

Quantum limits of exoplanet detection and localization

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Discovering exoplanets in orbit around distant stars via direct imaging is fundamentally impeded by the combined effect of optical diffraction and photon shot noise under extreme star-planet contrast. Coronagraphs strive to increase the signal-to-noise ratio of exoplanet signatures by optically suppressing light from the host star while preserving light from the exoplanet. However, it is unclear whether direct-imaging coronagraphs constitute an optimal strategy for attaining fundamental limits relevant to exoplanet discovery. In this work, we first review the quantum information limits of exoplanet detection and localization characterized by (1) the quantum Chernoff exponent for symmetric hypothesis testing, (2) the quantum relative entropy for asymmetric hypothesis testing, and (3) the quantum Fisher information matrix for multiparameter estimation. We demonstrate that coronagraphs designed to completely suppress light in the fundamental mode of the telescope, while perfectly transmitting higher-order orthogonal modes, indeed achieve these limits in the regime of high star-planet contrasts. Furthermore, we formulate coronagraphs as quantum channels, thus generalizing the classical framework of coronagraphy to the quantum setting. Using this framework, we compare the information-theoretic performance of leading coronagraph designs against the quantum limits. Our analysis indicates that quantum-optimal coronagraphs offer enhanced information efficiency in the subdiffraction regime compared to leading coronagraph designs and may significantly expand the domain of accessible exoplanets.

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I. INTRODUCTION

Exoplanets provide intriguing case studies for investigating extrasolar dynamics, planet formation, atmospheric science, and potentially the origins of life [1–3]. Direct imaging offers the most promising approach for measuring the spectra of an exoplanet, which can be used to analyze a planet's atmospheric composition and identify possible biosignatures [4,5]. However, the search for habitable exoplanets with direct imaging is fundamentally inhibited by the combined effect of optical diffraction and photon shot noise [6,7].

For an exoplanet whose angular separation from its parent star resides near or below the Rayleigh resolution limit of the telescope, $\sim \lambda/D$, it becomes increasingly difficult to discriminate whether a detected photon originated from the star or from the exoplanet. This uncertainty is a consequence of diffraction through the telescope aperture, which destroys the orthogonality between optical fields excited by the star and the exoplanet. The overlap between these optical fields in the image plane, combined with the relative scarcity of exoplanet

photons compared to stellar photons, places fundamental limits on our ability to detect exoplanets [8].

Additionally, fundamental shot noise induced by the Poissonian nature of photon arrivals for thermal sources prohibits obtaining exact knowledge of the underlying probability distribution governing the statistics of a particular measurement. The detriments of this statistical noise are accentuated under high contrast, where shot noise from the star completely overwhelms signal-bearing photons from the exoplanet. Ultimately, uncertainty in the underlying probability distribution translates into uncertainty in the estimate of the *parameters* of that distribution. Therefore, shot noise places fundamental limits on our ability to localize an exoplanet relative to its host star [9].

Earth-like exoplanets around Sun-like stars are expected to be relatively common, with recent estimates conservatively reporting between 0.37 and 0.6 exoplanets per star using data from Kepler surveys [10]. Most of these exoplanets reside below the diffraction limit of existing space-based observatories, requiring indirect detection methods such as radial velocity, astrometry, and gravitational microlensing [11], which cannot resolve spectra [12]. To illustrate the challenges of direct imaging, consider the James Webb space telescope operating at infrared wavelengths ($D = 6.5$ m, $\lambda = 1550$ nm). The population density of giant exoplanets peaks between 1 and 3 astronomical units (A.U.) away from their parent stars [13]. Meanwhile, Earth-like exoplanets in the habitable zone occupy even tighter orbits at typical distances of 0.1 to 1 A.U. [14]. For solar systems further than approximately 50 parsecs, all of these orbits fall below the diffraction limit of

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the telescope. Moreover, typical star-planet contrast ratios are extremely high, spanning from $10^5:1$ for hot Jupiters up to $10^{10}:1$ for colder Earth-like planets [1]. Taken in concert, the characteristic distance scales and contrast levels of exoplanets paired with the wave-particle nature of photons conspire to birth a formidable sensing challenge—one that motivates the development of high-resolution telescopes with coronagraphs operating at the fundamental limits imposed by physics.

A coronagraph is a passive linear optical instrument that reduces photon shot noise by rejecting light from the host star prior to photodetection. Various coronagraph designs have been proposed toward this aim, each with different working principles that lead to associated real-world advantages and disadvantages. Despite the vast landscape of designs, all coronagraphs can be modeled as a mode-selective optical filter [15,16], which provides a common mathematical framework for comparing their performance. The principal design objective of any coronagraph is to filter out optical modes excited by the star without unnecessarily discarding modes excited by the exoplanet. Historically, the high-contrast-imaging community has quantified the performance of different coronagraphs via operational metrics such as inner working angle and throughput. While these figures of merit have definitive practical utility, to the best of our knowledge, no rigorous information-theoretic analysis of coronagraphs in the context of exoplanet discovery has been undertaken.

Since the seminal work of Tsang and collaborators appeared in the mid-2010s [17], analysis on the quantum information limits of imaging subdiffraction incoherent source pairs has flourished. This body of literature maps directly to the problem of *exoplanet discovery*, which we formally regard as a detection task and a localization task [18]. Various papers from the quantum information community have already derived information-theoretic bounds for both of these tasks. In this work, our first objective is to consolidate known quantum information bounds on exoplanet discovery and elucidate noteworthy connections between optimal exoplanet detection and localization schemes. Moreover, we hope to offer a self-contained synthesis that bridges insights from the high-contrast-imaging and quantum information communities. To this end, we examine three quantum information measures: (1) the *quantum Chernoff exponent* (QCE), which bounds the asymptotic scaling of the error rate for detecting the presence or absence of an exoplanet planet under symmetric hypothesis testing [19,20] (2) the *quantum relative entropy* (QRE), which bounds the asymptotic scaling of false-negative detection rates under asymmetric hypothesis testing [21,22]; and (3) the *quantum Fisher information matrix* (QFIM), which provides a Cramér-Rao bound on the minimum achievable imprecision for an unbiased estimate of the planet position [23–26].

Our second objective is to formulate a general quantum model of coronagraphs. Taking inspiration from classical models [15,16], we define coronagraphs as pure loss multimode bosonic channels followed by a direct-imaging measurement. Using this model, we compare the information-theoretic performance of three state-of-the-art instruments: (1) the phase-induced amplitude apodization complex mask coronagraph (PIAACMC), (2) the vortex coronagraph (VC), and (3) the perfect coronagraph (PC). We find that the PIAACMC and VC approach the quantum information bounds

when star-planet separations are large relative to the diffraction limit, but they diverge from the bounds as the star-planet separation decreases below the diffraction limit. In comparison, we prove that the PC fully saturates the quantum bounds even at subdiffraction star-planet separations in the limit of extreme star-planet contrasts. As an alternative to direct-imaging coronagraphs, we also consider spatial mode demultiplexing (SPADE), which measures modal projections of the incident optical field. SPADE is also known to saturate all of the quantum information bounds [20,21,23], demonstrating the promise of mode-sorting solutions for future flagship space telescopes such as NASA’s Large Ultraviolet Optical Infrared Surveyor (LUVOIR), Habitable Exoplanet Observatory (HabEx), and Habitable Worlds Observatory (HWO) concepts [27–33].

Remarkably, the mathematical expressions for the quantum information bounds on exoplanet detection and localization admit intuitive interpretations, revealing that *exclusive* isolation or rejection of the telescope’s fundamental optical mode is critical to realizing an optimal measurement strategy. The quantum bounds also indicate that accessible information persists into deep subdiffraction star-planet separations, where current observational models predict an abundance of potential exoplanets [13,34–36].

II. PRELIMINARIES

A. Imaging system model

We consider a canonical optical imaging system with a circular pupil of radius R and focal length f operating at infinite object conjugate. For a wavelength λ , the traditional diffraction limit of spatial resolution (Rayleigh limit) is defined to coincide with the first zero of the Airy pattern located a radial distance $\mathcal{R} = 1.22 \frac{\lambda f}{2R}$ from the optical axis in the focal plane. Let the Cartesian coordinates of the pupil plane and focal plane be (X', Y') and (X, Y) , respectively, and define the dimensionless pupil plane $\mathbf{u} \equiv \frac{1}{R}(X', Y')$ and focal plane $\mathbf{r} \equiv \frac{R}{\lambda f}(X, Y)$ coordinate vectors. The pupil function $\tilde{\psi}(\mathbf{u})$ and the point spread function (PSF) $\psi(\mathbf{r})$ are given by

$$\tilde{\psi}(\mathbf{u}) \equiv \frac{1}{\sqrt{\pi}} \begin{cases} 1, & |\mathbf{u}| \leq 1 \\ 0, & |\mathbf{u}| > 1, \end{cases} \quad \psi(\mathbf{r}) = \frac{J_1(2\pi r)}{\sqrt{\pi} r}, \quad (1)$$

where both functions are related by a Fourier transform $\psi(\mathbf{r}) = \mathcal{F}[\tilde{\psi}(\mathbf{u})] \equiv \int \tilde{\psi}(\mathbf{u}) e^{-i2\pi \mathbf{u} \cdot \mathbf{r}} d\mathbf{u}$ and satisfy the normalization condition $\int |\tilde{\psi}(\mathbf{u})|^2 d\mathbf{u} = \int |\psi(\mathbf{r})|^2 d\mathbf{r} = 1$. Throughout this work, we refer to the PSF as the “fundamental mode” of the telescope and use $\sigma \equiv (\frac{R}{\lambda f})\mathcal{R} = 0.61$ as the dimensionless Rayleigh unit.

B. State-space and quantum measurements

For weak incoherent quasimonochromatic thermal sources (see Appendix B in Ref. [17]) such as astronomical objects, the quantum state of the optical field is defined on the complex Hilbert space $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1$. This space is a direct sum of the multimode vacuum state $\mathcal{H}_0 = \text{span}(|0\rangle)$ and single-photon states of the “pixel modes” $\mathcal{H}_1 = \text{span}(|\mathbf{r}\rangle : \mathbf{r} \in \mathbb{R}^2)$. The state $|\mathbf{r}\rangle = \hat{a}^\dagger(\mathbf{r})|0\rangle$ specifies the creation of a single photon at position $\mathbf{r}$ on the image plane, and $\{\hat{a}(\mathbf{r}) : \mathbf{r} \in$

$\mathbb{R}^2$ are bosonic annihilation operators satisfying the canonical commutation relation $[\hat{a}(\mathbf{r}), \hat{a}^\dagger(\mathbf{r}')] = \delta(\mathbf{r} - \mathbf{r}')$. Using this notation, a single-photon state of any square-integrable complex transverse spatial mode $\varphi(\mathbf{r}) \in \mathbb{L}^2(\mathbb{R}^2, \mathbb{C})$ can be expressed as

$$|\varphi\rangle = \int \varphi(\mathbf{r})|\mathbf{r}\rangle d\mathbf{r} \in \mathcal{H}_1. \quad (2)$$

As a result, it can be shown that the inner product between any two single-photon states satisfies

$$\langle\varphi|\phi\rangle = \int \varphi^*(\mathbf{r})\phi(\mathbf{r})d\mathbf{r}, \quad |\phi\rangle, |\varphi\rangle \in \mathcal{H}_1. \quad (3)$$

Meanwhile, the inner product between any single-photon state and the vacuum state is trivially zero: $\langle\phi|0\rangle = 0$.

For incoherent scenes, it is necessary to work with mixed states $\hat{\rho} \in \mathcal{D}(\mathcal{H})$ where $\mathcal{D}(\mathcal{H})$ is the set of valid density operators on $\mathcal{H}$ satisfying positive semidefiniteness $\hat{\rho} \geq 0$ and unit trace $\text{Tr}[\hat{\rho}] = 1$. Any sensing task is generically concerned with acquiring information about a state $\hat{\rho}$ from measurements of the state. Formally, a measurement is defined by a positive operator-valued measure (POVM) $\{\hat{\Pi}_j : j \in \mathcal{J}\}$ satisfying the conditions

$$\hat{\Pi}_j \geq 0 \quad \forall j \in \mathcal{J} \quad \text{and} \quad \sum_{j \in \mathcal{J}} \hat{\Pi}_j = \hat{I}, \quad (4)$$

where $\hat{I}$ is the identity operator on $\mathcal{H}$ and the set $\mathcal{J}$ is the collection of possible outcomes for the measurement (i.e., the event space). The probability of observing an outcome $j \in \mathcal{J}$ under the measurement $\{\hat{\Pi}_j\}$ is given by the Born rule $p_j = \text{Tr}[\hat{\rho}\hat{\Pi}_j]$. Throughout this work, we frequently make use of von Neumann measurements consisting of the vacuum state projector and projectors on a complete orthonormal basis of single-photon states,

$$\{\hat{\Pi}_{\text{vac}} = |0\rangle\langle 0| \cup \{\hat{\Pi}_j = |\pi_j\rangle\langle \pi_j| : j \in \mathcal{J}\}, \quad (5)$$

where $|\pi_j\rangle \in \mathcal{H}_1$ and $\langle\pi_j|\pi_{j'}\rangle = \delta_{jj'}$. Applying such a measurement to any optical field state $\hat{\rho} \in \mathcal{D}(\mathcal{H})$ may yield either a single photon in mode π_j or no photons at all with the following probabilities:

$$\text{Pr}(\text{one photon in mode } \pi_j) = p_{\pi_j} = \text{Tr}[\hat{\Pi}_j \hat{\rho}], \quad (6a)$$

$$\text{Pr}(\text{no photons}) = p_{\text{vac}} = \text{Tr}[\hat{\Pi}_{\text{vac}} \hat{\rho}]. \quad (6b)$$

These measurements are known as SPADE [17]. Direct imaging is a special instance of SPADE corresponding to a POVM involving “pixel mode” projectors $\{\hat{\Pi}_{\mathbf{r}} = |\mathbf{r}\rangle\langle \mathbf{r}| : \mathbf{r} \in \mathbb{R}^2\}$.

C. Star-exoplanet model

Following Ref. [17], we invoke an idealized model where the star and exoplanet are assumed to be quasimonochromatic point sources, ignoring their finite size and polychromatic emission spectra. We further assume that the scene is effectively static over the measurement period, ignoring the temporal dynamics of the exoplanet around the star. Under these assumptions, the scene is fully characterized by the position of the star, $\mathbf{r}_s$, the position of the exoplanet, $\mathbf{r}_e$, and the relative brightness of the exoplanet, $b \in (0, 1)$. Alternatively,

we may parametrize the scene in terms of the center of intensity, $\mathbf{R} = (1 - b)\mathbf{r}_s + b\mathbf{r}_e$, and the separation $\mathbf{r}_\Delta = \mathbf{r}_e - \mathbf{r}_s$. All subsequent analysis considers the optical axis (origin of the coordinate system) to be aligned with the center of intensity, $\mathbf{R} = (0, 0)$, as shown in Fig. 1. Then the star and planet positions become

$$\mathbf{r}_s = -b\mathbf{r}_\Delta, \quad \mathbf{r}_e = (1 - b)\mathbf{r}_\Delta. \quad (7)$$

The parameters b and $\mathbf{R}$ are taken to be known *a priori* such that the localization task amounts to estimating the unknown star-planet separation vector $\mathbf{r}_\Delta$. Simultaneous estimation of the separation vector, the centroid, and the relative brightness adds substantial complexity to the quantum information analysis, which we defer to future studies [37].

The optical field produced by the star-planet system over a temporal coherence interval is either in the vacuum state with probability $1 - \epsilon$ or occasionally in a single-photon state with probability $\epsilon \ll 1$. Occupation numbers of more than one photon are assumed to be negligibly rare. The state of the field is thus well approximated by

$$\hat{\rho} = (1 - \epsilon)|0\rangle\langle 0| + \epsilon\hat{\rho}_1 + O(\epsilon^2), \quad (8)$$

where $\hat{\rho}_1$ is the single-photon mixed state,

$$\hat{\rho}_1 \equiv (1 - b)|\psi(\mathbf{r}_s)\rangle\langle\psi(\mathbf{r}_s)| + b|\psi(\mathbf{r}_e)\rangle\langle\psi(\mathbf{r}_e)|, \quad (9)$$

encapsulating the possibility that the photon came from either the star or the exoplanet. The states $|\psi(\mathbf{r}_s)\rangle$ and $|\psi(\mathbf{r}_e)\rangle$ represent single-photon states of a shifted PSF given by

$$|\psi(\mathbf{s})\rangle \equiv \int \psi(\mathbf{r} - \mathbf{s})|\mathbf{r}\rangle d\mathbf{r}. \quad (10)$$

Additionally, we define the correlation function between any transverse mode $\varphi(\mathbf{r})$ and the shifted PSF as

$$\Gamma_\varphi(\mathbf{s}) \equiv \langle\varphi|\psi(\mathbf{s})\rangle. \quad (11)$$

For any binary pupil (hard aperture) of area A , the correlation of an arbitrary mode $\varphi(\mathbf{r})$ whose Fourier transform has the same support as the pupil is

$$\Gamma_\varphi(\mathbf{s}) = \frac{1}{\sqrt{A}}\varphi^*(\mathbf{s}) \quad \forall \varphi : \tilde{\varphi}(\mathbf{u})\tilde{\psi}(\mathbf{u}) = \frac{1}{\sqrt{A}}\tilde{\varphi}(\mathbf{u}), \quad (12)$$

a fact which we leverage extensively in the Appendixes and in numerical simulations.

III. QUANTUM LIMITS OF EXOPLANET DISCOVERY

A. Exoplanet detection

Exoplanet detection amounts to performing a binary hypothesis test between two possible quantum states of the incident optical field,

$$\text{exoplanet absent } H_0 : \hat{\rho}_0 \approx (1 - \epsilon)|0\rangle\langle 0| + \epsilon\hat{\rho}_0, \quad (13a)$$

$$\text{exoplanet present } H_1 : \hat{\rho}_1 \approx (1 - \epsilon)|0\rangle\langle 0| + \epsilon\hat{\rho}_1. \quad (13b)$$

If the exoplanet is absent (null hypothesis H_0), then the single-photon component of the field state is given by $\hat{\rho}_0 = |\psi\rangle\langle\psi|$ because the optical axis is aligned directly to the star. Otherwise, if the exoplanet is present (alternative hypothesis H_1), then the single-photon component is given by $\hat{\rho}_1$ defined in Eq. (9) [38].

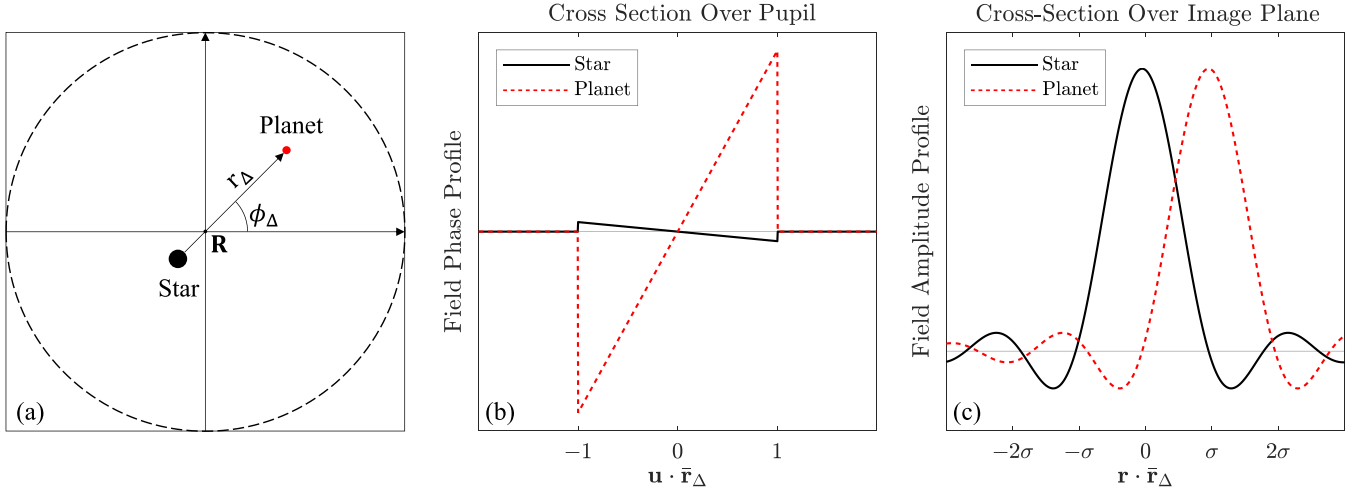


FIG. 1. (a) Geometry of the star-planet model with the optical axis aligned to the center of intensity, $\mathbf{R}$. (b) Cross section of the phase profiles over the pupil plane for the field generated by the star and planet independently. (c) Cross section of the amplitude profiles over the image plane for the fields produced by the star and planet independently. In the limit of high contrast, the optical axis nears coincidence with the star such that the star purely excites the fundamental mode of the telescope.

Suppose that we are supplied with M copies of an unknown state $\hat{\rho}_\gamma^{\otimes M}$, where it is guaranteed that $\hat{\rho}_\gamma$ may either be $\hat{\rho}_0$ or $\hat{\rho}_1$. Additionally, let q_0 and $q_1 = 1 - q_0$ be the prior probabilities of the hypotheses H_0 and H_1 , respectively. In the most general case, the goal of quantum hypothesis testing is to choose a measurement $\{\hat{\Pi}_j : j \in \mathcal{J}\}$ on $\mathcal{H}^{\otimes M}$ and a decision rule $\check{H} : \mathcal{J} \rightarrow \{\check{H}_0, \check{H}_1\}$, which maps a measurement outcome to an inference on the underlying hypothesis, such that the probability of error (symmetric or asymmetric) is minimized. Any pair of POVM and decision rule can be compressed into a single classification POVM via

$$\hat{T}_0 = \sum_{j \in \mathcal{J}} \hat{\Pi}_j \mathbb{1}[\check{H}(j) = \check{H}_0], \quad (14a)$$

$$\hat{T}_1 = \sum_{j \in \mathcal{J}} \hat{\Pi}_j \mathbb{1}[\check{H}(j) = \check{H}_1], \quad (14b)$$

where $\mathbb{1}[\cdot]$ is an indicator function. The total probability of error is given by the sum of false-positive (type I) and false-negative (type II) errors weighted by their priors,

$$P_e = q_0 P(\check{H}_1 | H_0) + q_1 P(\check{H}_0 | H_1) \\ = q_0 \text{Tr}[\hat{T}_1 \hat{\rho}_0^{\otimes M}] + q_1 \text{Tr}[\hat{T}_0 \hat{\rho}_1^{\otimes M}]. \quad (15)$$

The Helstrom bound provides a tight lower bound on the total probability of error achievable with any physically realizable joint measurement,

$$P_e \geq P_{e,\min} = \frac{1}{2} [1 - \|q_1 \hat{\rho}_1^{\otimes M} - q_0 \hat{\rho}_0^{\otimes M}\|], \quad (16)$$

where $\|\hat{O}\| = \text{Tr}[\sqrt{\hat{O}^\dagger \hat{O}}]$ is the trace norm. This bound is achieved by a binary POVM that projects onto the subspaces spanned by the positive and negative eigenstates of the operator $q_1 \hat{\rho}_1^{\otimes M} - q_0 \hat{\rho}_0^{\otimes M}$ [39]. However, joint measurements act on the full Hilbert space $\mathcal{H}^{\otimes M}$ which corresponds to measuring all of the received state copies at once (see Appendix A) [40]. In practice, we are often limited to copy-by-copy (separable) measurements that act on the smaller Hilbert space $\mathcal{H}$. For these measurements, the probability of error decreases

exponentially in the asymptotic limit of many copies,

$$P_e \sim e^{-\xi M}, \quad (17)$$

where $\xi > 0$ is called the Chernoff exponent. A useful metric would be to know the best asymptotic decay rate (largest ξ) achieved by optimizing over all separable measurements.

In the following two sections, we describe the quantum limits associated with symmetric and asymmetric hypothesis testing reported in Refs. [20] and [21], respectively. We note that the bounds for symmetric and asymmetric hypothesis testing converge in the regime of high contrast.

B. Exoplanet detection with symmetric hypothesis testing: The quantum Chernoff bound

In symmetric hypothesis testing, our goal is to choose a binary measurement $\{\hat{T}_0, \hat{T}_1\}$ that minimizes the probability of error, P_e . The problem is “symmetric” because we look to minimize the combined type-I and type-II error. The optimal probability of error for discriminating whether the field state is $\hat{\rho}_0$ or $\hat{\rho}_1$ asymptotically follows the quantum Chernoff bound (QCB) in large M [41],

$$P_{e,\min} \sim e^{-M \tilde{\xi}_Q}, \quad \tilde{\xi}_Q \equiv -\ln \left(\min_{0 \leq t \leq 1} \text{Tr}(\hat{\rho}_0^t \hat{\rho}_1^{1-t}) \right), \quad (18)$$

where $\tilde{\xi}_Q$ is the quantum Chernoff exponent (QCE). Under the weak thermal source model, Ref. [20] showed that

$$\tilde{\xi}_Q \approx \epsilon(1 - e^{-\xi_Q}), \quad \xi_Q \equiv -\ln \left(\min_{0 \leq t \leq 1} \text{Tr}(\hat{\rho}_0^t \hat{\rho}_1^{1-t}) \right), \quad (19)$$

where ξ_Q is the single-photon QCE. Furthermore, when specializing to the subdiffraction regime $|\mathbf{r}_\Delta| \ll \sigma$, we have (see Supplemental Material of Ref. [20])

$$\tilde{\xi}_Q \approx \epsilon \xi_Q. \quad (20)$$

Thus, in the context of subdiffraction exoplanet discovery the QCB depends entirely on ξ_Q :

$$\xi_Q = -\ln[(1-b)p_\psi(\mathbf{r}_s) + bp_\psi(\mathbf{r}_e)], \quad (21)$$

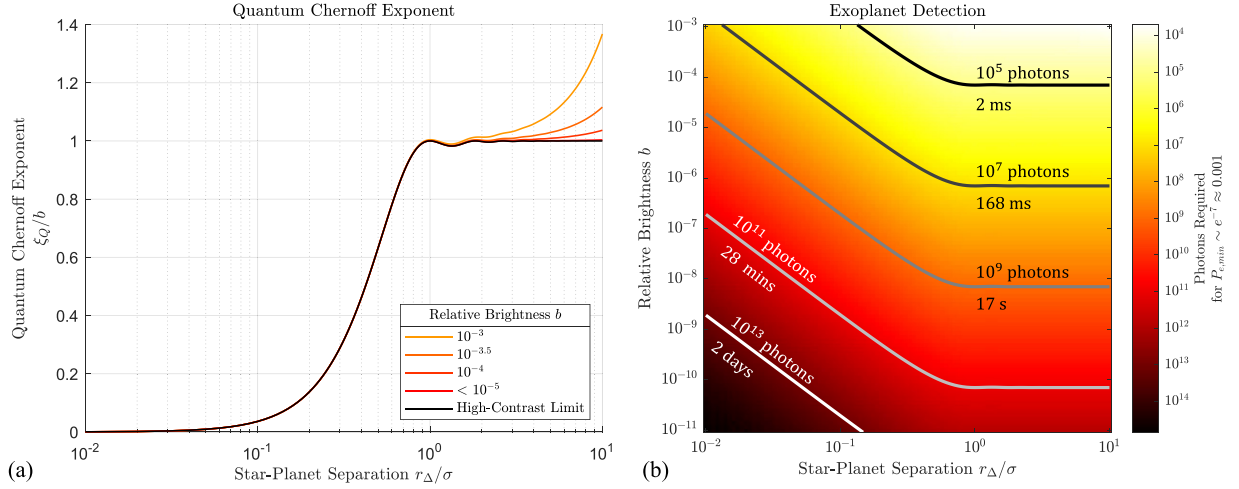


FIG. 2. (a) The quantum Chernoff exponent as a function of the star-planet separation converging toward the high-contrast limit derived in Eq. (22). (b) A map of the photon requirements for reaching an exoplanet detection confidence of 99.9% assuming quantum-limited performance. The conversion from photon counts to integration times assumes the telescope prescription, stellar magnitude, and waveband defined in Table I.

where $p_\psi(\mathbf{s}) = |\Gamma_\psi(\mathbf{s})|^2$ is the probability of detecting a photon in the fundamental mode given a point source located at $\mathbf{s}$. In the high-contrast regime where $b \rightarrow 0$, the single-photon QCE tends toward

$$\xi_Q(\mathbf{r}_\Delta) \xrightarrow{b \ll 1} b[1 - p_\psi(\mathbf{r}_\Delta)]. \quad (22)$$

In this limit, the QCE enjoys a simple interpretation: it represents the probability that a photon coming from the exoplanet arrives “outside” of the fundamental mode (i.e., in the orthogonal complement space of the fundamental mode). Thus, the optimal probability of error in the symmetric hypothesis testing decays as

$$P_{e,\min} \sim e^{-N\xi_Q}, \quad \text{when } b \ll 1, \quad r_\Delta \ll \sigma,$$

where $N = \epsilon M$ is the expected number of photons detected. The high-contrast QCE suggests that separating photons in the fundamental mode from photons in the orthogonal complement space constitutes an optimal strategy for exoplanet detection. Indeed, the so-called binary SPADE (BSPADE) measurement,

$$\hat{T}_0 = |\psi\rangle\langle\psi|, \quad \hat{T}_1 = \hat{I} - |\psi\rangle\langle\psi|, \quad (23)$$

achieves the QCB [20] and may be applied copy by copy. Performing a likelihood ratio test for this measurement amounts to declaring a positive exoplanet detection if any photons are observed in the orthogonal complement space $\hat{T}_1$ within the integration period.

Note that $\hat{T}_0$ and $\hat{T}_1$ are defined independently of the parameters $\mathbf{r}_\Delta$ and b . Therefore, we may apply this POVM to any star-planet configuration. Moreover, under the BSPADE measurement we have

$$P(\check{H}_1|H_0) = \text{Tr}[\hat{T}_1 \hat{\rho}_0]^N = 0,$$

$$P(\check{H}_0|H_1) = \text{Tr}[\hat{T}_0 \hat{\rho}_1]^N = [(1 - b)p_\psi(\mathbf{r}_s) + bp_\psi(\mathbf{r}_e)]^N.$$

We see that the type-I error probability is zero while the type-II error probability shrinks exponentially with the number of

photons received. This immediately lends some insight into why the symmetric and asymmetric hypothesis testing cases will be identical; namely, because there is no possibility of a false positive in this idealized setting. Figure 2(a) shows the single-photon QCE as a function of the star-planet separation converging to the high-contrast limit for a telescope with a circular pupil where $p_\psi(\mathbf{r}_\Delta) = (J_1(2\pi r_\Delta)/\pi r_\Delta)^2$. As the star-planet separation decreases below the diffraction limit $r_\Delta < \sigma$, the single-photon QCE approaches zero. This results from a growing majority of exoplanet photons coupling to the fundamental mode in the regime of small displacements from the optical axis. On the other hand, the single-photon QCE is maximal wherever the star-planet separation corresponds to a node of the PSF. At these points, no light from the exoplanet couples into the fundamental mode [a consequence of Eq. (12)]. Necessarily, the light must couple to higher-order modes if energy is to be conserved. Therefore, exoplanets residing near a node of the PSF may be detected more efficiently.

The exponential tails of the single-photon QCE observed for moderate contrasts (e.g., 10^{-3} to 10^{-4}) and large star-planet separations $r_\Delta > \sigma$ result from starlight coupling to higher-order modes when the system is aligned to the center of intensity. Under this alignment, any photon detected outside the fundamental mode, whether it be from the star or the

TABLE I. Reference values for calculating exposure times necessary to reach detection and localization performance isocontours in Figs. 2(b) and 3(b), respectively.

Telescope diameter	6.5 m
Center wavelength	1290 nm
Bandwidth	13 nm ($\sim 1\%$)
Visual magnitude (VM)	5.357 (Proxima Centauri J-Band)
Reference flux (VM = 0)	$1589 \times 10^{-26} \text{ W m}^{-2} \text{ Hz}^{-1}$
Photon flux	$6 \times 10^7 \text{ photons s}^{-1}$

exoplanet, is an indication that an exoplanet is present. Figure 2(b) quantifies how many photons are required to achieve a designated level of confidence in exoplanet detection for different star-planet contrasts and separation distances using a quantum-optimal instrument.

C. Exoplanet detection with asymmetric hypothesis testing:

The quantum Stein lemma

In asymmetric quantum hypothesis testing, we wish to choose a measurement that minimizes the false-negative rate $P_e^{II} \equiv P(\tilde{H}_0|H_1)$ subject to some required tolerance on the false-positive rate $P_e^I = P(\tilde{H}_1|H_0) < \delta$ for $\delta \in (0, 1)$. Here we review results from Ref. [21], where the authors assume that N identical copies of the single-photon states $\hat{\rho}_0$ or $\hat{\rho}_1$ were received. Let $P_{e,\min}^{II}(\delta) \leq P_e^{II}$ be the optimal false-negative error rate optimized over all POVMs on $\mathcal{H}$ that respect $P_e^I < \delta$. The quantum Stein lemma asserts that the optimal false-negative error scales asymptotically in N as

$$P_{e,\min}^{II}(\delta) \sim e^{-[N\alpha + \sqrt{N\beta}\Phi^{-1}(\delta) + O(\ln N)]}, \quad (24a)$$

$$\alpha = D(\hat{\rho}_0||\hat{\rho}_1) \equiv \langle \ln \hat{\rho}_0 - \ln \hat{\rho}_1 \rangle_{\hat{\rho}_0}, \quad (24b)$$

$$\beta = V(\hat{\rho}_0||\hat{\rho}_1) \equiv \langle [\ln \hat{\rho}_0 - \ln \hat{\rho}_1 - D(\hat{\rho}_0||\hat{\rho}_1)]^2 \rangle_{\hat{\rho}_0}, \quad (24c)$$

where $\langle \hat{A} \rangle_{\hat{\rho}} = \text{Tr}(\hat{A}\hat{\rho})$ is shorthand for the operator expectation, $D(\hat{\rho}_0||\hat{\rho}_1)$ is the quantum relative entropy, $V(\hat{\rho}_0||\hat{\rho}_1)$ is the variance of the quantity $(\ln \hat{\rho}_0 - \ln \hat{\rho}_1)$, and $\Phi(x)$ is the cumulative distribution function for a standard normal random variable [21,42]. In Appendix C, we provide computable forms for α and β .

In the asymptotic limit of many photons, $N \gg 1$, we are concerned only with the dominant term α in the exponential. Previously, Ref. [21] showed that in the limiting case of high contrast,

$$D(\hat{\rho}_0||\hat{\rho}_1) \xrightarrow{b \ll 1} b(1 - p_\psi(\mathbf{r}_\Delta)), \quad (25)$$

which exactly recovers the single-photon QCE of Eq. (22). Thus, in the high-contrast regime, $\alpha = \xi_Q$ and the quantum Stein lemma converges to the quantum Chernoff bound:

$$P_{e,\min}^{II} \sim P_{e,\min} \sim e^{-N\xi_Q}.$$

D. Exoplanet localization: The quantum Cramér-Rao bound

To localize an exoplanet, we consider applying an unbiased estimator over outcomes of measurements on the quantum state. In general, the covariance matrix of an unbiased estimator $\check{\theta}$ of the parameters $\theta = [\theta_1, \dots, \theta_n]^T$ characterizing a quantum state $\hat{\rho}(\theta)$ is lower bounded by the quantum Cramér-Rao bound (QCRB)

$$\text{Cov}[\check{\theta}] \geq \Sigma_Q = \frac{1}{N} \mathcal{K}^{-1}, \quad (26)$$

where $\mathcal{K}$ is the QFIM [43], N is the number of copies of $\hat{\rho}(\theta)$ measured, and the inequality indicates that $\text{Cov}[\check{\theta}] - \Sigma_Q \geq 0$ is a positive-semidefinite matrix. Thus, given a finite number of copies of the state, any set of parameters (or linear

combinations thereof) can be estimated up to a minimum uncertainty determined by the laws of physics. Applying this formalism to the exoplanet localization task, we specialize results from Refs. [24,25] to determine the QFIM for the star-planet separation parameter $\mathbf{r}_\Delta$. Assuming the optical axis is aligned to the center of intensity, the QFIM for any binary inversion-symmetric pupil $\tilde{\psi}(\mathbf{u}) = \tilde{\psi}(-\mathbf{u})$ is given by

$$\mathcal{K} = \frac{1}{4}(1 - \kappa^2)[\mathcal{K}_1 - \kappa^2 \mathcal{I}_\psi], \quad (27)$$

where the constant $\kappa \equiv 1 - 2b \in [-1, 1]$. Additionally, the symmetric matrices $\mathcal{K}_1, \mathcal{I}_\psi \in \mathbb{R}^{2 \times 2}$ are given by

$$[\mathcal{K}_1]_{ij} = 4 \langle \partial_{\theta_i}(2\pi\mathbf{u} \cdot \mathbf{r}_\Delta) \partial_{\theta_j}(2\pi\mathbf{u} \cdot \mathbf{r}_\Delta) \rangle, \quad (28a)$$

$$[\mathcal{I}_\psi]_{ij} = 4 \langle \partial_{\theta_i} \Gamma_\psi(\mathbf{r}_\Delta) \rangle \langle \partial_{\theta_j} \Gamma_\psi(\mathbf{r}_\Delta) \rangle, \quad (28b)$$

where (θ_1, θ_2) are general coordinate components of $\mathbf{r}_\Delta$ (e.g., Cartesian or polar) and $\langle \tilde{f}(\mathbf{u}) \rangle \equiv \int |\tilde{\psi}(\mathbf{u})|^2 \tilde{f}(\mathbf{u}) d\mathbf{u}$ indicates an average over the pupil function. Critically, $\mathcal{K}_1$ is the constant QFIM associated with localizing a single point source and $\mathcal{I}_\psi$ is the classical Fisher information (CFI) contribution of the fundamental mode.

Note that when the star and planet brightnesses are balanced ($\kappa = 0$), the second term of Eq. (27) vanishes such that $\mathcal{K} = \frac{1}{4}\mathcal{K}_1$, implying that the symmetry of the system gives rise to an equivalence with the single-point-source localization problem. The 1/4 prefactor arises from the normalization constant of the mixed state describing two balanced incoherent point sources, as opposed to the pure state for a single point source. This symmetry recovers Tsang's landmark result that the quantum Fisher information (QFI) of the separation between two balanced point sources remains constant, irrespective of the separation itself [17,26].

In the high-contrast limit, we keep terms of lowest order in b to arrive at the approximation

$$\mathcal{K} \xrightarrow{b \ll 1} b[\mathcal{K}_1 - \mathcal{I}_\psi]. \quad (29)$$

The QFIM of Eq. (29) lends itself to an intuitive interpretation: it is the information associated with localizing a single point source if all information in the fundamental mode were unavailable. This interpretation reinforces a long-held intuition within the high-contrast-imaging community that the fundamental mode is too noisy to be of value in exoplanet searches since it is dominated by photons from the star. In hindsight, it is remarkable that the high-contrast QCE of Eq. (22), the high-contrast QRE of Eq. (25), and the high-contrast QFIM of Eq. (29) allude to identical messages despite being information-theoretic measures for different sensing tasks; namely, the fundamental mode of the imaging system may readily be dispensed with in the high-contrast regime without incurring information losses about the exoplanet that were otherwise recoverable. This theoretical insight is highly suggestive of optimal coronagraph designs, which we explore in the later sections.

Assuming a circular pupil, the QFIM under polar parametrization of the star-planet separation vector $(\theta_1, \theta_2) =$

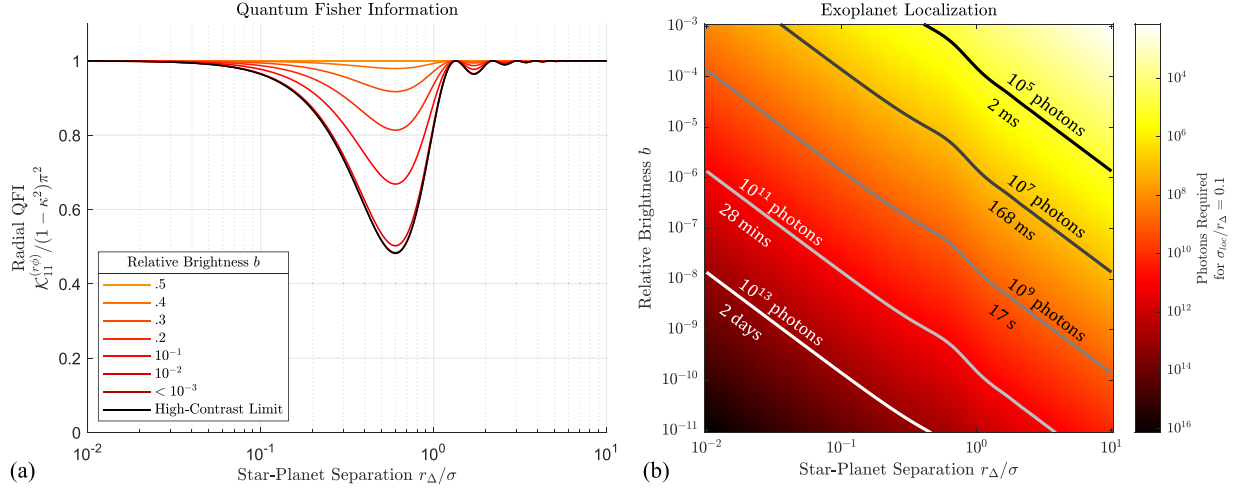


FIG. 3. (a) Quantum Fisher Information (QFI) of the radial parameter r_Δ converging to the high-contrast limit expressed in Eq. (31). The prominent dip in the QFI appears where the classical Fisher information of the fundamental mode is maximal. The presence of the star suppresses information in the fundamental mode by a factor of $\sim b$ compared to higher-order modes, resulting in a parameter-dependent reduction of the QFI. (b) Heat map indicating the number of photons required to achieve a relative error in exoplanet localization of $\sigma_{\text{loc}}/r_\Delta = 0.1$. The conversion from photon counts to integration times assumes the telescope prescription, stellar magnitude, and waveband defined in Table I.

(r_Δ, ϕ_Δ) is explicitly given by

$$\mathcal{K}^{(r\phi)} = (1 - \kappa^2)\pi^2 \left(\begin{bmatrix} 1 & 0 \\ 0 & r_\Delta^2 \end{bmatrix} - \kappa^2 \left(\frac{2J_2(2\pi r_\Delta)}{\pi r_\Delta} \right)^2 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \right). \quad (30)$$

The QFIM becomes diagonal due to the rotational symmetry of the pupil. When the source brightnesses are unequal ($\kappa \neq 0$), the QFI of the radial component r_Δ acquires a non-trivial parameter dependence associated with the CFI of the fundamental mode. Meanwhile, the QFI for the angular component ϕ_Δ grows quadratically in r_Δ , matching a general intuition that angular sensitivity scales with the length of the lever arm. In the high-contrast limit, the QFIM for a circular pupil approaches

$$\mathcal{K}^{(r\phi)} \xrightarrow{b \ll 1} 4\pi^2 b \left(\begin{bmatrix} 1 & 0 \\ 0 & r_\Delta^2 \end{bmatrix} - \left(\frac{2J_2(2\pi r_\Delta)}{\pi r_\Delta} \right)^2 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \right). \quad (31)$$

Figure 3(a) shows the convergence of the radial parameter QFI toward the high-contrast limit of Eq. (31). The dominant dip in the radial QFI coincides with the classical information lost to the fundamental mode. To convert the QFIM to a practical measure of the localization precision, we consider summing the QCRB variances of the estimation parameters. In polar coordinates, this imprecision is characterized by an uncertainty area patch,

$$\sigma_{\text{loc}}^2 \equiv \sigma_{r_\Delta}^2 + (r_\Delta \sigma_{\phi_\Delta})^2 = \frac{1}{N} \left[\frac{1}{\mathcal{K}_{11}^{(r\phi)}} + \frac{r_\Delta^2}{\mathcal{K}_{22}^{(r\phi)}} \right]. \quad (32)$$

A related figure of merit is the number of photons needed to localize the exoplanet up to a fraction of the star-planet separation, $\frac{\sigma_{\text{loc}}}{r_\Delta}$ (i.e., relative error). Figure 3(b) shows a heat map indicating the number of photons required to reach 10%

relative localization error over a range of star-planet separations and contrasts.

IV. AN INFORMATION-THEORETIC APPROACH TO CORONAGRAPHY

A. Coronagraphs as quantum channels

In the classical setting, a coronagraph is modeled as a passive linear map $C : \mathbb{L}^2(\mathbb{R}^2, \mathbb{C}) \rightarrow \mathbb{L}^2(\mathbb{R}^2, \mathbb{C})$ between the space of square-integrable two-dimensional complex functions. In discrete numerical calculations, this map is typically represented as a matrix $\mathbf{C} = \mathbf{U}_n \mathbf{D}_n \cdots \mathbf{U}_1 \mathbf{D}_1 \mathbf{U}_0$, where alternating $\mathbf{D}_i$ (diagonal matrix) and $\mathbf{U}_i$ (unitary matrix) describe cascaded phase-amplitude modulations of the field interspaced with plane-to-plane free-space propagation. The singular-value decomposition (SVD) of the coronagraph map $C = \mathbf{U} \mathbf{T} \mathbf{V}^\dagger$ lends the interpretation that a coronagraph is a mode-selective optical filter between a bijective input and output mode space [16,44]. The orthonormal input modes $\{v_k(\mathbf{r}) : k \in \mathbb{N}\}$ and orthonormal output modes $\{\mu_k(\mathbf{r}) : k \in \mathbb{N}\}$ are the eigenvectors of unitary operators V and $U^\dagger$, respectively. The singular values $\{\sqrt{\eta_k} \in [0, 1] : k \in \mathbb{N}\}$ of T capture the transmissivity between corresponding input and output modes. Thus, any coronagraph is fully characterized by the quantities $\{\eta_k, v_k(\mathbf{r}), \mu_k(\mathbf{r}) : k = 0, 1, \dots, \infty\}$. Acting the classical coronagraph operator on an arbitrary coherent field $\varepsilon(\mathbf{r})$ gives

$$\varepsilon'(\mathbf{r}) = C(\varepsilon(\mathbf{r})) = \sum_{k=0}^{\infty} \sqrt{\eta_k} \langle v_k | \varepsilon \rangle \mu_k(\mathbf{r}). \quad (33)$$

We extend this operation to the quantum setting by defining a coronagraph as a lossy quantum channel given by the completely positive trace-preserving map $\mathcal{C} : \mathcal{D}(\mathcal{H}) \rightarrow \mathcal{D}(\mathcal{H})$. Sending a state $\hat{\rho}$ through the channel results in the state $\hat{\rho}'$ given by

$$\hat{\rho}' = \mathcal{C}(\hat{\rho}) = \hat{C}_1 \hat{\rho} \hat{C}_1^\dagger + \sum_{k=0}^{\infty} \hat{C}_{0k} \hat{\rho} \hat{C}_{0k}^\dagger + \langle 0 | \hat{\rho} | 0 \rangle | 0 \rangle \langle 0 |, \quad (34)$$

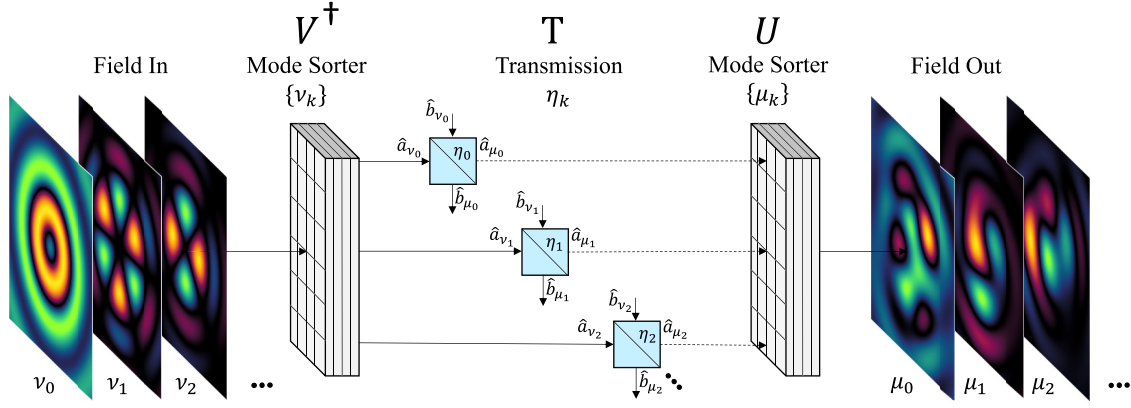


FIG. 4. A photonic diagram illustrating the singular-value decomposition of an arbitrary coronagraph. The input transverse spatial modes $\hat{a}_{v_k}$ are mapped bijectively to the output spatial modes $\hat{a}_{\mu_k}$ with loss η_k modeled by a beam-splitter array. We introduce ancillary modes $\hat{b}_{v_k}, \hat{b}_{\mu_k}$ to account for energy conservation. One recovers the Kraus operators for the channel by tracing out the ancillary modes.

where $\{\hat{C}_1, \hat{C}_{0k}, |0\rangle\langle 0|\}$ are the Kraus operators [45,46] of the channel,

$$\hat{C}_1 = \sum_{k=0}^{\infty} \sqrt{\eta_k} |\mu_k\rangle \langle v_k|, \quad \hat{C}_{0k} = \sqrt{1 - \eta_k} |0\rangle \langle v_k|. \quad (35)$$

Physically, the operator $\hat{C}_1$ describes the coherent transmission of modes v_k to modes μ_k . The operator $\hat{C}_{0,k}$ describes loss in mode v_k , resulting in the vacuum state. Critically, for any coronagraph which achieves complete nulling of an on-axis source, the fundamental mode of the telescope necessarily maps to the vacuum state after propagating through channel $\mathcal{C}(|\psi\rangle\langle\psi|) = |0\rangle\langle 0|$ such that, without loss of generality, $|v_0\rangle = |\psi\rangle$ with $\eta_0 = 0$. Applying a direct-imaging measurement to the output state $\mathcal{C}(\hat{\rho})$ gives

$$p_{\mathbf{r}} = \text{Tr}[\hat{\Pi}_{\mathbf{r}} \hat{\rho}'] = \sum_{k,k'=0}^{\infty} \sqrt{\eta_k \eta_{k'}} \langle v_k | \hat{\rho} | v_{k'} \rangle \mu_k(\mathbf{r}) \mu_{k'}^*(\mathbf{r}), \quad (36a)$$

$$p_{\text{vac}} = \text{Tr}[\hat{\Pi}_{\text{vac}} \hat{\rho}'] = \langle 0 | \hat{\rho} | 0 \rangle + \sum_{k=0}^{\infty} (1 - \eta_k) \langle v_k | \hat{\rho} | v_k \rangle. \quad (36b)$$

Figure 4 provides a photonic model of the coronagraph channel. Two spatial mode sorters are designed to (de)multiplex spatial modes $\hat{a}_{v_k} = \int v_k^*(\mathbf{r}) \hat{a}(\mathbf{r}) d\mathbf{r}$ and $\hat{a}_{\mu_k} = \int \mu_k^*(\mathbf{r}) \hat{a}(\mathbf{r}) d\mathbf{r}$, respectively. The input and output modes couple pairwise to a beam splitters parametrized by transmission coefficients η_k . We introduce an ancillary system via the beam-splitter ports $\{\hat{b}_{v_k}, \hat{b}_{\mu_k}\}$ to account for lost energy such that the photonic model defines a unitary operation. By injecting vacuum in ports $\hat{b}_{v_k}$ and tracing out the ancillary system, one recovers the Kraus operators of Eqs. (35).

B. Quantum limits after propagating through a coronagraph channel

Given the quantum limits derived for the received state $\hat{\rho}_1$ and the formulation of a coronagraph channel, a natural question arises: what are the quantum limits of exoplanet detection and localization after propagating $\hat{\rho}_1$ through a coronagraph? If the QCE, QRE, and QFIM are preserved after propagation, then the coronagraph is equipped to achieve the

quantum limits of exoplanet discovery, assuming an optimal measurement is made. In subsequent analysis, we specialize to coronagraphs that completely null the fundamental mode (and potentially attenuate higher-order modes) such that $|v_0\rangle = |\psi\rangle$ with $\eta_0 = 0$. In the high-contrast limit, the single-photon component of the state available after propagation is

$$\hat{\rho}'_1 = \hat{C}_1 \hat{\rho}_1 \hat{C}_1^\dagger \xrightarrow{b \ll 1} b \sum_{k,k' > 0} \sqrt{\eta_k \eta_{k'}} \Gamma_{v_k}(\mathbf{r}_\Delta) \Gamma_{v_{k'}}^*(\mathbf{r}_\Delta) |\mu_k\rangle \langle \mu_{k'}|. \quad (37)$$

Tailoring our information analysis to this surviving single-photon component, the postpropagation QCE and QFIM are (Appendix E)

$$\xi'_Q \xrightarrow{b \ll 1} b \sum_{k > 0} \eta_k |\Gamma_{v_k}(\mathbf{r}_\Delta)|^2, \quad (38)$$

$$\mathcal{K}'_{ij} \xrightarrow{b \ll 1} 4b \sum_{k > 0} \eta_k \text{Re}(\partial_{\theta_i} \Gamma_{v_k}(\mathbf{r}_\Delta))^* (\partial_{\theta_j} \Gamma_{v_k}(\mathbf{r}_\Delta)). \quad (39)$$

Both postpropagation limits appear as a sum of terms involving the correlation functions of the input modes $\Gamma_{v_k}(\mathbf{r}_\Delta)$ weighted by the transmission of each mode. In particular, the postpropagation QCE of Eq. (38) is indeed the throughput of the coronagraph scaled by the relative brightness of the exoplanet. Thus, ξ'_Q is the probability that a photon originated from the exoplanet and successfully avoided rejection by the coronagraph. Meanwhile, the postpropagation QFIM of Eq. (39) is a sum of classical Fisher information content within each input mode weighted by its transmission. In general, the postpropagation quantum limits are bounded from above as $\xi'_Q \leq \xi_Q$ and $\mathcal{K}' \leq \mathcal{K}$, where equality occurs when $\eta_{k>0} = 1$ such that all modes orthogonal to the fundamental mode are perfectly transmitted.

In Appendix E, we show that direct imaging and SPADE achieve the postpropagation QCB. In addition, both measurements achieve the postpropagation QFI if the SVD of the coronagraph satisfies certain conditions. For direct imaging, the function $\sum_k \Gamma_{v_k}(\mathbf{r}_\Delta) \mu_k(\mathbf{r})$ must be purely real or imaginary. For SPADE, one must measure with the POVM $|\mu_k\rangle \langle \mu_k|$ and the phase of the correlation coefficients must be parameter independent such that $\partial_{\theta_i} \arg(\Gamma_{v_k}(\mathbf{r}_\Delta)) = 0$.

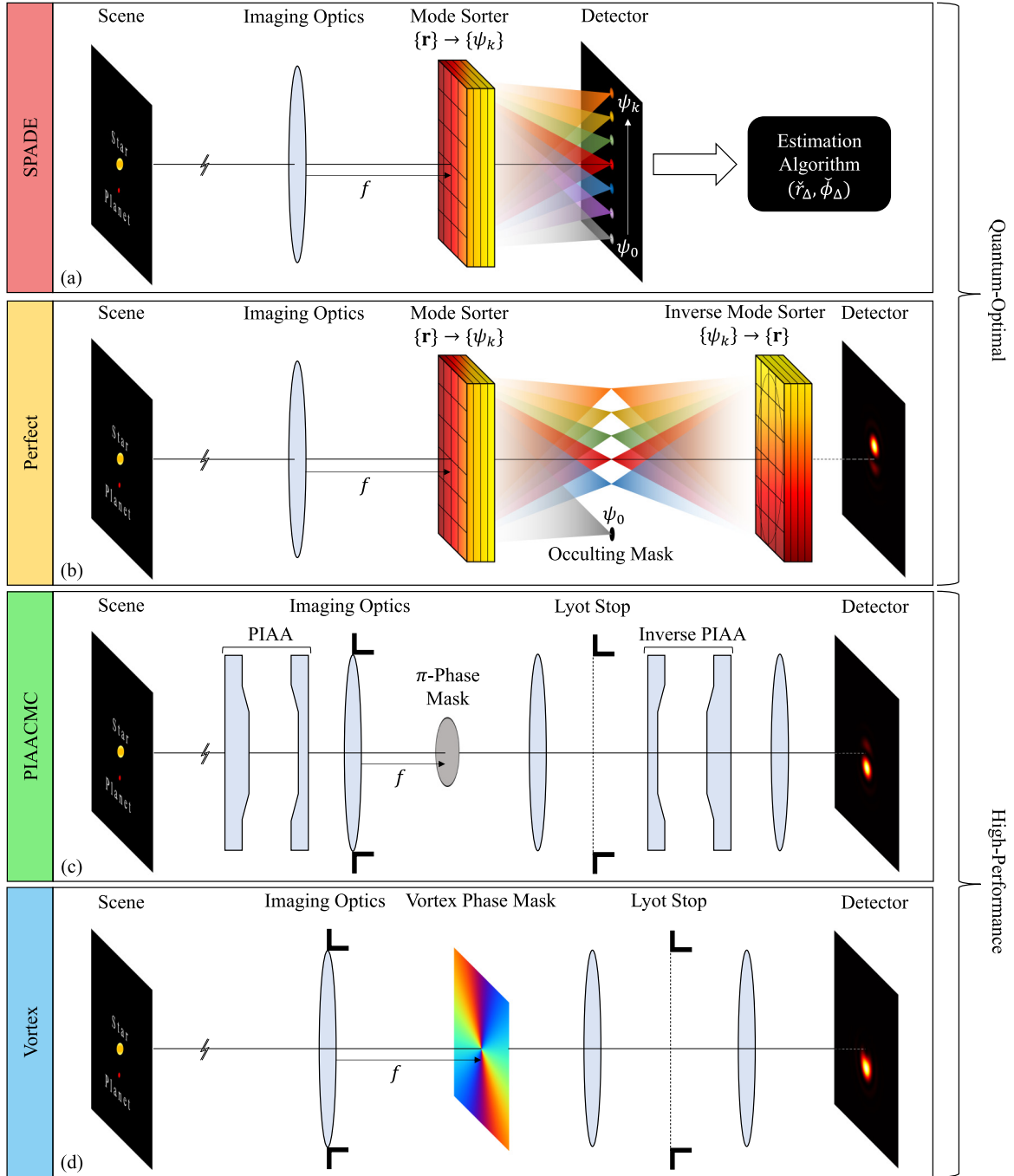


FIG. 5. (a) PSF-adapted SPADE: A system that decomposes the incident optical field into a set of PSF-adapted orthonormal transverse spatial modes and counts the number of photons in each mode channel. (b) PC: A direct-imaging coronagraph that exclusively eliminates the fundamental mode of the telescope prior to direct detection. (c) PIAACMC: A direct-imaging coronagraph that employs aspheric optics to induce a lossless prolate spheroidal apodization. The complex phase mask causes the apodized fundamental mode to destructively interfere with itself over the clear aperture of the Lyot stop. (d) VC: A direct-imaging coronagraph that diffracts the fundamental mode outside the clear aperture of the Lyot stop using a vortex phase mask with integer topological charge.

V. COMPARATIVE SURVEY OF CORONAGRAPH DESIGNS

In this section we introduce four unique coronagraph systems (SPADE, PC, PIAACMC, and VC) configured to isolate photons in the fundamental mode of a telescope with a circular pupil (see Fig. 5). In analyzing and comparing these systems, we seek to understand where quantum-optimal coronagraph

designs stand to offer valuable performance improvements over existing coronagraphs for exoplanet discovery tasks.

A. Quantum-optimal coronagraphs

SPADE and the PC are two systems that saturate the quantum limits of exoplanet detection and localization in

the high-contrast regime. Figures 5(a) and 5(b) shows possible implementations of either system using abstract spatial mode-sorting elements. Similar implementations of mode-sorter-based coronagraphs have been proposed that employ photonic lanterns [27] and multiplane light converters [47,48].

1. PSF-adapted SPADE

A PSF-adapted SPADE measurement of the state $\hat{\rho}_1$ is defined through the orthogonal projectors $\{|\psi_k\rangle\langle\psi_k| : k \in \mathbb{N}\}$ where $|\psi_0\rangle = |\psi\rangle$ is the single-photon state of the fundamental mode. Measuring multiple copies of the state amounts to counting the total number of photons arriving in each mode over a given integration time. A positive exoplanet detection is heralded if a photon arrives in any mode other than the PSF mode. An exoplanet is localized by applying an estimator (e.g., maximum likelihood) to the complete set of measurement outcomes. The real Fourier-Zernike modes (Appendix F) constitute one such PSF-adapted basis and are used ubiquitously throughout the numerical simulations underpinning this work.

2. Perfect coronagraph

The PC is characterized by a direct-imaging system that exclusively removes the fundamental mode from the field prior to detection without affecting higher-order modes. As such, its Kraus operators are

$$\hat{C}_1 = \hat{I}_1 - |\psi_0\rangle\langle\psi_0|, \quad \hat{C}_{0,k} = \delta_{0k}|0\rangle\langle\psi_k|, \quad (40)$$

where $\hat{I}_1$ is the identity on $\mathcal{H}_1$. Figure 5(b) depicts one possible implementation of the PC using mode sorters. Like the SPADE system, light is sorted into a set of PSF-adapted modes. However, rather than count photons in each mode channel, the PSF mode channel is rejected with a mask while all other modes are left to propagate freely past the sorting plane. Subsequently, an inverse mode sorter is applied to reconstitute the field in the sorting plane back into an image. Thus, a positive exoplanet detection is heralded if a photon arrives at the detector plane at all. An exoplanet is localized by applying an estimator on the irradiance distribution measured at the detector. Unlike SPADE measurements, direct imaging preserves perceptual resemblance to the scene itself, which could be useful for a broader variety of astronomical imaging tasks such as inferring the presence of multiple exoplanets and exozodiacal dust disks.

B. High-performance coronagraphs

As comparisons to the quantum-optimal coronagraphs, we consider two state-of-the-art coronagraphs: the PIAACMC [49] and the VC [50]. Both the PIAACMC and the VC theoretically achieve complete rejection of the fundamental mode by modulating the phase of the field at an intermediate focal plane using a (generally complex) transmission mask, as shown in Figs. 5(c) and 5(d). The mask is designed to diffract light in the fundamental mode outside the clear aperture support of a downstream Lyot stop. In simulations, we resort to numerical methods for determining the SVD of these coronagraphs. For completeness we provide a brief overview of each of these coronagraphs. The interested reader is referred

to Ref. [15] for a more comprehensive explanation of such advanced Lyot-style coronagraphs.

1. Phase-induced amplitude apodization complex mask coronagraph

The PIAACMC achieves complete extinction of an on-axis star by applying a pupil apodization in combination with a π -phase mask placed at the intermediate focal plane of the objective (i.e., Roddier and Roddier coronagraph [51]). The appropriate spatial profile of the apodization function is a member of the prolate spheroids specific to the geometry of the pupil [52–54]. These are eigenfunctions of the finite Fourier transform over the aperture support. The PIAACMC employs aspheric optics to redistribute the light over the pupil and achieve the required spatial apodization for complete star extinction [49]. The use of refractive elements to induce apodization as opposed to a spatial transmission mask preserves light throughput past the pupil. Phase modulating the apodized fundamental mode at the focal plane with a π -phase mask causes the field to destructively interfere with itself such that the amplitude is zero within the clear aperture area of a downstream Lyot stop. For off-axis point sources, self-interference induced by the phase mask does not completely cancel the field amplitude over the clear aperture. Thus, some light successfully propagates past the Lyot stop. Any distortion generated by the first set of aspheric surfaces is compensated by a set of identical but inverted aspheric surfaces before imaging the field onto a detector.

2. Vortex coronagraph

The VC first proposed in Ref. [50] introduces a vortex phase plate with even integer topological charge $\ell \in \mathbb{Z}$ at the focal plane of the objective. This element applies a helical phase modulation of the field. Like the PIAACMC, the phase modulation diffracts light in the fundamental mode outside the clear aperture area of a downstream Lyot stop. Off-axis sources will instead acquire a near-constant phase over the bulk of their field amplitude and pass through the system relatively unaltered.

Vortex coronagraphs and vortex fiber nullers have been the subject of several subdiffraction exoplanet imaging demonstrations in recent years [55–60]. Recently, Ref. [61] found a closed-form expression for the action of even-charge vortex coronagraphs on the *complex* Fourier-Zernike polynomials $z_n^m(\mathbf{r})$. The Fourier transforms of these modes define a complete orthonormal spatial mode basis over the circular pupil and the zeroth mode is equal to the fundamental mode of the telescope: $|z_0^0\rangle = |\psi\rangle$. For a given radial index $n = 0, 1, 2, \dots$, the orbital angular momentum (OAM) index of the modes ranges over $m \in \mathcal{S}_n = \{-n, -n+2, \dots, n-2, n\}$. The action of the vortex is to increment the OAM of a mode by ℓ such that $z_n^m \rightarrow z_n^{m+\ell}$. Subsequently, any complex Fourier-Zernike mode whose OAM is not within the permissible set $\mathcal{S}_n$ is rejected by the Lyot stop. This elegant behavior allows us to write the Kraus operators

$$\hat{C}_1 = \sum_{n=0}^{\infty} \sum_{m \in \mathcal{S}_n} \mathbb{1}[n \geq |m + \ell|] |z_n^{m+\ell}\rangle\langle z_n^m|, \quad (41a)$$

$$\hat{C}_{0,n}^m = \mathbb{1}[n < |m + \ell|] |0\rangle\langle z_n^m|. \quad (41b)$$

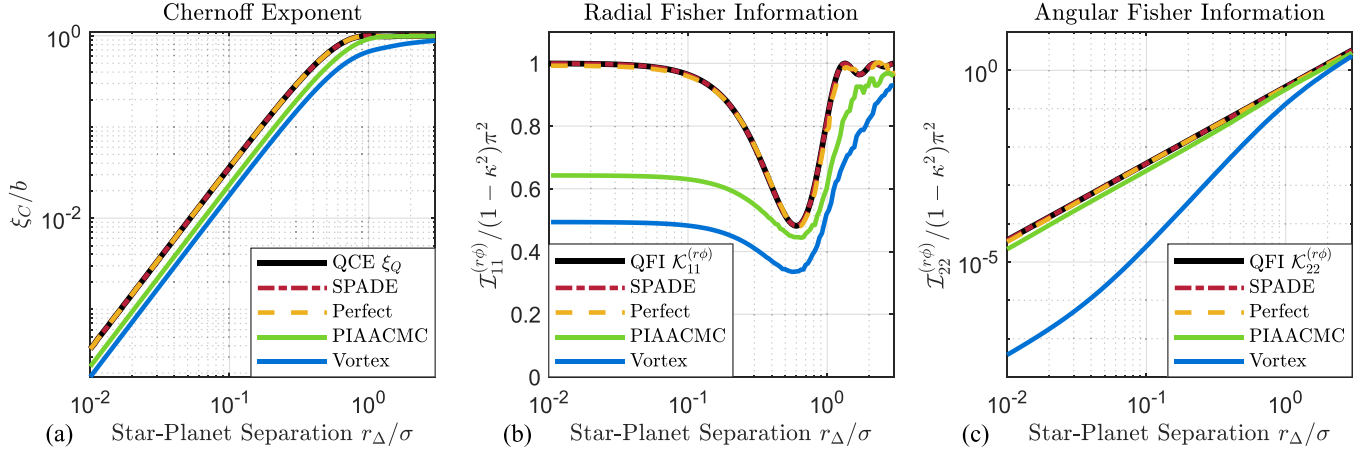


FIG. 6. Numerical results for the (a) Chernoff exponent, (b) radial Fisher information, and (c) angular Fisher information of each coronagraph as a function of the star-planet separation. The relative brightness was set to $b = 10^{-9}$. Quantum-optimal systems (SPADE and PC) offer salient improvements over high-performance coronagraphs (PIAACMC and VC) in the subdiffraction regime.

We note that the null space of the VC is infinite dimensional since all modes lying on an edge of the Zernike pyramid are rejected. This has dramatic consequences for its performance, as we later show. For all figures and comparative performance analysis presented herein, we consider a charge $\ell = 2$ vortex.

VI. SIMULATION RESULTS

A. Exoplanet detection performance comparison

In Fig. 6(a) we compare the classical Chernoff exponent of each coronagraph against the QCE at an extreme star-planet contrast ($b = 10^{-9}$). The SPADE and PC saturate the QCE at all contrasts due to complete rejection of the null hypothesis $\hat{\rho}_0$ and minimal attenuation of hypothesis $\hat{\rho}_1$. While the PIAACMC and the VC also completely reject the null hypothesis, they demonstrate suboptimal performance due to excessive attenuation of higher-order modes that couple light from the exoplanet under hypothesis $\hat{\rho}_1$. The quantum-optimal SPADE and PC systems achieve about 1.6- and 2-fold enhancement in the Chernoff exponent over PIAACMC and VC, respectively. Under shot-noise-limited conditions, quantum-optimal systems are therefore predicted to reach the same level of exoplanet detection confidence as the PIAACMC and VC with roughly 40–50 % lower integration time for exoplanets located below the diffraction limit. Table II compares the integration time required for each coronagraph to reach several detection error probabilities. The superior performance of the PIAACMC compared to the VC at small star-planet separations stems from its comparatively higher transmission of tip-tilt modes, which are dominantly active for small off-axis displacements of the exoplanet.

B. Exoplanet localization performance comparison

In Figs. 6(b) and 6(c) we directly compare the radial and angular CFI of SPADE, PC, PIAACMC, and VC against the QFI at an extreme star-planet contrast ($b = 10^{-9}$). When the star and exoplanet are well separated ($r_\Delta \gtrsim 2\sigma$), then the CFI for all systems converges to the quantum bound. In this

large-separation regime, light from the exoplanet predominantly couples to higher-order modes that experience minimal attenuation. However, at subdiffraction star-planet separations ($r_\Delta < \sigma$), only SPADE and PC fully saturate the QFI. Furthermore, the SPADE and PC information curves are strictly greater than those of the PIAACMC and VC. For the radial parameter r_Δ , Fig. 6(b) indicates that SPADE and PC are roughly twice as information efficient as PIAACMC and VC in the deep subdiffraction regime ($r_\Delta \lesssim \sigma/10$), thereby reducing the localization error by a factor of $\sqrt{2}$ for the same photon budget. This performance difference arises due to lower-order mode attenuation in the PIAACMC and VC.

For the angular parameter ϕ_Δ , Fig. 6(c) shows that both SPADE and PC saturate the QFI limit. In the subdiffraction regime, SPADE and the PC are approximately 2 and 100 times more information efficient than the PIAACMC and VC, respectively. The drastic insensitivity of the VC to angular orientation at small star-planet separations can be understood by inspecting simulated intensity images for each coronagraph shown Fig. 7. The PC images exhibit a double-lobed structure characteristic of strongly coupled tip-tilt modes after removal

TABLE II. Integration times required to reach particular detection error probabilities P_e and relative localization errors $\sigma_{\text{loc}}/r_\Delta$ with different coronagraphs. These times are calculated for an example star-planet system with $r_\Delta/\sigma = 0.1$ and $b = 10^{-9}$. The star photon flux and telescope specifications used are given in Table I.

	Perfect/SPADE	PIAACMC	Vortex
P_e	Detection integration times (s)		
10^{-1}	1 073	1 663	2 167
10^{-2}	2 146	3 326	4 334
10^{-3}	3 220	4 990	6 501
10^{-4}	4 293	6 653	8 668
$\sigma_{\text{loc}}/r_\Delta$	Localization integration times (s)		
1	6	9	33
1/2	25	37	134
1/10	626	927	3 350
1/100	62 596	92 749	335 000

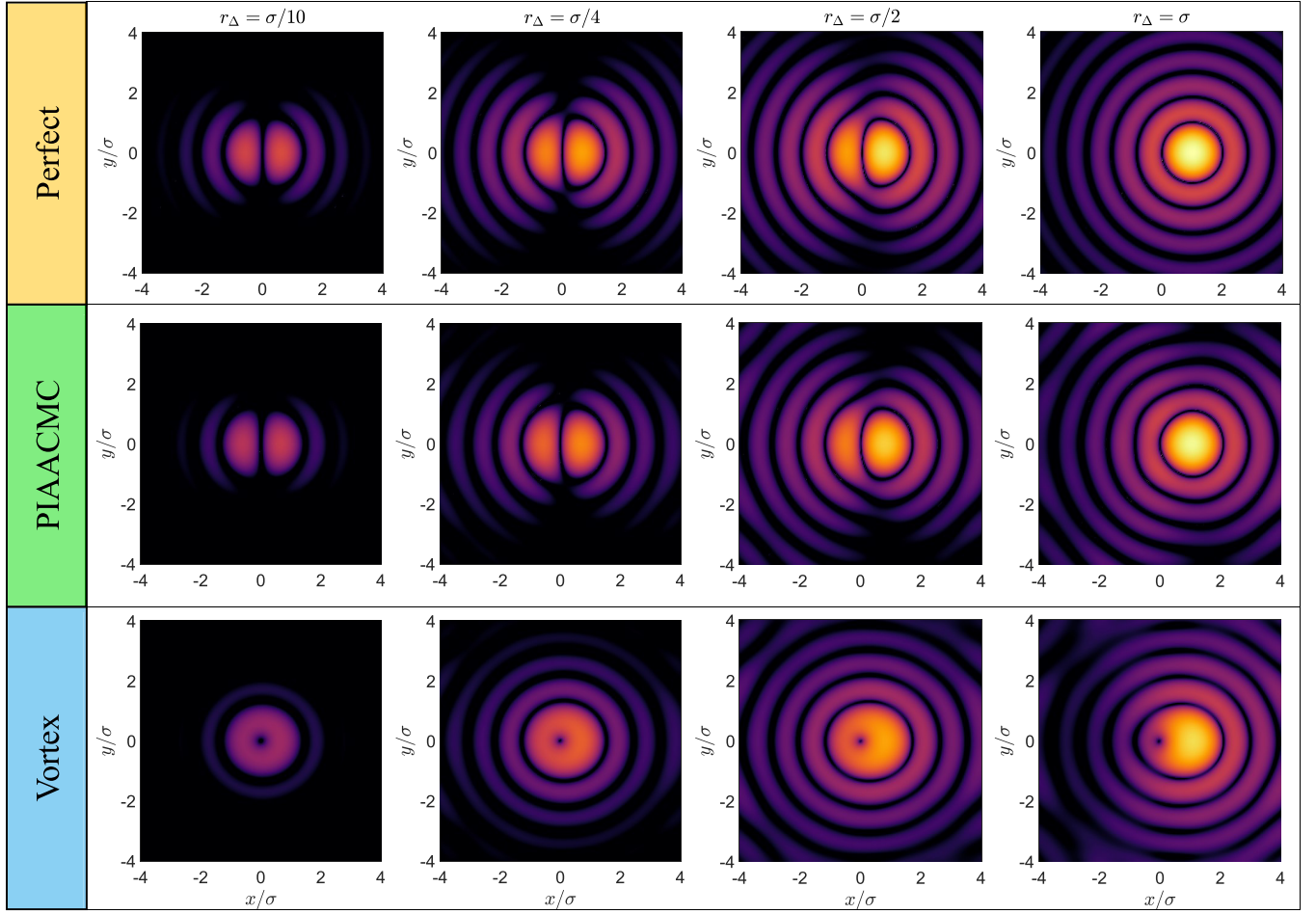


FIG. 7. Theoretical intensity distributions (logarithmic scale) for direct-imaging coronagraphs pointed at the center of intensity of a star-planet system with $b = 10^{-9}$. Each column corresponds to a different star-planet separation along the x axis. The two-lobe structure visible in the subdiffraction regime (column 1) for the PC and PIAACMC emerges from dominant coupling to the tip-tilt modes at small separations. Note that the vortex coronagraph does not exhibit a two-lobe structure as the tilt modes are highly attenuated. Consequently, the sensitivity of the VC to radial separation r_Δ and the angular orientation of the star-planet system suffers, as shown by the suboptimal CFI curves of Fig. 6.

of the fundamental mode. The PIAACMC images exhibit nearly identical structure as those of the PC with notably lower transmission, implying that the PIAACMC nulls the fundamental mode and small amounts of the tip-tilt modes. Conversely, the VC creates doughnutlike structures at subdiffraction separations, which increase uncertainty about the angular orientation of the star-planet axis.

The relative disparity in the Fisher information associated with each system translates to different exposure times required to achieve a particular exoplanet localization precision, which is of practical significance. Table II provides a ready comparison of these exposure times between each coronagraph. Overall, we observe that that SPADE and PC offer significant performance improvements over PIAACMC and VC primarily in the subdiffraction regime in terms of 1.5- to 2-fold lower exposure times, respectively. To illustrate the prospective value of quantum-optimal systems, Fig. 8(a) visually depicts the size of the localization uncertainty patch associated with each coronagraph for different numbers of photons allotted to localizing a subdiffraction exoplanet. Figure 8(b) depicts Monte Carlo simulations of a SPADE system localizing an exoplanet positioned about various lo-

cations in the subdiffraction regime. In particular, we apply maximum-likelihood estimates to the SPADE measurements in order to recover the exoplanet positions. The distribution of maximum-likelihood estimates is tightly concentrated around the ground-truth positions of the exoplanet.

VII. DISCUSSION

In review, we report the fundamental quantum bounds of exoplanet detection (QCE and QRE) and localization (QFIM). We also identify intuitive interpretations of these bounds in the high-contrast limit. In the case of exoplanet detection, the QCE and QRE are simply the probability that a photon originates from the exoplanet and is detected outside the fundamental mode. Meanwhile, the QFI can be thought of as the constant information available for localizing a single point source (i.e., the exoplanet) minus the classical Fisher information supplied by the fundamental mode. We present two quantum-optimal coronagraphs, SPADE and the PC, which saturate the QCE and QFI bounds in the high-contrast regime. As points of comparison, we numerically show that two competitive coronagraphs, the PIAACMC and the VC,

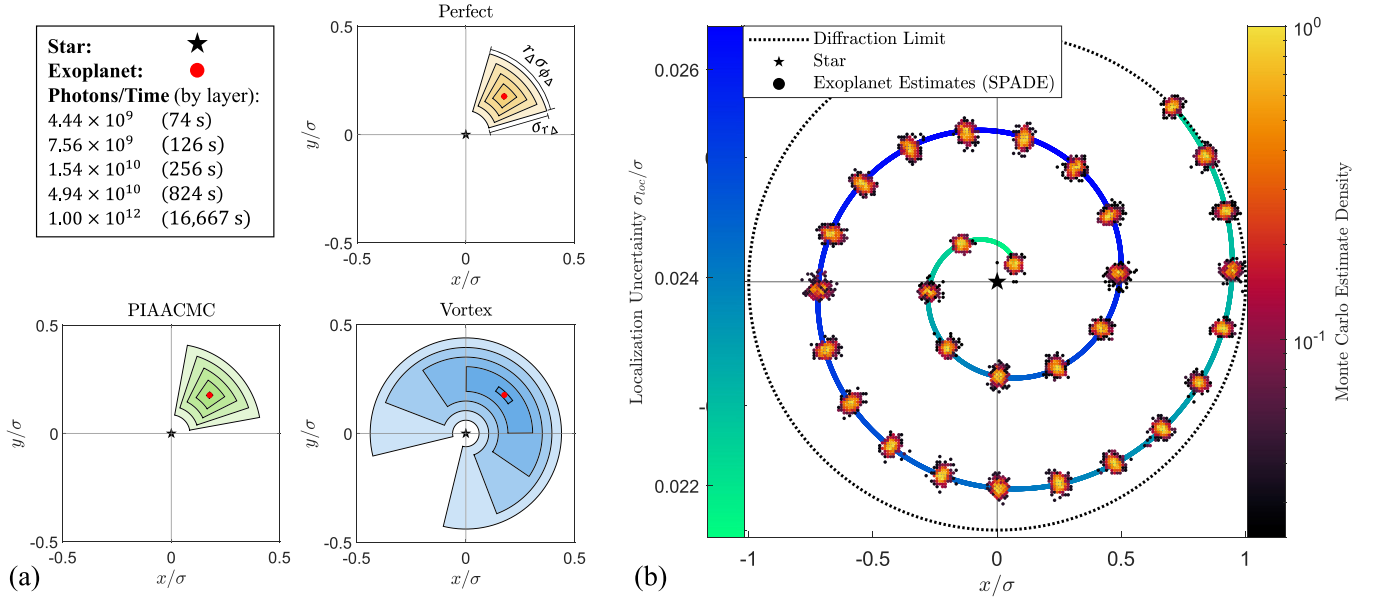


FIG. 8. (a) Visualization of the variation in localization uncertainty across different coronagraphs for a subdiffraction exoplanet with relative contrast brightness 10^{-9} situated $r_{\Delta} = \sigma/4$ away from its host star. The uncertainty patches are computed from the classical Fisher information matrix (CFIM) of each coronagraph. Each layer of localization refinement corresponds to a longer integration time under the star and telescope model detailed in Table I. (b) Monte Carlo simulations showing localization of an exoplanet at various subdiffraction distances from the star location using maximum-likelihood estimation on SPADE measurements in the Fourier-Zernike basis, truncated to maximum radial order $\max(n) = 10$. For each exoplanet location, 500 measurement and estimation trials were simulated. The brightness contrast was set to $b = 10^{-9}$ and the mean photon count per measurement was taken to be 300×10^9 such that, on average, about 300 photons from the exoplanet were collected per measurement. We have chosen exoplanet locations spaced equally over a spiral to illustrate variations in uncertainty that may depend on radial or angular position in the field of view.

are suboptimal in the regime of small star-planet separations due to excess attenuation of higher-order modes beyond the fundamental mode of the telescope.

Encouraged by these results, several future research directions appear ripe for exploration. On a theoretical front, a QFIM analysis remains to be done involving simultaneous estimation of the star-planet separation $\mathbf{r}_{\Delta}$, the centroid $\mathbf{R}$, and the relative brightness b in two dimensions. The centroid and brightness parameters are considered contextual nuisance parameters. Nevertheless, imperfect knowledge of these nuisance parameters may ultimately affect the imprecision limits of the star-planet separation $\mathbf{r}_{\Delta}$. In Ref. [23], the QFIM for simultaneous estimation of the centroid, relative brightness, and separation is derived in one dimension. The authors also propose a numerically optimized measurement for simultaneously saturating the QCRB. Meanwhile, in Ref. [62] the authors posit an optimal multistage receiver where the initial stages are dedicated to collecting sufficient prior information on nuisance parameters so as to ensure the primary estimation task is executed optimally under constrained photon resources. It would also be valuable to perform a QFIM analysis for a resolved star with finite extent. This analysis would help determine whether classical propositions for even-order coronagraphs [15,16] are optimal strategies.

Analysis quantifying the robustness of quantum-optimal instruments in the context of real-world imperfections will also be required to assess their prospective utility as an emerging technology. In particular, we expect low-order wave-front instability [56], finite spectral bandwidths, and detector noise to present the most salient challenges for optimal direct-

imaging coronagraphs. Meanwhile, for SPADE devices we expect modal crosstalk to also hamper performance. Fortunately, related analysis is already underway.

In the context of exoplanet detection, Ref. [63] showed that any amount of modal crosstalk in SPADE devices immediately demands introducing *a priori* knowledge of the exoplanet position in order to execute a maximum-likelihood hypothesis test. Later, Ref. [64] evaluated the impact of crosstalk and dark current on the performance of SPADE and PC, showing that these sources of error cause departure from the quantum Stein lemma at similar rates. Here, the author also formulates a decision rule for the hypothesis test that accounts for starlight leakage induced by modal crosstalk.

In the context of exoplanet localization, Ref. [65] recently reported the QFI for the position of an exoplanet against a uniformly bright disk. Such analysis targets the quantum limits of transit photometry. In yet earlier work, Ref. [66] derived the QFI for the two-source separation with a general spectral power density, while Ref. [67] discovered an important trade space while performing a Fisher information analysis of SPADE in the context of finite spectral bandwidths. As the authors show, a broader spectral bandwidth increases photon flux at the expense of greater mode mismatch. Thus, the optimal spectral bandpass filter for a SPADE receiver scales as $\frac{\Delta\lambda}{\lambda_0} \propto \frac{r_{\Delta}}{\sigma}$. In a related vein, numerical demonstrations of simultaneous spectral and spatial mode sorting using multiplane light conversion have recently been reported [68]. Such techniques may enable attainment of quantum-optimal exoplanet detection and localization while concurrently performing spectroscopy.

In this work, we have considered a scene consisting of a star and a single exoplanet. However, it is conceivable that a star is surrounded by multiple exoplanets. In this setting, a direct-imaging coronagraph like the PC would appear to be favorable as it provides a measurement that bears perceptual resemblance to the scene itself (minus the star). However, if the pairwise separation between multiple exoplanets is below the diffraction limit, then spatial mode-sorting measurements are likely to provide better localization estimates. A poignant example is the adaptive Bayesian SPADE technique developed in Ref. [69] for localizing multiple subdiffraction point sources and estimating their brightness. This technique substantially outperforms direct imaging and could be applied to coronagraphy by adaptively updating the mode basis spanning the orthogonal complement space of the fundamental mode. Adaptive mode-sorting techniques may also enable integration between wave-front error correction and SPADE measurements, as recently suggested in Refs. [70,71].

The quantum limits of exoplanet detection and localization can be achieved simultaneously by separating the fundamental mode of the imaging system without attenuating orthogonal modes in the process. The astronomy community has long intuited this to be an optimal strategy. By conducting a quantum information analysis, we ground this intuition on stronger theoretical footing. In particular, we introduce a quantum channel formalism which provides a systematic mathematical treatment of any coronagraph and facilitates comparative analyses. Finally, we find that the QFI for localization persists deep into the subdiffraction regime, where many undiscovered exoplanets are suspected to exist. The quantum-optimal instruments discussed in this work may therefore expand the domain of accessible exoplanets.

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DATA AVAILABILITY

The data that support the findings of this article are not publicly available. The data are available from the authors upon reasonable request.

APPENDIX A: TOWARD HELSTROM BOUND ON EXOPLANET HYPOTHESIS TESTING

In this section we consider the quantum limits of exoplanet detection given by the Helstrom bound where joint measurements are allowed across multicopy states. We will specialize to the discrimination of single-photon states $\hat{\rho}_0^{\otimes N}$ and $\hat{\rho}_1^{\otimes N}$ (as opposed to $\hat{\rho}_0^{\otimes M}$ and $\hat{\rho}_1^{\otimes M}$), which practically amounts to storing only those states for which a photon interacted with a quantum register; a more general derivation would consider the vacuum contributions as well. Moreover, we will make the simplifying assumption that we are working in the high-contrast regime wherein $|\psi(\mathbf{r}_s)\rangle \approx |\psi\rangle$ and

$|\psi(\mathbf{r}_e)\rangle \approx |\psi(\mathbf{r}_\Delta)\rangle$. For notational convenience we will assign $|\underline{0}\rangle = |\psi\rangle$ and $|\underline{1}\rangle \equiv |\psi(\mathbf{r}_\Delta)\rangle$. Here, $|\underline{0}\rangle$ is not to be confused with the vacuum state from the main text but is instead a single-photon excitation of the fundamental mode, while $|\underline{1}\rangle$ is a single-photon state of the shifted PSF mode associated with the position of the exoplanet. The overlap between these states is defined as

$$\Delta = \langle \psi(\mathbf{r}_s) | \psi(\mathbf{r}_e) \rangle = \Gamma_\psi(\mathbf{r}_\Delta).$$

The span of these vectors forms a two-dimensional Hilbert space $\mathcal{H} = \text{span}(|\underline{0}\rangle, |\underline{1}\rangle) \subset \mathcal{H}$. We now consider the tensor product space $\mathcal{H}^{\otimes N}$, which represents the joint system of N photon arrivals—one photon per temporal coherence interval. Each temporal coherence interval constitutes an independent quantum system such that the space $\mathcal{H}^{\otimes N}$ is the product space of multiple systems. Under this formulation, the tensor product density operators are given by

$$\hat{\rho}_0^{\otimes N} = |\underline{0}\rangle\langle\underline{0}|, \quad |\underline{0}\rangle = \underbrace{|\underline{0}, \dots, \underline{0}\rangle}_N,$$

$$\hat{\rho}_1^{\otimes N} = \sum_{\mathbf{x} \in \mathcal{X}} p_0^{n_0(\mathbf{x})} p_1^{n_1(\mathbf{x})} |\mathbf{x}\rangle\langle\mathbf{x}|, \quad p_0 = 1 - b, \quad p_1 = b,$$

where $\mathcal{X} = \{\underline{0}, \underline{1}\}^N$ is the Hamming space of binary codes of length N . One instance of such a code might be $|\mathbf{x}\rangle = |\underline{0}, \underline{1}, \underline{1}, \dots, \underline{0}\rangle$. Additionally, $n_0(\mathbf{x})$ and $n_1(\mathbf{x})$ are the numbers of $\underline{0}$'s and $\underline{1}$'s in $\mathbf{x}$, respectively. Physically, each $\mathbf{x}$ represents a possible sequence of photon arrivals where a $\underline{0}$ or $\underline{1}$ distinguishes which mode each photon arrived in. Note that the inner product between any pair of joint states $|\mathbf{x}\rangle, |\mathbf{y}\rangle$ for $\mathbf{x}, \mathbf{y} \in \mathcal{X}$ is given by

$$\langle \mathbf{x} | \mathbf{y} \rangle = \Delta^{d(\mathbf{x}, \mathbf{y})}, \quad d(\mathbf{x}, \mathbf{y}) = \text{sum}[\mathbf{x} \oplus \mathbf{y}] \in [0 : N],$$

where $d(\mathbf{x}, \mathbf{y})$ is the Hamming distance between two binary codes defined with $\oplus$ representing the XOR logical operation. That is, $d(\mathbf{x}, \mathbf{y})$ is simply the number of places where the vectors $\mathbf{x}$ and $\mathbf{y}$ differ. The Helstrom bound on the minimum probability of error is given by

$$P_{e,\min} = \frac{1}{2} [1 - \|q_1 \hat{\rho}_1^{\otimes N} - q_0 \hat{\rho}_0^{\otimes N}\|],$$

$$||\hat{A}|| \equiv \text{Tr}|\hat{A}|, \quad |\hat{A}| \equiv \sqrt{\hat{A}^\dagger \hat{A}}.$$

Computing the Helstrom bound numerically runs quickly into a curse of dimensionality as the dimension of $\mathcal{H}^{\otimes N}$ grows as 2^N . To make matters worse, in high-contrast-imaging contexts it is expected that the number of photons we must consider is on the order $N \geq 10^9$ to detect Earth-like faint exoplanets with $b \leq 10^{-9}$. Here we do not find a closed-form expression for the Helstrom bound, but we do determine upper and lower bounds of the Helstrom limit itself and provide a simple separable measurement that achieves identical asymptotic error scaling. As we will show, this measurement is, at worst, within a factor of 2 from the Helstrom bound.

Bounds on the Helstrom limit

Consider the trace distance between two quantum states $\hat{\rho}$ and $\hat{\sigma}$ given by

$$D_{tr}(\hat{\rho}, \hat{\sigma}) = \frac{1}{2} \text{Tr}|\hat{\rho} - \hat{\sigma}|.$$

The trace distance is bounded from below and above by the quantum superfidelity $G(\hat{\rho}, \hat{\sigma})$ and the quantum fidelity $F(\hat{\rho}, \hat{\sigma})$, respectively [39,72], via

$$1 - G(\hat{\rho}, \hat{\sigma})^2 \leq D_{tr}(\hat{\rho}, \hat{\sigma}) \leq \sqrt{1 - F(\hat{\rho}, \hat{\sigma})^2}, \quad (\text{A1})$$

where

$$G(\hat{\rho}, \hat{\sigma})^2 = \text{Tr}[\hat{\rho}\hat{\sigma}] + \sqrt{1 - \text{Tr}[\hat{\rho}^2]}\sqrt{1 - \text{Tr}[\hat{\sigma}^2]},$$

$$F(\hat{\rho}, \hat{\sigma})^2 = (\text{Tr}[\sqrt{\hat{\rho}^{\frac{1}{2}}\hat{\sigma}\hat{\rho}^{\frac{1}{2}}}]^2).$$

In the case where one of the states is pure, the superfidelity and the fidelity are equal such that the lower bound on the trace distance becomes

$$1 - F(|\psi\rangle, \hat{\sigma})^2 \leq D_{tr}(|\psi\rangle, \hat{\sigma}) \leq \sqrt{1 - F(|\psi\rangle, \hat{\sigma})^2}.$$

In the exoplanet detection task the fidelity between both multicopy hypotheses is

$$\begin{aligned} F(\hat{\rho}_0^{\otimes N}, \hat{\rho}_1^{\otimes N})^2 &= F(|\mathbf{0}\rangle, \hat{\rho}_1^{\otimes N})^2 \\ &= \langle \mathbf{0} | \left[\sum_{\mathbf{x} \in \mathcal{X}} p_0^{n_0(\mathbf{x})} p_1^{n_1(\mathbf{x})} |\mathbf{x}\rangle \langle \mathbf{x}| \right] | \mathbf{0} \rangle \\ &= \sum_{\mathbf{x} \in \mathcal{X}} p_0^{n_0(\mathbf{x})} p_1^{n_1(\mathbf{x})} (|\Delta|^2)^{d(\mathbf{0}, \mathbf{x})} \\ &= \sum_{n=0}^N \binom{N}{n} p_0^{N-n} p_1^n (|\Delta|^2)^n. \end{aligned}$$

The last line is the expectation for a function $(|\Delta|^2)^n$ of a binomial random variable n . Writing $(|\Delta|^2)^n = e^{tn}$ (with $t \equiv \ln |\Delta|^2$) we find that squared quantum fidelity evaluates to the moment-generating function of the binomial distribution:

$$\begin{aligned} F^2 &\equiv F(\hat{\rho}_0^{\otimes N}, \hat{\rho}_1^{\otimes N})^2 = \langle e^{tn} \rangle_{\text{Binom}(n; p_1, N)} \\ &= (p_0 + p_1 e^t)^N = [(1-b) + b|\Delta|^2]^N \\ &= [1 - \xi_Q]^N, \end{aligned}$$

where $\xi_Q = b(1 - |\Delta|^2) = b(1 - p_\psi(\mathbf{r}_\Delta))$, as previously found in Eq. (22). Since ξ_Q is the probability that a single photon is observed outside the fundamental mode, the squared fidelity is equal to the probability that all photons appear in the fundamental mode. Our goal now is to use this result to establish upper and lower bounds on the Helstrom limit. Specializing to the case where the priors $q_0 = q_1 = \frac{1}{2}$ are balanced, the Helstrom bound reduces to

$$P_{e,\min} = \frac{1}{2} [1 - D_{tr}(\hat{\rho}_0^{\otimes N}, \hat{\rho}_1^{\otimes N})],$$

whereupon applying the inequalities of Eq. (A1) gives the bounds

$$\frac{1}{2} [1 - \sqrt{1 - F^2}] \leq P_{e,\min} \leq \frac{1}{2} F^2.$$

In the asymptotic limit of many copies, $N \rightarrow \infty$, we Taylor-expand the lower bound to arrive at

$$\frac{1}{4} F^2 \leq P_{e,\min} \leq \frac{1}{2} F^2 \quad \text{as } N \rightarrow \infty.$$

Whether the lower bound is asymptotically tight remains an open question. Next, consider the simple binary joint measurement $\hat{\Pi}_0 \equiv |\mathbf{0}\rangle\langle\mathbf{0}|$ and $\hat{\Pi}_1 = \hat{I} - \hat{\Pi}_0$, where $\hat{I}$ is the

identity on $\mathcal{H}^{\otimes N}$. It is straightforward to show that the probability of detection error under balanced priors is given by

$$\begin{aligned} P_e(|\mathbf{0}\rangle, \hat{\Pi}_1) &= \frac{1}{2} [\text{Tr}[\hat{\Pi}_0 \hat{\rho}_1^{\otimes N}] + \text{Tr}[\hat{\Pi}_1 \hat{\rho}_0^{\otimes N}]] \\ &= \frac{1}{2} \langle \mathbf{0} | \hat{\rho}_1^{\otimes N} | \mathbf{0} \rangle = \frac{1}{2} F^2. \end{aligned}$$

Therefore, the probability of error for this measurement is, at worst, a factor of 2 above the optimal joint measurement on multicopy states in the asymptotic limit $N \rightarrow \infty$. Fortunately, this measurement is separable and can be implemented as a copy-by-copy measurement where one continuously monitors the number of photons arriving in the fundamental mode across all N temporal coherence intervals.

APPENDIX B: DERIVATION OF THE QUANTUM CHERNOFF EXPONENT

The single-photon QCE for discriminating between a single point source located on axis and an arbitrary normalized incoherent exitance distribution $m(\mathbf{r}) \geq 0$ is given by [20]

$$\xi_Q = -\ln \left(\int m(\mathbf{r}) |\Gamma_\psi(\mathbf{r})|^2 d\mathbf{r} \right). \quad (\text{B1})$$

Moreover, the authors prove that this bound is achieved using a binary spatial mode sorter that discriminates the PSF mode $\hat{\Pi}_\psi = |\psi\rangle\langle\psi|$ and its orthogonal complement $\hat{\Pi}_\perp = \hat{I}_1 - \hat{\Pi}_\psi$. In our star-planet model we have the following exitance distribution:

$$m(\mathbf{r}) = (1-b)\delta(\mathbf{r} + b\mathbf{r}_\Delta) + b\delta(\mathbf{r} - (1-b)\mathbf{r}_\Delta),$$

whereupon inserting into Eq. (B1) we resolve Eq. (21) from the main text,

$$\xi_Q(\mathbf{r}_\Delta) = -\ln[(1-b)p_\psi(-b\mathbf{r}_\Delta) + bp_\psi((1-b)\mathbf{r}_\Delta)].$$

To determine the high-contrast QCE, we Taylor-expand Eq. (21) to first order in b . Note that $p_\psi(0) = 1$, since all light emitted by an on-axis point source couples to the fundamental mode. Additionally, $\partial_{\mathbf{r}} p_\psi(\mathbf{r})|_{\mathbf{r}=0} = 0$, which follows directly from the recognition that $p_\psi(0)$ is necessarily a local maximum. Applying these observations to the Taylor expansion, it can be shown that

$$\xi_Q(\mathbf{r}_\Delta) \approx b(1 - p_\psi(\mathbf{r}_\Delta)),$$

which is the high-contrast QCE of Eq. (22).

APPENDIX C: DERIVATION OF THE QUANTUM STEIN LEMMA AND QUANTUM RELATIVE ENTROPY

In this section we derive the quantum Stein lemma for asymmetric hypothesis testing. From Eqs. (24) we see that this involves calculating the quantum relative entropy $D(\hat{\rho}_0||\hat{\rho}_1)$, as well as the variance $V(\hat{\rho}_0||\hat{\rho}_1)$. Defining $|\psi_1\rangle = |\psi(\mathbf{r}_e)\rangle$ and $|\psi_2\rangle = |\psi(\mathbf{r}_s)\rangle$, one can verify that the density operator $\hat{\rho}_1$ is diagonalized by the states $|\lambda_1\rangle, |\lambda_2\rangle$ and eigenvalues λ_1, λ_2 given by

$$\begin{aligned} |\lambda_1\rangle &= a_{11}|\psi_1\rangle + a_{12}|\psi_2\rangle, & \lambda_1 &= (1-\gamma)/2, \\ |\lambda_2\rangle &= a_{21}|\psi_1\rangle + a_{22}|\psi_2\rangle, & \lambda_2 &= (1+\gamma)/2, \end{aligned}$$

where $\gamma \equiv \sqrt{\Delta^2 + \tau^2}$ and $\tau^2 \equiv \kappa^2(1 - \Delta^2)$. The expansion coefficients are explicitly given by

$$\begin{aligned} a_{11} &= N_-(M_+u_- - M_- \tau), & a_{12} &= N_-(M_+u_- + M_- \tau), \\ a_{21} &= N_+(M_+u_+ - M_- \tau), & a_{22} &= N_+(M_+u_+ + M_- \tau), \end{aligned}$$

with definitions

$$u_{\pm} \equiv \Delta \pm \gamma, \quad N_{\pm} = 1/\sqrt{\tau^2 + u_{\pm}^2}, \quad M_{\pm} \equiv 1/\sqrt{2(1 \pm \Delta)}.$$

Therefore, the quantum relative entropy is

$$\begin{aligned} D(\hat{\rho}_0 || \hat{\rho}_1) &= \langle \ln \hat{\rho}_0 - \ln \hat{\rho}_1 \rangle_{\hat{\rho}_0} = \text{Tr}[\rho_0 \ln \rho_0] - \text{Tr}[\rho_0 \ln \rho_1] \\ &= -\text{Tr}[\rho_0 \ln \rho_1] \\ &= (\ln \lambda_1) \text{Tr}[\hat{\rho}_0 |\lambda_1\rangle \langle \lambda_1|] - (\ln \lambda_2) \text{Tr}[\hat{\rho}_0 |\lambda_2\rangle \langle \lambda_2|] \\ &= -\ln \lambda_1 |\langle \psi | \lambda_1 \rangle|^2 - \ln \lambda_2 |\langle \psi | \lambda_2 \rangle|^2, \end{aligned}$$

where

$$\begin{aligned} \langle \psi | \lambda_1 \rangle &= a_{11} \Gamma_0(\mathbf{r}_e) + a_{12} \Gamma_0(\mathbf{r}_s), \\ \langle \psi | \lambda_2 \rangle &= a_{21} \Gamma_0(\mathbf{r}_e) + a_{22} \Gamma_0(\mathbf{r}_s). \end{aligned}$$

Given the derived quantities, the variance is then straightforward to compute:

$$\begin{aligned} V(\hat{\rho}_0 || \hat{\rho}_1) &= \langle [(\ln \hat{\rho}_0 - \ln \hat{\rho}_1) - D(\hat{\rho}_0 || \hat{\rho}_1)]^2 \rangle_{\hat{\rho}_0} \\ &= \langle [\ln \hat{\rho}_0 - \ln \hat{\rho}_1]^2 \rangle_{\hat{\rho}_0} - D(\hat{\rho}_0 || \hat{\rho}_1)^2. \end{aligned}$$

The first term evaluates to

$$\begin{aligned} \langle [\ln \hat{\rho}_0 - \ln \hat{\rho}_1]^2 \rangle_{\hat{\rho}_0} &= 2D(\hat{\rho}_0 || \hat{\rho}_1) + \text{Tr}[\hat{\rho}_0 (\ln \hat{\rho}_1)^2], \\ \text{Tr}[\hat{\rho}_0 (\ln \hat{\rho}_1)^2] &= (\ln \lambda_1)^2 |\langle \psi | \lambda_1 \rangle|^2 + (\ln \lambda_2)^2 |\langle \psi | \lambda_2 \rangle|^2. \end{aligned}$$

Hence, the variance is

$$\begin{aligned} V(\hat{\rho}_0 || \hat{\rho}_1) &= 2D(\rho_0 || \rho_1) - D^2(\rho_0 || \rho_1) \\ &\quad + (\ln \lambda_1)^2 |\langle \psi | \lambda_1 \rangle|^2 + (\ln \lambda_2)^2 |\langle \psi | \lambda_2 \rangle|^2. \quad (\text{C1}) \end{aligned}$$

In Fig. 9 we plot the quantum relative entropy for different values of exoplanet contrast and star-planet separation.

APPENDIX D: DERIVATION OF THE QUANTUM FISHER INFORMATION MATRIX

References [24,25] provide the QFIM for estimation of the separation between two unbalanced point sources in three-dimensional space. We invoke this equation for the special case where both point sources are known to be located at a resolved object plane. Assuming an inversion-symmetric pupil, the QFIM is given by

$$\begin{aligned} [\mathcal{K}]_{ij} &= (1 - \kappa^2) [\langle \partial_{\theta_i} (2\pi \mathbf{u} \cdot \mathbf{r}_{\Delta}) \partial_{\theta_j} (2\pi \mathbf{u} \cdot \mathbf{r}_{\Delta}) \rangle \\ &\quad - \kappa^2 \partial_{\theta_i} \Delta \partial_{\theta_j} \Delta]. \quad (\text{D1}) \end{aligned}$$

We will show that the first and second terms in Eq. (D1) are associated with the single-source localization QFIM $\mathcal{K}_1$ and the classical Fisher information matrix (CFIM) contribution of the fundamental mode $\mathcal{I}_0$, respectively.

1. QFIM for localizing a single point source

Here we derive the QFIM for localization of a single point source. Let $\mathbf{r}_{\Delta}$ be the location of a single point source relative

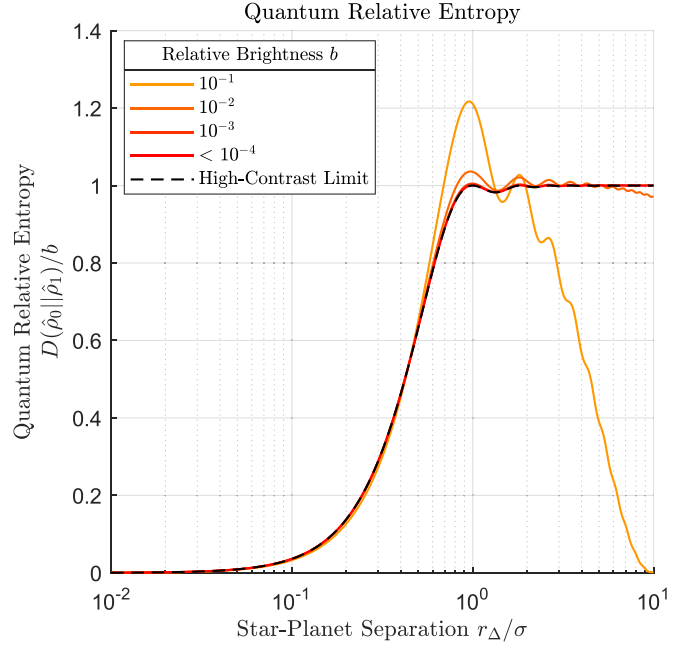


FIG. 9. Plots of the quantum relative entropy under different levels of contrast. In the high-contrast limit the quantum relative entropy converges to the quantum Chernoff exponent. For the results shown, the number of photons detected is $N = 10^9$ and the false-negative error tolerance is $\delta = 0.2$.

to the optical axis such that the single-photon state of the optical field is given by the pure state,

$$|\psi(\mathbf{r}_{\Delta})\rangle \langle \psi(\mathbf{r}_{\Delta})|.$$

The QFIM for pure states is given by [43]

$$\begin{aligned} [\mathcal{K}_1]_{ij} &= 4\text{Re} \left\{ \langle \partial_{\theta_i} \psi(\mathbf{r}_{\Delta}) | \partial_{\theta_j} \psi(\mathbf{r}_{\Delta}) \rangle \right. \\ &\quad \left. - \langle \partial_{\theta_i} \psi(\mathbf{r}_{\Delta}) | \psi(\mathbf{r}_{\Delta}) \rangle \langle \psi(\mathbf{r}_{\Delta}) | \partial_{\theta_j} \psi(\mathbf{r}_{\Delta}) \rangle \right\}. \quad (\text{D2}) \end{aligned}$$

We will evaluate each inner product in Eq. (D2) separately. The first inner product gives

$$\langle \partial_{\theta_i} \psi(\mathbf{r}_{\Delta}) | \partial_{\theta_j} \psi(\mathbf{r}_{\Delta}) \rangle = \langle \partial_{\theta_i} (2\pi \mathbf{u} \cdot \mathbf{r}_{\Delta}) \partial_{\theta_j} (2\pi \mathbf{u} \cdot \mathbf{r}_{\Delta}) \rangle,$$

while the remaining inner products evaluate to zero: $\langle \partial_{\theta_i} \psi(\mathbf{r}_{\Delta}) | \psi(\mathbf{r}_{\Delta}) \rangle = 0$. Therefore, the QFIM for single-source localization is

$$[\mathcal{K}_1]_{ij} = 4 \langle \partial_{\theta_i} (2\pi \mathbf{u} \cdot \mathbf{r}_{\Delta}) \partial_{\theta_j} (2\pi \mathbf{u} \cdot \mathbf{r}_{\Delta}) \rangle,$$

which we immediately recognize as the first term in Eq. (D1) up to a proportionality constant. We also point out that since the derivatives in $\mathcal{K}_1$ act on terms of first order in $\mathbf{r}_{\Delta}$, the Cartesian parametrization of $\mathcal{K}_1$ is independent of the distance r_{Δ} . A dependence on r_{Δ} does arise for a polar parametrization; however, it only plays the role of a geometric factor intrinsic to the Jacobian. Therefore, the uncertainty patch σ_{loc} for localizing a single point source is invariant with respect to the position of the source.

2. CFIM contribution of the fundamental mode

Again, we consider a single point source located at $\mathbf{r}_{\Delta}$ relative to the optical axis. For a real-valued PSF, the probability

of detecting a photon in the fundamental mode is given by

$$p_\psi(\mathbf{r}_\Delta) = \Gamma_\psi^2(\mathbf{r}_\Delta) = \Delta^2.$$

The classical Fisher information contained in the fundamental mode is therefore

$$[\mathcal{I}_\psi]_{ij} = \frac{(\partial_{\theta_i} p_\psi)(\partial_{\theta_j} p_\psi)}{p_\psi} = 4(\partial_{\theta_i} \Delta(\mathbf{r}_\Delta))(\partial_{\theta_j} \Delta(\mathbf{r}_\Delta)),$$

which we immediately recognize as the second term in Eq. (D1) up to a proportionality constant. Therefore, we may write the QFIM as

$$\mathcal{K} = \frac{1}{4}(1 - \kappa^2)(\mathcal{K}_1 - \kappa^2 \mathcal{I}_\psi),$$

which is the QFIM of the star-planet separation vector $\mathbf{r}_\Delta$ given in Eq. (27).

APPENDIX E: POSTPROPAGATION QUANTUM LIMITS

In this section we consider the quantum information limits of the optical state remaining after propagating through a coronagraph. Consider a coronagraph that achieves complete nulling of the fundamental mode such that $|\psi_0\rangle = |\psi\rangle$ and $\eta_0 = 0$. In the high-contrast limit, it is straightforward to see that the single-photon component of the state remaining after propagation through the coronagraph is

$$\hat{\rho}'_1 = b \sum_{k,k' > 0} \sqrt{\eta_k \eta_{k'}} \Gamma_{v_k}(\mathbf{r}_\Delta) \Gamma_{v_{k'}}^*(\mathbf{r}_\Delta) |\mu_k\rangle \langle \mu_{k'}| = |\Psi\rangle \langle \Psi|,$$

where we have defined the *non-normalized* state

$$|\Psi\rangle \equiv \sqrt{b} \sum_{k > 0} \sqrt{\eta_k} \Gamma_{v_k}(\mathbf{r}_\Delta) |\mu_k\rangle.$$

The single-photon output state of the coronagraph is clearly a pure state (up to a normalization constant). Therefore, the QFIM for the postpropagation state is given by

$$\begin{aligned} [\mathcal{K}(\hat{\rho}'_1)]_{ij} &= 4\text{Re}\langle \partial_{\theta_i} \Psi | \partial_{\theta_j} \Psi \rangle - \langle \partial_{\theta_i} \Psi | \Psi \rangle \langle \Psi | \partial_{\theta_j} \Psi \rangle \\ &= 4\text{Re} \left\{ b \sum_{k > 0} \eta_k (\partial_{\theta_i} \Gamma_{v_k})^* (\partial_{\theta_j} \Gamma_{v_k}) \right. \\ &\quad \left. - b^2 \sum_{k,k' > 0} \eta_k \eta_{k'} (\partial_{\theta_i} \Gamma_{v_{k'}})^* (\Gamma_{v_k}) (\Gamma_{v_{k'}})^* (\partial_{\theta_j} \Gamma_{v_{k'}}) \right\}, \end{aligned}$$

where we have omitted the arguments of the correlation functions $\Gamma_{v_k}(\mathbf{r}_\Delta) \rightarrow \Gamma_{v_k}$ for notational convenience. In the high-contrast limit, we drop the second term of order $O(b^2)$ to recover Eq. (39) of the main text,

$$[\mathcal{K}(\hat{\rho}'_1)]_{ij} \xrightarrow{b \ll 1} 4b \sum_{k > 0} \eta_k \text{Re}(\partial_{\theta_i} \Gamma_{v_k})^* (\partial_{\theta_j} \Gamma_{v_k}).$$

Alternatively, we may express the postpropagation QFIM in terms of amplitude and phase contributions:

$$\begin{aligned} [\mathcal{K}(\hat{\rho}'_1)]_{ij} &= 4b \sum_{k > 0} \eta_k [(\partial_{\theta_i} |\Gamma_{v_k}|)(\partial_{\theta_j} |\Gamma_{v_k}|) \\ &\quad + |\Gamma_{v_k}|^2 (\partial_{\theta_i} \arg(\Gamma_{v_k}))(\partial_{\theta_j} \arg(\Gamma_{v_k}))], \end{aligned} \quad (\text{E1})$$

which will be useful for understanding the conditions in which SPADE and direct imaging (DI) achieve the postpropagation

QFIM for a particular coronagraph. In particular, we will show that if the phase of the input mode correlation coefficients Γ_{v_k} are parameter independent [such that the second term inside the summation of Eq. (E1) disappears], then SPADE and DI measurements can saturate the postpropagation QFIM.

1. CFIM of SPADE after a coronagraph

Performing a SPADE measurement in the output mode basis μ_k of the coronagraph operator gives probabilities

$$p_{\mu_k} = \langle \mu_k | \hat{\rho}'_1 | \mu_k \rangle = b \sum_{k > 0} \eta_k |\Gamma_{v_k}|^2.$$

The CFIM of the SPADE measurement on the coronagraph channel output is

$$[\mathcal{I}^{\text{SPADE}}(\hat{\rho}'_1)]_{ij} = 4b \sum_{k > 0} \eta_k (\partial_{\theta_i} |\Gamma_{v_k}|)(\partial_{\theta_j} |\Gamma_{v_k}|). \quad (\text{E2})$$

Note that if the phases of the correlation coefficients Γ_{v_k} are parameter independent, then Eq. (E2) is equal to Eq. (E1) such that SPADE saturates the postpropagation QFIM of the coronagraph.

2. CFIM of direct imaging after a coronagraph

Performing a direct-imaging measurement on the output state of the coronagraph channel gives single-photon detection probability

$$p_{\mathbf{r}} = |\langle \mathbf{r} | \Psi \rangle| = |\Psi(\mathbf{r})|^2 = \left| \sqrt{b} \sum_{k > 0} \sqrt{\eta_k} \Gamma_{v_k} \mu_k(\mathbf{r}) \right|^2.$$

The CFIM of direct imaging on the coronagraph channel output is

$$[\mathcal{I}^{\text{DI}}(\hat{\rho}'_1)]_{ij} = \int \frac{(\partial_{\theta_i} p_{\mathbf{r}})(\partial_{\theta_j} p_{\mathbf{r}})}{p_{\mathbf{r}}} d\mathbf{r}.$$

If $\Psi(\mathbf{r})$ is either purely real or imaginary, then we may exploit the orthogonality of the $\mu_k(\mathbf{r})$'s to find that

$$[\mathcal{I}^{\text{DI}}(\hat{\rho}'_1)]_{ij} = 4b \sum_{k > 0} \eta_k (\partial_{\theta_i} \Gamma_{v_k})(\partial_{\theta_j} \Gamma_{v_k}),$$

which is equal to the postpropagation QFIM. Otherwise, if $\Psi(\mathbf{r})$ has both real and imaginary parts, the direct-imaging CFIM is given by

$$[\mathcal{I}^{\text{DI}}(\hat{\rho}'_1)]_{ij} = 4 \int (\partial_{\theta_i} |\Psi(\mathbf{r})|)(\partial_{\theta_j} |\Psi(\mathbf{r})|) d\mathbf{r},$$

and the postpropagation QFIM is not necessarily saturated. This last result may be derived from the relation $\partial_{\theta} |\Psi|^2 = 2|\Psi|(\partial_{\theta} |\Psi|)$.

APPENDIX F: THE FOURIER-ZERNIKE MODES

The Zernike modes $\tilde{\psi}_{nm}$ constitute a PSF-matched basis over a circular pupil,

$$\tilde{\psi}_{nm}(u, \theta) \equiv R_{nm}(u) \Theta_m(\theta) \text{circ}(u), \quad (\text{F1a})$$

$$R_{nm}(u) \equiv \sum_{j=0}^{(n-|m|)/2} \frac{(-1)^j \sqrt{n+1} (n-j)! u^{n-2j}}{j! [(n+m)/2 - j]! [(n-m)/2 - j]!}, \quad (\text{F1b})$$

$$\Theta_m(\theta) \equiv \begin{cases} \sqrt{2} \cos(|m|\theta) & (m > 0) \\ 1 & (m = 0) \\ \sqrt{2} \sin(|m|\theta) & (m < 0), \end{cases} \quad (\text{F1c})$$

$$\text{circ}(u) \equiv \begin{cases} 1, & u \leq 1 \\ 0, & u > 1, \end{cases} \quad (\text{F1d})$$

where the radial index range is $n = 0, 1, 2, \dots, \infty$. The angular index range is $m \in \mathcal{S}_n = \{-n, -n+2, \dots, n-2, n\}$ for a given radial index. These modes are defined to satisfy orthonormality,

$$\int \tilde{\psi}_{nm}^*(u, \theta) \tilde{\psi}_{n'm'}(u, \theta) u du d\theta = \delta_{nn'} \delta_{mm'}.$$

The Fourier transform of the Zernike modes over the pupil are found in Ref. [73] to be

$$\psi_{nm}(r, \phi) = i^{n+2|m|} \sqrt{n+1} \frac{J_{n+1}(2\pi r)}{\sqrt{\pi} r} \Theta_m(\phi). \quad (\text{F2})$$

APPENDIX G: SPADE CFIM IN THE FOURIER-ZERNIKE MODES

In this section we derive the CFIM for a SPADE system configured to sort the Fourier-Zernike modes. Here the SPADE system acts directly on the state $\hat{\rho}_1$ rather than on the postpropagation state $\hat{\rho}'_1$ resulting from a coronagraph channel. In the first section, we express the complete SPADE CFIM free of approximations. In the second section, we consider the CFIM in the high-contrast limit and prove that it equals the high-contrast QFIM.

1. Complete SPADE CFIM

The polar coordinates for the star and planet vectors are given by

$$\begin{aligned} (r_s, \phi_s) &= (br_\Delta, \phi_\Delta + \pi), \\ (r_e, \phi_e) &= ((1-b)r_\Delta, \phi_\Delta), \end{aligned}$$

where $r_\Delta \in [0, \infty)$ and $\phi_\Delta \in [0, 2\pi)$. Let

$$P_{\psi_{nm}} = (1-b)p_{\psi_{nm}}(r_s, \phi_s) + bp_{\psi_{nm}}(r_e, \phi_e)$$

be the probability of detecting a photon in Fourier-Zernike mode ψ_{nm} under the star-planet configuration. Explicitly,

$$p_{\psi_{nm}}(r, \phi) = |\Gamma_{\psi_{nm}}(r, \phi)|^2 = \frac{1}{\pi} \left[\sqrt{n+1} \frac{J_{n+1}(2\pi r)}{\sqrt{\pi} r} \Theta_m(\phi) \right]^2$$

is the probability of detecting a photon in mode ψ_{nm} if a single point source is located at position $\mathbf{r} = (r, \phi)$. The CFIM

contribution of each mode is given by

$$[\mathcal{I}_{\psi_{nm}}^{\text{SPADE}}(\hat{\rho}_1)]_{ij} = \frac{(\partial_{\theta_i} P_{\psi_{nm}})(\partial_{\theta_j} P_{\psi_{nm}})}{P_{\psi_{nm}}},$$

where $\theta_i \in \{r_\Delta, \phi_\Delta\}$. The partial derivatives of the single-source mode probabilities are

$$\begin{aligned} p_{\psi_{nm}}^{(r)}(r, \phi) &\equiv \partial_r p_{\psi_{nm}}(r, \phi) = 2 \left(\frac{J_{n+1}(2\pi r)}{r} \right) \\ &\times [J_{n-1}(2\pi r) - J_{n-3}(2\pi r)] (\Theta_m(\phi))^2, \end{aligned} \quad (\text{G1a})$$

$$\begin{aligned} p_{\psi_{nm}}^{(\phi)}(r, \phi) &\equiv \partial_\phi p_{\psi_{nm}}(r, \phi) = 4 \left(\frac{J_{n+1}(2\pi r)}{r} \right)^2 \\ &\times (n+1)(-m) \cos(|m|\phi) \sin(|m|\phi). \end{aligned} \quad (\text{G1b})$$

The partial derivatives of the total mode probability are

$$\begin{aligned} \partial_{r_\Delta} P_{\psi_{nm}} &= b(1-b)[p_{\psi_{nm}}^{(r)}(br_\Delta, \phi_\Delta + \pi) \\ &+ p_{\psi_{nm}}^{(r)}((1-b)r_\Delta, \phi_\Delta)], \\ \partial_{\phi_\Delta} P_{\psi_{nm}} &= (1-b)p_{\psi_{nm}}^{(\phi)}(br_\Delta, \phi_\Delta + \pi) \\ &+ bp_{\psi_{nm}}^{(\phi)}((1-b)r_\Delta, \phi_\Delta). \end{aligned}$$

Figure 10 shows the Fisher information contribution for each Zernike mode, $[\mathcal{I}_{\psi_{nm}}]_{ij}$. Drawing attention to Eq. (G1b), we see that $p_{\psi_{nm}}^{(\phi)}(r, \phi) = 0$ for all n, m when ϕ is an integer multiple of $\frac{\pi}{2}$. This implies that the CFI of the angular coordinate is zero when the separation axis of the star-planet system exactly coincides with the x axis or the y axis. These apparent singularities in the CFI can be neglected in practice because the probability of exact alignment along the x and y axes is infinitesimally small. That said, the disappearing CFI has non-negligible effects when considering a finite Zernike mode basis that is truncated at some arbitrarily large order. We find that a good rule of thumb for simulating the localization of point sources near the x and y axes is to use a truncated Zernike mode basis with the maximum radial being bounded by $n \gtrsim \lceil \frac{\pi}{2|\phi_{\min}|} \rceil$ where $\phi_{\min}$ is the minimum orientation angle (in radians) that the star-planet separation axis makes with either x or y axis. Alternatively, one may choose a lower mode truncation order and simply rotate the basis by $\pi/4$ rad halfway through the photon collection period to cover the singularities near the x and y axes.

2. SPADE CFIM in the high-contrast limit

To compute the CFIM of SPADE in the high-contrast limit, we introduce the OSA standard linear index of the Zernike $k = (n(n+2) + m)/2$ for later convenience. Additionally, inspecting Eq. (F2), we see that the Fourier-Zernike modes are either purely imaginary or purely real based on the parity of $n + 2|m|$. Since this is independent of the argument $\mathbf{r}$, we may redefine $\psi_k(\mathbf{r}) \leftarrow i^{-(n+2|m|)} \psi_{nm}(\mathbf{r})$ without loss of generality to make the Fourier-Zernike modes an orthonormal basis of real functions. This allows us to write the mode-coupling probability functions as the square of the correlation function,

$$p_{\psi_k}(\mathbf{r}) = \Gamma_{\psi_k}(\mathbf{r})^2.$$

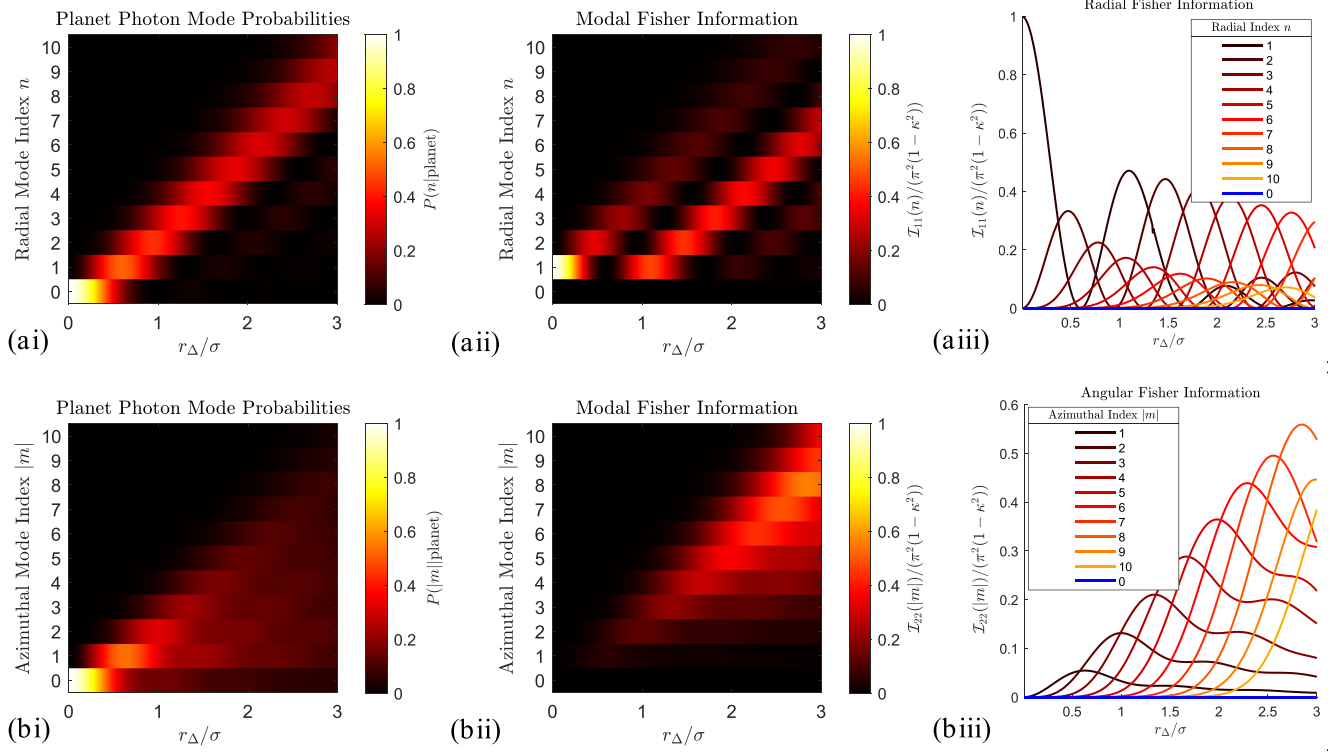


FIG. 10. Decomposition of the CFIM diagonal entries by Zernike mode over different star-planet separations with relative brightness $b = 10^{-9}$ and orientation angle $\phi_\Delta = 45^\circ$. [(ai), (bi)] Exoplanet coupling probabilities over the Zernike modes as a function of the star-planet separation. The probability mass spreads to higher-order modes as the star-planet separation increases. [(aii), (aiii), (bii), (biii)] Concentration of information about the radial parameter r_Δ and angular parameter ϕ_Δ evolves with respect to the star-planet separation over Zernike modes with radial index n and angular index $|m|$, respectively. The fundamental mode (index pair $n = 0, m = 0$) carries virtually no information about the separation parameters.

In the high-contrast limit where $b \rightarrow 0$, $\mathbf{r}_s \rightarrow 0$, and $\mathbf{r}_e \rightarrow \mathbf{r}_\Delta$, the probability of detection in a particular mode approaches

$$P_{\psi_k} \rightarrow (1 - b)p_{\psi_k}(0) + bp_{\psi_k}(\mathbf{r}_\Delta).$$

Critically, for a PSF-matched SPADE basis, we have $p_{\psi_k}(0) = \delta_{k0}$ such that

$$P_{\psi_k} \rightarrow (1 - b)\delta_{k0} + bp_{\psi_k}(\mathbf{r}_\Delta).$$

Then the CFI per mode adopts the form

$$[\mathcal{I}_{\psi_k}^{\text{SPADE}}(\hat{\rho}_1)]_{ij} \rightarrow 4b^2\Gamma_{\psi_k}^2(\mathbf{r}_\Delta) \frac{(\partial_{\theta_i}\Gamma_{\psi_k}(\mathbf{r}_\Delta))(\partial_{\theta_j}\Gamma_{\psi_k}(\mathbf{r}_\Delta))}{\delta_{k0} + b\Gamma_{\psi_k}^2(\mathbf{r}_\Delta)}.$$

Note that the presence of the Kronecker delta in the denominator means the information contribution of the fundamental mode $k = 0$ scales as b^2 . Meanwhile, the information contribution of higher-order modes $k > 0$ scales as b . Thus, in the high-contrast limit the SPADE CFIM approaches

$$[\mathcal{I}_{\psi_k}^{\text{SPADE}}(\hat{\rho}_1)]_{ij} \rightarrow 4b \sum_{k>0} (\partial_{\theta_i}\Gamma_{\psi_k}(\mathbf{r}_\Delta))(\partial_{\theta_j}\Gamma_{\psi_k}(\mathbf{r}_\Delta)). \quad (\text{G2})$$

3. Asymptotic optimality of SPADE

To prove the optimality of SPADE for localization, we show that Eq. (G2) is equal to the high-contrast QFIM of Eq. (31) for a circular aperture. We note that this derivation

also constitutes a proof for the quantum optimality of the Zernike modes in the canonical balanced two-source localization problem involving a circular aperture. We begin by evaluating the diagonal terms of the CFIM.

The radial derivatives of the Fourier-Zernike modes are

$$\partial_r \Gamma_{\psi_{nm}}(r, \phi) = \frac{\pi}{\sqrt{n+1}} [J_{n-1}(2\pi r) - J_{n+3}(2\pi r)] \Theta_n(\phi),$$

and the sum of their squares resolves,

$$\sum_{n=0}^{\infty} \sum_{m \in \mathcal{S}_n} (\partial_r \Gamma_{\psi_{nm}}(r, \phi))^2 = \pi^2,$$

which can be shown by invoking the identity $\sum_{n=0}^{\infty} [J_{n-1}(x) - J_{n+3}(x)]^2 = 1$, $\forall x \geq 0$. The angular derivatives of the Fourier-Zernike modes are

$$\partial_\phi \Gamma_{\psi_{nm}}(r, \phi) = \sqrt{n+1} \frac{J_{n+1}(2\pi r)}{\pi r} [-m\Theta_{-m}(\phi)],$$

and the sum of their squares resolves,

$$\sum_{n=0}^{\infty} \sum_{m \in \mathcal{S}_n} (\partial_\phi \Gamma_{\psi_{nm}}(r, \phi))^2 = (\pi r)^2,$$

which can be shown by invoking the identity $\frac{4}{3} \sum_{n=0}^{\infty} n(n+2)[J_n(x) + J_{n+2}(x)]^2 = x^2$, $\forall x \geq 0$. Applying these terms to

the high-contrast CFI of Eq. (G2), we see that

$$[\mathcal{I}^{\text{SPADE}}(\hat{\rho}_1)]_{11} = 4b(\pi^2 - (\partial_{r_\Delta} \Gamma_0(r_\Delta, \phi_\Delta))^2) \\ = 4\pi^2 b \left(1 - \left(\frac{2J_2(2\pi r_\Delta)}{\pi r_\Delta} \right)^2 \right), \quad (\text{G3})$$

$$[\mathcal{I}^{\text{SPADE}}(\hat{\rho}_1)]_{22} = 4b(\pi r_\Delta)^2 - (\partial_{\phi_\Delta} \Gamma_0(r_\Delta, \phi_\Delta))^2 \\ = 4\pi^2 b r_\Delta^2, \quad (\text{G4})$$

which are precisely the diagonal terms of the high-contrast QFIM shown in Eq. (31). The off-diagonal terms of the CFIM are equal to zero,

$$[\mathcal{I}^{\text{SPADE}}(\hat{\rho}_1)]_{12} = 4b \sum_{n=0}^{\infty} \sum_{m \in \mathcal{S}_n} \partial_r \Gamma_{\psi_{nm}} \partial_\phi \Gamma_{\psi_{nm}} = 0,$$

which can be easily shown by recognizing that the sums over the angular indices evaluate to zero.

APPENDIX H: ASYMPTOTIC OPTIMALITY OF THE PERFECT CORONAGRAPH

In this section we will prove that the PC asymptotically approaches the QFIM of Eq. (29) in the high-contrast regime $b \ll 1$ between the star and the exoplanet. The coronagraph operator for the PC can be written as $\hat{C}_1 = \hat{I}_1 - |\psi\rangle\langle\psi|$. The postpropagation field state after nulling the fundamental PSF mode is given by

$$\hat{\rho}'_1 = (\hat{I}_1 - |\psi\rangle\langle\psi|)\hat{\rho}_1(\hat{I}_1 - |\psi\rangle\langle\psi|) \\ = \hat{\rho}_1 + \langle\psi|\hat{\rho}_1|\psi\rangle|\psi\rangle\langle\psi| - |\psi\rangle\langle\psi|\hat{\rho}_1 - \hat{\rho}_1|\psi\rangle\langle\psi|,$$

where $|\psi\rangle$ is the single-photon state in the fundamental mode. Our goal is to determine the CFI for a direct-imaging measurement $\{\hat{\Pi}_{\mathbf{r}} = |\mathbf{r}\rangle\langle\mathbf{r}| : \mathbf{r} \in \mathbb{R}^2\}$ on the nulled density operator. First we find the probability of detecting a photon at location $\mathbf{r}$ on the image plane:

$$p(\mathbf{r}) = \langle\mathbf{r}|\hat{\rho}'_1|\mathbf{r}\rangle \\ = \rho_1(\mathbf{r}) + \langle\psi|\hat{\rho}_1|\psi\rangle|\psi(\mathbf{r})|^2 \\ - \psi(\mathbf{r})\langle\psi|\hat{\rho}_1|\mathbf{r}\rangle - \psi_0^*(\mathbf{r})\langle\mathbf{r}|\hat{\rho}_1|\psi\rangle \\ = \rho_1(\mathbf{r}) + \langle\psi|\hat{\rho}_1|\psi\rangle|\psi(\mathbf{r})|^2 - 2\text{Re}\psi(\mathbf{r})\langle\psi|\hat{\rho}_1|\mathbf{r}\rangle,$$

which expands to

$$p(\mathbf{r}) = (1-b)|\psi(\mathbf{r}-\mathbf{r}_s)|^2 + b|\psi(\mathbf{r}-\mathbf{r}_e)|^2 \\ + |\psi(\mathbf{r})|^2((1-b)|\Gamma_\psi(\mathbf{r}_s)|^2 + b|\Gamma_\psi(\mathbf{r}_e)|^2)$$

$$- 2\text{Re}\{\psi(\mathbf{r})((1-b)\Gamma_\psi(\mathbf{r}_s)\psi^*(\mathbf{r}-\mathbf{r}_s) \\ + b\Gamma_\psi(\mathbf{r}_e)\psi^*(\mathbf{r}-\mathbf{r}_e))\}.$$

In the limit of high contrast $b \ll 1$ where $\mathbf{r}_s \rightarrow 0$ and $\mathbf{r}_e \rightarrow \mathbf{r}_\Delta$, the probability is well approximated by

$$p(\mathbf{r}) \approx 2(1-b)|\psi(\mathbf{r})|^2 + b|\psi(\mathbf{r}-\mathbf{r}_\Delta)|^2 \\ + b|\psi(\mathbf{r})|^2|\Gamma_\psi(\mathbf{r}_\Delta)|^2 \\ - 2\text{Re}\psi(\mathbf{r})((1-b)\psi^*(\mathbf{r}) + b\Gamma_\psi(\mathbf{r}_\Delta)\psi^*(\mathbf{r}-\mathbf{r}_\Delta)).$$

Assuming a real-valued PSF (inversion-symmetric pupil), this further reduces to

$$p(\mathbf{r}) \approx b[\psi^2(\mathbf{r}-\mathbf{r}_\Delta) + \psi^2(\mathbf{r})\Gamma_\psi^2(\mathbf{r}_\Delta) \\ - 2\psi(\mathbf{r})\Gamma_\psi(\mathbf{r}_\Delta)\psi(\mathbf{r}-\mathbf{r}_\Delta)] \\ \approx b[\psi(\mathbf{r}-\mathbf{r}_\Delta) - \Gamma_\psi(\mathbf{r}_\Delta)\psi(\mathbf{r})]^2.$$

Let us now define the real-valued proxy correlation function

$$\Gamma(\mathbf{r}; \mathbf{r}_\Delta) \equiv \sqrt{b}[\psi(\mathbf{r}-\mathbf{r}_\Delta) - \Gamma_\psi(\mathbf{r}_\Delta)\psi(\mathbf{r})],$$

such that $p(\mathbf{r}) \approx \Gamma^2(\mathbf{r}; \mathbf{r}_\Delta)$. Doing so allows us to write the CFIM of the PC in the convenient form

$$[\mathcal{I}^{\text{PC}}(\hat{\rho}_1)]_{ij} \approx \int \frac{(\partial_{\theta_i} \Gamma^2(\mathbf{r}; \mathbf{r}_\Delta))(\partial_{\theta_j} \Gamma^2(\mathbf{r}; \mathbf{r}_\Delta))}{\Gamma^2(\mathbf{r}; \mathbf{r}_\Delta)} d\mathbf{r} \\ = 4 \int (\partial_{\theta_i} \Gamma(\mathbf{r}; \mathbf{r}_\Delta))(\partial_{\theta_j} \Gamma(\mathbf{r}; \mathbf{r}_\Delta)) d\mathbf{r}.$$

Note that $\Gamma(\mathbf{r}; \mathbf{r}_\Delta)$ is proportional to the optical field induced by a shifted PSF minus its component in the fundamental mode. Moreover, the shifted PSF admits an expansion in a PSF-matched orthonormal basis,

$$\psi(\mathbf{r}-\mathbf{r}_\Delta) = \sum_{k=0}^{\infty} \Gamma_{\psi_k}(\mathbf{r}_\Delta)\psi_k(\mathbf{r}),$$

such that we may express the proxy correlation function as

$$\Gamma(\mathbf{r}; \mathbf{r}_\Delta) = \sqrt{b} \sum_{k>0} \Gamma_{\psi_k}(\mathbf{r}_\Delta)\psi_k(\mathbf{r}).$$

Substituting this expression into the CFI and invoking the orthogonality of the modes, we have

$$[\mathcal{I}^{\text{PC}}(\hat{\rho}_1)]_{ij} \rightarrow 4b \sum_{k>0} (\partial_{\theta_i} \Gamma_{\psi_k}(\mathbf{r}_\Delta))(\partial_{\theta_j} \Gamma_{\psi_k}(\mathbf{r}_\Delta)),$$

which we immediately recognize as Eq. (G2), the CFIM of SPADE in the high-contrast limit. We proved Eq. (G2) approaches QFIM in Appendix G 2. Hence, the PC is quantum optimal.

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