

Complete classification and additional saddle-point solutions for high-order above-threshold ionization induced by a strong laser field. III. Two-component fields

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The complete classification of the saddle-point solutions for high-order above-threshold ionization, presented in *Phys. Rev. A* **111**, 023103 (2025) and *Phys. Rev. A* **111**, 053110 (2025) for a linearly polarized laser field, is generalized to the case of an arbitrary bichromatic elliptically polarized field. We first present the classification of the saddle-point solutions for the case of a monochromatic elliptically polarized driving field, which is the simplest example of the field that has two components, i.e., that evolves in the plane. For a bichromatic laser field whose elliptically polarized components have the frequencies $r\omega$ and $s\omega$ (r and s are integers, $s > r$, and ω is the fundamental frequency), the system of the saddle-point equations has $8s^2$ solutions per optical cycle. One-half of these solutions are the so-called backward-scattering solutions for which the direction of the electron motion is significantly affected by the rescattering. The other half are the forward-scattering solutions for which the electron is only slightly deflected during the rescattering event. For some specific field configurations, the number of saddle-point solutions can be smaller. For example, for a bicircular field, which consists of two counterrotating circularly polarized components, there are $4s(r + s)$ solutions, while for the corotating configuration there are $4s^2$ solutions. As an application, we have shown that for a monochromatic elliptically polarized laser field, all four threshold anomalies appear in the spectra of the rescattered photoelectrons.

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I. INTRODUCTION

Processes driven by intense laser fields are central to strong-field physics [1–16] and attoscience [17–25]. The most prominent examples of such phenomena include high-order harmonic generation [26,27] and high-order above-threshold ionization (HATI) [28,29]. These phenomena can be described using the three-step model [30,31], which provides an intuitive picture. The first step involves the tunnel ionization of an electron from an atomic or molecular potential under the influence of a strong laser field. In the second step, the liberated electron is accelerated by the oscillating laser field and may subsequently return to the vicinity of its parent ion. In the third step, the returning electron can either recombine with the ion, emitting a high-energy photon (as in high-order harmonic generation), or scatter elastically off the ion, thereby gaining substantial kinetic energy. In the latter case, the resulting photoelectron energy can greatly exceed the ionization threshold, so that the process is referred to as HATI.

In the regime of high intensity and low-frequency, the HATI process can effectively be modeled within the framework of the strong-field approximation [32–35]. This assumes that the electron, once liberated from the core, propagates freely under the influence of the laser field, and further

interaction with the parent ion is neglected until the rescattering (and after it). The HATI transition amplitude has a form of the two-dimensional integral over the ionization time t_0 and the rescattering time t . This integral can be calculated using asymptotic techniques for oscillatory integrals [36]. Specifically, the saddle-point (SP) approximation is employed, in which the integral is approximated by a sum of the contributions from discrete saddle points. Each term in the sum corresponds to a pair (t_0, t) that satisfies the SP equations, which correspond to the energy-conservation conditions at the ionization and rescattering times. These contributions can be interpreted as partial transition amplitudes in the sense of the Feynman path integral formalism [37].

In recent years, considerable scientific effort has been devoted to achieving precise control over various properties of laser light leading to the emergence of the field of tailored light [38–41]. Early advancements in this field focused on tuning fundamental pulse characteristics such as carrier frequency, spectral bandwidth, pulse duration, and envelope shape [42–45]. Over time, control strategies have expanded to more complex aspects, including the angular momentum of light [46,47]. A particularly powerful method of manipulating light's symmetry involves the coherent combination of multiple frequencies into a single beam. This approach initially gained attention through the use of a bicircular field [48–51], which enabled, for example, the generation of circularly polarized high-order harmonics and coherent x-ray emission [52–57]. More recently, such structured light fields have proven instrumental in designing tailored waveforms

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for probing chiral molecules [58,59]. Similarly, the field configuration that consists of the linearly polarized components with mutually orthogonal polarizations (the so-called OTC field) has also attracted interest and has been theoretically investigated and experimentally utilized for various purposes [56,60–70].

In our recent publications [71,72], we found and classified all the solutions of the system of the HATI SP equations for the case of a linearly polarized driving field. In particular, we separately analyzed the so-called backward-scattering (BS) and forward-scattering (FS) SP solutions, as well as the solutions with very short (or even negative) electron travel times. In the present paper, we generalize our analysis to the case of two-component fields. We first consider the case of a monochromatic elliptically polarized field which has the x and y components. Then we generalize our results to the case of an arbitrary elliptically polarized bichromatic field. We illustrate our findings using the examples of the fields that are frequently employed in strong-field physics.

The paper is organized as follows: After defining our problem in Sec. II, in Sec. III we present the classification of the SP solutions for HATI by a monochromatic elliptically polarized field. In Sec. IV, classification of the SP solutions for a two-component bichromatic field is illustrated using the examples of the OTC and bicircular fields. As an application, in Sec. V we analyze the channel closing effect and threshold anomalies for HATI by a monochromatic elliptically polarized field. Finally, our conclusions are presented in Sec. VI. Atomic units are used.

II. THEORY

A. Background

The double integral that appears in the rescattering part of the T -matrix element [see Eq. (2) in Ref. [71]] can be calculated numerically or by using the SP method. In the latter case, the condition that the action is stationary with respect to the ionization and rescattering times leads to the SP equations

$$[\mathbf{k}_{\text{st}}(t, t_0) + \mathbf{A}(t_0)]^2 = -2I_p, \quad (1)$$

$$[\mathbf{k}_{\text{st}}(t, t_0) + \mathbf{A}(t)]^2 = [\mathbf{p} + \mathbf{A}(t)]^2, \quad (2)$$

which correspond to the energy-conservation condition at the moment of ionization and rescattering, respectively. Here, t_0 and t are the ionization and rescattering times, respectively, $\mathbf{A}(t)$ is the vector potential of the driving field, $\mathbf{k}_{\text{st}}$ is the intermediate stationary momentum [71], $\mathbf{p}$ is the final photoelectron momentum ($E_p = \mathbf{p}^2/2$ is the photoelectron energy), and I_p is the ionization potential. The rescattering T -matrix element is $T_p^{\text{res}} \approx \sum_j C_j e^{iS_j}$, where C_j is the subintegral function multiplied by the factor which appears due to the application of the SP method, and S_j is the action calculated at the SP times t_{0j} and t_j .

B. Number of SP solutions for HATI by a two-component field

Our driving field $\mathbf{E}(t) = \mathbf{E}_1(t) + \mathbf{E}_2(t)$ has two components with frequencies $\omega_1 = r\omega$ and $\omega_2 = s\omega$ (r and s are integer, $s > r$) and it evolves in the plane defined by the unit

vectors $\hat{\mathbf{e}}_x$ and $\hat{\mathbf{e}}_y$ according to

$$\begin{aligned} E_x(t) &= \mathcal{E}_1[\sin(r\omega t + \phi_1) - \varepsilon_1 c_\eta \cos(r\omega t + \phi_1)] \\ &\quad + \mathcal{E}_2[c_\eta \sin(s\omega t + \phi_2) - \varepsilon_2 \cos(s\omega t + \phi_2)], \\ E_y(t) &= s_\eta[-\mathcal{E}_1 \varepsilon_1 \cos(r\omega t + \phi_1) + \mathcal{E}_2 \sin(s\omega t + \phi_2)]. \end{aligned} \quad (3)$$

Here, $\mathcal{E}_i = E_i/\sqrt{1 + \varepsilon_i^2}$, where E_i and ε_i , $i = 1, 2$, are the electric-field amplitudes and ellipticities, respectively, η is the crossing angle between the semimajor axes of the field components, $c_\eta = \cos \eta$, $s_\eta = \sin \eta$, and ϕ_i , $i = 1, 2$, are the phases. This field configuration represents a general case of a bichromatic field which consists of two elliptically polarized copropagating components with the angle η between their semimajor axes. Introducing the notation

$$\mathcal{A}_1 = \frac{\mathcal{E}_1}{r\omega} = \mathcal{A}_0, \quad \mathcal{A}_2 = \frac{\mathcal{E}_2}{s\omega} = a\mathcal{A}_0, \quad a = \frac{r}{s} \frac{\mathcal{E}_2}{\mathcal{E}_1}, \quad (4)$$

the components of the vector potential of our driving field are

$$\begin{aligned} A_x(t) &= \mathcal{A}_0\{\cos(r\omega t + \phi_1) + \varepsilon_1 c_\eta \sin(r\omega t + \phi_1) \\ &\quad + a[c_\eta \cos(s\omega t + \phi_2) + \varepsilon_2 \sin(s\omega t + \phi_2)]\}, \\ A_y(t) &= \mathcal{A}_0 s_\eta[\varepsilon_1 \sin(r\omega t + \phi_1) + a \cos(s\omega t + \phi_2)], \end{aligned} \quad (5)$$

while the stationary momentum is

$$k_{\text{st},j}(t, t_0) = [\alpha_j(t_0) - \alpha_j(t)]/(t - t_0), \quad j = x, y, \quad (6)$$

where

$$\begin{aligned} \alpha_x(t) &= \frac{\mathcal{A}_0}{\omega} \{[\sin(r\omega t + \phi_1) - \varepsilon_1 c_\eta \cos(r\omega t + \phi_1)]/r \\ &\quad + a[c_\eta \sin(s\omega t + \phi_2) - \varepsilon_2 \cos(s\omega t + \phi_2)]/s\}, \\ \alpha_y(t) &= \frac{\mathcal{A}_0}{\omega} s_\eta [-\varepsilon_1 \cos(r\omega t + \phi_1)/r + a \sin(s\omega t + \phi_2)/s]. \end{aligned} \quad (7)$$

The ponderomotive energy is

$$U_p = \frac{1}{2T} \int_0^T dt \mathbf{A}^2(t) = \frac{E_1^2}{4r^2\omega^2} + \frac{E_2^2}{4s^2\omega^2}, \quad T = \frac{2\pi}{\omega}. \quad (8)$$

Notice that the ponderomotive energy does not depend on the crossing angle η and the ellipticities ε_1 and ε_2 .

Introducing the dimensionless quantities $\gamma = \sqrt{2I_p}/\mathcal{A}_0$, $k_j \equiv k_j(\varphi_0, \varphi) = k_{\text{st},j}/\mathcal{A}_0$, $q_j = p_j/\mathcal{A}_0$, $j = x, y$, $\varphi_0 = \omega t_0$, and $\varphi = \omega t$, the SP equations become

$$\begin{aligned} \{k_x + c_r(\varphi_0) + \varepsilon_1 c_\eta s_r(\varphi_0) + a[c_\eta c_s(\varphi_0) + \varepsilon_2 s_s(\varphi_0)]\}^2 \\ + \{k_y + s_\eta[\varepsilon_1 s_r(\varphi_0) + a c_s(\varphi_0)]\}^2 + \gamma^2 = 0, \end{aligned} \quad (9)$$

$$\begin{aligned} (k_x - q_x)\{c_r(\varphi) + \varepsilon_1 c_\eta c_r(\varphi) + a[c_\eta c_s(\varphi) + \varepsilon_2 s_r(\varphi)]\} \\ + (k_y - q_y)s_\eta[\varepsilon_1 s_r(\varphi) + a s_s(\varphi)] = (q^2 - k^2)/2. \end{aligned} \quad (10)$$

Here we also introduced the notation $s_r(\varphi) = \sin(r\varphi + \phi_1)$, $c_r(\varphi) = \cos(r\varphi + \phi_1)$, $s_s(\varphi) = \sin(s\varphi + \phi_2)$, and $c_s(\varphi) = \cos(s\varphi + \phi_2)$.

For the direct above-threshold ionization, the SP equation is $[\mathbf{p} + \mathbf{A}(t_0)]^2 = -2I_p$, and, in our notation, it becomes

$$\begin{aligned} \{q_x + c_r(\varphi_0) + \varepsilon_1 c_\eta s_r(\varphi_0) + a[c_\eta c_s(\varphi_0) + \varepsilon_2 s_s(\varphi_0)]\}^2 \\ + \{q_y + s_\eta[\varepsilon_1 s_r(\varphi_0) + a c_s(\varphi_0)]\}^2 + \gamma^2 = 0. \end{aligned} \quad (11)$$

This equation can be written in the form $F(\varphi_0) = 0$, where $F(\varphi_0)$ is a trigonometric polynomial with real coefficients. The order of this polynomial and thus the number of solutions of Eq. (11) depends on the laser-field parameters [36,73,74]. For example, for a monochromatic linearly or elliptically polarized driving field, Eq. (11) has two solutions for $\text{Im } \varphi_0 > 0$. Furthermore, for the bicircular counterrotating $r\omega$ - $s\omega$ field, there are $r + s$ solutions with $\text{Im } \varphi_0 > 0$, while for the corotating combination there are s solutions. Finally, the most general configuration accounts for the combination of two elliptically (or linearly) polarized components with frequencies $r\omega$ and $s\omega$ and an arbitrary value of the angle η between their semimajor axes, in which case Eq. (11) has $2s$ solutions with $\text{Im } \varphi_0 > 0$.

1. Backward-scattering SP solutions

First, we investigate the number of BS SP solutions of the system of Eqs. (9) and (10). In this case, the electron, liberated at the time t_0 , is first driven away from the core and then returned and rescattered off the core. Consequently, the travel time $\text{Re}(t - t_0)$, which appears in the denominator of the stationary momentum (6), is long, and, in the lowest-order approximation, we assume that $k_{x,y}^{(0)} \propto 1/\text{Re}(\varphi - \varphi_0) \ll 1$. In this approximation, Eqs. (9) and (10) become

$$\{c_r(\varphi_0^{(0)}) + \varepsilon_1 c_\eta s_r(\varphi_0^{(0)}) + a[c_\eta c_s(\varphi_0^{(0)}) + \varepsilon_2 s_s(\varphi_0^{(0)})]\}^2 + s_\eta^2 [\varepsilon_1 s_r(\varphi_0^{(0)}) + a c_s(\varphi_0^{(0)})]^2 + \gamma^2 = 0, \quad (12)$$

$$-q_x \{c_r(\varphi_0^{(0)}) + \varepsilon_1 c_\eta s_r(\varphi_0^{(0)}) + a[c_\eta c_s(\varphi_0^{(0)}) + \varepsilon_2 s_s(\varphi_0^{(0)})]\} - q_y s_\eta [\varepsilon_1 s_r(\varphi_0^{(0)}) + a c_s(\varphi_0^{(0)})] = q^2/2, \quad (13)$$

and they are not coupled, i.e., Eq. (12) is the equation for the variable $\varphi_0^{(0)}$, while Eq. (13) depends only on the variable $\varphi^{(0)}$. Consequently, these equations can be solved independently. Both equations have the form of a trigonometric polynomial with real coefficients and can be solved in the same way as Eq. (11). We denote by n_1 and n_2 the orders of the trigonometric polynomials (12) and (13), respectively. The order of the polynomial depends on the values of the laser-field parameters. Equation (12) is in the form of a polynomial of the order n_1 , and thus it has n_1 solutions per optical cycle with $\text{Im } \varphi_0^{(0)} > 0$. These solutions are denoted by $\varphi_{0\beta}^{(0)}$, $\beta = 1, \dots, n_1$. Let us explain why it should be $\text{Im } \varphi_0^{(0)} > 0$, i.e., $\text{Im } t_0 > 0$. The factor with the t_0 -dependent exponent in the subintegral function of the rescattering T -matrix element is $e^{iS_{\mathbf{k}_{\text{st}}}(t_0) + iI_p t_0}$ [36]. Expanding $S_{\mathbf{k}_{\text{st}}}(t_0)$ in the Taylor series about the point $\text{Re } t_0$, and using the SP Eq. (1), we get that this factor is proportional to $e^{-[A^2(\text{Re } t_0) + 2I_p] \text{Im } t_0/3}$, so that for $\text{Im } t_0 < 0$ the ionization probability would exponentially increase, which is unacceptable.

Equation (13) is a polynomial of the order n_2 , and thus there are $2n_2$ solutions $\varphi^{(0)}$ per optical cycle (we do not require that $\text{Im } \varphi > 0$, since the solutions with $\text{Im } \varphi < 0$ also contribute [71,75]). In particular, one-half of these solutions is in the upper half of the complex plane, while the other half is in the lower half of the complex plane, i.e., the solutions are $\varphi_j^{(0)}$ and $\varphi_j^{(0)*}$, $j = 1, \dots, n_2$. The solutions are jointly denoted as $\varphi_{\alpha j}^{(0)}$, $\alpha = \pm 1$, where $\varphi_{1j}^{(0)} = \varphi_j^{(0)}$, while $\varphi_{-1j}^{(0)} = \varphi_j^{(0)*}$. To

obtain the next-order approximation for the stationary momentum, we introduce n_1 solutions $\varphi_{0\beta}^{(0)}$ and $2n_2$ solutions $\varphi_{\alpha j}^{(0)}$ in the expression for the stationary momentum, i.e., $k_{x,y}^{(1)} = k_{x,y}(\varphi_{\alpha j}^{(0)}, \varphi_{0\beta}^{(0)})$. Introducing $k_{x,y}^{(1)}$ into Eqs. (9) and (10), we get a system of equations for $\varphi_0^{(1)}$ and $\varphi^{(1)}$. Using the solutions $\varphi_0^{(1)}$ and $\varphi^{(1)}$, we obtain the next-order approximation for the stationary momentum, $k_{x,y}^{(2)} = k_{x,y}(\varphi_{\alpha j}^{(1)}, \varphi_{0\beta}^{(1)})$. The method of successive approximations continues until desired precision is achieved. In the present paper, we are interested in the number of solutions of the system of the SP equations. These solutions for φ_0 can be in the m th cycle before the rescattering so that the BS solutions can be classified using the following multi-index:

$$(\alpha, \beta, m)_j, \quad \alpha = \pm 1, \quad \beta = 1, \dots, n_1, \quad j = 1, \dots, n_2. \quad (14)$$

This means that there are $2n_1 n_2$ BS solutions per laser-field period. The orders n_1 and n_2 of the polynomials depend on the values of the laser-field parameters. For example, for a monochromatic linearly/elliptically polarized field, it is $n_1 = 2$ and $n_2 = 1$ so that the total number of solutions is $2n_1 n_2 = 4$. Furthermore, for a general case of a bichromatic linearly/elliptically polarized field (η is arbitrary), it is $n_1 = 2s$ and $n_2 = s$ so that there are $2n_1 n_2 = 4s^2$ BS solutions. This general type of the field as special cases includes the bichromatic linearly polarized field ($\eta = 0^\circ$, $\varepsilon_1 = \varepsilon_2 = 0$; for detailed analysis, see Refs. [71,72]) and the orthogonally polarized two-color field (OTC) ($\eta = 90^\circ$, $\varepsilon_1 = \varepsilon_2 = 0$). However, for some bichromatic field configurations, the total number of solutions is different. For example, for an $r\omega$ - $s\omega$ bicircular counterrotating field, we have $n_1 = r + s$ and $n_2 = s$ so that the total number of BS solutions is $2n_1 n_2 = 2s(r + s)$. For the corotating configuration, we have $n_1 = s$ and $n_2 = s$ so that $2n_1 n_2 = 2s^2$.

2. Forward-scattering SP solutions

In the FS case, the electron does not change its direction during the rescattering, i.e., $\mathbf{k}_{\text{st}} \approx \mathbf{p}$. Consequently, the second SP equation, Eq. (2), is approximately satisfied, while the first one [Eq. (1)] takes the form

$$\{q_x + c_r(\varphi_0) + \varepsilon_1 c_\eta s_r(\varphi_0) + a[c_\eta c_s(\varphi_0) + \varepsilon_2 s_s(\varphi_0)]\}^2 + \{q_y + s_\eta [\varepsilon_1 s_r(\varphi_0) + a c_s(\varphi_0)]\}^2 + \gamma^2 = 0. \quad (15)$$

This equation is a trigonometric polynomial whose order n_1 depends on the values of the field parameters. In general, such a polynomial has n_1 zeros $\varphi_{0\rho}$, $\rho = 1, \dots, n_1$, in the upper half of the complex plane ($\text{Im } \varphi_{0\rho} > 0$). Introducing the solutions $\varphi_{0\rho}$ into $k_j = q_j$, $j = x, y$, we get

$$\begin{aligned} & \{s_r(\varphi_{0\rho}) - s_r(\varphi) - \varepsilon_1 c_\eta [c_r(\varphi_{0\rho}) - c_r(\varphi)]\}/r \\ & + a\{c_\eta [s_s(\varphi_{0\rho}) - s_s(\varphi)] - \varepsilon_2 [c_s(\varphi_{0\rho}) - c_s(\varphi)]\}/s \\ & = q_x(\varphi - \varphi_{0\rho}), \end{aligned} \quad (16)$$

$$\begin{aligned} & -\varepsilon_1 s_\eta [c_r(\varphi_{0\rho}) - c_r(\varphi)]/r + a s_\eta [s_s(\varphi_{0\rho}) - s_s(\varphi)]/s \\ & = q_y(\varphi - \varphi_{0\rho}). \end{aligned} \quad (17)$$

By dividing Eqs. (16) and (17), we get the equation in the form of the trigonometric polynomial in the variable φ , and we denote its order by n_2 . If we neglect $\text{Im } \varphi_{0\rho}$, the coefficients of this polynomial are real so that there are $2n_2$ solutions φ per optical cycle. One-half of these solutions is in the upper half of the complex plane, while the other half is in the lower half of the complex plane, i.e., the solutions are φ_j and φ_j^* , $j = 1, \dots, n_2$. Analogously to the BS index α , we use the index $v = \pm 1$ in such a way that $\varphi_{v=-1,j} = \varphi_{v=+1,j}^*$, $j = 1, \dots, n_2$. We also introduce the index μ , which corresponds to the BS index m and which counts the optical cycles. Taking everything together, the FS solutions can be classified using the multi-index:

$$(\nu, \rho, \mu)_j, \quad \nu = \pm 1, \quad \rho = 1, \dots, n_1, \quad j = 1, \dots, n_2. \quad (18)$$

C. Backscattering-like and forward-scattering-like solutions

In our notation, we assign the index $m = 0$ (or $\mu = 0$) to the BS (or FS) SP solutions that exhibit either very short or negative travel time. These particular solutions often deviate from the standard semiclassical three-step model, and, when analyzed using the conventional quantum-orbit formalism, may produce artificial discontinuities in the electron trajectories. In such cases, the complex-time quantum-orbit formalism [76,77] can be employed. Consequently, we refer to the $m = 0$ and $\mu = 0$ SP solutions as backscattering-like and forward-scattering-like solutions, respectively, to reflect the ambiguity of the (semi)classical motion described by real electron trajectories between the real ionization time $\text{Re } t_0$ and the real rescattering time $\text{Re } t$.

For each SP solution corresponding to the times t_0 and t , there exists an associated solution with the negative travel time. We add the prime symbol to the indices m and μ for the solutions where $\text{Re}(t - t_0) < 0$ for all photoelectron energies, denoting them as (α, β, m') and (ν, ρ, μ') . In contrast, for the solutions where $\text{Re}(t - t_0)$ changes sign with the change of the photoelectron energy, we use the double-prime notation: (α, β, m'') and (ν, ρ, μ'') .

III. RESULTS FOR THE MONOCHROMATIC FIELD

We present our numerical results for the Ne atom, which has a $2p$ ground state and the ionization potential $I_p = 21.56$ eV. The ground state is obtained by solving the Roothan-Hartree-Fock equations expanded in terms of Slater-type orbitals, while the rescattering is modeled using the independent-particle model potential represented by the double Yukawa potential [36,78]. Comparison of the results obtained using our approach with the results obtained using the solutions of the time-dependent Schrödinger equation is provided in Refs. [5,79–81]. The impact of the Coulomb effects on our quantum-orbit formalism was recently analyzed in Ref. [82]. In this paper, the differential ionization rate is calculated using the formula $w_p(n) = 2\pi |\mathbf{p}| |T_p^{\text{res}}(n)|^2$, where the rescattering T -matrix element is calculated as described in [71] and references therein.

In this section, we illustrate our findings using the example of a monochromatic elliptically polarized field ($\eta = 90^\circ$, $\varepsilon_1 = \varepsilon$, $\phi_1 = 0^\circ$, $E_2 = 0$) for which $n_1 = 2$ and $n_2 = 1$, and

the solutions can be classified using the multi-indices (14) and (18) with $m, \mu = 0, 1, 2, \dots$. In Fig. 1 we present the BS [panels (a) and (c)] and the FS [panels (b) and (d)] SP solutions, together with the corresponding partial contributions to the differential ionization rate [panels (e) and (f)] for HATI induced by an elliptically polarized field. The emission is in the direction $\theta = 0^\circ$ (θ is the angle between $\hat{\mathbf{e}}_x$ and $\mathbf{p}$). Both, the BS and FS SP solutions can be classified in the same way as the SP solutions for HATI by a monochromatic linearly polarized field: $\alpha, v = \pm 1$, $\beta, \rho = 1, 2$, and, $m, \mu = 0, 1, 2, \dots$. In Fig. 1, the results are presented for $m, \mu \leq 3$. As for a monochromatic linearly polarized driving field, there are two pairs of the BS and two pairs of the FS solutions per optical cycle. The BS solutions for elliptical polarization are qualitatively similar to those obtained for linear polarization. The corresponding partial differential ionization rates are slightly lower for elliptical polarization. Again, out of two solutions of a given pair, one converges after the cutoff while the other is divergent and should be discarded in this energy region (these solutions and their contributions are denoted by dashed lines).

On the other hand, the differences between the FS SP solutions obtained for linearly and elliptically polarized driving fields are much more pronounced. For example, the values of the real part of the ionization time of the solutions with different ν and the same ρ and μ are not the same as for the case of a linearly polarized driving field [see Fig. 1(b)], and the difference becomes more apparent for larger ellipticity and higher photoelectron energy. More importantly, for a linearly polarized driving field the partial contributions of all FS SP solutions are convergent after the cutoff. However, for the FS SP solutions with $\rho = 1$, as the ellipticity increases from zero, the partial contribution of one SP solution from a pair becomes divergent after the cutoff so that its contribution should be discarded. The problematic situation happens for the driving field with small nonzero ellipticity (around 0.15) for which it is not clear from which value of the photoelectron energy the partial contribution of this SP solution should be discarded. This problem is not present if the coherent sum of the partial contributions is calculated using the uniform approximation [75,83–85], which does not eliminate the divergent solution but rather treats properly both solutions of a single pair. Furthermore, the other solution with $\rho = 1$ remains convergent for all values of the driving-field ellipticity. However, for a large ellipticity of the driving field ($\varepsilon \gtrsim 0.42$), the corresponding imaginary part of the ionization time becomes negative for some values of the photoelectron energy. The dependence of the $\rho = 2$ SP solutions on the ellipticity is also interesting. In particular, as the ellipticity increases, both of the $\rho = 2$ solutions become divergent. One solution becomes divergent already for small ellipticity of the driving field (around 0.1), while for the other this happens for larger ellipticity (around 0.3).

As for the case of a linearly polarized driving field, the SP solutions with short or negative travel time exist. We call these solutions the backscattering-like and the forward-scattering-like solutions, and we denote them by $(\alpha, \beta, m) = (\pm 1, 2, 0)$ and $(\alpha, \beta, m) = (\pm 1, 1, 0')$, and by $(\nu, \rho, \mu) = (\pm 1, 2, 0)$ and $(\nu, \rho, \mu) = (\pm 1, 1, 0')$, where the prime means that the corresponding travel time is negative. For a linearly

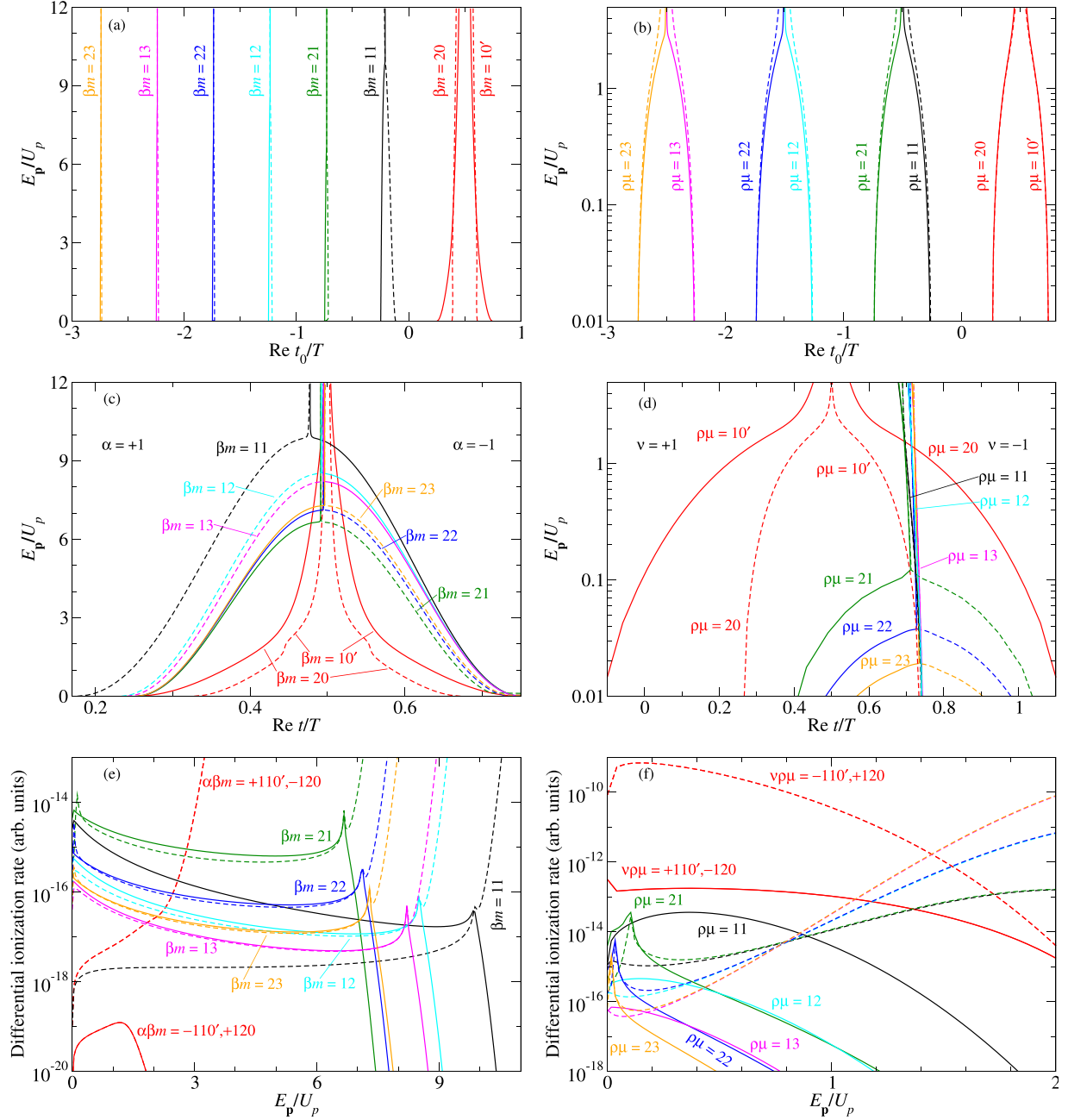


FIG. 1. Backscattering (left-hand panels) and forward-scattering (right-hand panels; notice the logarithmic scale) SP solutions for the ionization and rescattering times, and the partial contributions to the differential ionization rate (bottom panels) for $\theta = 0^\circ$ and for ionization of neon atoms ($I_p = 21.56$ eV) by an elliptically polarized monochromatic laser field with the wavelength 1300 nm, the intensity $E_1^2 = 4 \times 10^{14}$ W/cm², and the ellipticity $\varepsilon = 0.2$. The photoelectron energy, in units of the ponderomotive energy U_p , is presented as a function of the real part of (a), (b) the ionization time $\text{Re } t_0$, and (c), (d) rescattering time $\text{Re } t$, expressed in optical cycles T . (e), (f) The partial contributions to the differential ionization rate as a function of the photoelectron energy (in units of U_p).

polarized field, if (t_0, t) is a solution of the SP equations, then $(T - t_0^*, T - t^*)$ is also a solution of these equations with the same imaginary part of the ionization and rescattering times. For the case with $m = 0$ or $\mu = 0$, these solutions are within the same optical cycle (the ionization and rescattering times are symmetric with respect to $T/2$) and the corresponding partial contributions are the same. The same holds for an

elliptically polarized driving field (see the red solid and dashed lines in Fig. 1), with the exception that in this case the partial contributions are the same only for the emission in the directions $\theta = 0^\circ$ or 180° . Otherwise, the symmetry with respect to $T/2$ is broken and the corresponding partial contributions are not the same (for details, see Fig. 3 in Ref. [76] and the corresponding text).

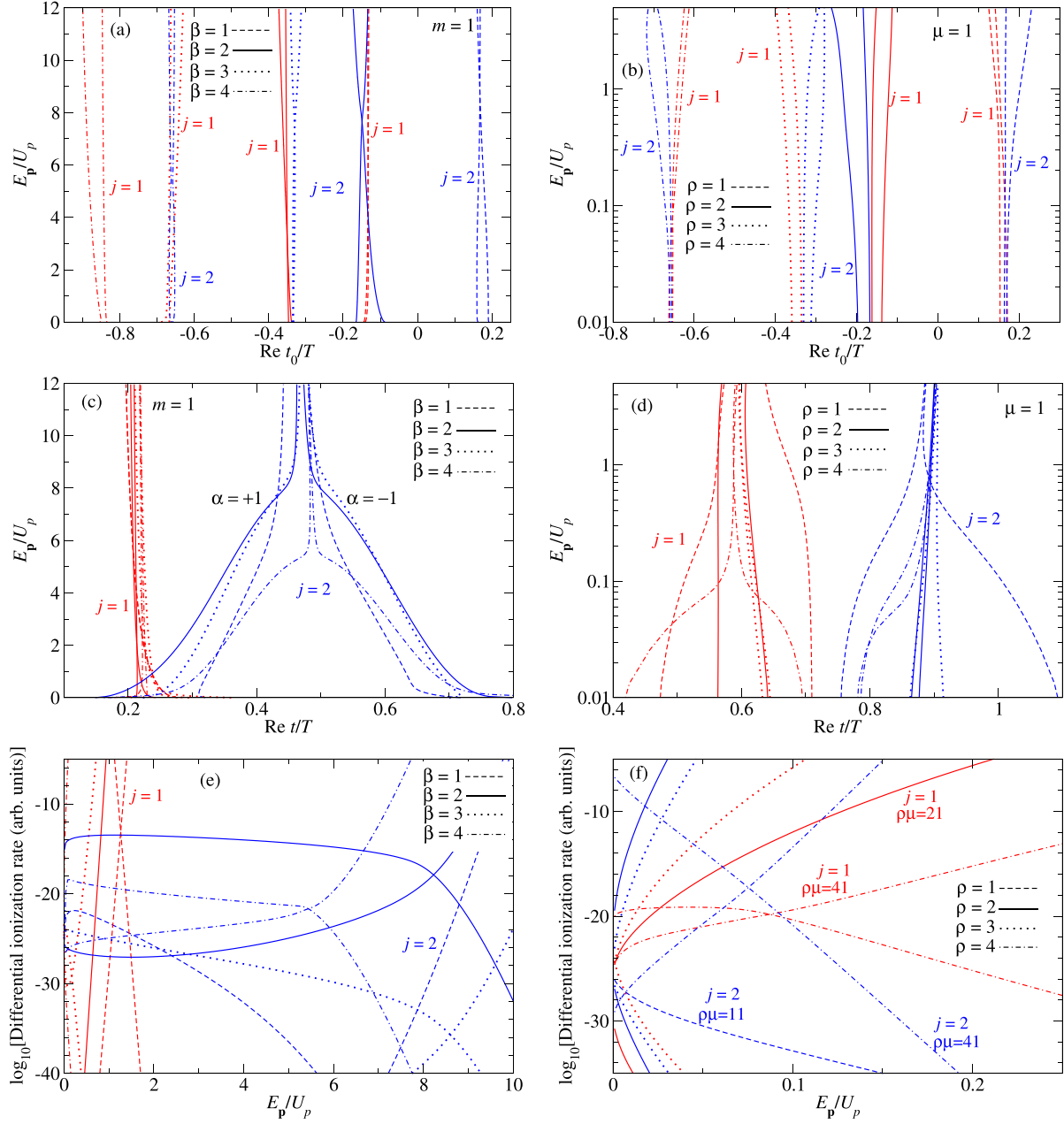


FIG. 2. Same as in Fig. 1 but for an ω - 2ω OTC laser field with the fundamental wavelength 1300 nm, the component intensities $E_1^2 = E_2^2 = 2 \times 10^{14}$ W/cm², and the phases $\phi_1 = \phi_2 = 0^\circ$.

IV. RESULTS FOR THE BICHROMATIC FIELD

After analyzing the case of a monochromatic field, we now focus on the bichromatic driving fields. As examples, we use the ω - 2ω OTC and bicircular fields, which are frequently employed to induce the strong-field processes.

A. Backscattering and forward-scattering solutions

1. ω - 2ω OTC field

First we investigate the case of the ω - 2ω OTC driving field ($\eta = 90^\circ$, $\varepsilon_1 = \varepsilon_2 = 0$) with equal component intensities $E_1^2 = E_2^2 = 2 \times 10^{14}$ W/cm², a fundamental wavelength of 1300 nm, and the phases $\phi_1 = \phi_2 = 0^\circ$. This field consists of

two linearly polarized components with mutually orthogonal polarizations. In Fig. 2 we present the SP solutions for HATI induced by our ω - 2ω OTC field for which $n_1 = 2s = 4$ and $n_2 = s = 2$. Both, the BS [panels (a) and (c)] and the FS [panels (b) and (d)] solutions are displayed. In total, there are $2n_1n_2 = 4s^2 = 16$ BS and $2n_1n_2 = 4s^2 = 16$ FS solutions. The solutions appear in pairs. For the BS solutions, the partial contribution of one solution of a given pair is divergent, while the contribution of the other is convergent after the cutoff. On the other hand, for the FS solutions, both solutions of a single pair can be divergent or convergent, or one solution of the pair is convergent while the other diverges. The BS and FS solutions can be classified into two groups denoted by the index $j = 1$ and 2. Only the BS group with $j = 2$

[blue lines in Figs. 2(a) and 2(c)] contributes in the medium- and high-energy parts of the spectrum. The BS solutions with $j = 1$ and all FS solutions contribute in the low-energy part of the spectrum. By looking at the shapes of the curves corresponding to the FS solutions, it is difficult to distinguish the BS group with $j = 1$ from the FS solutions. This is the case due to the fact that the shapes of the curves representing the photoelectron energy as a function of the real part of the ionization time are not specific for the FS solutions, as was the case for a monochromatic linearly or even elliptically polarized field. The FS solutions can be distinguished by the fact that for these solutions, $\mathbf{k}_{\text{st}} = \mathbf{p}$. This equation is not exactly but rather approximately satisfied because $\mathbf{k}_{\text{st}}(t_0, t)$ is complex while $\mathbf{p}$ is real.

The partial contributions of the BS solutions with $j = 2$ are responsible for the medium- and high-energy parts of the spectrum. In particular, for our choice of the values of the driving-field parameters of the employed ω - 2ω OTC field, only the BS solution with $(\alpha, \beta, m)_j = (1, 2, 1)_2$ is relevant for $E_p > 2U_p$ [see the blue solid line in Fig. 2(e) corresponding to the convergent contribution]. Other solutions lead to much smaller partial contributions to the differential ionization rate. On the other hand, the BS solutions with $j = 1$ and the FS solutions with $j = 1$ and 2 are responsible for the low-energy part of the spectrum. The partial contributions of these solutions are either convergent or divergent and usually do not have a clearly defined cutoff. Some exceptions include the solutions with $(\rho, \mu)_j = (4, 1)_1$ [see the red dash-dotted lines in Fig. 2(f)] whose partial contributions exhibit a short plateau.

2. ω - 2ω bicircular field

After analyzing the OTC driving field, we now turn our attention to the case of a bicircular counterrotating field ($\eta = 90^\circ$, $\varepsilon_1 = \varepsilon_2 = 1$) with equal component intensities $E_1^2 = E_2^2 = 2 \times 10^{14}$ W/cm², fundamental wavelength of 1300 nm, and the phases $\phi_1 = 0^\circ$ and $\phi_2 = -90^\circ$. For these values of the phases, the photoelectrons with high energy can be expected in the direction $\theta = 0^\circ$. In Fig. 3 we present the SP solutions for HATI induced by our ω - 2ω bicircular field for which $n_1 = r + s = 3$ and $n_2 = s = 2$ and the total number of expected solutions is $2n_1n_2 = 2s(r + s) = 12$. As for the OTC field, the FS solutions can be distinguished from the BS solutions by the fact that for these solutions, $\mathbf{k}_{\text{st}} = \mathbf{p}$. Again, only one group (with $j = 2$) of the BS solutions is responsible for the medium- and high-energy parts of the spectrum. However, contrary to the case of an OTC field, many SP solutions are relevant [see the blue lines in Fig. 3(e)]. The BS solutions with $j = 1$ as well as the FS solutions are responsible for the low-energy part of the spectrum. Contrary to the case of the OTC driving field, the partial contributions of these solutions for our bicircular field exhibit plateau structure much more frequently. They are essential for accurate calculations of the low-energy photoelectron spectrum. For example, the solutions $(\beta, m)_j = (1, 1)_1$ and $(\rho, \mu)_j = (3, 1)_1$ lead to the partial contributions, which are either comparable to or even larger than the contributions of the BS solutions with $j = 2$ [cf. the positions of the blue lines in Fig. 3(e) with the positions of the red lines in Fig. 3(e) and with the positions of the red and blue lines in Fig. 3(f)].

The general rule is that the partial contributions corresponding to the FS SP solutions are responsible for the low-energy part of the spectrum, while the BS contributions are relevant for all values of the photoelectron energy. Depending on the values of the driving-field parameters, the contributions of some BS solutions may have similar shapes to the FS solutions and contribute only in the low-energy part of the spectrum. In this case, the easiest way to identify the FS solutions is to check whether the condition $\mathbf{k}_{\text{st}} = \mathbf{p}$ is satisfied. Here, it is important to mention again that this condition cannot be exactly satisfied due to the fact that $\mathbf{k}_{\text{st}}(t_0, t)$ is complex (because t_0 and t are complex), while $\mathbf{p}$ is real. However, the difference between the real part of the stationary momentum and the final photoelectron momentum is much smaller for the FS solutions than for the BS solutions.

B. Backscattering-like and forward-scattering-like solutions

Let us now analyze the case of backscattering-like and forward-scattering-like solutions which have short or even negative travel time. In our classification, these solutions are denoted by the indices $m = 0, 0'$ and $\mu = 0, 0'$ and their ionization time is located in the interval $t_0 \in [0, T]$ [see Figs. 1(a) and 1(b) for the monochromatic elliptically polarized field case where there are two pairs of each of these solutions]. We will show (and illustrate in Fig. 4) that for the bichromatic elliptically polarized field, some of the $m = 0, 0'$ solutions are not always present (analogously this is valid for the $\mu = 0, 0'$ solutions). These solutions may be replaced by the $m = 1'$ solutions. Namely, for every BS SP solution $(\alpha, \beta, m)_j$ there is the corresponding negative-travel-time solution $(\alpha, \beta, m')_j$. The number of solutions per optical cycle is fixed to $2n_1n_2$, and, if the $m = 1'$ solutions appear in the optical cycle $[0, T]$, there is no place for some of the $m = 0, 0'$ solutions. This means that the time interval $[0, T]$ can be occupied partially by the $m = 0, 0'$ solutions and partially by the $m = 1'$ solutions. However, the mentioned solutions are always present for the one-component fields considered in the first two parts of our series of papers [71,72].

For illustration, we use our ω - 2ω bicircular counterrotating field with the parameters as in Fig. 3. For $\theta = 0^\circ$, for every SP solution (t_0, t) with the positive travel time τ there is a corresponding solution $(-t_0^*, -t^*)$ with the negative travel time $-\tau$. In Fig. 4 we present the SP solutions for the values of $\text{Re } t_0$ which include the region where the short-travel-time solutions with $m = 0, 0'$ and $\mu = 0, 0'$ are expected. For the BS solutions with $j = 1$ [panel (a)], the solutions with $m = 1$ ($m = 1'$) are presented with the red (green) lines. Clearly, the real parts of the ionization and rescattering times of these solutions are related via reflection symmetry with respect to zero. The values of $\text{Re } t_0$ for the $(\beta, m) = (1, 1)$ solutions are relatively close to the corresponding values of $\text{Re } t$. Consequently, the optical cycle consecutive to the one in which the $m = 1$ solutions are situated is occupied by the corresponding $m = 1'$ solutions, and the solutions with $m = 0$ and/or $m = 0'$ are not present at all.

The situation is different for the BS solutions with $j = 2$ [panel (b)]. These SP solutions with $m = 1$ are presented by the blue lines. In this case, the values of $\text{Re } t_0$ for the $(\beta, m) = (1, 1)$ solutions are not close to the corresponding values of

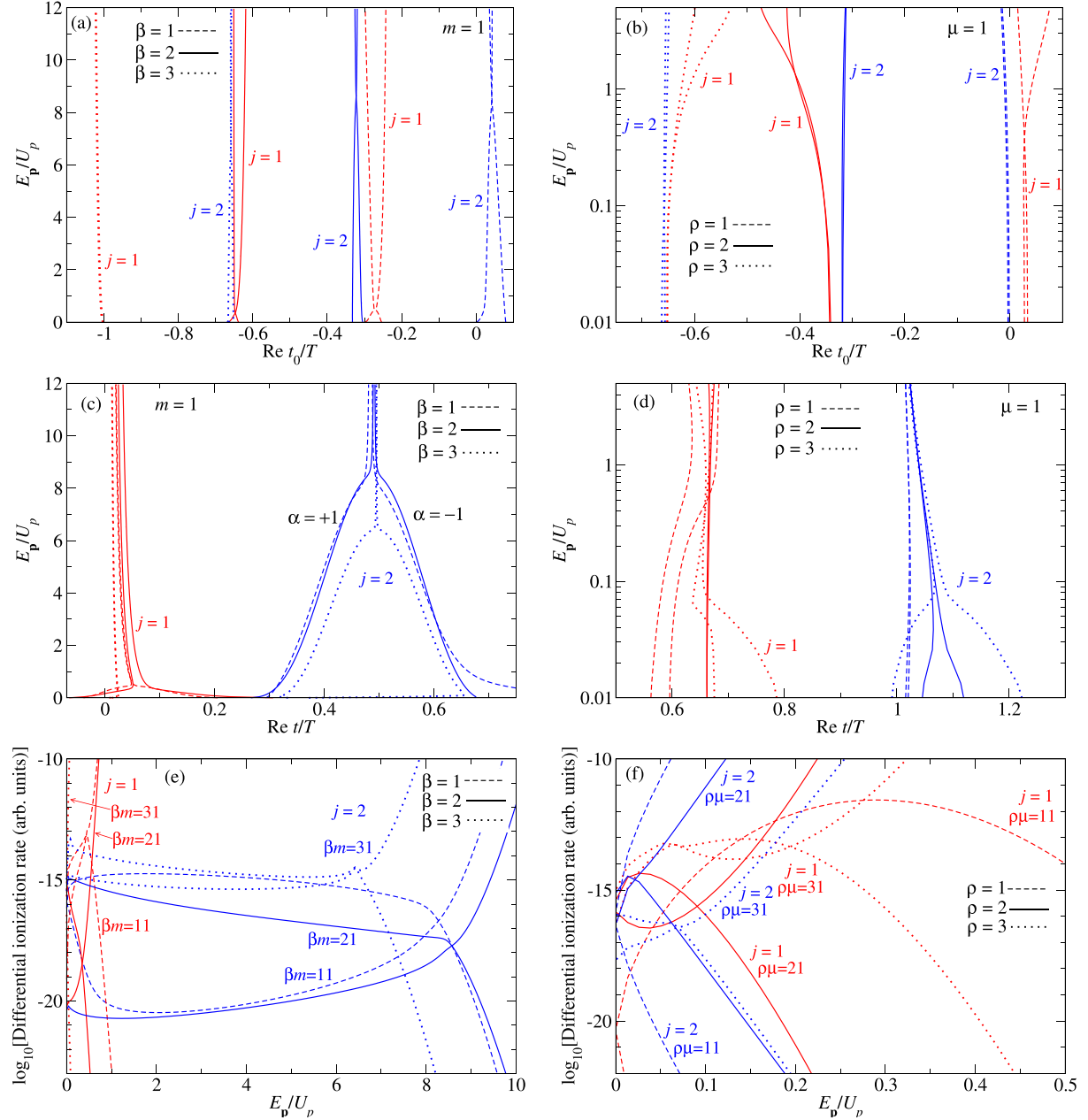


FIG. 3. Same as in Fig. 1 but for an ω - 2ω bicircular counterrotating laser field with the fundamental wavelength 1300 nm, the component intensities $E_1^2 = E_2^2 = 2 \times 10^{14} \text{ W/cm}^2$, and the phases $\phi_1 = 0^\circ$ and $\phi_2 = -90^\circ$.

$\text{Re } t$. Consequently, the optical cycle consecutive to the one in which the $m = 1$ solutions are situated is not fully occupied by the corresponding $m = 1'$ SP solutions. More precisely, only the solutions with $(\beta, m) = (1, 1')$ (green dashed line) are in this optical cycle, leaving “enough space” for two pairs of the $m = 0, 0'$ solutions (see the magenta dashed and solid lines). The pair with $(\beta, m) = (1, 0)$ has the positive travel time, while the pair with $(\beta, m) = (2, 0')$ has the negative travel time. Their partial contributions to the differential ionization rate are relevant for the photoelectron energy up to $3U_p$.

Similar conclusions hold for the forward-scattering-like solutions with $j = 1$ [panel (c)]. The SP solutions with $\mu = 1$ are presented by the red lines. The corresponding

consecutive optical cycle is occupied by two pairs of the forward-scattering-like solutions [with $(\rho, \mu) = (1, 0)$ and $(\rho, \mu) = (2, 0')$; see the magenta dashed and solid lines] and one pair of the negative-travel-time solutions with $(\rho, \mu) = (1, 1')$. For the forward-scattering-like solutions with $j = 2$ [panel (d)], the values of $\text{Re } t_0$ for the $(\rho, \mu) = (1, 1)$ solutions, presented by the blue dashed lines, are far away from the corresponding values of $\text{Re } t$. Therefore, the negative-travel-time solutions $(\rho, \mu) = (\rho, 1')$ are not in the optical cycle, which is consecutive to the optical cycle occupied by the $\mu = 1$ solutions, so that this cycle contains three pairs of the $\mu = 0$ solutions. In this case, the corresponding negative-travel-time counterparts of the $\mu = 0$ solutions have

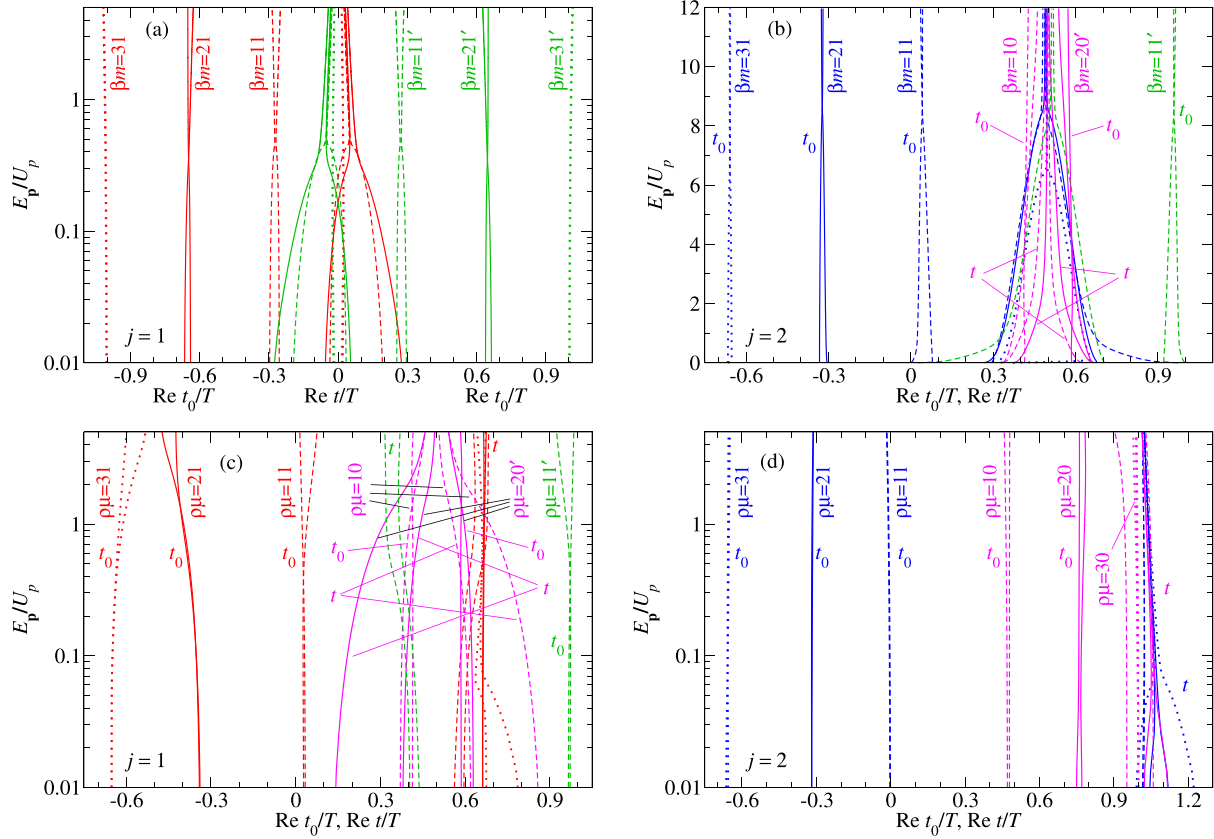


FIG. 4. Saddle-point solutions for HATI induced by the same field as in Fig. 3 for the values of the real part of the ionization time, which includes the region where the short-travel-time solutions $m = 0, 0'$ and $\mu = 0, 0'$ are expected. Panels (a) and (b) correspond to the backscattering and backscattering-like solutions with $j = 1$ and 2 , respectively, while panels (c) and (d) correspond to the forward-scattering and forward-scattering-like solutions with $j = 1$ and 2 , respectively.

$\text{Re } t_0 > T$. The partial contributions to the differential ionization rate of all forward-scattering-like solutions are restricted to the low-energy region of the photoelectron spectrum.

Finally, we mention that one might choose different intervals for the values of the real part of the ionization and rescattering times. This may lead to the association of different indices (compared to our classification) to particular SP solutions, but the number of solutions per optical cycle always remains unchanged.

V. APPLICATION—CHANNEL CLOSINGS FOR ELLIPTICAL POLARIZATION

To illustrate application of our classification of the SP solutions, we consider HATI by the monochromatic elliptically polarized laser field for the parameters of Fig. 1. In this case, only the BS solutions (α, β, m) , $\alpha = \pm 1$, $\beta = 1, 2$, $m = 1, 2, \dots$, contribute to the medium- and high-energy parts of the spectrum. We calculate the rescattering differential ionization rate using the uniform approximation (see [85] and references therein) and compare it with the exact result obtained by numerical integration. Our results are shown in Fig. 5. We see that the result obtained using 20 BS SP solutions ($m \leq 5$) is in excellent agreement with the exact result. On the other hand, the result obtained using only four solutions ($m = 1$) is satisfactory only in the cutoff region.

This is expected since only the shortest $m = 1$ BS solutions contribute in this region.

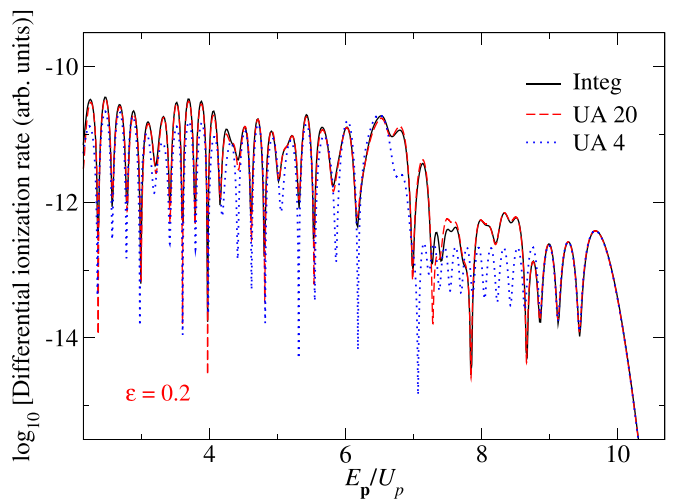


FIG. 5. The logarithm of the rescattering differential ionization rates as functions of the photoelectron energy (in units of U_p) for the same laser and atomic parameters as in Fig. 1. The results are obtained by numerical integration (black solid line) and using the uniform approximation with 20 (red dashed line) or 4 (blue dotted line) BS SP solutions.

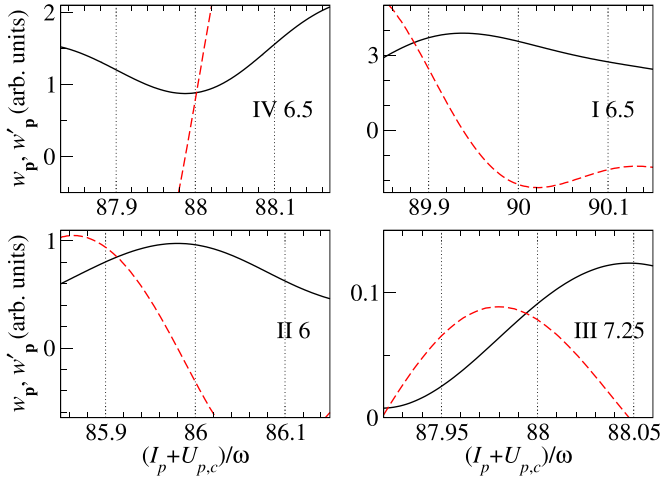


FIG. 6. Behavior of the differential ionization rates (black solid lines) and their first derivatives (red dashed lines) as functions of the channel-closing parameter $(I_p + U_{p,c})/\omega$ near particular integer values of this parameter. The results are for the Ne atom, $\theta = 0^\circ$, and the monochromatic laser field with the ellipticity $\epsilon = 0.2$, wavelength 1300 nm, and the intensities which correspond to the ponderomotive energy $U_{p,c}$. Upper left (right) panel: type IV (I) threshold anomaly for the photoelectron energy $E_p = 6.5U_p$. Lower left (right) panel: type II (III) threshold anomaly for $E_p = 6U_p$ ($E_p = 7.25U_p$). The ponderomotive energy U_p corresponds to the intensity $4 \times 10^{14} \text{ W/cm}^2$.

It is known that for linear polarization, resonant enhancements in the rescattering spectrum appear for particular values of the laser intensity which satisfy the channel closings condition $I_p + U_{p,c} = n_c\omega$, where n_c is an integer, and the ponderomotive energy $U_{p,c}$ corresponds to the channel closing intensity [75]. In this case, many quantum orbits interfere constructively, and more SP solutions should be taken into account. The explanation of the channel closing effect is the following: with the increase of the laser intensity, at some intensity the absorption of n_c photons no longer suffices for ionization, and $n_c + 1$ photons are required. In this case, we say that the n_c -photon channel has closed. In HATI this effect is manifested as the threshold anomalies which appear in the time-dependent scattering theory and reactions [86]. In Ref. [75] we expressed the rescattering part of the T -matrix element as a sum over the integer number n' , and we have shown that it contains the term $\sqrt{n'_c - n_c}$ where from all values of n' we explicitly separated the term $n' = n'_c$ with n'_c the integer nearest n_c . If we consider this T -matrix element as an analytical function of the parameter n_c , we see that the nonanalyticity (and the threshold anomaly) appears for $n_c = n'_c$. It is shown in [75] that for a monochromatic linearly polarized laser field, the differential ionization rate near the channel closing can exhibit a rounded downward step, which we called the type-I enhancement, or an upward cusp (we called this the type-II enhancement). We will now show that for an elliptically polarized field, additional types of the enhancements appear, i.e., all types of the threshold anomalies from [86] are manifested. This means that, in addition to the downward steps and upward cusps, we have upward steps (we call it type-III threshold anomaly) and downward cusps (type-IV).

The type of the threshold anomaly can be determined by looking at the first derivative of the differential ionization rate with respect to n_c . For the type-I enhancements, this derivative goes to a large negative value at the position of the channel closing, while for the type-II enhancements it goes from $+\infty$ just below n_c to $-\infty$ just above n_c . For the upward step (type-III), the derivative has a maximum at the channel closing, while for the downward cusp the derivative goes from $-\infty$ just below n_c to $+\infty$ just above n_c . In Fig. 6 we have shown all four types of threshold anomalies using our example of an elliptically polarized laser field. For calculation, we used the uniform approximation and 20 BS SP solutions. The obtained minima/maxima would be sharper if more SP solutions are taken into consideration.

In summary, for accurate investigation of the channel closing for two-component driving fields, all SP solutions from a certain time interval have to be taken into consideration. This can only be ensured if the classification of the SP solutions is sufficiently complete to capture all SP solutions within the relevant time interval. We have illustrated this using a monochromatic elliptically polarized driving field which has the x and y components. We expect that analogous results can be obtained for bichromatic fields, but this is beyond the scope of our paper. In this case, even for linear polarization, novel effects can appear [87], and the use of the uniform approximation or more advanced methods of asymptotic analysis of integrals [36], necessary for the analysis of the channel closing effects, is more demanding.

VI. CONCLUSIONS

Strong-field processes such as high-order above-threshold ionization lie at the core of attosecond science and ultrafast atomic physics. The ability to precisely manipulate temporal, spectral, and polarization characteristics of laser fields, particularly through tailored two-component field configurations, has opened new avenues for controlling electron dynamics on the subfemtosecond timescale. Bichromatic fields, including orthogonally polarized and bicircular combinations, offer a fine control over the ionization and rescattering steps of strong-field processes, leading to the new insights and improved resolution in probing and steering strong-field phenomena.

In this paper, we have provided a comprehensive generalization of the classification of the SP solutions for high-order above-threshold ionization to the case of arbitrary bichromatic elliptically polarized laser fields. Following our previous analysis for a linearly polarized driving field, we demonstrated that for a bichromatic field consisting of two elliptically polarized components with frequencies $r\omega$ and $s\omega$ ($s > r$) there are $8s^2$ SP solutions per optical cycle. These solutions are separated into two classes: the BS solutions usually responsible for the medium- and high-energy parts of the spectrum, and the FS solutions which contribute significantly in the low-energy domain.

Furthermore, we have explored how the number of SP solutions depends on the specific parameters of the laser field. Special cases such as counterrotating and corotating bicircular fields exhibit distinct symmetry properties and yield fewer solutions, namely $4s(r + s)$ and $4s^2$, respectively. For each of

TABLE I. Summary of the classification of the saddle-point solutions and their characteristics.

Backscattering solutions	Forward-scattering solutions
$(\alpha, \beta, m)_j$	$(\nu, \rho, \mu)_j$
$\beta = 1, \dots, n_1$ n_1 – degree of the trigonometric polynomial in $\varphi_{0\beta}$, Eq. (12), $\text{Im } \varphi_{0\beta} > 0$	$\rho = 1, \dots, n_1$ n_1 – degree of the trigonometric polynomial in $\varphi_{0\rho}$, Eq. (15), $\text{Im } \varphi_{0\rho} > 0$
$\alpha = \pm 1, \quad j = 1, \dots, n_2$ n_2 – degree of the trigonometric polynomial in $\varphi_{\alpha j}$, Eq. (13)	$\nu = \pm 1, \quad j = 1, \dots, n_2$ n_2 – degree of the trigonometric polynomial in $\varphi_{\nu j}$, Eqs. (16) and (17)
$m = 0, 1, 2, \dots$ ^a $m = 1, 2, \dots$: relevant for all energies ^b $m = 0$: backscattering-like, $E_p \lesssim 3U_p$ ^{c,d}	$\mu = 0, 1, 2, \dots$ ^a $\mu = 1, 2, \dots$: relevant for $E_p < 1U_p$ ^c $\mu = 0$: forward-scattering-like, $E_p \lesssim 3U_p$ ^{c,d}
monochromatic: $j = 1$ – relevant for all energies $\omega - 2\omega$: $j = 1, 2$ – all energies ^e	monochromatic: $j = 1$ – relevant for low energies $\omega - 2\omega$: $j = 1, 2$ – low energies
OTC or bicircular corotating $r\omega$ - $s\omega$: $2s^2$ solutions bicircular counterrotating $r\omega$ - $s\omega$: $2s(r+s)$ solutions	OTC or bicircular corotating $r\omega$ - $s\omega$: $2s^2$ solutions bicircular counterrotating $r\omega$ - $s\omega$: $2s(r+s)$ solutions

^a m, μ – approximate length of the travel time τ in optical cycles, $\tau = \text{Re}(t - t_0) \approx mT, \mu T$. The photoelectron wave packet spreads and the probability amplitude decreases as $\tau^{-3/2}$.

^bRelevant for oscillations in the medium- and high-energy parts of the spectrum.

^cRelevant for the low-energy and holographic structures.

^dThese solutions are characterized by very short or negative travel time.

^eContributions of some solutions may be restricted to the low energies.

these configurations, the total number of SP solutions remains consistent with the predicted trigonometric polynomial structure of the SP equations.

Apart from the BS and FS solutions, we have also analyzed the short-travel-time solutions, which we called the backscattering-like and forward-scattering-like solutions. These solutions, which include those with short or even negative travel times, often deviate from the standard semiclassical three-step model and require special attention in both numerical and analytical treatments. While their contributions are generally restricted to the low-energy part of the photoelectron spectrum, their inclusion is necessary for a complete description, particularly in the regimes where interference effects and subtle temporal dynamics play a significant role.

Finally, we summarize the relevance of different SP solutions (Table I). The BS solutions are generally relevant for all values of the photoelectron energy, and the interference of their partial contributions explains the oscillations in the medium- and high-energy parts of the spectrum. The partial contributions of some of these solutions may be restricted only to the low-energy part of the spectrum. On the other hand, the FS solutions, as well as the BS-like and FS-like solutions, are crucial for the low-energy part of the spectrum. The interference of these solutions with the partial contribution of the direct electrons is responsible for the low-energy and holographic structures. The partial contributions of all SP solutions decrease with the increase of the electron travel time (i.e., with the increase of m or μ) due to the electron wave-

packet dissipation. The number of optical cycles that should be taken into consideration depends on the required accuracy.

Overall, the classification presented here provides a unified theoretical framework for understanding and predicting the number of SP solutions for the case of a two-component driving field. This contributes to a deeper understanding of the HATI mechanism, and it also paves the way for precise control and analysis of different parts of the photoelectron spectrum using structured and/or multicolor light fields. Finally, we comment on the role of Coulomb corrections. The presented classification relies on the analytic structure of the saddle-point equations, which is determined solely by the external driving field, ionization potential, and the photoelectron momentum. Long-range Coulomb effects modify the action and the saddle-point equations, but whether, and to what extent, they alter the number and/or topology of the saddle-point solutions remains an open question. In particular, Coulomb corrections may lead to additional low-energy trajectories [82], while the backbone classification into backward- and forward-scattering families remains unaffected. This demonstrates that our classification provides a solid foundation for extending quantum-orbit approaches to incorporate Coulomb interaction.

DATA AVAILABILITY

The data that support the findings of this article are not publicly available. The data are available from the authors upon reasonable request.

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