

Analytical study of a finite-range impurity in a one-dimensional Bose gas

T. Alper Yoğurt^{*} and Matthew T. Eiles[†]

Max Planck Institute for the Physics of Complex Systems, Nöthnitzer Strasse 38, 01187 Dresden, Germany



(Received 15 August 2025; accepted 20 October 2025; published 7 November 2025)

One-dimensional Bose gases present an interesting setting to study the physics of Bose polarons, as density fluctuations play an enhanced role due to reduced dimensionality. Theoretical descriptions of this system have predominantly relied on contact pseudopotentials to model the impurity-bath interaction, leading to unphysical results in the strongly coupled limit. In this work, we analytically solve the Gross-Pitaevskii equation, using a square-well potential instead of a zero-range potential, for the ground-state wave function of a static impurity. We compute perturbative corrections arising from infinitesimally slow impurity motion. The polaron energy and effective mass remain finite in the strongly coupled regime, in contrast to the divergent behavior obtained using a contact potential. In this limit, we characterize the polaron properties in terms of the dimensionless ratio $\tilde{w} \equiv w/\xi$ between the interaction range w of the impurity-bath potential and the coherence length ξ of the Bose gas. The effective mass exhibits a $1/\tilde{w}$ scaling. The energy of the attractive polaron scales as $-1/\tilde{w}^3$, whereas the repulsive polaron features subleading corrections to the dark soliton energy at the order $\tilde{w}^3$.

DOI: [10.1103/gqd6-t881](https://doi.org/10.1103/gqd6-t881)

I. INTRODUCTION

The introduction of atomic impurities into a Bose gas has opened a new avenue for exploring Bose polarons, which were first realized experimentally in a quasi-one-dimensional (1D) system [1]. The ability to tune both the sign and the strength of the impurity-bath interaction has revealed new regimes that challenge conventional polaron theories [2–5]. Similar to solid-state systems [6–8], the leading theoretical framework describes the Bose polaron in terms of phonon excitations of the uniform bath and their interactions with both the impurity and with one another [9–16]. While this has proved successful in the weak- and intermediate-coupling regimes, it is inadequate at strong coupling due to the neglect of density modulations around the impurity [17–22]. These modulations become even more pronounced in lower dimensions [4,23,24], as evidenced by the fact of vanishing quasiparticle weight [13] and impurity self-trapping for any repulsive interaction in 1D at the thermodynamic limit [25–28].

To account for these density modulations near the impurity, the exact mean-field (MF) solutions to the Gross-Pitaevskii (GP) equation have been obtained [17,29] by treating the impurity-bath interaction as a contact pseudopotential with the coupling strength $g_{\text{IB}} \propto -1/a_{\text{IB}}$, where a_{IB} is the zero-energy scattering length. This framework has been applied

to study both the properties of a single Bose polaron [17] and the interactions between two impurities [30,31]. Remarkably, the analytical results have shown strong agreement—even at the MF level—with Monte Carlo simulations. However, in the strongly coupled limit $g_{\text{IB}} \rightarrow \pm\infty$, the MF results yielded diverging values of key polaron properties, including the energy of the attractive impurity and the effective mass for both attractive and repulsive cases. The Monte Carlo results displayed a similar unbounded growth as the number of particles included in the simulations increased [30,32]. This divergent behavior can be attributed to the fact that the pseudopotential itself diverges in the strong-coupling limit, inadequately representing the actual short-range impurity-bath interaction.

In this article, we investigate how a finite-range potential, namely, a square well, corrects this behavior in the strong-coupling limit. Within the MF framework, we obtain semianalytical ground-state solutions to the GP equation. Although this approach has been successfully employed to obtain physical results at strong coupling for the Bose polaron in 3D [18,33], it has not yet been employed in 1D [13,28,29,34–43]. Additionally, we compute the first-order corrections to the wave functions arising from the impurity’s infinitesimally slow motion. We compute the polaron energy and effective mass as a function of the coupling strength g_{IB} . In the weak- to intermediate-coupling regime, we observe that the polaron properties exhibit minimal sensitivity to the short-range character of the impurity interaction and closely align with those predicted for a contact potential. In contrast, in the strong-coupling limit, $g_{\text{IB}} \rightarrow \infty$, the polaron energy and effective mass remain finite and become analytical functions only of the dimensionless ratio of the interaction range relative to the Bose gas coherence length, $\tilde{w} \equiv w/\xi$. For attractive polarons, the energy scales as $-1/\tilde{w}^3$, while for repulsive polarons, we compute subleading corrections of order $\tilde{w}^3$ to

^{*}Contact author: ayogurt@pks.mpg.de

[†]Contact author: meiles@pks.mpg.de

Published by the American Physical Society under the terms of the Creative Commons Attribution 4.0 International license. Further distribution of this work must maintain attribution to the author(s) and the published article’s title, journal citation, and DOI. Open access publication funded by Max Planck Society.

the dark soliton energy. The effective mass in both cases scales as $1/\tilde{w}$.

This paper is organized as follows. Section II explains the general theoretical framework and the solutions to the GP equation. The polaron properties of the finite-ranged impurity are discussed in Sec. III. Section IV presents the characterization of the strong-coupling limit. Finally, we conclude in Sec. V.

II. THEORY

We consider a mobile impurity of mass m_I immersed in a 1D gas of bosonic atoms of mass m . The intraspecies interaction is described by a contact pseudopotential with coupling strength g . The Hamiltonian of such an impurity-bath system with the chemical potential μ is represented in the second-quantized formalism as

$$\hat{\mathcal{K}} = \frac{\hat{p}_I^2}{2m_I} + \int dx \hat{\Psi}^\dagger(x) \left[-\frac{\hbar^2 \hat{\partial}_x^2}{2m} + \frac{g}{2} \hat{\Psi}^\dagger(x) \hat{\Psi}(x) + U_{IB}(x - \hat{x}_I) - \mu \right] \hat{\Psi}(x), \quad (1)$$

where $\hat{\Psi}(x)$ is the field operator of the bath particle at position x . $U_{IB}(x)$ denotes the two-body potential between bath particles and the impurity localized at the position $\hat{x}_I$. The operator $\hat{p}_I$ denotes the momentum of the impurity.

In the absence of an impurity, Lieb and Liniger [44] demonstrated that the system (1) admits an exact solution by Bethe ansatz, characterized by the gas parameter $\gamma \equiv mg/n$, where n is the gas density. Despite the absence of long-range coherence in an infinite 1D Bose gas, the GP ansatz—which assumes the bosons macroscopically occupy the same single-particle state—accurately describes the ground state and excitations when $\gamma \lesssim 2$ [45]. Moreover, studies of the 1D polaron problem [4, 17, 30] found strong agreement between the GP theory and Monte Carlo simulations for sufficiently weak intraspecies interactions. In this work, we likewise focus on weakly interacting bosons and develop the GP framework for a finite-range impurity.

A. Gross-Pitaevskii equation

We move to the frame of the impurity by applying the Lee-Low-Pines coordinate transformation by $\hat{\mathcal{K}}_{LLP} = \hat{\mathcal{S}}^{-1} \hat{\mathcal{K}} \hat{\mathcal{S}}$, where $\hat{\mathcal{S}} = \exp -i\hat{x}_I \hat{p}_B / \hbar$, and $\hat{p}_B = -i\hbar \int dx \hat{\Psi}^\dagger(x) \partial_x \hat{\Psi}(x)$ is the total phonon momentum. This transformation results in a Hamiltonian:

$$\hat{\mathcal{K}}_{LLP} = \frac{(p_{\text{tot}} - \hat{p}_B)^2}{2m_I} + \int dx \hat{\Psi}^\dagger(x) \left[-\frac{\hbar^2 \hat{\partial}_x^2}{2m_r} + \frac{g}{2} \hat{\Psi}^\dagger(x) \hat{\Psi}(x) + U_{IB}(x) - \mu \right] \hat{\Psi}(x), \quad (2)$$

where $m_r \equiv m_I m / (m_I + m)$ is the reduced mass. The total momentum $p_{\text{tot}} \mathbf{e}_x$ of the impurity-bath system is conserved. We proceed by applying the MF approximation to the Hamiltonian in Eq. (2), replacing the operators by their expectation values, $\hat{\Psi}(x) \simeq \langle \hat{\Psi}(x) \rangle = \Psi(x)$, while neglecting any fluctuations. We apply the variational analysis $\frac{\delta \mathcal{K}_{LLP}}{\delta \Psi^*} = 0$ to derive the time-independent GP equation governing the Bose

field with the nonzero total system momentum p_{tot} :

$$\mu \Psi(x) = -\frac{\hbar^2 \partial_x^2 \Psi(x)}{2m_r} + U_{IB}(x) \Psi(x) + g |\Psi(x)|^2 \Psi(x) + \frac{i\hbar \partial_x \Psi(x) [p_{\text{tot}} - \int dx' \Psi^*(x') (-i\hbar) \partial_{x'} \Psi(x')]}{m_I}, \quad (3)$$

where the term in square brackets in the second line is the impurity momentum $p_I = p_{\text{tot}} - p_B$ at the MF level, with $p_B = \langle \hat{p}_B \rangle$.

We set the chemical potential $\mu = gn_0$, where n_0 is the uniform bath density far from the impurity. The length scale $\tilde{\xi} = \xi \sqrt{\frac{m}{m_r}}$ is set by the coherence length $\xi = \frac{\hbar}{\sqrt{2m\mu}}$. The dimensionless GP equation governing the Bose field $\phi(x) \equiv \Psi(x)/\sqrt{n_0}$ is

$$-\partial_x^2 \phi(x) + [|\phi(x)|^2 + \tilde{U}_{IB}(x) - 1] \phi(x) = -i\tilde{v} \partial_x \phi(x). \quad (4)$$

Here, $\tilde{v} = 2m_r \tilde{\xi} v / \hbar$ is the dimensionless impurity speed, with $v = p_I / m_I$. The dimensionless potential is $\tilde{U}_{IB}(x) = \frac{U_{IB}(x)}{gn_0}$.

B. Two-body potential

We model $U_{IB}(x)$ with a square well,

$$U_{IB}(x) = V_0 \Theta(w/2 - |x|), \quad (5)$$

because it admits an analytical solution and has a finite range. $\Theta(x)$ denotes the Heaviside step function, and V_0 and w are the well's depth and width, respectively. The zero-energy scattering length a_{IB} , corresponding to the even-parity solution of the two-body problem, is known analytically [46, 47] as

$$\frac{a_{IB}}{w/2} = \begin{cases} 1 + \frac{\cot z}{z} & V_0 < 0, \\ 1 - \frac{\coth z}{z} & V_0 > 0, \end{cases} \quad (6)$$

where $z = \sqrt{\frac{|V_0| 2m_r}{\hbar^2} \frac{w}{2}}$. In 1D, the relation between the coupling strength g_{IB} and the scattering length a_{IB} is given by $g_{IB} = -\frac{\hbar^2}{m_r a_{IB}}$ (see Appendix A for details).

C. Calculation of observables

Two key observables are the polaron energy and the effective mass of the impurity [1–3, 48, 49]. While the polaron energy can be obtained by solving Eq. (4) for the ground-state wave function with zero total momentum, $\phi_0(x)$, evaluating the effective mass requires solutions at $p_{\text{tot}} \neq 0$. To this end, we perform a low impurity velocity expansion of the solution $\phi(x)$ of the GP equation (4) around the stationary solution $\phi_0(x)$ in the form

$$\phi(x) \simeq \phi_0(x) + i\tilde{v} \phi_1(x), \quad (7)$$

which ensures mobilizing the localized impurity with an infinitesimal velocity $|\tilde{v}| \ll 1$. Upon inserting Eq. (7) into Eq. (4), we obtain

$$0 = -\partial_x^2 \phi_0 + (\phi_0^2 - 1) \phi_0 + \tilde{U}_{IB}(x) \phi_0, \quad (8)$$

$$-\partial_x \phi_0 = -\partial_x^2 \phi_1 + (\phi_0^2 - 1) \phi_1 + \tilde{U}_{IB}(x) \phi_1, \quad (9)$$

to first order in the impurity velocity $\tilde{v}$ [20]. The nonuniform phase $\theta(x)$ of the moving condensate is determined by ϕ_1 ,

via $\theta(x) \equiv \arctan[\tilde{v}\phi_1(x)/\phi_0(x)]$. Once the expansion (7) is replaced in the total energy functional (2), both the ground-state polaron energy and the correction due to a small impurity velocity, i.e., $E = E_{\text{pol}} + \frac{1}{2}m_{\text{eff}}v^2$, can be computed, yielding

$$\frac{E_{\text{pol}}}{gn_0^2\tilde{\xi}} = \frac{1}{2} \int dx (1 - |\phi_0|^4), \quad (10)$$

$$\frac{m_{\text{eff}}}{m_I} = 1 - \frac{4n_0\tilde{\xi}m_r}{m_I} \int dx \phi_1 \partial_x \phi_0, \quad (11)$$

where ϕ_1 is fixed to be a continuous function of $x \in (-\infty, \infty)$ and m_{eff} is the effective mass of the impurity. Note that E_{pol} is the difference between the ground-state energies of the many-body system with and without the impurity [50].

D. Semianalytical solution of ground-state wave function

We restrict our consideration to the ground-state solution, which must satisfy $\phi_0(-x) = \phi_0(x)$, $\partial_x \phi_0(0) = 0$, and $\phi_0(\pm\infty) = 1$, and (along with its derivative) must be continuous at the edges of the square well. As the specific solution depends on the sign of V_0 , we treat each case separately, beginning with $V_0 < 0$. The semianalytical solution for $\phi_0(x)$ can be obtained by integrating Eq. (4) twice. The first integration yields the constant of motion $K = (\partial_x \phi_0)^2/2 + V(\phi_0)$, where $V(\phi_0) = -\phi_0^4/4 + W_a \phi_0^2/2$ and $W_a = |\tilde{V}_0| + 1$ with $\tilde{V}_0 = V_0/(gn_0)$. We perform the second integration by putting the GP equation into the form $\partial_x \phi_0 = \sqrt{2[K - V(\phi_0)]}$, which can be related to Jacobi elliptic functions [51,52] (see Appendix B). The solutions inside and outside the square well are

$$\phi_0(x) = \begin{cases} \tanh^{-1}\left(\frac{|x|+x_a}{\tilde{\xi}\sqrt{2}}\right), & |x| > \frac{w}{2}, \\ \sqrt{\frac{2W_a v_a}{v_a+1}} \text{cd}\left(\sqrt{\frac{W_a}{v_a+1}} \frac{x}{\tilde{\xi}}, v_a\right), & |x| < \frac{w}{2}, \end{cases} \quad (12)$$

where $\text{cd}(x, v_a)$ is a Jacobi elliptic function and v_a is the square of the elliptic modulus. The solution exists for $v_a \in [0, 1]$. Imposing the boundary conditions leads to a transcendental equation [see Eq. (C1) in Appendix C] determining the value of v_a . When multiple roots exist, the root with the largest value of v_a corresponds to the lowest-energy solution [53].

When $V_0 > 0$, the solution for $|x| > w/2$ is

$$\phi_0\left(|x| > \frac{w}{2}\right) = \tanh\left(\frac{|x|+x_r}{\tilde{\xi}\sqrt{2}}\right). \quad (13)$$

The case $x_r = 0$ corresponds to the dark soliton profile [54–56], arising in the strongly repulsive zero-range limit where the vanishing density at the impurity position enforces a solitonic surrounding.

Within the well ($|x| < w/2$), the solution depends on the sign of $W_r \equiv \tilde{V}_0 - 1$:

$$\phi_0(x) = \begin{cases} \sqrt{\frac{2W_r(1-v_r)}{2v_r-1}} \text{nc}\left(\sqrt{\frac{W_r}{2v_r-1}} \frac{x}{\tilde{\xi}}, v_r\right), & W_r > 0, \\ \sqrt{\frac{2|W_r|}{v_r+1}} \text{dc}\left(\sqrt{\frac{|W_r|}{v_r+1}} \frac{x}{\tilde{\xi}}, v_r\right), & W_r < 0. \end{cases} \quad (14)$$

Here, $\text{nc}(x, v_r)$ and $\text{dc}(x, v_r)$ are Jacobi elliptic functions, and v_r is determined numerically after imposing boundary conditions [see Eqs. (C2) and (C3)].

Some clarification regarding the boundary conditions for $\phi_1(x)$ is necessary, as the phase $\theta(x)$ of the GP wave function

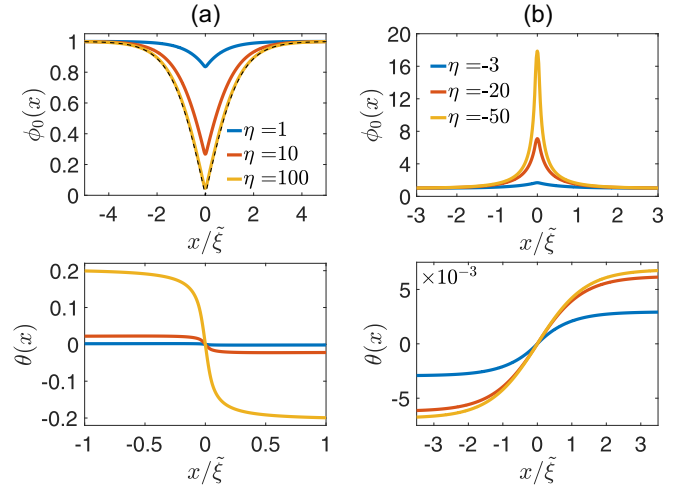


FIG. 1. The ground-state wave function $\phi_0(x)$ (upper panels) and the corresponding phase function $\theta(x) \equiv \arctan[\tilde{v}\phi_1(x)/\phi_0(x)]$ (lower panels) of the (a) repulsive and (b) attractive interactions for various coupling strengths $\eta \equiv g_{\text{IB}}/g$. The well size is fixed as $w = 0.1\tilde{\xi}$. The velocity is fixed as $\tilde{v} = 0.01$. The thin dashed line in the upper panel of part (a) represents the solitonlike profile $\tanh[|x|/(\sqrt{2}\tilde{\xi})]$, shown for comparison.

is only defined up to an arbitrary global phase. First, since the spatial derivative of the phase function corresponds to the velocity field of the Bose gas, $\phi_1(x)$ is assumed to converge to a constant at infinity, i.e., $\partial_x \phi_1(x \rightarrow \pm\infty) = 0$ to ensure that there is no flow far away from the impurity. Second, due to the gauge freedom in the definition of the phase, $\phi_1(x)$ can be shifted by an arbitrary multiple of $\phi_0(x)$. Without loss of generality, one can fix this freedom by imposing the condition $\phi_1(0) = 0$. Together, these two conditions uniquely determine $\phi_1(x)$ for a given $\phi_0(x)$.

III. RESULTS AND DISCUSSION

We now present the polaron energy and effective mass for a short-ranged impurity, $w/\tilde{\xi} \ll 1$. Relevant system parameters are constrained to those used in the one-dimensional polaron experiment [1], with the gas parameter fixed at $\gamma = 1/(2n_0^2\tilde{\xi}^2) = 0.438$ and the impurity-bath mass ratio set to $m_I/m = 0.47$.

In Fig. 1, we show the ground-state wave function $\phi_0(x)$ and the phase function $\theta(x)$ for both repulsive and attractive interactions. The width of the square well is fixed to $w = 0.1\tilde{\xi}$, while its depth V_0 is varied to give different coupling strengths $\eta \equiv g_{\text{IB}}/g$. As the magnitude of the coupling strength $|\eta|$ increases, the density at the impurity decreases (increases) for the repulsive (attractive) impurity. For the repulsive impurity at strong coupling strength, the density profile increasingly resembles, but never fully converges to, that of a dark soliton.

Two features should be highlighted in the behavior of $\theta(x)$. First, its derivative $\partial_x \theta(x)$, corresponding to the fluid flow field of the Bose gas, acquires a negative (positive) sign in the presence of a repulsive (attractive) impurity. The sign inversion in the case of the repulsive impurity can be understood as a consequence of the Bose gas flowing in the $-x$ direction in re-

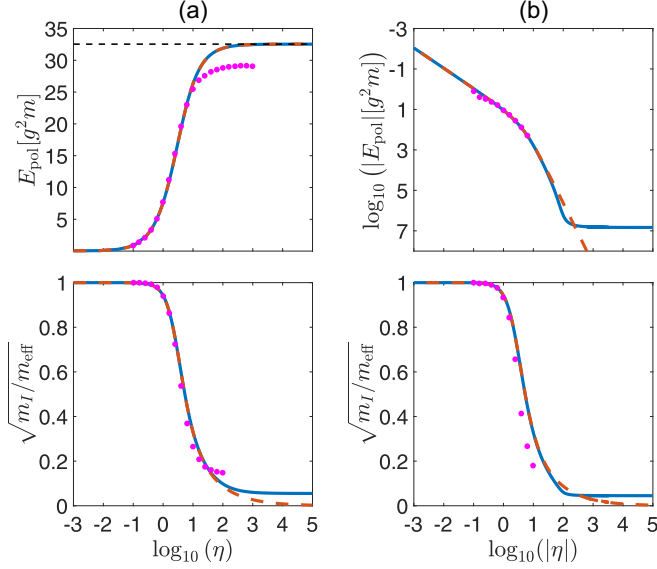


FIG. 2. The polaron energy and effective mass of an impurity assuming (a) a repulsive square-well potential and (b) an attractive square-well potential (blue solid line), and a contact potential (dashed red line) as a function of the coupling strength $\eta \equiv g_{\text{IB}}/g_{\text{BB}}$. The width is again $w = 0.1\xi$. The magenta points show the DMC results obtained for a system of 100 particles with a contact impurity-bath potential [17]. The black dashed line denotes the dark soliton energy.

sponse to the impurity moving in the $+x$ direction. Second, for the repulsive impurity, the phase contrast between $x \rightarrow \pm\infty$ is roughly associated with the ratio of the dip density n_{min} and the uniform density n_0 , similar to the solitons satisfying $\theta(+\infty) - \theta(-\infty) \propto -\cos^{-1}(\sqrt{n_{\text{min}}/n_0})$ [54].

In Fig. 2, we show the polaron energy and the effective mass over a broad range of coupling strengths $|\eta| \in [10^{-3}, 10^5]$. Our results for a square-well potential (blue solid curves) are compared with those obtained for a contact potential using either a MF treatment of the GP theory (red dashed lines) [17] or diffusion quantum Monte Carlo (DMC) (scattered points) [17,30]. The energy of the repulsive square-well impurity coincides precisely with that of the contact potential for all η and asymptotically approaches the energy of a dark soliton in the strong-coupling regime. The effective mass of the square-well impurity also agrees with the contact impurity for couplings $\eta \lesssim 100$; however, it saturates to a finite value at the strong-coupling limit $\eta \rightarrow \infty$ only for the finite-size impurity. This saturation is even more apparent for the attractive impurity. In this case, both the polaron energy and the effective mass initially agree with the contact potential results, but begin to deviate around $|\eta| \simeq 50$. While the polaron energy for the contact polaron diverges as $g_{\text{IB}} \rightarrow \infty$, the energy in the case of a finite-range potential saturates at a finite value. We note that the DMC data also saturate to finite values in the strong-coupling regime. However, this behavior cannot be attributed to finite-range effects, as the simulations [17] employ a zero-range interaction potential; it instead results from the finite number of particles used.

The convergence to a finite value in the strong-coupling limit can be understood by considering the relation between

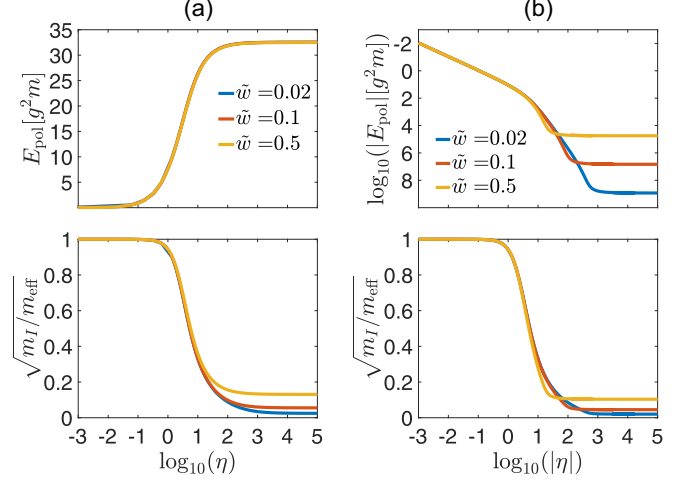


FIG. 3. The polaron energy and the effective mass of the (a) repulsive square-well impurity and the (b) attractive square-well impurity for various values of the well size w as a function of the coupling strength η .

the finite-range potential and the corresponding coupling strength $g_{\text{IB}} \propto -a_{\text{IB}}^{-1}$ of the pseudopotential. A divergent g_{IB} corresponds to a finite-valued finite-range potential, whose zero-energy scattering length satisfies $a_{\text{IB}} \rightarrow 0$ (see Fig. 5). Therefore, in this limit, the pseudopotential provides a poor representation of the actual impurity potential.

Finally, we examine how the polaron properties depend on w in Fig. 3. The polaron properties are largely insensitive to w in the weak-coupling regime $|\eta| \lesssim 3$. This persists for all η for the repulsive impurity's energy. However, deviations among the polaron properties begin to emerge at stronger couplings, leading to substantial differences in the saturated values of the effective mass and even to orders-of-magnitude variation in the energy of the attractive polaron.

IV. POLARON PROPERTIES IN THE STRONGLY COUPLED REGIME

Having noted the strong dependence of polaron properties on interaction range in the strong-coupling regime, we now ask whether a description based on the length scales of the problem applies to a short, but finite-range, impurity-bath interaction in this regime. To this end, we analyze $\phi_0(x)$ and $\phi_1(x)$ near the strongly coupled limit, taking the ratio of the range to the coherence length as a small expansion parameter, $\tilde{w} = w/\xi \ll 1$. This choice reflects the short-range nature of the impurity, enabling the derivation of analytical expressions in this regime.

We first consider the solution of the repulsive impurity in the strong-coupling limit $a_{\text{IB}} \rightarrow 0^-$. This occurs when $\tilde{V}_0 \simeq (\frac{z_r}{\tilde{w}/2})^2$, where $z_r = \coth z_r \simeq 1.199$ [see Eq. (6)]. In this limit, $\phi_0(x)$ is approximately $\tanh[|x|/(\xi\sqrt{2})]$ for $|x| > w/2$ and $\cosh(\sqrt{W_r}x/\xi)$ for $|x| < w/2$. Therefore, we can first determine the elliptic modulus $\nu_r = 1 - \epsilon_r$, where $\epsilon_r \ll 1$, by solving the boundary condition equation (C3):

$$\epsilon_r \simeq \frac{1}{4\sinh^2(u_r)W_r^2} = \frac{\tilde{w}^4}{64\sinh^2(u_r)}, \quad (15)$$

where $u_r \equiv \sqrt{W_r}\tilde{w}/2 \simeq z_r[1 + O(\tilde{w})]$. Imposing continuity at $x = w/2$ yields

$$\frac{x_r}{\tilde{\xi}} = \frac{\coth^2(z_r) - 1}{2z_r^2} \left(\frac{\tilde{w}}{2} \right)^3. \quad (16)$$

Once x_r is determined, the energy of the repulsive polaron can be approximated by evaluating the integral (10), assuming $\phi_0(x) = \tanh[(|x| + x_r)/(\sqrt{2}\tilde{\xi})]$ holds everywhere:

$$\frac{E_{\text{pol}}}{gn_0^2\tilde{\xi}} = \frac{4\sqrt{2}}{3} \left[1 - \frac{3\sqrt{2}}{8} \frac{\coth^2(z_r) - 1}{2z_r^2} \left(\frac{\tilde{w}}{2} \right)^3 \right]. \quad (17)$$

The leading term corresponds to the energy of the dark soliton in the zero-range limit $\tilde{w} \rightarrow 0$, and the corrections appear at third order in $\tilde{w}$. This approximation is expected to slightly overestimate the polaron energy at larger values of $\tilde{w}$, since the true solution inside the potential differs from the hyperbolic tangent. We derive an identity, valid for all V_0 , specifically useful in determining the phase function $\phi_1(x)$ in the region $|x| \leq w/2$. One can obtain the expression $\partial_x^2 \phi_0 / \phi_0 = \phi_0^2 - 1 + \tilde{U}_{\text{IB}}(x)$ from Eq. (8). Substituting this into Eq. (9) and integrating by parts over the domain $x \in [0, \infty)$ yields

$$\partial_x \phi_1(0) = \frac{\phi_0^2(0) - 1}{2\phi_0(0)}. \quad (18)$$

This result admits simple asymptotic approximations in the limits of strong impurity potentials. For a repulsive impurity, where $\phi_0(0) \ll 1$, $\partial_x \phi_1(0) \simeq -\frac{1}{2\phi_0(0)}$, while, for an attractive impurity, the expression simplifies to $\partial_x \phi_1(0) \simeq \frac{\phi_0(0)}{2}$.

Similar to the polaron energy, the dominant contribution to the effective mass of the repulsive polaron also originates from the region outside of the impurity. Noticing that $\partial_x \phi_0 = \frac{1}{\sqrt{2}}(1 - \phi_0^2)$ for $|x| > w/2$, we express the integral in the effective mass [Eq. (11)] as

$$\int_{w/2}^{\infty} dx \phi_1(x) \partial_x \phi_0(x) = \frac{\partial_x \phi_1(w/2) + 1 - \phi_0(w/2)}{\sqrt{2}}. \quad (19)$$

To compute the term $\partial_x \phi_1(w/2)$, we note that $\phi_0(|x| < w/2) \simeq \sqrt{2W_r\epsilon_r} \cosh(\sqrt{W_r}x/\tilde{\xi}) + O(\epsilon_r^{3/2})$. Therefore, the solution of Eq. (9) is

$$\phi_1(|x| < w/2) \simeq -\frac{\sinh(z_r)}{\sqrt{2}} \sinh(\sqrt{W_r}x/\tilde{\xi}) + O(\epsilon_r^{3/2}). \quad (20)$$

Hence, the effective mass is

$$\frac{m_{\text{eff}}}{m_I} \simeq 2n_0\tilde{\xi} \frac{m_r}{m_I} \frac{z_r \sinh(2z_r)}{\tilde{w}/2}. \quad (21)$$

For the attractive impurity in the strong-coupling limit, we have $\tilde{V}_0 \simeq (\frac{z_a}{\tilde{w}/2})^2$, where $z_a = -\cot z_a \simeq 2.798$ [see Eq. (6)]. The solution [Eq. (12)] approaches the form

$$\phi_0(x) \simeq \sqrt{W_a} \left\{ 1 - \frac{\epsilon_a}{4} \left[\cosh \left(\sqrt{\frac{W_a}{2}} \frac{x}{\tilde{\xi}} \right) - 1 \right] \right\} \quad (22)$$

for $|x| < w/2$, as the elliptic modulus satisfies $v_a = 1 - \epsilon_a$, with $\epsilon_a \ll 1$. To second order in $\tilde{w}$, the boundary condition

equation (C1) reads

$$\epsilon_a \simeq \frac{2[1 - \tanh(z_a/\sqrt{2})]}{1 + \frac{1}{4}[\sqrt{2}z_a - \sinh(\sqrt{2}z_a)][\tanh^2(z_a/\sqrt{2}) - 1]}, \quad (23)$$

which is only a constant at leading order, with corrections appearing at order $\tilde{w}^2$. Once ϵ_a is known, the wave function is matched at $x = w/2$ in order to approximate x_a to first order in $\tilde{w}$:

$$\frac{x_a}{\tilde{\xi}} = \frac{\tilde{w}}{2} \left(\frac{\sqrt{2}}{z_a} \frac{1}{1 + \frac{\epsilon_a}{4}[1 - \cosh(\sqrt{2}z_a)]} - 1 \right). \quad (24)$$

The dominant contribution to the attractive polaron energy arises from the impurity region, since it is proportional to ϕ_0^4 . Using Eq. (22), the polaron energy $\propto \int_0^{w/2} (1 - \phi_{\text{in}}^4)$ is

$$\frac{E_{\text{pol}}}{gn_0^2\tilde{\xi}} \simeq -\frac{z_a^4}{(\tilde{w}/2)^3} [1 + O(\epsilon_a)], \quad (25)$$

which, to leading order, scales as $\tilde{w}^{-3}$ due to the nearly flat density inside the impurity.

The effective mass calculation requires integrating over all x , as both the interior and exterior regions contribute similarly. Hence, we evaluate the integral (11) for both regions. We find $\phi_1(x) \simeq x\phi_0(x)/2$ for $|x| < w/2$ by solving Eq. (9) with the boundary condition $\partial_x \phi_1(0) \simeq \phi_0(0)/2$, following from Eq. (18), and substituting Eq. (22) for $\phi_0(x)$. Then, the contributions from the exterior and interior regions can be calculated using Eq. (19) and evaluating $\int_0^{w/2} x\phi_0\partial_x\phi_0 dx$, respectively. Combining these results yields

$$\frac{m_{\text{eff}}}{m_I} \simeq 4n_0\tilde{\xi} \frac{m_r}{m_I} \left(\frac{\epsilon_a z_a^2 \cosh(\sqrt{2}z_a)}{4} + \frac{z_a}{\sqrt{2}} \right) \frac{2}{\tilde{w}}, \quad (26)$$

where the first (second) terms in the parentheses are contributions from inside (outside) the impurity range and are approximately equal in magnitude.

In Fig. 4, we compare these expressions with the exact numerics. They show excellent agreement, with deviations appearing as expected when the ratio $w/\tilde{\xi}$ approaches unity. Both the polaron energy and the effective mass can be expressed solely as functions of the interaction range. For the attractive polaron, the energy scales as $\tilde{w}^{-3}$ at leading order, whereas in the repulsive case the interaction range determines the subleading corrections—of order $\tilde{w}^3$ —to the dark soliton energy. The effective mass for both types of impurities scales inversely with the interaction range, $m_{\text{eff}} \propto \tilde{w}^{-1}$, with distinct prefactors, reflecting the inherent asymmetry between the two branches.

V. CONCLUSION

We derived an analytical solution to the ground-state wave function of the GP equation for a one-dimensional Bose gas with a finite-ranged impurity and calculated perturbative corrections arising from the impurity's slow motion. This approach resolves the divergence of polaron properties in the strongly coupled regime observed with contact potentials. Moreover, we found that in this regime the polaron properties

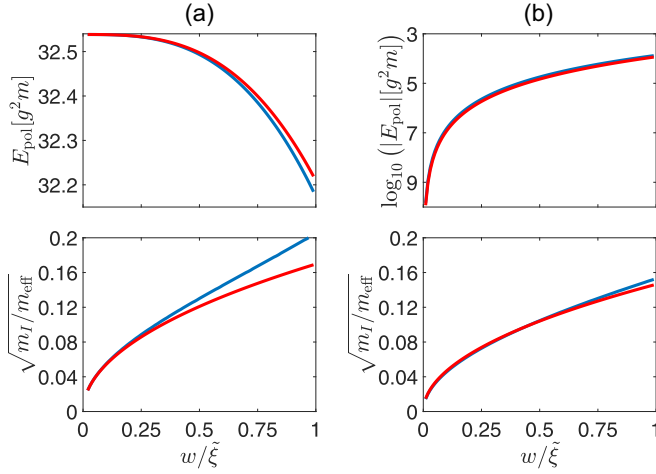


FIG. 4. The polaron properties of (a) repulsive and (b) attractive impurities at the strong-coupling limit $\eta \rightarrow \infty$ as a function of the well size w/ξ . The blue lines represent the exact numerical results, while the red lines correspond to the analytical expressions derived from expansions valid in the limit of a small interaction range, $w/\xi \ll 1$.

admit a description determined by the ratio of the interaction range to the coherence length. The developed formalism can also be applied to study polarons in the long-range regime, $\xi \gg w$, and to obtain excited-state solutions of the GP equation, enabling investigation of the connection between polaron branches and GP equation solutions in one dimension.

ACKNOWLEDGMENTS

We gratefully acknowledge Nikolay Yegovtsev and Michael Fleischhauer for valuable discussions, and we thank Gregory E. Astrakharchik and Fabian Grusdt for kindly providing the DMC data.

DATA AVAILABILITY

The data that support the findings of this article are not publicly available. The data are available from the authors upon reasonable request.

APPENDIX A: SCATTERING LENGTH OF THE FINITE-WELL IMPURITY

Here, we briefly summarize the bound and scattering states of the square-well potential. We restrict our analysis to the even-parity solution, since the ground-state wave function exhibits even parity. In Fig. 5, we illustrate the mapping between the square-well parameters and the scattering length a_{IB} in Eq. (6), as well as the coupling strength g_{IB} .

An infinitesimally shallow attractive square well always supports an even-parity bound state. Additional odd- and even-parity states appear as z crosses $(n - \frac{1}{2})\pi$ and $n\pi$, respectively, with n being a positive integer. The emergence of a new even-parity state coincides with a scattering-length divergence from $-\infty$ to $+\infty$. A repulsive square well supports no bound states and thus no such transition. Notably, the

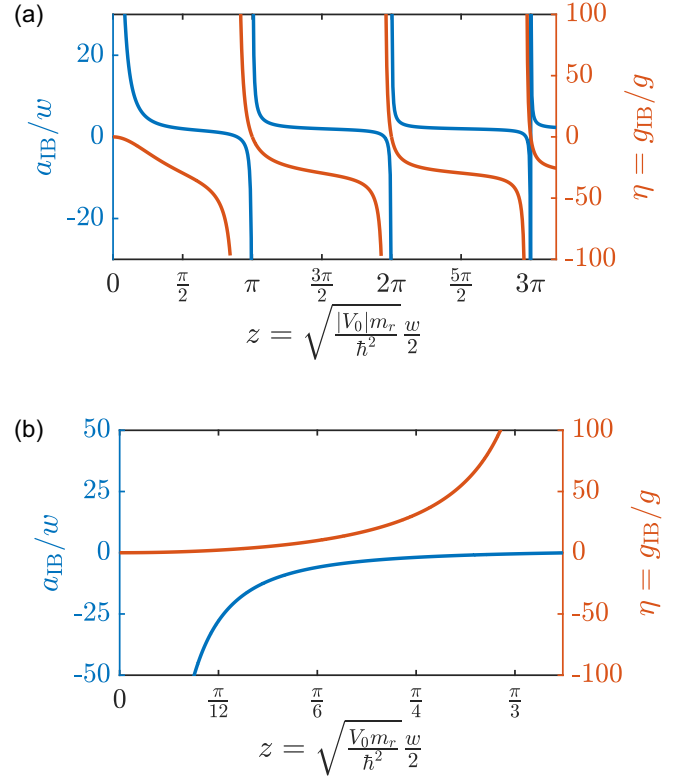


FIG. 5. The mapping between the parameters $z = \sqrt{|V_0|2m_r}/\hbar^2 \cdot w/2$ of the square-well potential and the corresponding zero-energy scattering length a_{IB} (blue lines), as well as the coupling constant $g_{\text{IB}} = -\hbar^2/(m_r a_{\text{IB}})$ (red lines), is shown for (a) attractive and (b) repulsive impurities. The gas parameter $\gamma = 1/(2n_0^2 \xi^2) = 0.438$ and the impurity-bath mass ratio $m_I/m = 0.47$ are used to obtain the coupling strength η .

strongly coupled limit ($g_{\text{IB}} \rightarrow \pm\infty$) does not coincide with the unitarity points where new bound states form.

We model the repulsive impurity using a repulsive square-well potential ($V_0 > 0$) as a simplified representation of the impurity-bath interaction, interpreting the repulsive polaron as the ground-state solution under this potential. While this approach does not capture the metastable nature of repulsive polarons—typically decay into the ground state arising from attractive short-range interactions—it remains valuable for gaining physical insights into their behavior and exploring how their properties depend on the parameters of a finite-range potential. Analogous use of repulsive potentials to approximate short-range interactions between bath particles has yielded physically consistent results [57].

APPENDIX B: PROPERTIES OF JACOBI ELLIPTIC FUNCTIONS

The integral representations of Jacobi elliptic functions related to this work are as follows [51,52]:

$$u = \int_1^{\text{cd}(u,v)} \frac{dt}{\sqrt{(1-t^2)(1-vt^2)}}, \quad (\text{B1})$$

$$u = \int_{\text{dc}(u,v)}^1 \frac{dt}{\sqrt{(t^2-1)(t^2-v)}}, \quad (\text{B2})$$

$$u = \int_1^{\text{nc}(u, \nu)} \frac{dt}{\sqrt{(t^2 - 1)[(1 - \nu)t^2 + \nu]}}, \quad (\text{B3})$$

where ν is the square of the elliptic modulus. The standard Jacobi elliptic functions satisfy the following identities:

$$\text{sn}^2(u, \nu) + \text{cn}^2(u, \nu) = 1, \quad (\text{B4})$$

$$\nu \text{sn}^2(u, \nu) + \text{dn}^2(u, \nu) = 1. \quad (\text{B5})$$

Some useful derivatives and expansions are:

$$\frac{\partial \text{sn}(u, \nu)}{\partial u} = \text{cn}(u, \nu) \text{dn}(u, \nu), \quad (\text{B6})$$

$$\frac{\partial \text{cn}(u, \nu)}{\partial u} = -\text{sn}(u, \nu) \text{dn}(u, \nu), \quad (\text{B7})$$

$$\frac{\partial \text{dn}(u, \nu)}{\partial u} = -\nu \text{sn}(u, \nu) \text{cn}(u, \nu), \quad (\text{B8})$$

$$\text{nc}(x, 1 - \epsilon) \simeq \cosh(x) - \frac{\epsilon}{8} [2x - \sinh(2x)] \sinh(x), \quad (\text{B9})$$

$$\text{cn}(x, 1 - \epsilon) \simeq \text{sech}(x) + \frac{\epsilon}{8} [2x - \sinh(2x)] \tanh(x), \quad (\text{B10})$$

$$\text{sn}(x, 1 - \epsilon) \simeq \tanh(x) + \frac{\epsilon}{8} [2x - \sinh(2x)] [\tanh^2(x) - 1], \quad (\text{B11})$$

$$\text{dn}(x, 1 - \epsilon) \simeq \text{sech}(x) + \frac{\epsilon}{4} \text{sech}(x) \left([\cosh(2x) - 1] + \frac{1}{2} \tanh(x) [2x - \sinh(2x)] \right), \quad (\text{B12})$$

$$\text{cd}(x, 1 - \epsilon) \simeq 1 + \frac{\epsilon}{4} [1 - \cosh(2x)]. \quad (\text{B13})$$

APPENDIX C: THE EQUATIONS OF BOUNDARY CONDITIONS

For the attractive well ($V_0 < 0$), the boundary conditions yield the following transcendental equation:

$$0 = \left(1 - \frac{2W_a \nu_a}{\nu_a + 1} \right) - \frac{2W_a(\nu_a - 1)\sqrt{\nu_a}}{\nu_a + 1} \text{sn}(u_a, \nu_a) + \left(\frac{2W_a \nu_a}{\nu_a + 1} - \nu_a \right) \text{sn}^2(u_a, \nu_a), \quad (\text{C1})$$

where $u_a = \sqrt{\frac{W_a}{\nu_a + 1}} \frac{w}{2\xi}$. The solutions are defined in the interval $\nu_a \in [0, 1]$. For the repulsive well ($V_0 > 0$), the corresponding equations become

$$0 = \left(1 - \frac{2|W_r|}{1 + \nu_r} \right) - \frac{2|W_r|(1 - \nu_r)}{1 + \nu_r} \text{sn}(u_r, \nu_r) + \left(\frac{2|W_r|\nu_r}{1 + \nu_r} - 1 \right) \text{sn}^2(u_r, \nu_r), \quad (\text{C2})$$

$$0 = \frac{2W_r(1 - \nu_r)}{2\nu_r - 1} + \frac{2W_r\sqrt{1 - \nu_r}}{2\nu_r - 1} \text{sn}(u_r, \nu_r) \text{dn}(u_r, \nu_r) - \text{cn}^2(u_r, \nu_r), \quad (\text{C3})$$

for $W_r < 0$ and $W_r > 0$, respectively. Here $u_r = \sqrt{\frac{|W_r|}{1 + \nu_r}} \frac{w}{2\xi}$ ($u_r = \sqrt{\frac{W_r}{2\nu_r - 1}} \frac{w}{2\xi}$) for $W_r < 0$ ($W_r > 0$). The solutions are restricted to the intervals $\nu_r \in [0.5, 1]$ for $W_r > 0$ and $\nu_r \in [-1, 1]$ for $W_r < 0$.

-
- [1] J. Catani, G. Lamporesi, D. Naik, M. Gring, M. Inguscio, F. Minardi, A. Kantian, and T. Giamarchi, Quantum dynamics of impurities in a one-dimensional Bose gas, *Phys. Rev. A* **85**, 023623 (2012).
 - [2] N. B. Jørgensen, L. Wacker, K. T. Skalmstang, M. M. Parish, J. Levinsen, R. S. Christensen, G. M. Bruun, and J. J. Arlt, Observation of attractive and repulsive polarons in a Bose-Einstein condensate, *Phys. Rev. Lett.* **117**, 055302 (2016).
 - [3] M.-G. Hu, M. J. Van de Graaff, D. Kedar, J. P. Corson, E. A. Cornell, and D. S. Jin, Bose polarons in the strongly interacting regime, *Phys. Rev. Lett.* **117**, 055301 (2016).
 - [4] F. Grusdt, N. Mostaan, E. Demler, and L. A. P. Ardila, Impurities and polarons in bosonic quantum gases: A review on recent progress, *Rep. Prog. Phys.* **88**, 066401 (2025).
 - [5] P. Massignan, R. Schmidt, G. E. Astrakharchik, A. Imamoglu, M. Zwierlein, J. J. Arlt, and G. M. Bruun, Polarons in atomic gases and two-dimensional semiconductors, *arXiv:2501.09618*.
 - [6] H. Fröhlich, Electrons in lattice fields, *Adv. Phys.* **3**, 325 (1954).
 - [7] T. D. Lee, F. E. Low, and D. Pines, The motion of slow electrons in a polar crystal, *Phys. Rev.* **90**, 297 (1953).
 - [8] J. T. Devreese, Fröhlich polarons: Lecture course including detailed theoretical derivations – 10th edition, *arXiv:1611.06122*.
 - [9] W. Li and S. Das Sarma, Variational study of polarons in Bose-Einstein condensates, *Phys. Rev. A* **90**, 013618 (2014).
 - [10] A. Shashi, F. Grusdt, D. A. Abanin, and E. Demler, Radio-frequency spectroscopy of polarons in ultracold Bose gases, *Phys. Rev. A* **89**, 053617 (2014).
 - [11] F. Grusdt and E. Demler, New theoretical approaches to Bose polarons, in *Quantum Matter at Ultralow Temperatures* (IOS, Amsterdam, 2016), pp. 325–411.
 - [12] Y. E. Shchadilova, R. Schmidt, F. Grusdt, and E. Demler, Quantum dynamics of ultracold Bose polarons, *Phys. Rev. Lett.* **117**, 113002 (2016).
 - [13] F. Grusdt, G. E. Astrakharchik, and E. Demler, Bose polarons in ultracold atoms in one dimension: Beyond the Fröhlich paradigm, *New J. Phys.* **19**, 103035 (2017).
 - [14] B. Kain and H. Y. Ling, Analytical study of static beyond-Fröhlich Bose polarons in one dimension, *Phys. Rev. A* **98**, 033610 (2018).
 - [15] S. Van Loon, W. Casteels, and J. Tempere, Ground-state properties of interacting Bose polarons, *Phys. Rev. A* **98**, 063631 (2018).
 - [16] F. Scazza, M. Zaccanti, P. Massignan, M. M. Parish, and J. Levinsen, Repulsive Fermi and Bose polarons in quantum gases, *Atoms* **10**, 55 (2022).
 - [17] J. Jäger, R. Barnett, M. Will, and M. Fleischhauer, Strong-coupling Bose polarons in one dimension: Condensate

- deformation and modified Bogoliubov phonons, *Phys. Rev. Res.* **2**, 033142 (2020).
- [18] P. Massignan, N. Yegovtsev, and V. Gurarie, Universal aspects of a strongly interacting impurity in a dilute Bose condensate, *Phys. Rev. Lett.* **126**, 123403 (2021).
- [19] N. Yegovtsev, P. Massignan, and V. Gurarie, Strongly interacting impurities in a dilute Bose condensate, *Phys. Rev. A* **106**, 033305 (2022).
- [20] N. Yegovtsev and V. Gurarie, Effective mass and interaction energy of heavy Bose polarons at unitarity, *Phys. Rev. A* **108**, L051301 (2023).
- [21] R. Alhyder, V. Colussi, M. Čufar, J. Brand, A. Recati, and G. M. Bruun, Lattice Bose polarons at strong coupling and quantum criticality, *SciPost Phys.* **19**, 002 (2025).
- [22] T. Hartweg, T. Gupta, and G. Pupillo, The Bose-Hubbard polaron from weak to strong coupling, [arXiv:2508.00486](https://arxiv.org/abs/2508.00486).
- [23] T. Giamarchi, *Quantum Physics in One Dimension*, International Series of Monographs on Physics Vol. 121 (Clarendon, Oxford, 2003).
- [24] M. A. Cazalilla, R. Citro, T. Giamarchi, E. Orignac, and M. Rigol, One dimensional bosons: From condensed matter systems to ultracold gases, *Rev. Mod. Phys.* **83**, 1405 (2011).
- [25] F. M. Cucchiatti and E. Timmermans, Strong-coupling polarons in dilute gas Bose-Einstein condensates, *Phys. Rev. Lett.* **96**, 210401 (2006).
- [26] M. Bruderer, W. Bao, and D. Jaksch, Self-trapping of impurities in Bose-Einstein condensates: Strong attractive and repulsive coupling, *Europhys. Lett.* **82**, 30004 (2008).
- [27] K. Sacha and E. Timmermans, Self-localized impurities embedded in a one-dimensional Bose-Einstein condensate and their quantum fluctuations, *Phys. Rev. A* **73**, 063604 (2006).
- [28] D. Breu, E. Vidal Marcos, M. Will, and M. Fleischhauer, Impurities in a trapped one-dimensional Bose gas of arbitrary interaction strength: Localization-delocalization transition and absence of self-localization, *Phys. Rev. A* **111**, 023326 (2025).
- [29] A. G. Volosniev and H.-W. Hammer, Analytical approach to the Bose-polaron problem in one dimension, *Phys. Rev. A* **96**, 031601(R) (2017).
- [30] M. Will, G. E. Astrakharchik, and M. Fleischhauer, Polaron interactions and bipolarons in one-dimensional Bose gases in the strong coupling regime, *Phys. Rev. Lett.* **127**, 103401 (2021).
- [31] A. Petković and Z. Ristivojević, Mediated interaction between polarons in a one-dimensional Bose gas, *Phys. Rev. A* **105**, L021303 (2022).
- [32] L. Parisi and S. Giorgini, Quantum Monte Carlo study of the Bose-polaron problem in a one-dimensional gas with contact interactions, *Phys. Rev. A* **95**, 023619 (2017).
- [33] R. Schmidt and T. Enss, Self-stabilized Bose polarons, *SciPost Phys.* **13**, 054 (2022).
- [34] W. Casteels, J. Tempere, and J. Devreese, Polaronic properties of an impurity in a Bose-Einstein condensate in reduced dimensions, *Phys. Rev. A* **86**, 043614 (2012).
- [35] A. S. Dehkharghani, A. G. Volosniev, and N. T. Zinner, Quantum impurity in a one-dimensional trapped Bose gas, *Phys. Rev. A* **92**, 031601(R) (2015).
- [36] J. Akram and A. Pelster, Numerical study of localized impurity in a Bose-Einstein condensate, *Phys. Rev. A* **93**, 033610 (2016).
- [37] V. Pastukhov, Impurity states in the one-dimensional Bose gas, *Phys. Rev. A* **96**, 043625 (2017).
- [38] S. I. Mistakidis, A. G. Volosniev, and P. Schmelcher, Induced correlations between impurities in a one-dimensional quenched Bose gas, *Phys. Rev. Res.* **2**, 023154 (2020).
- [39] Z. Ristivojević, Exact result for the polaron mass in a one-dimensional Bose gas, *Phys. Rev. A* **104**, 052218 (2021).
- [40] G. Panochko and V. Pastukhov, Mean-field construction for spectrum of one-dimensional Bose polaron, *Ann. Phys.* **409**, 167933 (2019).
- [41] F. K. Abdullaev and R. Galimzyanov, Bosonic impurity in a one-dimensional quantum droplet in the Bose–Bose mixture, *J. Phys. B: At., Mol. Opt. Phys.* **53**, 165301 (2020).
- [42] P. Leboeuf and N. Pavloff, Bose-Einstein beams: Coherent propagation through a guide, *Phys. Rev. A* **64**, 033602 (2001).
- [43] S. Majumdar and A. Petković, Relaxation dynamics of a mobile impurity injected in a one-dimensional Bose gas, [arXiv:2507.10402](https://arxiv.org/abs/2507.10402).
- [44] E. H. Lieb and W. Liniger, Exact analysis of an interacting Bose gas. I. The general solution and the ground state, *Phys. Rev.* **130**, 1605 (1963).
- [45] E. H. Lieb, Exact analysis of an interacting Bose gas. II. The excitation spectrum, *Phys. Rev.* **130**, 1616 (1963).
- [46] V. E. Barlette, M. M. Leite, and S. K. Adhikari, Quantum scattering in one dimension, *Eur. J. Phys.* **21**, 435 (2000).
- [47] A. Farrell and B. P. van Zyl, S-wave scattering and the zero-range limit of the finite square well in arbitrary dimensions, *Can. J. Phys.* **88**, 817 (2010).
- [48] Z. Z. Yan, Y. Ni, C. Robens, and M. W. Zwierlein, Bose polarons near quantum criticality, *Science* **368**, 190 (2020).
- [49] J. Etrych, G. Martirosyan, A. Cao, C. J. Ho, Z. Hadzibabic, and C. Eigen, Universal quantum dynamics of Bose polarons, *Phys. Rev. X* **15**, 021070 (2025).
- [50] Note that one also needs to subtract the interaction energy between the background particles of the uniform density $\sqrt{n_0}$ and the particles repelled (attracted) from (into) the vicinity of the impurity.
- [51] M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*, Vol. 55 (U.S. Government printing office, Washington, DC, 1948).
- [52] D. F. Lawden, *Elliptic Functions and Applications*, Applied Mathematical Sciences Vol. 80 (Springer, Berlin, 2013).
- [53] The function $\text{cd}(x, \nu)$ interpolates between the $\cos(x)$ in the limit $\nu \rightarrow 0$ and a constant in the limit $\nu \rightarrow 1$. The ground-state solution is given by the larger elliptic parameter ν , corresponding to a nodeless wave function. The solution with the smaller ν corresponds to an excited-state solution, with at least one node in the range of the impurity.
- [54] C. J. Pethick and H. Smith, *Bose–Einstein Condensation in Dilute Gases* (Cambridge University, Cambridge, England, 2008).
- [55] L. Pitaevskii and S. Stringari, *Bose-Einstein Condensation and Superfluidity*, International Series of Monographs on Physics Vol. 164 (Oxford University, Oxford, 2016).
- [56] P. G. Kevrekidis, D. J. Frantzeskakis, and R. Carretero-Gonzalez, *Emergent Nonlinear Phenomena in Bose-Einstein Condensates: Theory and Experiment*, Vol. 45 (Springer, Berlin, 2008).
- [57] M. Drescher, M. Salmhofer, and T. Enss, Theory of a resonantly interacting impurity in a Bose-Einstein condensate, *Phys. Rev. Res.* **2**, 032011(R) (2020).