \mathbf{C} & -\mathbf{1} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{A}^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} \end{bmatrix} + \begin{bmatrix} \mathbf{A}^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}, \end{aligned} \quad (\text{B3})$$

where $\mathbf{0}$ and $\mathbf{1}$ are the null and the identity matrices, respectively. In the main text, we evaluate $(\mathbf{V}_{\text{pt}}^{vv} + i\mathbf{K}^v)^{-1}$. Then, we can use the expression derived above, by substituting

$$\begin{aligned} \mathbf{A} &= V^{vv}(\tau), \\ \mathbf{B} &= V_{12}^{vv}(\tau) + i\delta(\tau), \\ \mathbf{C} &= \mathbf{B}^\dagger = V_{21}^{vv}(\tau) - i\delta(\tau), \\ \mathbf{D} &= V_{22}^{vv}(\tau). \end{aligned} \quad (\text{B4})$$

The result can be found in Eq. (14).

APPENDIX C: CONVENTIONS AND THE WIENER-HOPF METHOD

Here, we first restate the conventions used in dot products between function operators and functions. Assume real functions of t , $f(t)$, $g(t)$, and a real operator $A(t-t')$, for $t, t' > 0$. First, let us write down the inner product between the functions f, g : assuming that the functions are column

vectors, we have

$$\begin{aligned} f^T \cdot g &:= \int_0^\infty f(t)g(t)dt \\ &= \int_{-\infty}^\infty f(t)g(t)dt \quad \text{if } f(t) \text{ or } g(t) \text{ is causal} \\ &= \int_{-\infty}^\infty f(\Omega)^\dagger g(\Omega) \frac{d\Omega}{2\pi}, \end{aligned} \quad (\text{C1})$$

where T denotes the transpose operation such that f^T is a row vector, $\dagger$ denotes the complex conjugate operation, and $f(\Omega)$ is the Fourier transform of $f(t)$. Note that we used the Plancherel theorem in the last line. We use the following Fourier transform convention:

$$\begin{aligned} f(\Omega) &= \mathcal{F}\{f(t)\} = \int_{-\infty}^\infty dt f(t)e^{i\Omega t}, \\ f(t) &= \mathcal{F}^{-1}\{f(\Omega)\} = \int_{-\infty}^\infty \frac{d\Omega}{2\pi} f(\Omega)e^{-i\Omega t} \end{aligned} \quad (\text{C2})$$

such that, for example,

$$\begin{aligned} f(\Omega) &= \frac{1}{b + i(a - \Omega)}, \quad a > 0, b > 0 \\ \Rightarrow f(t) &= e^{-(ia+b)t} H(t), \end{aligned} \quad (\text{C3})$$

where $H(t)$ is the Heaviside step function. Then, we see that the spectrum of a causal (anticausal) function has poles located on the lower (upper) half of the complex plane.

The dot product between the operator A and the function f is given as

$$\begin{aligned} A \cdot f &= h(t) := \int_0^\infty A(t' - t)f(t')dt', \quad t > 0, \\ f^T \cdot A &= (A^T \cdot f)^T = l(t) := \int_0^\infty A(t - t')f(t')dt', \quad t > 0, \end{aligned} \quad (\text{C4})$$

where $h(t)$ and $l(t)$ are column and row vectors, respectively. Then, we see that the transpose operation corresponds to a time inversion for operators. Furthermore, if $f(t)$ is a causal function of t [i.e., $f(t) = 0$ for $t < 0$], we can change the integration to be between $-\infty$ and ∞ , and can write $h(\Omega) = A(-\Omega)f(\Omega) = A(\Omega)^\dagger f(\Omega)$ and $l(\Omega) = A(\Omega)f(\Omega)$. Note that $A(-\Omega) = A(\Omega)^\dagger$ since A is a real operator.

Now, assume an operator M , and functions h, g , which are related to each other with

$$\int_0^\infty M(t - t')h(t')dt' = g(t), \quad t > 0. \quad (\text{C5})$$

Equation (C5) implies that $h = M^{-1}g$. In the main text, we want to find h given M and g , in order to evaluate the characteristic polynomial [Eq. (11)]. For this purpose, we use the Wiener-Hopf method (See Ref. [57] for a review). First, let us define $M(\Omega), g(\Omega)$ as the spectrum of M and g , respectively, and assume that they are rational functions of Ω . Notice that Eq. (C5) can be rewritten as

$$\int_{-\infty}^\infty dt' M(t - t')h(t')_+ dt' = g(t), \quad t > 0, \quad (\text{C6})$$

where

$$f(t)_+ = [f(t)]_+ = \begin{cases} f(t) & \text{if } t > 0 \\ 0 & \text{otherwise,} \end{cases} \quad (\text{C7})$$

which implies that

$$\begin{aligned} \left[\int_{-\infty}^\infty dt' [M(t - t')h(t')_+ - g(t)] \right]_+ &= [s(t)]_+ = 0, \\ s(t) &:= \int_{-\infty}^\infty dt' [M(t - t')h(t')_+ - g(t)], \end{aligned} \quad (\text{C8})$$

i.e., $s(t)$ is an anticausal function of time. In the frequency domain, we have that

$$s(\Omega) = \sum_k \frac{r(\Omega_k, \sigma_k)}{(\Omega - \Omega_k)^{\sigma_k}} = s(\Omega)_+ + s(\Omega)_-, \quad (\text{C9})$$

for any rational function $s(\Omega)$, where the sum is the partial fraction decomposition of the spectrum, and it is over the poles Ω_k of $s(\Omega)$, with multiplicity σ_k . $r(\Omega_k, \sigma_k)$ is the residue at $\Omega = \Omega_k$. Furthermore, $s(\Omega)_+$ is given by

$$\begin{aligned} s(\Omega)_+ &= \mathcal{F}\{s(t)_+\} \\ &= \sum_k \frac{r(\Omega_k, \sigma_k)}{(\Omega - \Omega_k)^{\sigma_k}} \quad \text{with } \text{Im}(\Omega_k) < 0, \end{aligned} \quad (\text{C10})$$

where $\text{Im}(\Omega_k)$ is the imaginary part of Ω_k . From Eq. (C8), we know that

$$\begin{aligned} s(\Omega) &= M(\Omega)h(\Omega)_+ - g(\Omega) \\ \Rightarrow s(\Omega)_+ &= [M(\Omega)h(\Omega)_+ - g(\Omega)]_+ = 0. \end{aligned} \quad (\text{C11})$$

Since $M(\Omega)$ is a rational function, we can write it as $M(\Omega) = M(\Omega)_+M(\Omega)_-$, where $M(\Omega)_+ [M(\Omega)_-]$ contains the causal (anticausal) zeros and poles of $M(\Omega)$. Furthermore, $M(\Omega)_- = M(\Omega)_+^\dagger$ since $M(-t) = M(t)^\dagger$ in the main text. We have

$$\begin{aligned} [M(\Omega)_+M(\Omega)_-h(\Omega)_+ - g(\Omega)]_+ &= \\ = \left[M(\Omega)_+h(\Omega)_+ - \frac{g(\Omega)}{M(\Omega)_-} \right]_+ &= 0, \end{aligned} \quad (\text{C12})$$

where the second equality comes from the fact that multiplying the spectrum of an anticausal function with a rational function with anticausal roots and poles does not change the causality of the function. Therefore, we find

$$\begin{aligned} h(\Omega)_+ &= \frac{1}{M(\Omega)_+} \left[\frac{g(\Omega)}{M(\Omega)_-} \right]_+, \\ h(t) &= \mathcal{F}^{-1} \left\{ \frac{1}{M(\Omega)_+} \left[\frac{g(\Omega)}{M(\Omega)_-} \right]_+ \right\}, \quad t > 0. \end{aligned} \quad (\text{C13})$$

APPENDIX D: EXISTENCE OF ENTANGLEMENT FOR ALL FORCE NOISES IN ABSENCE OF SENSING NOISE

Here, we want to obtain a condition for optomechanical entanglement in the presence of force noise only, i.e., in the absence of any sensing noise. Let us parametrize the spectrum of the force noise in the system as $\alpha_F S_{n_F}$, where α_F is a dimensionless constant scaling the overall strength of the noise; and $\alpha_F = 0$ indicates that there is no force noise. If the system is entangled for α_F , it will also be entangled for $\alpha_F - \epsilon$, since the entanglement criterion gets modified as $\mathbf{V}_{\text{pt}} + i\mathbf{K} - \epsilon\mathbf{V}' < 0$,

where $\mathbf{V}_{\text{pt}} + i\mathbf{K}$ is negative definite, and $-\epsilon\mathbf{V}'$ is negative semidefinite (see Appendix E). Then, the criterion will be satisfied for $\alpha_F - \epsilon$. Therefore, we set $S_{n_s}(\Omega) = 0$ for the rest of this Appendix and characterize the force noise as $\alpha_F S_{n_F}$, assuming that $\alpha_F \gg 1$. Due to the discussion above, if the system is entangled in this limit, it will be entangled for all force noise spectra, regardless of the amplitude.

In this limit, from Eq. (16c),

$$\begin{aligned} M(\Omega) &= \Omega_q^2 |\chi(\Omega)|^2 [\omega_m \alpha_F S_{n_F}(\Omega) + 2\gamma_m \Omega] \\ &= M_R(\Omega) + M_I(\Omega), \end{aligned} \quad (\text{D1})$$

where

$$\begin{aligned} M_R(\Omega) &= \Omega_q^2 |\chi(\Omega)|^2 \omega_m \alpha_F S_{n_F}(\Omega) = \Omega_q^2 \chi_F(\Omega), \\ M_I(\Omega) &= 2\Omega_q^2 |\chi(\Omega)|^2 \gamma_m \Omega = 2\Omega_q^2 \Theta(\Omega) \gamma_m \Omega. \end{aligned} \quad (\text{D2})$$

Note that in the time domain, $M_R(t) = \Omega_q^2 \chi_F(t)$, $M_I(t) = 2i\Omega_q^2 \gamma_m \dot{\Theta}(t)$. Since $M_R(\Omega)$ is a real and even function of Ω , $M_R(t)$ is a real and symmetric function of time. Similarly, since $M_I(\Omega)$ is a real and odd function of Ω , $M_I(t)$ is a purely imaginary and antisymmetric function. Now, we want to express the inverse of the operator M as a function of the operators M_R and M_I , where M_R is real and symmetric, and M_I is imaginary and Hermitian [as $M_I(\tau) = -M_I(-\tau)$]. In the limit of $\alpha_F \gg 1$, we have

$$\begin{aligned} M^{-1} &= M_R^{-1} \cdot (1 + M_I \cdot M_R^{-1})^{-1} \\ &\approx M_R^{-1} \cdot (1 - M_I \cdot M_R^{-1}) \\ &= M_R^{-1} - M_R^{-1} \cdot M_I \cdot M_R^{-1}, \end{aligned} \quad (\text{D3})$$

where the approximation used that $M_R^{-1} \cdot M_I \propto 1/\alpha_F \ll 1$.

With this approximation of M^{-1} , we evaluate Eq. (27). In order to calculate the dot products with M_R^{-1} using the Wiener-Hopf method, we write M_R^{-1} as a product of its causal and anticausal parts. Note that we omit writing α_F for the rest of this Appendix. First, we express $\chi_F(\Omega)$ as a product of its causal and anticausal parts:

$$\begin{aligned} \chi_F(\Omega) &= \chi_F(\Omega)_+ \chi_F(\Omega)_-, \\ \chi_F(\Omega)_+ &= \sqrt{\omega_m} \chi(\Omega) S_{n_F}(\Omega)_+, \end{aligned} \quad (\text{D4})$$

and $\chi_F(\Omega)_- = \chi_F(\Omega)_+^\dagger$. Here, $S_{n_F} = S_{n_F}(\Omega)_+ S_{n_F}(\Omega)_-$, where $S_{n_F}(\Omega)_+$ contains the causal roots and zeros of $S_{n_F}(\Omega)$. This decomposition is possible for any power noise spectrum $S_{n_F}(\Omega)$ that is a rational function of Ω . Then, we have $M_R(\Omega)_\pm = \Omega_q \chi_F(\Omega)_\pm$.

We now want to calculate $\tilde{\mathbf{V}}_{\text{pt}}^{bv} \cdot \mathbf{V}_M \cdot (\tilde{\mathbf{V}}_{\text{pt}}^{bv})^T$ in Eq. (27). For this purpose, we can define $f \star g = f \cdot M^{-1} \cdot g$ for functions f and g . Using this notation, $\tilde{\mathbf{V}}_{\text{pt}}^{bv} \cdot \mathbf{V}_M \cdot (\tilde{\mathbf{V}}_{\text{pt}}^{bv})^T$ is given by (we drop the superscripts bv)

$$\begin{aligned} &\begin{bmatrix} V_{11} \star V_{11}^T & -V_{11} \star V_{21}^T \\ -V_{21} \star V_{11}^T & V_{21} \star V_{21}^T \end{bmatrix} + \begin{bmatrix} \tilde{V}_{12} \star \tilde{V}_{12}^T & -\tilde{V}_{12} \star \tilde{V}_{22}^T \\ -\tilde{V}_{22} \star \tilde{V}_{12}^T & \tilde{V}_{22} \star \tilde{V}_{22}^T \end{bmatrix} \\ &- i \begin{bmatrix} V_{11} \star \tilde{V}_{12}^T - \tilde{V}_{12} \star V_{11}^T & \tilde{V}_{12} \star V_{21}^T - V_{11} \star \tilde{V}_{22}^T \\ \tilde{V}_{22} \star V_{11}^T - V_{21} \star \tilde{V}_{12}^T & V_{21} \star \tilde{V}_{22}^T - \tilde{V}_{22} \star V_{21}^T \end{bmatrix}. \end{aligned} \quad (\text{D5})$$

We first calculate Eq. (D5) by substituting M^{-1} with M_R^{-1} from Eq. (D3). We notice that

$$\begin{aligned} \tilde{V}_{12}^{bv}(\Omega) &= \sqrt{\omega_m} \Omega_q \chi_F(\Omega)_+ \chi_F(\Omega)_-, \\ \tilde{V}_{22}^{bv}(\Omega) &= -i\Omega \frac{\Omega_q}{\sqrt{\omega_m}} \chi_F(\Omega)_+ \chi_F(\Omega)_-. \end{aligned} \quad (\text{D6})$$

We proceed with the Wiener-Hopf method to calculate $\tilde{V}_{12}^{bv} \cdot M_R^{-1}$. In the frequency domain, we have

$$\begin{aligned} (\tilde{V}_{12}^{bv} \cdot M_R^{-1})(\Omega) &= \frac{\sqrt{\omega_m} \Omega_q}{\Omega_q^2} \frac{1}{\chi_F(\Omega)_+} \left[\frac{\chi_F(\Omega)_+ \chi_F(\Omega)_-}{\chi_F(\Omega)_-} \right]_+ \\ &= \frac{\sqrt{\omega_m}}{\Omega_q}; \end{aligned} \quad (\text{D7})$$

therefore, in the time domain,

$$\tilde{V}_{12}^{bv} \cdot M_R^{-1} = 2\delta(t) \sqrt{\omega_m} / \Omega_q \quad (\text{D8})$$

for $t > 0$. The factor of 2 arises from the causality of the delta function. Then, the inner product $\tilde{V}_{12}^{bv} \cdot M_R^{-1} \cdot (\tilde{V}_{12}^{bv})^T$ is given by

$$\begin{aligned} \tilde{V}_{12}^{bv} \cdot M_R^{-1} \cdot (\tilde{V}_{12}^{bv})^T &= \frac{2\sqrt{\omega_m}}{\Omega_q} \int_0^\infty \tilde{V}_{12}^{bv}(t) \delta(t) dt \\ &= \omega_m \chi_F(0) = (\tilde{\mathbf{V}}^{bb})_{11}. \end{aligned} \quad (\text{D9})$$

Similarly,

$$\tilde{V}_{22}^{bv} \cdot M_R^{-1} = \frac{2\dot{\delta}(t)}{\sqrt{\omega_m} \Omega_q}, \quad (\text{D10})$$

which implies that the second term in Eq. (D5) is given by

$$\begin{aligned} &\begin{bmatrix} \tilde{V}_{12}^{bv} \cdot M_R^{-1} \cdot (\tilde{V}_{12}^{bv})^T & -\tilde{V}_{12}^{bv} \cdot M_R^{-1} \cdot (\tilde{V}_{22}^{bv})^T \\ -\tilde{V}_{22}^{bv} \cdot M_R^{-1} \cdot (\tilde{V}_{12}^{bv})^T & \tilde{V}_{22}^{bv} \cdot M_R^{-1} \cdot (\tilde{V}_{22}^{bv})^T \end{bmatrix} \\ &= \begin{bmatrix} \omega_m \chi_F(0) & 0 \\ 0 & \frac{1}{\omega_m} \dot{\chi}_F(0) \end{bmatrix} = \tilde{\mathbf{V}}^{bb}, \end{aligned} \quad (\text{D11})$$

which cancels out with $\tilde{\mathbf{V}}^{bb}$ in Eq. (27). Next, let us evaluate the third term in Eq. (D5). From Eqs. (D8) and (D10), we have

$$\begin{aligned} &\tilde{V}_{12}^{bv} \cdot M_R^{-1} \cdot (V_{21}^{bv})^T - \tilde{V}_{11}^{bv} \cdot M_R^{-1} \cdot (\tilde{V}_{22}^{bv})^T \\ &= \left(\frac{\sqrt{\omega_m}}{\Omega_q} V_{21}^{bv}(0^+) + \frac{\dot{V}_{11}^{bv}(0^+)}{\sqrt{\omega_m} \Omega_q} \right) = 2\dot{\chi}(0^+) = 2 \end{aligned} \quad (\text{D12})$$

and the entries in the diagonals are zero, since M_R is a real and symmetric operator. Then, the third term in Eq. (D5) will be

$$\begin{bmatrix} 0 & -2i \\ 2i & 0 \end{bmatrix}. \quad (\text{D13})$$

Finally, we want to calculate the first term in Eq. (D5). However, one cannot directly take the dot products of M_R^{-1} with V_{11}^{bv} and V_{21}^{bv} : indeed, if the force noise spectrum decays faster than $1/\Omega^2$, the expressions are not well defined. However, (quantum) physically, we expect that any (bosonic) continuous variable force noise will remain finite, as we shall see at the end of this Appendix. Then, we work in the limit of $M_R(t) = \epsilon + \Omega_q^2 \chi_F(t)$, $\epsilon \ll 1$. Similar to above, we need to write $M_R(t)$ as a product of a causal and anticausal function in the Fourier domain. For this purpose, notice that the spectrum

of M_R is—up to some constant—the spectrum of the force noise, $S_{n_F}(\Omega)$, multiplied by the absolute value squared of the mechanical susceptibility, $|\chi(\Omega)|^2$, due to the response of the harmonic oscillator to the external forces caused by the force noise. The force noise spectra can be written as a product of rational functions of Ω^2 as it is a real and even function of Ω . We assume that $S_{n_F}(\Omega)$ has z zeros and p poles, $k = p - z \geq 0$, k even. First, notice that we can always write $S_{n_F}(\Omega)$ in the form of

$$S_{n_F}(\Omega) \propto \frac{A(\Omega)_+ A(\Omega)_-}{B(\Omega)_+ B(\Omega)_-} = \frac{|A(\Omega)_+|^2}{|B(\Omega)_+|^2} \quad (\text{D14})$$

since $S_{n_F}(\Omega)$ is real. We set the coefficient multiplying Ω^2 to the largest power in $B(\Omega)_+ B(\Omega)_-$ to unity, since any coefficient can be absorbed into $A(\Omega)_\pm$. Furthermore, we define $D(\Omega)_+ = \Omega^2 + i\gamma_m \Omega - \omega_m^2 = -1/\chi(\Omega)$, $D(\Omega)_- = D(\Omega)_+^\dagger$, which contain the poles of the mechanical susceptibility. Combining the new definitions, we write

$$M_R(\Omega) = \epsilon + \frac{1}{D(\Omega)_+ D(\Omega)_-} \frac{A(\Omega)_+ A(\Omega)_-}{B(\Omega)_+ B(\Omega)_-} \approx \frac{\epsilon \Omega^{4+p} + A(\Omega)_+ A(\Omega)_-}{D(\Omega)_+ D(\Omega)_- B(\Omega)_+ B(\Omega)_-}, \quad (\text{D15})$$

where the approximation holds in the limit of $\epsilon \ll 1$. Now, in order to find the roots of the numerator, notice that in the limit $\epsilon \rightarrow 0$, the numerator goes to $A(\Omega)_+ A(\Omega)_-$. Then, we can write it as a product of $A(\Omega)_+ A(\Omega)_-$ and the rest of the roots, of which there are $4 + k$ of them. To find the remaining roots, notice that they will be solutions to

$$\epsilon \Omega^{4+p} + L \Omega^z = 0, \quad (\text{D16})$$

where L is a real number since the roots will be at large Ω in the limit of $\epsilon \rightarrow 0$, and only the term $L \Omega^z$ in $A(\Omega)_+ A(\Omega)_-$ will be of importance. The roots of Eq. (D16) are $\Omega_n = L \epsilon^{-1/(4+k)} \exp[(i2\pi n + i\pi)/(4+k)]$, $n = 0, 1, \dots, 3 + k$. Then,

$M_R(\Omega)$

$$\approx \frac{A(\Omega)_+ A(\Omega)_- \prod_{n=0}^{3+k} (\epsilon^{1/(4+k)} \Omega - L e^{i\pi/(4+k) + i2\pi n/(4+k)})}{D(\Omega)_+ D(\Omega)_- B(\Omega)_+ B(\Omega)_-},$$

$M_R(\Omega)_\pm$

$$\approx \frac{A(\Omega)_\pm \prod_{n=0}^{1+k/2} (\epsilon^{1/(4+k)} \Omega \pm L e^{i\pi/(4+k) + i2\pi n/(4+k)})}{D(\Omega)_\pm B(\Omega)_\pm}. \quad (\text{D17})$$

While using the Wiener-Hopf method, one needs to calculate $1/M_R(\Omega)_+$. We see that thanks to the terms with ϵ , the number of poles and zeros of $1/M_R(\Omega)_+$, in the limit of $\epsilon \rightarrow 0$, are equal. Therefore, using Eq. (C13), the spectrum of $g(\Omega)_+$ is guaranteed to be bounded for bounded $h(\Omega)$.

We now return to the evaluation of the first term in Eq. (D5). In taking the dot products of M_R^{-1} with V_{11}^{bv} and V_{21}^{bv} one encounters expressions such as

$$V_{11}^{bv} \cdot M_R^{-1} \cdot V_{11}^{bv} = \int \frac{d\Omega}{2\pi} \frac{V_{11}^{bv}(\Omega)^\dagger}{M_R(\Omega)_+} \left[\frac{V_{11}^{bv}(\Omega)}{M_R(\Omega)_-} \right]_+. \quad (\text{D18})$$

Since the number of poles and zeros of $M_R(\Omega)_+$ in the limit of $\epsilon \rightarrow 0$ are equal from the discussion, and $V_{11}^{bv}(\Omega)$ has two

poles, we can write this as the dot product of two bounded, causal functions of time. Or, in the frequency domain, as

$$\int \frac{d\Omega}{2\pi} \left[\frac{V_{11}^{bv}(\Omega)^\dagger}{M_R(\Omega)_+} \right]_- \left[\frac{V_{11}^{bv}(\Omega)}{M_R(\Omega)_-} \right]_+. \quad (\text{D19})$$

Then, we only need the residues of $1/M_R(\Omega)_-$ at the poles of the mechanical susceptibility to compute the dot products in the limit of $\epsilon \rightarrow 0$. In the end, the first term in Eq. (D5) is

$$\begin{bmatrix} V_{11}^{bv} \cdot M_R^{-1} \cdot (V_{11}^{bv})^T & -V_{11}^{bv} \cdot M_R^{-1} \cdot (V_{21}^{bv})^T \\ -V_{21}^{bv} \cdot M_R^{-1} \cdot (V_{11}^{bv})^T & V_{21}^{bv} \cdot M_R^{-1} \cdot (V_{21}^{bv})^T \end{bmatrix}. \quad (\text{D20})$$

Now that we have finished discussing the terms with M_R^{-1} , we want to calculate $\hat{\mathbf{y}}_{\text{pt}}^{bv} \cdot \mathbf{V}_M \cdot \hat{\mathbf{y}}_{\text{pt}}^{vb}$ [Eq. (D5)] again, substituting M^{-1} with $M_R^{-1} \cdot M_I \cdot M_R^{-1}$ [cf. Eq. (D3)]. First, note that after this substitution, the first term in Eq. (D5) will be proportional to $1/\alpha_F^2 \ll 1$, therefore we can ignore this term. From Eqs. (D8) and (D10), the second term will be given by

$$\frac{1}{\Omega_q^2} \begin{bmatrix} \omega_m M_I(0) & \dot{M}_I(0) \\ -\dot{M}_I(0) & \frac{-1}{\omega_m} \ddot{M}_I(0) \end{bmatrix} = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \quad (\text{D21})$$

since $M_I(0) = \dot{M}_I(0) = 0$. In order to calculate the third term in Eq. (D5), we notice that

$$\begin{aligned} \tilde{V}_{12}^{bv} \cdot M_R^{-1} \cdot M_I &= \delta(t) \cdot M_I \frac{\sqrt{\omega_m}}{\Omega_q} = M_I(t) \frac{\sqrt{\omega_m}}{\Omega_q}, \\ \tilde{V}_{22}^{bv} \cdot M_R^{-1} \cdot M_I &= \dot{\delta}(t) \cdot M_I \frac{1}{\sqrt{\omega_m} \Omega_q} = \frac{\dot{M}_I(t)}{\sqrt{\omega_m} \Omega_q} \end{aligned} \quad (\text{D22})$$

for $t > 0$. Then, the third term in Eq. (D5) will involve inner products with $M_I(t)$ and $\dot{M}_I(t)$. Since these products will have support only on the half real line (i.e., the integration limits will be from 0 to $+\infty$), we can replace $M_I(t)$ and $\dot{M}_I(t)$ with their causal counterparts. Notice that

$$M_I(t) \frac{\sqrt{\omega_m}}{\Omega_q} = -i \Omega_q \sqrt{\omega_m} \chi(t) = -i V_{11}^{bv}(t) \quad (\text{D23})$$

for $t > 0$. Similarly, $\dot{M}_I(t)/\sqrt{\omega_m} \Omega_q = -i V_{21}^{bv}(t)$ for $t > 0$. Therefore, the third term in Eq. (D5) can be written as

$$2 \begin{bmatrix} V_{11}^{bv} \cdot M_R^{-1} \cdot (V_{11}^{bv})^T & -V_{11}^{bv} \cdot M_R^{-1} \cdot (V_{21}^{bv})^T \\ -V_{21}^{bv} \cdot M_R^{-1} \cdot (V_{11}^{bv})^T & V_{21}^{bv} \cdot M_R^{-1} \cdot (V_{21}^{bv})^T \end{bmatrix}. \quad (\text{D24})$$

Therefore, summing all of the matrices above, the condition to be entangled in the absence of sensing noise is given by

$$\det \left\{ \begin{bmatrix} V_{11}^{bv} \cdot M_R^{-1} \cdot (V_{11}^{bv})^T & -V_{11}^{bv} \cdot M_R^{-1} \cdot (V_{21}^{bv})^T \\ -V_{21}^{bv} \cdot M_R^{-1} \cdot (V_{11}^{bv})^T & V_{21}^{bv} \cdot M_R^{-1} \cdot (V_{21}^{bv})^T \end{bmatrix} + \begin{bmatrix} 0 & 2i \\ -2i & 0 \end{bmatrix} \right\} < 0 \quad (\text{D25})$$

using the Wiener-Hopf method and the expressions for V_{11}^{bv} , V_{21}^{bv} , and M_R , where we calculate the following elegant

expression:

$$\begin{aligned} & \left[V_{11}^{bv} \cdot M_R^{-1} \cdot (V_{11}^{bv})^T \right] \left[V_{21}^{bv} \cdot M_R^{-1} \cdot (V_{12}^{bv})^T \right] \\ & - \left[V_{11}^{bv} \cdot M_R^{-1} \cdot (V_{21}^{bv})^T \right]^2 = \frac{4\gamma_m^2}{|S_{n_F-}(\Omega_*)|^4}, \end{aligned} \quad (D26)$$

where $\Omega_* = -i\gamma_m/2 \pm \sqrt{\omega_m^2 - \gamma_m^2/4}$ is one of the poles of the mechanical susceptibility. From Eqs. (D25) and (D26), the condition for optomechanical entanglement reduces to

$$|S_{n_F-}(\Omega_*)|^2 > \gamma_m \iff \text{entanglement.} \quad (D27)$$

There exists a lower limit to $|S_{n_F-}(\Omega_*)|$ from the fluctuation-dissipation theorem [51,52], since $S_{n_F}(\Omega)$ includes the thermal fluctuations that give rise to velocity damping on the mechanical oscillator, with a damping rate γ_m defined in the main text. Then, assuming that there is no additional force noise in the system, $S_{n_F}(\Omega)$ is connected to the mechanical susceptibility with

$$S_{n_F}(\Omega) |\chi(\Omega)|^2 = \frac{2\gamma_m \Omega}{\omega_m} |\chi(\Omega)|^2 \coth\left(\frac{\hbar \Omega}{2k_B T}\right), \quad (D28)$$

where T is the temperature of the bath, and k_B is the Boltzmann constant. Eliminating the mechanical susceptibility from each side, and writing S_{n_F} as $S_{n_F}(\Omega) = S_{n_F}(\Omega) + S_{n_F}(\Omega_-)$, the minimum value of $|S_{n_F-}(\Omega_*)|^2$ is given by

$$|S_{n_F-}(\Omega_*)|^2 \geq 2\gamma_m. \quad (D29)$$

We therefore see that all physical force noise spectra ensure optomechanical entanglement, from Eq. (D27).

APPENDIX E: UNIQUENESS OF THE ENTANGLING-DISENTANGLING TRANSITION

Here, we assume a force and sensing noise spectrum $\alpha_F S_{n_F}(\Omega)$, $\beta_S S_{n_S}(\Omega)$, respectively, where α_F and β_S are real, positive constants. We want to show that given α_F , the value of β_S for which the entangling-disentangling transition occurs is unique. In other words, if the system is not entangled for a pair (α_F, β_S) , entanglement cannot be achieved by increasing β_S . Note that when we increase β_S , we introduce more sensing noise to the system; however, we do not change the functional form of the spectrum [i.e., $S_{n_S}(\Omega)$].

We write the covariance matrix as $\mathbf{V}$, and add some additional sensing noise to the system, modifying the covariance matrix in the form of $\mathbf{V} \rightarrow \mathbf{V} + r\mathbf{V}'$, where $r \in \mathbb{R}$, and $\mathbf{V}'$ is positive-semidefinite as it is a covariance matrix. $\mathbf{V}'$ will be nonzero only in the sector of the covariance matrix containing the variance of $\hat{v}_2$ due to the structure of the equations of motion, given in Eq. (1). This modification corresponds to modifying β_S , such that $\beta_S \rightarrow \beta_S + r$.

We recall from Sec. II B that optomechanical entanglement exists when $\mathbf{V}_{\text{pt}} + i\mathbf{K} < 0$, where $\mathbf{V}_{\text{pt}}$ is the covariance matrix of the system. If the system is separable for a given β_S , it will remain separable for $\beta_S + r$, $r > 0$. This is because the entanglement criterion becomes $\mathbf{V}_{\text{pt}} + r\mathbf{V}' + i\mathbf{K} < 0$, where $\mathbf{V}_{\text{pt}} + i\mathbf{K}$ is positive definite, and $\mathbf{V}'$ is positive semidefinite (since it is a covariance matrix). Therefore, the criterion is not satisfied, and optomechanical entanglement does not exist. Conversely, if optomechanical entanglement exists for β_S ,

it will also exist for $\beta_S + r$, $r > 0$, since the criterion gets modified as $\mathbf{V}_{\text{pt}} + i\mathbf{K} - r\mathbf{V}' < 0$, where $\mathbf{V}_{\text{pt}} + i\mathbf{K}$ is negative definite, and $-\mathbf{V}'$ is negative semidefinite. Then, the criterion will be satisfied for $\beta_S + r$.

Therefore, if we introduce some sensing noise to an initially entangled system, there exists a critical amount of additional sensing noise for which an entangling-disentangling transition occurs, and the system becomes separable. Beyond this point, the system cannot get entangled again as we continue introducing more sensing noise. Conversely, before the entangling-disentangling transition (when the system is entangled), we cannot induce separability by removing sensing noise from the system. These statements prove our initial claim that the entangling-disentangling transition is unique with respect to the amount of sensing noise present.

APPENDIX F: ANOTHER APPROACH TO PROVING THE UNIVERSALITY OF ENTANGLEMENT

Let us provide another approach to proving the universality of entanglement when the input light field is the vacuum state, as was shown in Sec. III. The entanglement criterion that we use in the main text is

$$\det(\mathbf{V}_{\text{pt}} + i\mathbf{K}) < 0 \iff \text{entanglement.} \quad (F1)$$

We can apply a transformation $\mathbf{T}$ to this expression that is not necessarily symplectic, such that the entanglement criterion is modified as

$$\det(\mathbf{T}\mathbf{V}_{\text{pt}}\mathbf{T}^\dagger + i\mathbf{T}\mathbf{K}\mathbf{T}^\dagger) < 0 \iff \text{entanglement.} \quad (F2)$$

since $\mathbf{T}$ is not symplectic, we cannot use the transformed expression to quantify the amount of optomechanical entanglement; however, we can still inquire the existence of entanglement. We choose $\mathbf{T}$ to be

$$\begin{aligned} \mathbf{T} &= \begin{bmatrix} \mathbf{T}^{bb} & \mathbf{T}^{bv} \\ \mathbf{T}^{vb} & \mathbf{T}^{vv} \end{bmatrix}, \\ \mathbf{T}^{bb} &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \\ \mathbf{T}^{bv} &= \sqrt{\omega_m \Omega_q} \begin{bmatrix} -\chi(t) & 0 \\ \frac{1}{\omega_m} \dot{\chi}(t) & 0 \end{bmatrix}, \\ \mathbf{T}^{vb} &= \mathbf{0}, \\ \mathbf{T}^{vv} &= \begin{bmatrix} \delta(\tau) & 0 \\ -\Omega_q^2 \chi(\tau) & \delta(\tau) \end{bmatrix}. \end{aligned} \quad (F3)$$

Then, $\mathbf{T}\mathbf{V}_{\text{pt}}\mathbf{T}^\dagger$ is calculated as

$$\begin{bmatrix} \omega_m \chi_F(0) & 0 & 0 & \sqrt{\omega_m \Omega_q} \chi_F(t') \\ 0 & \frac{-1}{\omega_m} \ddot{\chi}_F(0) & 0 & -\frac{\Omega_q}{\sqrt{\omega_m}} \dot{\chi}_F(t') \\ 0 & 0 & \delta(\tau) & 0 \\ \sqrt{\omega_m \Omega_q} \chi_F(t) & -\frac{\Omega_q}{\sqrt{\omega_m}} \dot{\chi}_F(t) & 0 & \delta(\tau) + \Omega_q^2 \chi_{\text{EN}}(\tau) \end{bmatrix}, \quad (F4)$$

where $t, t' > 0$ are the column and the row indices, respectively, and $\tau = t - t'$. This matrix can be viewed as the partially transposed covariance matrix of the quadratures

$\hat{c}_i, \hat{\omega}_i(t)$, $i = 1, 2$, $t < 0$. These obey the following equations of motion:

$$\hat{\omega}_1(t) = \hat{u}_1(t), \quad (\text{F5a})$$

$$\hat{\omega}_2(t) = \hat{u}_2(t) + \frac{\Omega_q}{\sqrt{\omega_m}}[\hat{c}_1(t) + \hat{n}_S(t)], \quad (\text{F5b})$$

$$\dot{\hat{c}}_2(t) = -\gamma_m \hat{c}_2(t) - \omega_m \hat{c}_1(t) + \hat{n}_F(t), \quad (\text{F5c})$$

$$\dot{\hat{c}}_1(t) = \omega_m \hat{c}_2(t). \quad (\text{F5d})$$

Comparing to the original equations of motion, Eq. (1), we notice that here, the mechanical mode $\hat{c}_1, \hat{c}_2$ is driven only by the force noise, $\hat{n}_F(t)$.

We compute $\mathbf{TKT}^\dagger$ as

$$\begin{bmatrix} 0 & 1 & 0 & -\sqrt{\omega_m}\Omega_q\chi(t') \\ -1 & 0 & 0 & \frac{\Omega_q}{\sqrt{\omega_m}}\dot{\chi}(t') \\ 0 & 0 & 0 & \delta(\tau) \\ \sqrt{\omega_m}\Omega_q\chi(t) & -\frac{\Omega_q}{\sqrt{\omega_m}}\dot{\chi}(t) & -\delta(\tau) & \Omega_q^2[\chi(-\tau) - \chi(\tau)] \end{bmatrix}, \quad (\text{F6})$$

where, again, $t, t' > 0$, and $\tau = t - t'$. Therefore, we can write $\mathbf{TV}_{\text{pt}}\mathbf{T}^\dagger + i\mathbf{TKT}^\dagger$ as (using the same row and column indices t and t' , as well as $\tau = t - t'$)

$$\begin{bmatrix} \langle \hat{c}_1^2(0) \rangle & i & 0 & \langle \hat{c}_1(0)\hat{\omega}_2(t') \rangle_s - i\sqrt{\omega_m}\Omega_q\chi(t') \\ -i & \langle \hat{c}_2^2(0) \rangle & 0 & -\langle \hat{c}_2(0)\hat{\omega}_2(t') \rangle_s + i\frac{\Omega_q}{\sqrt{\omega_m}}\dot{\chi}(t') \\ 0 & 0 & \delta(\tau) & i\delta(\tau) \\ \langle \hat{c}_1(0)\hat{\omega}_2(t) \rangle_s + i\sqrt{\omega_m}\Omega_q\chi(t) & -\langle \hat{c}_2(0)\hat{\omega}_2(t) \rangle_s - i\frac{\Omega_q}{\sqrt{\omega_m}}\dot{\chi}(t) & -i\delta(\tau) & \delta(\tau) + M(\tau) \end{bmatrix}, \quad (\text{F7})$$

where we define the symmetrized expectation value $\langle \hat{a}\hat{b} \rangle_s := \langle \hat{a}\hat{b} + \hat{b}\hat{a} \rangle / 2$, and $M(\tau) := \langle \hat{c}_1(t)\hat{c}_1(t') \rangle \Omega_q^2 / \omega_m + \langle \hat{n}_S(t)\hat{n}_S(t') \rangle$, which is also equivalent to the expression that we had for $M(\tau)$ in the main text, given in Eq. (16c).

We can simplify this expression by relating some of its terms to the commutation relations between the quadratures $\hat{c}_j(0)$ and $\hat{\omega}_j(t)$, $j = 1, 2$. First, we realize that

$$[\hat{b}_j(0), \hat{v}_2(t)] = 0, \quad j = 1, 2, \quad t < 0 \quad (\text{F8})$$

due to causality. Furthermore, we can write

$$\begin{aligned} \hat{b}_1(t) &= \hat{c}_1(t) + \sqrt{\omega_m}\Omega_q \int_{-\infty}^t \chi(-t')\hat{u}_1(t')dt', \\ \hat{b}_2(t) &= \hat{c}_2(t) + \frac{\Omega_q}{\sqrt{\omega_m}} \int_{-\infty}^t \dot{\chi}(-t')\hat{u}_1(t')dt'. \end{aligned} \quad (\text{F9})$$

Therefore, using Eqs. (F8) and (F9), as well as the original equations of motion in Eq. (1), we find

$$\begin{aligned} [\hat{b}_j(0), \hat{v}_2(t)] &= \left[\hat{c}_j(0) + \Omega_q\omega_m^{(3/2)-j} \int_{-\infty}^{\infty} \chi(-t')\hat{u}_1(t')dt', \right. \\ &\quad \left. \hat{\omega}_2(t) + \Omega_q^2 \int_{-\infty}^t \chi(-t')\hat{u}_1(t')dt' \right]. \end{aligned} \quad (\text{F10a})$$

Since $\hat{c}_j$ does not contain any vacuum fluctuations, we can simplify this expression to obtain for $[\hat{b}_j(0), \hat{v}_2(t)]$

$$\begin{aligned} &[\hat{c}_j(0), \hat{\omega}_2(t)] + \Omega_q\omega_m^{(3/2)-j} \int_{-\infty}^{\infty} dt' \chi(-t')[\hat{u}_1(t'), \hat{\omega}_2(t)] \\ &= [\hat{c}_j(0), \hat{\omega}_2(t)] + \Omega_q\omega_m^{(3/2)-j} \int_{-\infty}^{\infty} dt' \chi(-t')2i\delta(t-t') \\ &= [\hat{c}_j(0), \hat{\omega}_2(t)] + 2i\Omega_q\omega_m^{(3/2)-j}\chi(-t) = 0. \end{aligned} \quad (\text{F11})$$

Therefore, we have

$$[\hat{c}_1(0), \hat{\omega}_2(t)] = -2i\sqrt{\omega_m}\Omega_q\chi(-t), \quad t < 0, \quad (\text{F12a})$$

$$[\hat{c}_2(0), \hat{\omega}_2(t)] = -2i\frac{\Omega_q}{\sqrt{\omega_m}}\dot{\chi}(-t), \quad t < 0. \quad (\text{F12b})$$

We can then reexpress the symmetrized expectation values in the form of

$$\begin{aligned} \langle \hat{c}_1(0)\hat{\omega}_2(t) \rangle_s &= \frac{\langle \hat{c}_1(0)\hat{\omega}_2(t) + \hat{\omega}_2(t)\hat{c}_1(0) \rangle}{2} \\ &= \langle \hat{c}_1(0)\hat{\omega}_2(t) \rangle + i\sqrt{\omega_m}\Omega_q\chi(-t). \end{aligned} \quad (\text{F13})$$

We can write a similar expression for $\langle \hat{c}_2(0)\hat{\omega}_2(t) \rangle_s$. Plugging these equations into Eq. (F7), we obtain

$$\begin{bmatrix} \langle \hat{c}_1^2(0) \rangle & i & 0 & \langle \hat{c}_1(0)\hat{\omega}_2(t') \rangle \\ -i & \langle \hat{c}_2^2(0) \rangle & 0 & -\langle \hat{c}_2(0)\hat{\omega}_2(t') \rangle \\ 0 & 0 & \delta(\tau) & i\delta(\tau) \\ \langle \hat{\omega}_2(t)\hat{c}_1(0) \rangle & -\langle \hat{\omega}_2(t)\hat{c}_2(0) \rangle & -i\delta(\tau) & \delta(\tau) + M(\tau) \end{bmatrix}. \quad (\text{F14})$$

To inquire about optomechanical entanglement, we need to assess whether the smallest eigenvalue of this matrix is negative. Using the Schur complement of the top left block, we can rewrite the entanglement criterion as

$$\lambda_{1,1}\lambda_{2,2} - \lambda_{1,2}^2 < 1 \iff \text{entanglement}, \quad (\text{F15})$$

where

$$\lambda_{i,j} = \langle \hat{c}_i(0)\hat{c}_j(0) \rangle - \int dt dt' \langle \hat{c}_i(0)\hat{\omega}_2(t') \rangle M^{-1}(\tau) \langle \hat{\omega}_2(t)\hat{c}_j(0) \rangle \quad (\text{F16})$$

with $\tau = t - t'$, $i, j = 1, 2$. The physical picture here contains a mechanical oscillator driven solely by a force noise $\hat{n}_F$, sensed by a field that contains a sensing noise $\hat{n}_S$. Then,

optomechanical entanglement is observed only if the conditional quantum state of the mechanical oscillator, given its initial state and the output light field during $t < 0$, violates the Heisenberg uncertainty principle. Accordingly, Eq. (F15) contains the second-order moments of the conditional state, and tests whether the uncertainty principle is held. Mathematically, this statement is exactly equivalent to Eq. (27) in the main text, and the interaction strength Ω_q cancels out of Eq. (F15), as we had observed, giving rise to the universality of optomechanical entanglement.

APPENDIX G: UNIVERSAL ENTANGLEMENT IN THE PRESENCE OF FREQUENCY-INDEPENDENT SQUEEZING

We consider two classes of symplectic transformations on the input light field: (i) phase shifts and (ii) squeezing the amplitude and phase quadratures with a constant squeezing factor. We show that both of these transformations result in the same determinant condition for optomechanical entanglement computed for the case when the input light field consists of vacuum fluctuations, given in Eq. (27). First, let us consider phase shift operations. Such a transformation can be represented by

$$\begin{bmatrix} \hat{u}_1(\Omega) \\ \hat{u}_2(\Omega) \end{bmatrix} \rightarrow e^{i\phi(\Omega)} \begin{bmatrix} \cos \theta(\Omega) & -\sin \theta(\Omega) \\ \sin \theta(\Omega) & \cos \theta(\Omega) \end{bmatrix} \begin{bmatrix} \hat{u}_1(\Omega) \\ \hat{u}_2(\Omega) \end{bmatrix}, \quad (\text{G1})$$

where $\theta(\Omega)$ is a real function of Ω , and the phase factor $e^{i\phi(\Omega)}$ ensures the causality of the transformation. We compute the light-field sector of the covariance matrix in the frequency domain as

$$\begin{aligned} V_{11}^{vv}(\Omega) &= \cos^2 \theta(\Omega) + \sin^2 \theta(\Omega) = 1, \\ V_{12}^{vv}(\Omega) &= \cos^2 \theta(\Omega) \Omega_q^2 \chi(-\Omega) - \cos \theta(\Omega) \sin \theta(\Omega) \\ &\quad + \cos \theta(\Omega) \sin \theta(\Omega) + \sin^2 \theta(\Omega) \Omega_q^2 \chi(-\Omega) \\ &= \Omega_q^2 \chi(-\Omega), \\ V_{22}^{vv}(\Omega) &= |\cos [\theta(\Omega)] \Omega_q^2 \chi(\Omega) - \sin [\theta(\Omega)]|^2 \\ &\quad + |\sin [\theta(\Omega)] \Omega_q^2 \chi(\Omega) + \cos [\theta(\Omega)]|^2 + \Omega_q^2 \chi_{\text{EN}}(\Omega) \\ &= 1 + \Omega_q^2 \chi_{\text{EN}}(\Omega) + \Omega_q^4 |\chi(\Omega)|^2. \end{aligned} \quad (\text{G2})$$

Then, we see that the light-field sector of the covariance matrix is unchanged with respect to the case where the input light field is the vacuum state. Similarly, we find that all of the elements of the covariance matrix are unchanged. Hence, the condition for the force and the sensing noise spectra for which the entangling-disentangling transition occurs is unaltered.

Now, we consider squeezing the quadratures of the light field with a constant squeezing factor. Such a transformation looks like the following:

$$\begin{bmatrix} \hat{u}_1(\Omega) \\ \hat{u}_2(\Omega) \end{bmatrix} \rightarrow \begin{bmatrix} e^r & 0 \\ 0 & e^{-r} \end{bmatrix} \begin{bmatrix} \hat{u}_1(\Omega) \\ \hat{u}_2(\Omega) \end{bmatrix}, \quad (\text{G3})$$

where r is a real number. In order to utilize the formalism of the main text, we can perform a symplectic transformation that simply rescales $\hat{v}_1(t)$ and $\hat{v}_2(t)$ as $\hat{v}_1(t) \rightarrow e^{-r} \hat{v}_1(t)$, $\hat{v}_2(t) \rightarrow e^r \hat{v}_2(t)$. Then, the light-field sector of the covariance matrix is computed after the symplectic transformation as

$$\begin{aligned} V_{11}^{vv}(\Omega) &= 1, \quad V_{12}^{vv}(\Omega) = e^{2r} \Omega_q^2 \chi(-\Omega), \\ V_{22}^{vv}(\Omega) &= 1 + e^{2r} \Omega_q^2 \chi_{\text{EN}}(\Omega) + e^{4r} \Omega_q^4 |\chi(\Omega)|^2. \end{aligned} \quad (\text{G4})$$

The covariances of the quadratures of the mechanical oscillator are

$$\begin{aligned} V_{11}^{bb}(\Omega) &= \omega_m \chi_F(\Omega) + \omega_m \Omega_q^2 |\chi(\Omega)|^2 e^r, \\ V_{22}^{bb}(\Omega) &= \frac{\Omega^2}{\omega_m^2} V_{11}^{bb}(\Omega), \end{aligned} \quad (\text{G5})$$

and $V_{12}^{bb}(0) = V_{21}^{bb}(0) = 0$ due to symmetry. Lastly, the correlations between the quadratures of the mechanical oscillator and the light field are computed as

$$\begin{aligned} V_{11}^{bv}(\Omega) &= \sqrt{\omega_m} e^r \Omega_q \chi(\Omega), \\ V_{12}^{bv}(\Omega) &= \sqrt{\omega_m} e^r \Omega_q \chi_F(\Omega) + \sqrt{\omega_m} e^{3r} \Omega_q^3 |\chi(\Omega)|^2, \end{aligned} \quad (\text{G6})$$

and $V_{21}^{bv}(\Omega) = -\frac{i\Omega}{\omega_m} V_{11}^{bv}(\Omega)$, $V_{22}^{bv}(\Omega) = -\frac{i\Omega}{\omega_m} V_{12}^{bv}(\Omega)$. From Eqs. (G4)–(G6), we notice that the effect of the frequency-independent squeezing is such that the interaction strength Ω_q is rescaled to $e^r \Omega_q$. Since we have proved the universality of the entangling-disentangling transition with respect to the interaction strength (see Sec. III), rescaling it will not affect the transition. Therefore, we conclude that the condition for the force and the sensing noise spectra for which the entangling-disentangling transition occurs is unaltered if the input light field is squeezed by a constant squeezing factor.

APPENDIX H: GENERAL SYMPLECTIC TRANSFORMATIONS ON THE LIGHT FIELD

We want to write down an expression for a general, causal symplectic transformation applied to the light field. Such a transformation can be written down as a 2-by-2 matrix of operators $\mathbf{S}$, with

$$\begin{bmatrix} \hat{u}_1(t) \\ \hat{u}_2(t) \end{bmatrix} \rightarrow \mathbf{S} \begin{bmatrix} \hat{u}_1(t) \\ \hat{u}_2(t) \end{bmatrix}, \quad \mathbf{S} = \begin{bmatrix} S_{11}(\tau) & S_{12}(\tau) \\ S_{21}(\tau) & S_{22}(\tau) \end{bmatrix}, \quad (\text{H1})$$

where $S_{ij}(t)$ are causal functions of time, $i, j = 0, 1$. In order to be a valid symplectic transformation, $\mathbf{S}$ must satisfy [58]

$$\mathbf{S}^T \mathbf{K} \mathbf{S} = \mathbf{K}, \quad \mathbf{K} = \begin{bmatrix} 0 & \delta(\tau) \\ -\delta(\tau) & 0 \end{bmatrix}, \quad (\text{H2})$$

where $\mathbf{K}$ is the commutator matrix. Writing $\mathbf{S}^T \mathbf{K} \mathbf{S} = \mathbf{K}$ explicitly,

$$\int_0^\infty dt' \begin{bmatrix} S_{11}(\tau_1) S_{21}(\tau_2) - S_{21}(\tau_1) S_{11}(\tau_2) & S_{11}(\tau_1) S_{22}(\tau_2) - S_{21}(\tau_1) S_{12}(\tau_2) \\ S_{12}(\tau_1) S_{21}(\tau_2) - S_{22}(\tau_1) S_{11}(\tau_2) & S_{12}(\tau_1) S_{22}(\tau_2) - S_{22}(\tau_1) S_{12}(\tau_2) \end{bmatrix} = \begin{bmatrix} 0 & \delta(t - t'') \\ -\delta(t - t'') & 0 \end{bmatrix}, \quad (\text{H3})$$

with $\tau_1 := t' - t$, $\tau_2 := t'' - t'$. We can extend the lower limit of this integral to $-\infty$ due to the causality of $S_{ij}(t)$, $i, j = 1, 2$. Thus, we can express these constraints in the frequency domain, by defining $S_{ij}(\Omega) = \int_{-\infty}^{\infty} d\tau S_{ij}(\tau) e^{i\Omega\tau}$, $i, j = 1, 2$. We have

$$S_{11}^\dagger(\Omega)S_{21}(\Omega), S_{12}^\dagger(\Omega)S_{22}(\Omega) \in \mathbb{R} \quad \forall \Omega \in \mathbb{R}, \quad (\text{H4a})$$

$$S_{11}^\dagger(\Omega)S_{22}(\Omega) - S_{12}^\dagger(\Omega)S_{21}(\Omega) = 1 \quad \forall \Omega \in \mathbb{R}. \quad (\text{H4b})$$

We realize that we can write down a symplectic transformation satisfying these constraints in the following manner:

$$S(\Omega) = e^{i\phi(\Omega)} \begin{bmatrix} A(\Omega) & B(\Omega) \\ C(\Omega) & D(\Omega) \end{bmatrix}, \quad A(\Omega), B(\Omega), C(\Omega), D(\Omega), \phi(\Omega) \in \mathbb{R} \quad \forall \Omega, \quad A(\Omega)D(\Omega) - B(\Omega)C(\Omega) = 1 \quad \forall \Omega, \quad (\text{H5})$$

where the “phase” $\phi(\Omega)$ ensures causality by canceling out the anticausal poles of $A(\Omega)$, $B(\Omega)$, $C(\Omega)$ and $D(\Omega)$. Furthermore, we can replace $A(\Omega)$, $B(\Omega)$, $C(\Omega)$, and $D(\Omega)$ with other real functions of Ω such that

$$\begin{bmatrix} A(\Omega) & B(\Omega) \\ C(\Omega) & D(\Omega) \end{bmatrix} = \begin{bmatrix} \cos \theta(\Omega) \cosh r(\Omega) + \cos \rho(\Omega) \sinh r(\Omega) & -\sin \theta(\Omega) \cosh r(\Omega) + \sin \rho(\Omega) \sinh r(\Omega) \\ \sin \theta(\Omega) \cosh r(\Omega) + \sin \rho(\Omega) \sinh r(\Omega) & \cos \theta(\Omega) \cosh r(\Omega) - \cos \rho(\Omega) \sinh r(\Omega) \end{bmatrix}, \quad (\text{H6})$$

where $r(\Omega), \theta(\Omega), \rho(\Omega) \in \mathbb{R} \quad \forall \Omega$. It is straightforward to check that this form satisfies the constraints in Eq. (H4).

APPENDIX I: REALIZATION OF FREQUENCY-DEPENDENT SQUEEZING

We aim to find the parameters of a filter that squeezes the input light field such that the spectrum of the quantum noise in the output is squeezed by the same amount at every frequency, i.e., the shot noise and the quantum radiation pressure noise (QRPN) are squeezed simultaneously [59,60]. First, we write down the input-output relations for the light field only,

$$\begin{bmatrix} \hat{v}_1(\Omega) \\ \hat{v}_2(\Omega) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \Omega_q^2 \chi(\Omega) & 1 \end{bmatrix} \begin{bmatrix} \hat{u}_1(\Omega) \\ \hat{u}_2(\Omega) \end{bmatrix}. \quad (\text{I1})$$

Assuming that the input light field is in the vacuum state, the spectrum of the quantum noise $S(\Omega)$ in the phase quadrature $\hat{v}_2$ is given by

$$S(\Omega) = 1 + \Omega_q^4 |\chi(\Omega)|^2. \quad (\text{I2})$$

We observe that the first term contributes as a shot noise, whereas the second term is the QRPN since it is transduced by the mechanical oscillator. In order to realize frequency-dependent squeezing, we first squeeze the input light field in a frequency-independent manner (e.g., with propagation through a bulk crystal), and filter it through a detuned Fabry-Perot cavity. Such an operation will have the following transfer function in the frequency domain:

$$\frac{1}{\tilde{C}} \begin{bmatrix} \Omega^2 + \gamma^2 - \Delta^2 & -2\gamma\Delta \\ 2\gamma\Delta & \Omega^2 + \gamma^2 - \Delta^2 \end{bmatrix} \cdot \begin{bmatrix} e^r & 0 \\ 0 & e^{-r} \end{bmatrix}, \quad (\text{I3})$$

where $\tilde{C} = \Delta^2 - (\Omega + i\gamma)^2$. γ and Δ are the cavity linewidth and detuning, respectively, and r is the squeezing factor, with $r, \gamma, \Delta \in \mathbb{R}$, $r, \gamma > 0$. Plugging this into Eq. (I1), assuming a high-Q oscillator (i.e., $\omega_m \gg \gamma_m$), $\hat{v}_2$ will be in the form of

$$\begin{aligned} \hat{v}_2 = & \frac{e^r}{\tilde{C}} \left(-2\gamma\Delta + \frac{\Omega_q^2(\gamma^2 + \Omega^2 - \Delta^2)}{\Omega^2 - \omega_m^2} \right) \hat{u}_1 \\ & + \frac{e^{-r}}{\tilde{C}} \left(\Delta^2 - \gamma^2 - \Omega^2 - \frac{\Omega_q^2 2\gamma\Delta}{\Omega^2 - \omega_m^2} \right) \hat{u}_2, \end{aligned} \quad (\text{I4})$$

where $\hat{u}_1, \hat{u}_2$ are the vacuum fluctuations. We notice that the coefficient multiplying the amplitude quadrature in Eq. (I4) can be set to zero with an appropriate choice of γ, Δ . More specifically, if we set

$$\gamma = \sqrt{\frac{-\omega_m^2 + \sqrt{\omega_m^4 + \Omega_q^4}}{2}}, \quad \Delta = \frac{\Omega_q^2}{2\gamma}, \quad (\text{I5})$$

we obtain that the spectrum of the quantum noise is modified as

$$S(\Omega) \rightarrow e^{-2r} S(\Omega). \quad (\text{I6})$$

Therefore, the spectrum is squeezed with a fixed factor of e^{-2r} .

APPENDIX J: PASSIVE LOSSES

Here, we consider passive (detection) losses, affecting the readout of the output light field. Such losses occur due to processes such as the finite quantum efficiency of the photodetector. They can be modeled as [61]

$$\hat{v}'_i(t) = \sqrt{\eta} \hat{v}_i(t) + \sqrt{1-\eta} \hat{e}_i(t), \quad i = 1, 2, \quad (\text{J1})$$

where $\hat{v}'_i(t)$ is the measured quadrature of the output light field, $\hat{e}_i(t)$ is an additional quantum noise that is assumed to be in the vacuum state, and $0 < \eta < 1$ is the efficiency of the photodetector. Then, $S_{e_i e_i} = 1$, $i = 1, 2$, and $S_{e_1 e_2} = 0$. We can rewrite the equations of motion in Eqs. (1) in terms of $\hat{v}'_{1,2}(t)$ as

$$\dot{\hat{v}}'_1(t) = \sqrt{\eta} \dot{\hat{u}}_1(t) + \sqrt{1-\eta} \dot{\hat{e}}_1(t), \quad (\text{J2a})$$

$$\dot{\hat{v}}'_2(t) = \sqrt{\eta} \left[\dot{\hat{u}}_2(t) + \frac{\Omega_q}{\sqrt{\omega_m}} [\hat{b}_1(t) + \hat{n}_s(t)] \right] + \sqrt{1-\eta} \dot{\hat{e}}_2(t), \quad (\text{J2b})$$

$$\dot{\hat{b}}_2(t) = -\gamma_m \hat{b}_2(t) - \omega_m \hat{b}_1(t) + \frac{\Omega_q}{\sqrt{\omega_m}} \dot{\hat{u}}_1(t) + \hat{n}_F(t), \quad (\text{J2c})$$

$$\dot{\hat{b}}_1(t) = \omega_m \hat{b}_2(t). \quad (\text{J2d})$$

If the ingoing light field is not frequency-dependently squeezed, such that $\hat{u}_{1,2}(t)$ denote vacuum fluctuations, we can combine the operators $\hat{u}_{1,2}(t)$ and $\hat{e}_{1,2}(t)$ in the form

of a white noise. In this way, Eqs. (J2) will resemble the initial equations of motion in the absence of passive losses [Eqs. (1)]. Defining $\hat{u}'_{1,2}(t) = \sqrt{\eta} \hat{u}_{1,2}(t) + \sqrt{1-\eta} \hat{e}_{1,2}(t)$, we observe that $\hat{u}'_{1,2}$ is a white noise with $S_{u'_i u'_i} = 1$, $i = 1, 2$, and $S_{u'_1 u'_2} = 0$, since $\hat{u}_{1,2}$ and $\hat{e}_{1,2}$ are independent noise sources. Making sure to preserve the dynamics of the oscillator in terms of the amount of driving noise, we will obtain

$$\hat{v}'_1(t) = \hat{u}'_1(t), \quad (\text{J3a})$$

$$\hat{v}'_2(t) = \hat{u}'_2(t) + \frac{\Omega_q \sqrt{\eta}}{\sqrt{\omega_m}} [\hat{b}_1(t) + \hat{n}_S(t)], \quad (\text{J3b})$$

$$\begin{aligned} \dot{\hat{b}}_2(t) = & -\gamma_m \hat{b}_2(t) - \omega_m \hat{b}_1(t) + \frac{\Omega_q \sqrt{\eta}}{\sqrt{\omega_m}} \hat{u}'_1(t) \\ & + \hat{n}_F(t) + \frac{\Omega_q \sqrt{1-\eta}}{\sqrt{\omega_m}} \hat{\xi}(t), \end{aligned} \quad (\text{J3c})$$

$$\dot{\hat{b}}_1(t) = \omega_m \hat{b}_2(t), \quad (\text{J3d})$$

where $\hat{\xi}$ is an additional noise in the form of vacuum fluctuations. We notice that Ω_q is scaled by $\sqrt{\eta}$, and there appears to be an additional white force noise on the mechanical oscillator with a constant spectrum of $\Omega_q^2(1-\eta)/\omega_m$, due to the rescaling of Ω_q by η . Note that the total force noise on the oscillator is, as expected, unchanged. Due to the dependency of this additional force noise to the coherent coupling Ω_q , the entangling-disentangling transition is no longer universal with respect to Ω_q .

Then, we encounter two distinct limits: if $S_{n_F}(\Omega) \gg \Omega_q^2(1-\eta)/\omega_m$, the entangling-disentangling transition is approximately unaffected. However, we expect the amount of entanglement in the system—if any exists—to reduce because of the rescaling of Ω_q by η . On the other hand, if $S_{n_F}(\Omega) \ll \Omega_q^2(1-\eta)/\omega_m$, it is more advantageous to have a small coupling Ω_q in order to achieve entanglement, as the additional force noise in the system is approximately proportional to Ω_q .

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Correction: The address in the byline footnote for the third author contained a typographical error and has been fixed.