

Unspeakable coherence concentration

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Unspeakable coherence is a key feature separating quantum and classical physics. Modeled as asymmetry with respect to a continuous transformation generated by a physically relevant observable, such as the Hamiltonian or angular momentum, unspeakable coherence has been shown to be the relevant notion of coherence for achieving quantum advantage in the tasks of metrology, reference frame alignment, and work extraction, among others. A question of both practical and foundational value is: Given some copies of a state with low coherence, can we prepare a more coherent state via coherence nonincreasing operations? Here, we study this question in the minimal limiting case: Given two uncorrelated copies of a coherent state, can one, via globally coherence nonincreasing unitaries, increase the coherence in a subsystem? We fully solve this problem for qubits, identifying the optimal unitaries and revealing the existence of bound coherence. This is then used to create a completely constructive multiqubit coherence enhancement protocol, where only effective-qubit unitaries are used. Unexpectedly, in this protocol, we show that there exist states for which the ratio of the input-output coherence can be amplified unboundedly. Extending beyond qubits, we derive two fundamental upper bounds on the amount of local coherence that can be increased and prove a no-go theorem showing that certain global correlations cannot be converted to local coherence.

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I. INTRODUCTION

Coherence is a key quantum feature. Whilst it is necessary for many quantum-enhanced tasks—such as those in metrology [1], communication [2], computation [3], and more—it is highly nontrivial to generate, maintain, and manipulate. Consequently, as with many other quantum properties, it has proven fruitful to treat it as a resource [4–6].

Coherence comes in two forms: speakable and unspeakable [5,6]. In both cases, coherence is only meaningful when considered with respect to a known decomposition of the Hilbert space into subspaces. Colloquially, for unspeakable coherence, this decomposition is performed by considering the eigenspaces of some physically relevant observable—such as the Hamiltonian, angular momentum, or photon number—whereas, for speakable coherence, an arbitrary basis is used.

The notion of unspeakable coherence has attracted significant attention as it provides a natural framework for describing coherence manipulation under symmetry-constrained dynamics, such as those conserving energy or momentum [7–16]. As a result, it has been instrumental in building our understanding of the role of coherence in thermodynamics [7–9]. Moreover,

unspeakable coherence has proven to be the relevant notion of coherence in the important tasks of quantum metrology [1], reference frame alignment [17], and work extraction [7], among others [18]. Therefore, we here focus on unspeakable coherence, henceforth referring to it as just coherence.

Given some partially coherent state, the question of whether its coherence can be increased under coherence nonincreasing operations is of both foundational and practical interest [10,19]. Foundationally, it allows us to understand the interplay of coherence and some conserved quantity, such as how coherence can be manipulated by energy-conserving dynamics [7–10,20]. Practically, these insights can guide the development of methods for generating highly coherent states, which are an essential requirement for quantum technologies.

In this work, we look to answer the resource concentration problem [21] for coherence: given two uncorrelated copies of a coherent state, can one, via globally coherence nonincreasing unitaries, i.e., closed-system dynamics, increase the coherence in a subsystem? This is the minimally feasible, and therefore the most practically implementable, scenario where coherence enhancement can be considered. Whilst the manipulation and enhancement of unspeakable coherence has been well studied in the asymptotic and catalytic conversion scenarios [8,10–16], an understanding of this minimalist case—where access to almost all available freedoms is limited—is missing. Understanding this limiting case sets a “lower bound” on what coherence enhancement is possible. This complements the “upper bound” set by considering coherence enhancement under catalytic and asymptotic distillation protocols [10] where all available freedoms are instead

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maximized, i.e., where one assumes access to an infinite number of copies of the state, infinite free states, and access to some catalytic state. Moreover, considering resource concentration allows a bottom-up understanding of what is necessary for coherence enhancement to be built.

Here, we fully solve the problem in the qubit case and use this complete characterization to build a fully constructive protocol for multiqubit coherence amplification, where only effective qubit unitaries are used. Unexpectedly, in this protocol, we find that the input-output ratio of the unspeakable coherence can be unboundedly increased for some input states. We then go beyond qubits to provide multiple, incomparable, fundamental limitations as well as no-go results on coherence enhancement.

II. UNSPEAKABLE COHERENCE

Consider a finite-dimensional system and let L be a Hermitian operator acting on it, such that L describes a physical observable. A state ρ is said to be coherent with respect to L if and only if $[\rho, L] \neq 0$ [5], as this condition is equivalent to ρ containing superpositions between different eigenspaces of L . If this is not the case, we say ρ is incoherent. This is equivalent to $e^{-iLx}\rho e^{iLx} = \rho \forall x \in \mathbb{R}$, meaning that ρ is “translation invariant” under the continuous translation generated by L [5].

As we aim to study coherence enhancement from one system to another, it is essential to characterize operations that cannot generate coherence. This ensures that no additional coherence can be added into the system. Formally, these are called *allowed operations*, which are completely positive trace-preserving linear maps $\mathcal{E}$ satisfying

$$\mathcal{E}[e^{-iLx}(\cdot)e^{iLx}] = e^{-iLx}\mathcal{E}(\cdot)e^{iLx} \quad \forall x \in \mathbb{R}; \quad (1)$$

namely, they are translationally covariant with respect to L . If one considers only unitary dynamics $\mathcal{E}(\cdot) = U(\cdot)U^\dagger$, Eq. (1) simplifies to $[U, e^{-iLx}] = 0 \forall x \in \mathbb{R}$, which holds if and only if $[U, L] = 0$. Hence, a unitary U gives an allowed operation, termed an *allowed unitary*, if and only if it is block diagonal with respect to the eigenspaces of L , i.e., $V = \sum_k V_k \Pi_k$, where each V_k is an independent unitary and $\{\Pi_k\}_k$ are projectors onto the individual eigenspaces of L such that $V_k V_l^\dagger = \delta_{k,l} \Pi_k$. From here, it is easy to see that these allowed unitaries cannot generate coherence.

We can apply this approach to a bipartite system AB by considering $L_{AB} = L_A \otimes \mathbb{I}_B + \mathbb{I}_A \otimes L_B$ (capital letters in subscripts denote separate spaces; $\mathbb{I}$ is the identity operator). If $[\rho_{AB}, L_{AB}] = 0$, then the bipartite state ρ_{AB} is incoherent with respect to the eigenspaces of L_{AB} ; also, a bipartite allowed unitary V_{AB} satisfies $[V_{AB}, L_{AB}] = 0$ (see Appendix A for further discussions). If L_{AB} is nondegenerate, then all allowed unitaries are diagonal in the joint eigenbasis of L_{AB} . Such unitaries do not facilitate any interaction between the two subsystems, and no coherence concentration is possible. If L_{AB} contains degeneracies, then the allowed unitaries are arbitrary unitaries within the degenerate subspaces of L_{AB} . Physically, this means that no matter what property L_{AB} represents, it can be freely exchangeable between A and B , as long as its total value is conserved. For example, if L_{AB} is the total Hamiltonian, then $[U_{AB}, L_{AB}] = 0$ ensures the total energy is conserved under the dynamics U_{AB} . If one can move energy

between A and B whilst conserving total energy, then coherence can be locally increased; otherwise, it cannot.

From now, we consider our *local* physical observables to be the nondegenerate, truncated number operators $L_X = \sum_{n=0}^{d-1} n|n\rangle\langle n|_X$ for $X = A, B$. This ensures that coherence cannot be locally increased, but that L_{AB} contains degeneracies so that coherence concentration is possible.

III. MODES OF COHERENCE

Under translationally covariant operations, quantum states can be decomposed into a set of modes [22]. Each mode is a set of components of the state that transform independently due to the symmetry of the covariant dynamics. Formally, the j th mode of a d -dimensional state ρ is given by

$$\rho^{(j)} := \sum_{n=0}^{d-1} |n+j\rangle\langle n+j| \rho |n\rangle\langle n|, \quad (2)$$

where we take $|n+j\rangle = 0$ if $n+j \geq d$. This is an operator supported in the j th mode subspace $\mathcal{B}^{(j)} := \text{span}\{|n+j\rangle\langle n+j|\}_{n=0}^{d-1}$. By definition, we have $-d+1 \leq j \leq d-1$ and $\rho = \sum_{j=-d+1}^{d-1} \rho^{(j)}$. We note that since ρ is a state, $[\rho^{(j)}]^\dagger = \rho^{(-j)} \forall j$, meaning it suffices to focus on $\rho^{(j)}$'s with $j \geq 0$. In addition, it can be seen that $\rho = \rho^{(0)}$ if and only if ρ is incoherent, i.e., $[\rho, L] = 0$. Finally, given two states ρ and σ , if $\sigma = \mathcal{E}(\rho)$ for some allowed operation $\mathcal{E}$, then $\sigma^{(j)} = \mathcal{E}(\rho^{(j)}) \forall j$. Hence, under symmetry-constrained dynamics, the output state's j th mode depends only on the input state's j th mode. Consequently, $\rho^{(j)} = 0$ implies $\sigma^{(j)} = 0$, which can be generalized to $\text{modes}(\sigma) \subseteq \text{modes}(\rho)$, where $\text{modes}(\rho) := \{j \in \mathbb{N} \mid \rho^{(j)} \neq 0\}$ is the set of all integers labeling modes for which ρ has a nonzero component.

In the bipartite setting, the global modes of a bipartite product state are given by a combination of local modes [22]: $(\rho_A \otimes \rho_B)^{(j)} = \sum_k \rho_A^{(k)} \otimes \rho_B^{(j-k)}$. It can further be seen that the j th-mode subspace reads $\mathcal{B}_{AB}^{(j)} := \text{span}\{|n+k, m+j-k\rangle\langle n+k, m+j-k| \mid n, m, k \in \mathbb{N}\}$. Using this, in Appendix A we show the following fact:

Lemma 1. Let $\rho_A := \text{tr}_B[\rho_{AB}]$ be the reduced state of ρ_{AB} in A . Then, $\rho_A^{(j)} = \text{tr}_B[\rho_{AB}^{(j)}]$, meaning the j th local mode depends only on the j th global mode.

Proof Sketch. The j th mode of the bipartite state σ_{AB} can be written as

$$\sigma_{AB}^{(j)} = \sum_k \sum_{n,m} q_{n+k, m+j-k, n, m} |n+k, m+j-k\rangle\langle n, m|_{AB}, \quad (3)$$

which is an operator supported in $\mathcal{B}_{AB}^{(j)}$. Taking the partial trace over the B system of this operator gives

$$\text{tr}_B(\sigma_{AB}^{(j)}) = \sum_n \tilde{q}_{n+j, n} |n+j\rangle\langle n|_A \in \mathcal{B}_A^{(j)}, \quad (4)$$

where $\tilde{q}_{n+j, n} = \sum_m q_{n+j, m, n, m}$ and $\mathcal{B}_A^{(j)}$ is the j th-mode subspace of the A system. From the definition of $\tilde{q}_{n+j, n}$, it can be seen that the matrix elements associated to each basis state are those one would expect from taking the partial trace of the whole system, $\tilde{q}_{n+j, n} = \langle n+j| \text{tr}_B[\sigma_{AB}] |n\rangle$. As the operator exists only in $\mathcal{B}_A^{(j)}$ and has the correct matrix elements, it is $\sigma_A^{(j)}$. ■

Given that the number of available modes is a function of dimension, there are more available modes globally than locally. Hence, Lemma 1 implies that tracing over a global mode with $|j| > d_A - 1$ equals zero, with it making no contribution to the local modes in the A subsystem, where d_{AB} and d_A are the dimensions of AB and A , respectively.

IV. COHERENCE CONCENTRATION PROBLEMS

To study coherence concentration, we first quantify the coherence within a specific mode. To do this, we use $M^{(j)}(\rho) := \|\rho^{(j)}\|_1$ [22], where $\|P\|_1 := \text{tr}\sqrt{PP^\dagger}$ is the *one norm* of the operator P . For all $j \neq 0$, this family of measures are monotonic under allowed operations and are faithful, i.e., $M^{(j)}(\rho) = 0 \forall j$ iff $[\rho, L] = 0$. It is also faithful for a specific mode: $M^{(j)}(\rho) = 0$ iff $\rho^{(j)} = 0$. For a given integer $j \neq 0$ and a state ρ , the key figure of merit we will be concerned with is

$$\Delta M_A^{(j)} := \max_{V_{AB}} M^{(j)}(\sigma_A) - M^{(j)}(\rho_A), \quad (5)$$

which maximizes over all allowed unitaries V_{AB} ($[V_{AB}, L_{AB}] = 0$) where $\sigma_A := \text{tr}_B[V_{AB}(\rho_A \otimes \rho_B)V_{AB}^\dagger]$. Our central question then amounts to asking: for a given j , is $\Delta M_A^{(j)} \geq 0$?

V. OPTIMAL QUBIT COHERENCE CONCENTRATION

To better understand the physics, we first focus on performing coherence concentration using two qubits, where we can set the local generator to be $L = |1\rangle\langle 1|$ without loss of generality. Each qubit has only two modes, with support on the first mode, $|1\rangle\langle 0|$, indicating the presence of coherence. This first mode then only has a single component—the single off-diagonal element of ρ —which is the sole parameter of a qubit state that determines its coherence. Hence, any coherence measure on a qubit will depend only on this parameter. Therefore, whilst we consider $\Delta M^{(j)}$, our results can be applied to *any* qubit coherence measure.

Any qubit state ρ can be completely characterized by $p_{00} := \langle 0|\rho|0\rangle$ and $p_{01} := \langle 0|\rho|1\rangle$, such that initially $M^{(1)}(\rho_A) = |p_{01}|$. From here, we can explicitly write down the analytical form for the optimal coherence concentration possible (see Appendix B for sketched proof and Appendix B 1 for a full proof).

Result 1. For a qubit state ρ , we have

$$\Delta M_A^{(1)} = |p_{01}| (\sqrt{1 + (2p_{00} - 1)^2} - 1), \quad (6)$$

which is achieved by applying a unitary rotation of angle $\cos^{-1}(1/\sqrt{1 + (2p_{00} - 1)^2})$ in the $\{|01\rangle, |10\rangle\}$ degenerate subspace of L_{AB} .

Proof Sketch. In the two-qubit case, an allowed unitary applies an arbitrary qubit unitary in the $\{|01\rangle, |10\rangle\}$ degenerate subspace. This unitary can be decomposed into two Z rotations (with respect to $|01\rangle$ and $|10\rangle$), a Y rotation, and some global phase. The Z rotations and global phase can be shown to only decrease local coherence, leaving only the Y rotation. One can apply this rotation to two copies ρ , and then use a geometric argument to find the optimal rotation angle. See Appendix B 1 for a full proof. ■

Result 1 implies that if ρ has no initial coherence, i.e., $p_{01} = 0$, then no coherence can be concentrated, as expected.

Interestingly, it can also be seen that if $p_{00} = 1/2$ (i.e., the diagonal of ρ is maximally mixed in the eigenbasis of L), then no coherence can be concentrated. Put alternately, if ρ sits on the plane through the equator of the Bloch sphere defined by L , then no coherence can be concentrated. Hence, qubit states having $p_{00} = 1/2$ and $p_{01} > 0$ carry *bound coherence* with respect to this restricted set of allowed operations, i.e., they do not have concentratable coherence, despite having nonzero and nonmaximal initial coherence [21].

VI. QUBIT CONCATENATION

Result 1 fully solves the coherence concentration problem for two qubits. We now assume access to many copies of a qubit, but still aim to maximally increase the coherence of a single qubit. Whilst this puts us outside of the resource concentration framework [21], we still utilize Result 1 to develop a multiqubit resource amplification protocol. Specifically, we consider sequentially applying the optimal qubit concentration unitaries detailed in Result 1, feeding the output of two optimal concentration processes into the input of the next (see Appendix C for a schematic example).

We equate such a concatenation protocol to a greedy optimization algorithm: the locally optimal choice is always performed, potentially at the expense of reaching the global optimum. Performing a global operation on all available copies of ρ simultaneously will be at least as powerful, and likely more efficient in terms of the local resource increase per number of copies of ρ used. However, the approach presented here offers a fully constructive method for generating a multicopy coherence amplification process, with all performed unitaries being effective qubit unitaries acting only within the degenerate subspace of L_{AB} .¹

Assuming the input qubits are characterized by p_{00}^0 and p_{01}^0 , after applying m steps of this concatenation protocol, the output qubit state of the $(m + 1)$ th step will be characterized by

$$\begin{aligned} p_{00}^{m+1} &= p_{00}^m - \frac{2(2p_{00}^m - 1)|p_{01}^m|^2}{1 + (2p_{00}^m - 1)^2}, \\ p_{01}^{m+1} &= |p_{01}^m| \sqrt{1 + (2p_{00}^m - 1)^2}, \end{aligned} \quad (7)$$

which are coupled, nonlinear recurrence relations. See Appendix B 2 for a proof. Note, after the m th step, 2^m copies of ρ will have been consumed. Hence, each additional layer leads to an exponential increase in the number of consumed states.

As a first step, we find the fixed points of these recurrence relations: values that remain unchanged when the recurrence is applied (i.e., $p_{00}^{m+1} = p_{00}^m$ and $p_{01}^{m+1} = p_{01}^m$). Physically, they represent qubit states for which no coherence can be concentrated, with $p_{00}^m = 1/2$ (part of the condition for bound coherence) and $p_{01}^m = 0$ (no initial coherence) both found to be fixed points.

¹Treating subspaces as virtual qubits was shown to be an effective strategy for simplifying the description of dynamics in Ref. [23].

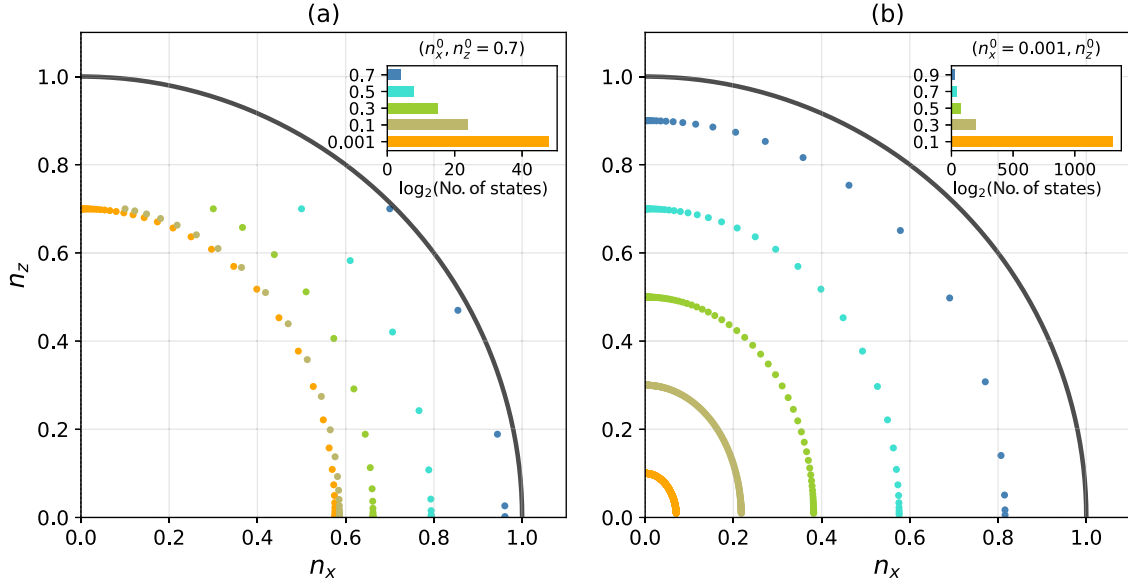


FIG. 1. Qubit concatenation paths through the Bloch sphere. (a) Paths for $n_z^0 = 0.7$ and various n_x^0 . (b) Paths for $n_x^0 = 0.001$ and various n_z^0 . The insert shows the log base 2 of the number of states needed to converge to within 0.001 of $n_z = 0$, which dictates the maximum reachable coherence.

By setting $p_{00} = (1 + n_z)/2$ and $p_{01} = (n_x + in_y)/2$, where $\mathbf{n} = (n_x, n_y, n_z)$ is the Bloch vector, we plot the output states from each step on the Bloch sphere in Fig. 1. This is done for multiple different input states. As Z rotations are locally allowed, we can set $n_y = 0$ without loss of generality, leaving $M^{(1)}(\rho) = |n_x|$ to completely characterize the coherence of ρ . Since the Zaxis and X axis are fixed points of the recurrence relations, we also only need to consider one quarter of the $n_y = 0$ plane of the Bloch sphere, as the same behavior will be replicated in the other quarters.

It can be seen in Fig. 1 that even when given access to infinitely many copies of ρ , the concatenation protocol still cannot reach a maximally coherent state ($n_x = 1$) if ρ is not a pure state. In fact, the maximum increase in coherence that can be achieved as $m \rightarrow \infty$ can be shown to depend on the purity of ρ :

Result 2. For an initial qubit state ρ , let $\sigma_{(m)}$ be the qubit state output from the concatenation protocol's m th step. Then, $M^{(1)}(\sigma_{(m)}) \leq \sqrt{2\text{tr}(\rho^2) - 1}$.

Proof Sketch. It can be shown that under the optimal concentrating unitary, the purity of the state in the A subsystem is reduced. Hence, under each step of the concatenation protocol, the purity is reduced. Relating the Bloch vector to purity via its 2-norm and using the formula of $M^{(1)}(\rho)$ completes the proof. See Appendix B 3 for a full proof. ■

The above analysis reveals an interesting feature of coherence amplification. From Eq. (7), and exemplified in Fig. 1(B), even for an initial qubit state with an *arbitrarily small amount* of coherence, coherence can still be amplified via the concatenation protocol. Unlike in the asymptotic and catalytic case [10,15], this is not arbitrary amplification, as the achievable coherence is bounded by Result 2. However, there do exist states for which the *ratio* of the final to single-copy initial coherence is *unbounded*. In Appendix B 4, we prove that:

Result 3. For any number of states $n = 2^N$ where $N \in \mathbb{N}$ and $\epsilon > 0$, there exists an initial state ρ (dependent on n, ϵ)

such that

$$M^{(1)}(\rho) < \frac{1}{n} \ \& \ \frac{M^{(1)}(\sigma_{(m)})}{M^{(1)}(\rho)} > 2^{-\epsilon} \sqrt{n} \ \forall m \geq \log_2(n). \quad (8)$$

Therefore, if n is very large, then one has a large number of initial states with low initial coherence. Result 3 then states that $\log_2(n)$ number of steps of the concatenation protocol is sufficient to amplify the ratio of the coherence $n^{3/2}$ times. As n can be arbitrarily large, the ratio of initial to final coherence can be unbounded when given access to arbitrarily many copies of the initial state. This finding is complementary to those reported in Refs. [10,15], where arbitrary amplifications of unspeakable coherence with unbounded rates are demonstrated in the asymptotic and catalytic transformation. Here, we show that, rather surprisingly, in a very limited setting where only single-qubit states and effective qubit unitaries are used, an unbounded input-output ratio can already be found.

A. Example

In this section, we give an example of a two-layer qubit concatenation protocol.

The protocol takes four copies of the state ρ as input. A two-qubit unitary, denoted by $U_{\text{opt}}(\rho)$, is applied to the first and second copy in spaces A and B, and the third and fourth copies in spaces C and D. Here, $U_{\text{opt}}(\rho)$ is the optimal concentrating unitary for ρ ; i.e., if $p_{00}^0 = \langle 0 | \rho | 0 \rangle$ and $p_{01}^0 = \langle 0 | \rho | 1 \rangle$, then

$$U_{\text{opt}}(\rho) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos \theta_{\text{opt}} & -\sin \theta_{\text{opt}} & 0 \\ 0 & \sin \theta_{\text{opt}} & \cos \theta_{\text{opt}} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad (9)$$

where $\theta_{\text{opt}} = \cos^{-1}(1/\sqrt{1 + (2p_{00} - 1)^2})$. Let $\sigma_B = \text{tr}_A[U_{\text{opt}}(\rho)(\rho_A \otimes \rho_B)U_{\text{opt}}(\rho)^\dagger]$ be the output state into space B from the first concentration step, which is also the output into

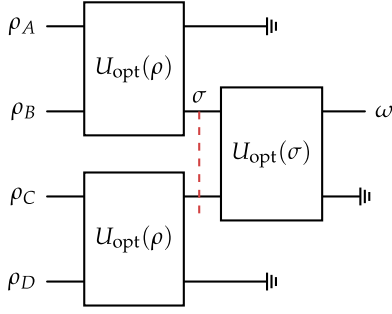


FIG. 2. A quantum circuit diagram of the qubit concatenation protocol for four input states.

space C . Let $p_{00}^1 = \langle 0 | \sigma | 0 \rangle$ and $p_{01}^1 = \langle 0 | \sigma | 1 \rangle$, which can be explicitly calculated via the recurrence relations in Eq. (7). From here, the optimal unitary for performing coherence concentration given two copies of σ , denoted by $U_{\text{opt}}(\sigma)$, can be found and is of the same form as above. Applying $U_{\text{opt}}(\sigma)$ to the states in wires B and C will output an even more coherent state $\omega_B = \text{tr}_A[U_{\text{opt}}(\sigma)(\sigma_B \otimes \sigma_C)U_{\text{opt}}(\sigma)^\dagger]$. One can repeat this process an arbitrary number of times. See Fig. 2 for a graphical depiction of the protocol.

In Fig. 3, we plot the vector field of the recurrence relations in Eq. (7). The increase in coherence in each step is smallest near the fixed points (along the x axis and z axis), and it can be seen that a smaller n_x (closer to the z axis) leads to smaller step sizes than a smaller n_z (closer to the x axis). It can also be seen in Fig. 3 that the change in n_z and n_x in a single step is highly dependent on the input state, with n_x or n_z close to the fixed points leading to smaller steps. When close to the n_z axis, such that n_x is close to zero, the small step size results from

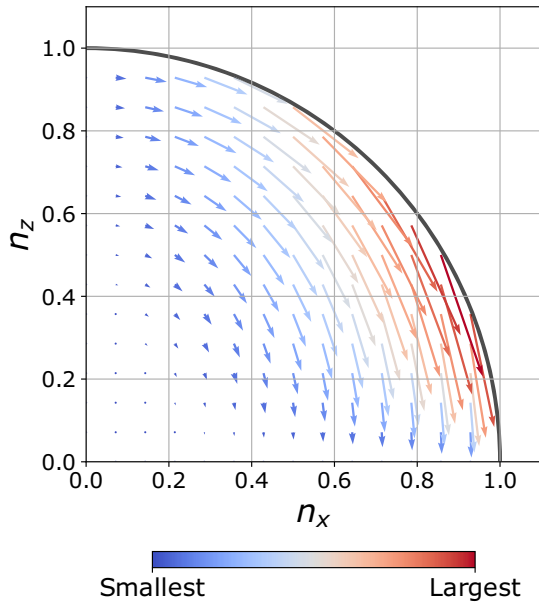


FIG. 3. Vector field for the recurrence relations in Eq. (7). The length of the vector and its color detail how much a given point will change under the recurrence relation. For example, a short blue arrow means a small change will occur, a long red arrow means a large change will occur.

both input states into the concentration having low coherence, and hence there not being much coherence to relocate from system B into system A . When close to the n_x axis, however, the step sizes remain small, despite both input states having a (potentially) large amount of coherence. The small step size here instead arises from the fact that increasing coherence when the Bloch vector points predominantly along the X axis is equivalent to increasing the purity of the state. Since it is impossible to increase the local purity given two qubits and unitary operations [21], each step size is the small change in coherence that can be managed whilst not increasing the purity.

VII. COHERENCE CONCENTRATION BEYOND QUBITS: FUNDAMENTAL LIMITATIONS

We now investigate the general concentration cases, considering states with arbitrary finite local dimension. Given that modes transform independently under allowed operations, in conjugation with Lemma 1, one only needs to consider the j th global mode to concentrate coherence on the j th local mode. However, not all components of the global mode will contribute to the local mode. Specifically, one can separate $\mathcal{B}_{AB}^{(j)}$ into two subspaces,

$$\begin{aligned}\mathcal{V}_{\text{in}}^{(j)} &:= \text{span}\{|n+j, m\rangle\langle n, m|\}_{n,m} \\ \mathcal{V}_{\text{out}}^{(j)} &:= \text{span}\{|n+k, m+j-k\rangle\langle n, m|\}_{n,m,k \neq j},\end{aligned}\quad (10)$$

where $\mathcal{B}_{AB}^{(j)} = \mathcal{V}_{\text{in}}^{(j)} \cup \mathcal{V}_{\text{out}}^{(j)}$ and $\mathcal{V}_{\text{in}}^{(j)} \cap \mathcal{V}_{\text{out}}^{(j)} = \emptyset$, such that the basis operators in $\mathcal{V}_{\text{in}}^{(j)}$ contribute to the local mode, whereas those in $\mathcal{V}_{\text{out}}^{(j)}$ do not. The aim of mode concentration can therefore be loosely summarized as maximizing the coefficients of the basis operators in $\mathcal{V}_{\text{in}}^{(j)}$ under allowed unitaries. It is noted that the projector onto $\mathcal{V}_{\text{in}}^{(j)}$ reads $\Pi_A^{(j)} \otimes \Pi_B^{(0)}$, where $\Pi_X^{(0)}$ is the projector onto the j th local mode ($X = A, B$). Using this, we find the following upper bound on the ability to concentrate coherence in the j th local mode.

Result 4. Let $d^{(j)} := \dim(\mathcal{V}_{\text{in}}^{(j)})$. For any state ρ , we have

$$\Delta M_A^{(j)} \leq \|(\rho_A \otimes \rho_B)^{(j)}\|_{d^{(j)}-\text{KF}} - \|\rho^{(j)}\|_1. \quad (11)$$

Proof Sketch. The figure of merit $M_A^{(j)}(\sigma_A)$ can be seen to be upper bounded by the sum of the absolute values of the matrix elements of $\sigma_{AB}^{(j)}$ associated to basis operators in $\mathcal{V}_{\text{in}}^{(j)}$. All these matrix elements sit along a diagonal of length $\tilde{d}$. It can separately be shown that for any operator, the sum of the absolute values of the matrix elements along any diagonal of length k is upper-bounded by the k th Ky-Fan norm of the operator. Using the unitary invariance of the Ky-Fan norm and taking away $M^{(j)}(\rho_A)$ completes the proof. See Appendix B 5 for a full proof. ■

Here, $\|P\|_{\alpha-\text{KF}}$ is the α -Ky-Fan norm [24], which is the sum of the α largest singular values of the operator P . Hence, as for informational non-equilibrium concentration [21], the Ky-Fan norm emerges as a tool for quantifying how much of the global resource content of a state can be relocated into a subsystem.

We now present an alternative (and more technical) bound on coherence concentration, which is sometimes tighter than the one reported in Result 4 (see also Appendix C for

details). Moreover, see Appendix E for a qutrit example of both bounds.

We achieve this alternative bound by noting that a global mode can be separated further into independently transforming components. Within the j th mode, we introduce the $(\lambda_c + j, \lambda_c)$ *left-right-degenerate* (LR-D) operator of the mode as $\Pi_{c+j}(\rho \otimes \rho)^{(j)}\Pi_c$, where Π_c is the projection onto the eigenspace of L_{AB} with eigenvalue λ_c (i.e., $L_{AB} = \sum_c \lambda_c \Pi_c$ with spectrum $\{\lambda_c\}_c$). Under any bipartite allowed unitary V_{AB} , the $(\lambda_c + j, \lambda_c)$ LR-D operator of the j th mode then evolves independently as $V_{c+j}(\Pi_{c+j}(\rho \otimes \rho)^{(j)}\Pi_c)V_c$, where V_c is the unitary that acts within the c th degenerate subspace of L_{AB} . One can then perform a similar analysis to Result 4 and define the $(\lambda_c + j, \lambda_c)$ th LR-D subspace, $\mathcal{B}_{AB}^{(j,c)}$, and

$$\begin{aligned}\mathcal{V}_{\text{in}}^{(j,c)} &:= \text{span}\{|n+j, m\rangle\langle n, m|\}_{n+m=c} \\ \mathcal{V}_{\text{out}}^{(j,c)} &:= \text{span}\{|n+k, m+j-k\rangle\langle n, m|\}_{n+m=c, k \neq j}\end{aligned}\quad (12)$$

with $\mathcal{B}_{AB}^{(j,c)} = \mathcal{V}_{\text{in}}^{(j,c)} \cup \mathcal{V}_{\text{out}}^{(j,c)}$ and $\mathcal{V}_{\text{in}}^{(j,c)} \cap \mathcal{V}_{\text{out}}^{(j,c)} = \emptyset$. The aim of mode concentration is again to maximize the coefficients associated with the basis operators in $\mathcal{V}_{\text{in}}^{(j,c)}$. Applying this approach leads to the following upper bound:

Result 5. Let $d^{(j,c)} := \dim(\mathcal{V}_{\text{in}}^{(j,c)})$. For any state ρ , we have

$$\Delta M_A^{(j)} \leq \sum_{c=0}^{2d-2-j} \|\Pi_{c+j}(\rho \otimes \rho)^{(j)}\Pi_c\|_{d^{(j,c)}\text{-KF}} - \|\rho^{(j)}\|_1. \quad (13)$$

The proof technique is the same as for Result 4, but now applied individually to each LR-D subspace. See Appendix B 6 for the full proof.

Interestingly, in Appendix C, we show that bounds from Results 4 and 5 are incomparable—each provides a tighter upper bound in a variety of cases considered.

VIII. THE ROLE OF INITIAL CORRELATIONS AND NO-GO RESULTS FOR COHERENCE CONCENTRATION

Finally, we expand the concentration problem beyond two initially uncorrelated copies, and ask when initial correlations can aid in local coherence enhancement and when they cannot. In AB with equal local dimension d , consider a state $\rho_{AB} = \rho_{AB}^{(0)} + \sum_{i \geq d} \rho_{AB}^{(i)}$ such that there exists at least one $k \geq d$ where $\rho_{AB}^{(k)} \neq 0$ and $\rho_{AB}^{(k)} = 0 \forall k \in [1, d-1]$. In Appendix D, we show that such a mode distribution implies correlations exist between A and B . Now, applying any allowed operation to this state will map it to an output state supported on the same modes. Then, using Lemma 1, it can be concluded that both local states are supported only on the zeroth mode (as the partial trace over modes with $k \geq d$ is zero, making no contribution to the reduced states). Hence, both local states are always symmetric, regardless of the applied allowed operation. Therefore, despite the global state having coherence between the eigenspaces of L_{AB} , there exists no allowed operation—not just no allowed unitary—that can increase the local coherence. Furthermore, this argument is *independent* of the choice of coherence measure. We thus summarize this as the following no-go result.

Result 6. If $[1, d-1] \cap \text{modes}(\rho_{AB}) = \{0\}$, then there exists no allowed operation to concentrate coherence with respect to any coherence measure.

Result 6 demonstrates that not all coherence in a bipartite state is able to be converted into local coherence by covariant (allowed) operations, despite the covariant operations acting globally on the state. As a consequence, it also demonstrates that there exist correlations between the A and B subsystems that are unable to improve one's ability to perform local resource enhancement. As an explicit example, consider the two-qubit isotropic state, $\rho_{AB}^{\text{iso}} = p|\Phi^+\rangle\langle\Phi^+|_{AB} + (1-p)\mathbb{I}/2$, where $0 \leq p \leq 1$ and $|\Phi^+\rangle_{AB} = (|00\rangle + |11\rangle)_{AB}/\sqrt{2}$. As this state has support only on the zeroth and second bipartite modes, no coherence can be locally concentrated for all $p > 0$, despite there existing a correlation between A and B for all $p > 0$. This also means global entanglement—even maximal entanglement—may not always be exchangeable for local coherence, despite the ability to perform global allowed operations.

IX. DISCUSSION

Here, we have studied the concentration problem for (un)speakable coherence. A complete characterization was found for two qubits, and upper bounds were given beyond qubits. From this, we detailed a completely constructive, multiqubit coherence amplification protocol that uses only effective qubit unitaries involved. It was shown to be able to unboundedly amplify the ratio of the input-output coherence for certain initial states—a finding that is complementary to the recently reported unbounded coherence amplification in the asymptotic and catalytic regime [10]. It would be instructive to quantitatively understand how far from the global optimal this protocol is, as it would inform whether the increased practicality of the concatenation protocol is worth the reduction in coherence concentrated. In addition, it would be interesting to introduce finitely many incoherent states into the concentration protocol, and assess whether more coherence enhancement can then be achieved given access to these incoherent states.

Finally, a family of states with bound coherence with respect to the resource concentration scenario was shown to exist, i.e., states that are not maximally coherent, nor incoherent, but for which no allowed unitary exists to perform coherence concentration. We leave it open to determine whether such states exist in higher dimensions.

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DATA AVAILABILITY

The data that support the findings of this article are openly available [25].

APPENDIX A: RELATIONSHIP BETWEEN LOCAL AND GLOBAL SYMMETRY

A few observations can be made about the relationship between the local and global (bipartite) symmetry and, hence, the local and global coherence. Firstly, we prove the following Lemma.

Lemma A1 (Global symmetry implies local symmetry).

Consider a finite-dimensional bipartite system AB . Let ρ_{AB} be a bipartite state in AB and L_A and L_B be Hermitian operators acting on systems A and B , respectively. Then,

$$[\rho_{AB}, L_{AB}] = 0 \Rightarrow [\rho_A, L_A] = 0, \quad (\text{A1})$$

where $\rho_A = \text{tr}_B[\rho_{AB}]$ and $L_{AB} = L_A \otimes \mathbb{I}_B + \mathbb{I}_A \otimes L_B$, where L_A, L_B are nondegenerate, truncated number operators.

Proof. This can be seen from the following equivalence for a given Hermitian operator L_X :

$$[\rho_X, L_X] = 0 \iff e^{-iL_X\theta} \rho_X e^{iL_X\theta} = \rho_X \quad \forall \theta \in \mathbb{R}. \quad (\text{A2})$$

Given the bipartite state ρ_{AB} and bipartite Hermitian operator L_{AB} , such that $[\rho_{AB}, L_{AB}] = 0$, taking a partial trace over B then gives

$$\begin{aligned} \rho_A &= \text{tr}_B[\rho_{AB}] \\ &= \text{tr}_B[(e^{-iL_{AB}\theta})\rho_{AB}(e^{iL_{AB}\theta})] \quad \forall \theta \in \mathbb{R} \\ &= \text{tr}_B[(e^{-iL_A\theta} \otimes e^{-iL_B\theta})\rho_{AB}(e^{iL_A\theta} \otimes e^{iL_B\theta})] \quad \forall \theta \in \mathbb{R} \\ &= e^{-iL_A\theta} \text{tr}_B[\rho_{AB}] e^{iL_A\theta} \quad \forall \theta \in \mathbb{R}, \\ &= e^{-iL_A\theta} \rho_A e^{iL_A\theta} \quad \forall \theta \in \mathbb{R}, \end{aligned} \quad (\text{A3})$$

where we have used the fact that the partial trace over B is linear and cyclic for operators on only B . Hence, from Eq. (A2), it can be concluded that $[\rho_A, L_A] = 0$.

Lemma A1 means that global symmetry implies local symmetry. However, the reverse is not true: the presence of local symmetry does not imply global symmetry. To see this, consider $\rho_{AB} = |\Phi^+\rangle\langle\Phi^+|_{AB}$ with $|\Phi^+\rangle_{AB} = (1/\sqrt{2})(|00\rangle + |11\rangle)_{AB}$, and $L_X = \lambda_0|0\rangle\langle 0|_X + \lambda_1|1\rangle\langle 1|_X$ for $\lambda_0 \neq \lambda_1$ ($X = A, B$). Locally both systems are maximally mixed and hence symmetric, while the global state is not symmetric with respect to L_{AB} . In the specific case that ρ_{AB} is a product state, $\rho_{AB} = \rho_A \otimes \rho_B$, then local symmetry in both subsystems ($[\rho_A, L_A] = 0$ and $[\rho_B, L_B] = 0$) does imply global symmetry ($[\rho_A \otimes \rho_B, L_{AB}] = 0$). This can then be extended to hold for any convex mixture of product states that are locally symmetric. Using these ideas, we now argue that one cannot exploit the differing global and local symmetries in the concentration scenario to increase the local coherence unless there is some initial local coherence present. If ρ_{AB} is a product state and both local systems contain no coherence, then ρ_{AB} is incoherent. Since V_{AB} cannot increase coherence, $V_{AB}(\rho_A \otimes \rho_B)V_{AB}^\dagger$ must also be incoherent. Hence, according to Lemma A1, the output local systems will be incoherent, too.

APPENDIX B: PROOFS OF MAIN RESULTS

1. Proof of Result 1

We first state Result 1 here for clarity.

Result 1. For a qubit state ρ , we have

$$\Delta M_A^{(1)} = |p_{01}| (\sqrt{1 + (2p_{00} - 1)^2} - 1), \quad (\text{B1})$$

where the maximum is over all resource nonincreasing unitaries, with the optimal achieved by applying a rotation of angle $\cos(\sqrt{1 + (2p_{00} - 1)^2})^{-1}$ in the degenerate subspace.

Proof. Firstly, we note that an arbitrary single-qubit state ρ can be written as

$$\rho = \begin{bmatrix} p_{00} & p_{01} \\ p_{01}^* & 1 - p_{00} \end{bmatrix}, \quad (\text{B2})$$

where the decomposition is with respect to the eigenbasis of L , and $p_{00} = \langle 0 | \rho | 0 \rangle$, $p_{01} = \langle 0 | \rho | 1 \rangle$. To two copies of ρ , we apply an allowed unitary V_{AB} on it (i.e., $[V_{AB}, L_{AB}] = 0$). The most general form of such a unitary is

$$V_{AB} = \begin{bmatrix} e^{i\omega_0} & 0 & 0 & 0 \\ 0 & U_{00} & U_{01} & 0 \\ 0 & U_{10} & U_{11} & 0 \\ 0 & 0 & 0 & e^{i\omega_1} \end{bmatrix}, \quad (\text{B3})$$

where $0 \leq \omega_i \leq 2\pi$ for $i \in \{0, 1\}$ and

$$\begin{aligned} \begin{bmatrix} U_{00} & U_{01} \\ U_{10} & U_{11} \end{bmatrix} &= e^{-i\alpha} \begin{bmatrix} e^{-i\phi_1} & 0 \\ 0 & e^{i\phi_1} \end{bmatrix} \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \\ &\times \begin{bmatrix} e^{-i\phi_2} & 0 \\ 0 & e^{i\phi_2} \end{bmatrix} \end{aligned} \quad (\text{B4})$$

is a general two-qubit unitary in the $\{|01\rangle, |10\rangle\}$ degenerate subspace, which has been decomposed into a combination of two rotations about the Z axis (defined by the states $\{|01\rangle, |10\rangle\}$), a rotation about the Y axis, and a phase factor α . Hence, the two-qubit unitary V_{AB} can be written as a product of unitaries,

$$V_{AB} = S(\omega_0)T(\omega_1)R^z(2\phi_1)R^y(2\theta)R^z(2\phi_2), \quad (\text{B5})$$

where

$$\begin{aligned} S(\omega_0) &= \begin{bmatrix} e^{i\omega_0} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \\ T(\omega_1) &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & e^{i\omega_1} \end{bmatrix}, \\ R^z(2\phi) &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & e^{-i\phi} & 0 & 0 \\ 0 & 0 & e^{i\phi} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \\ R^y(2\theta) &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos \theta & -\sin \theta & 0 \\ 0 & \sin \theta & \cos \theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \end{aligned} \quad (\text{B6})$$

with all decompositions with respect to the eigenbasis of L_{AB} .

To achieve coherence concentration, we need to evaluate the output state $\sigma_A = \text{tr}_B[V_{AB}(\rho_A \otimes \rho_B)V_{AB}^\dagger]$, which is characterized by $q_{00} := \langle 0 | \sigma_A | 0 \rangle$ and $q_{01} := \langle 0 | \sigma_A | 1 \rangle$. To this end, we now sequentially apply each unitary in Eq. (B5) to $\rho_A \otimes \rho_B$ and assess each's ability to increase local coherence. Firstly, if applying $S(\omega_0)$,

$$\begin{aligned} |q_{01}| &= |p_{01}(1 - p_{00} + e^{i\omega_0} p_{00})| \\ &\leq |p_{01}|(|1 - p_{00}| + |e^{i\omega_0} p_{00}|) \\ &= |p_{01}|, \end{aligned} \quad (\text{B7})$$

where the equality occurs when $e^{i\omega_0} = 1$. Hence, under unitaries of this form, we have $\Delta M^{(j)} \leq 0 \forall \omega_0$, with equality if $\omega_0 = 2\pi n$ for integers n . The same can be shown for $T(\omega_1)$, meaning that when looking to concentrate coherence, it is always sufficient to set $\omega_0 = 0 = \omega_1$ and hence apply the identity in the $|00\rangle$ and $|11\rangle$ subspaces.

Secondly, when applying $R^z(2\phi)$ to $\rho_A \otimes \rho_B$ and taking the partial trace, it can be seen that $|q_{01}| = |p_{01}| \forall \phi$, meaning that $\Delta M^{(j)} = 0$ for all unitaries of this form. Hence, when looking to concentrate coherence, it is sufficient to always set $\phi = 0$. It is noted that $[S(\omega_0), R^z(2\phi_1)R^y(2\theta)R^z(2\phi_1)] = 0$ and $[T(\omega_1), R^z(2\phi_1)R^y(2\theta)R^z(2\phi_1)] = 0$, meaning that changing the order in which these unitaries are applied cannot help one concentrate coherence. One therefore only has to consider the local coherence increasing power of $R^y(2\theta)$.

Applying $R^y(2\theta)$ to $\rho_A \otimes \rho_B$ and tracing out the B system then gives $q_{01} = p_{01}[\cos \theta + (2p_{00} - 1)\sin \theta]$. From here, the maximum $M^{(j)}(\sigma_A)$ that can be achieved under allowed unitaries is given by

$$\max_{V_{AB}} M^{(j)}(\sigma_A) = \max_{\theta} |p_{01}[\cos \theta + \sin \theta(2p_{00} - 1)]|$$

$$= |p_{01}| \max_{\theta} |\mathbf{u} \cdot \mathbf{v}|, \quad (\text{B8})$$

where

$$\mathbf{u} = \begin{bmatrix} \cos \theta \\ \sin \theta \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} 1 \\ 2p_{00} - 1 \end{bmatrix}. \quad (\text{B9})$$

The dot product will be maximized when $\mathbf{u}$ points in the same direction as $\mathbf{v}$. Hence, the maximum is achieved by

$$\mathbf{u}_{\text{opt}} = \frac{1}{\sqrt{1 + (2p_{00} - 1)^2}} \begin{bmatrix} 1 \\ 2p_{00} - 1 \end{bmatrix}, \quad (\text{B10})$$

from which the optimal angle of rotation to increase the local coherence can be found:

$$\theta_{\text{opt}} := \cos^{-1} \left(\frac{1}{\sqrt{1 + (2p_{00} - 1)^2}} \right). \quad (\text{B11})$$

Finally, using $\mathbf{u}_{\text{opt}}$, the maximum $\Delta M_A^{(j)}$ can be found to be

$$\begin{aligned} \Delta M_A^{(j)} &:= \max_{V_{AB}} M^{(j)}(\sigma_A) - M^{(j)}(\rho_A) \\ &= |p_{01}|(\sqrt{1 + (2p_{00} - 1)^2}) - |p_{01}|, \end{aligned} \quad (\text{B12})$$

completing the proof. $\blacksquare$

2. Proof of Eq. (7)

Proof. Here, we will show how the coupled nonlinear recurrence relationships, namely, Eq. (7) in the main text, are formed from the unitary found in Result 1. Assuming one initially has two copies of the state ρ :

$$\begin{aligned} \rho_A \otimes \rho_B &= \begin{bmatrix} p_{00} & p_{01} \\ p_{01}^* & (1 - p_{00}) \end{bmatrix} \otimes \begin{bmatrix} p_{00} & p_{01} \\ p_{01}^* & (1 - p_{00}) \end{bmatrix} \\ &= \begin{bmatrix} p_{00}^2 & p_{00}p_{01} & p_{01}p_{00} & p_{01}^2 \\ p_{00}p_{01}^* & p_{00}(1 - p_{00}) & |p_{01}|^2 & p_{01}(1 - p_{00}) \\ p_{01}^*p_{00} & |p_{01}|^2 & (1 - p_{00})p_{00} & (1 - p_{00})p_{01} \\ (p_{01}^*)^2 & p_{01}^*(1 - p_{00}) & (1 - p_{00})p_{01}^* & (1 - p_{00})^2 \end{bmatrix}. \end{aligned} \quad (\text{B13})$$

To this bipartite state, the following optimal concentrating unitary, found in Result 1, is applied:

$$U_{\text{opt}}(\rho) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos \theta_{\text{opt}} & -\sin \theta_{\text{opt}} & 0 \\ 0 & \sin \theta_{\text{opt}} & \cos \theta_{\text{opt}} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad (\text{B14})$$

where, from Eq. (B11), we have

$$\cos \theta_{\text{opt}} = \frac{1}{\sqrt{1 + (2p_{00} - 1)^2}}, \quad \sin \theta_{\text{opt}} = \frac{2p_{00} - 1}{\sqrt{1 + (2p_{00} - 1)^2}}, \quad (\text{B15})$$

which are dependent on ρ . The B system is then traced over, giving an output state

$$\sigma_A = \text{tr}_B[U_{\text{opt}}(\rho)(\rho_A \otimes \rho_B)U_{\text{opt}}(\rho)^\dagger] =: \begin{bmatrix} q_{00} & q_{01} \\ q_{01}^* & (1 - q_{00}) \end{bmatrix}. \quad (\text{B16})$$

By explicitly performing this calculation, it can be seen that

$$\begin{aligned} q_{00} &= p_{00}^2 + p_{00}(1 - p_{00})\cos^2 \theta_{\text{opt}} - |p_{01}|^2 \sin \theta_{\text{opt}} \cos \theta_{\text{opt}} \\ &\quad - |p_{01}|^2 \sin \theta_{\text{opt}} \cos \theta_{\text{opt}} \\ &= p_{00} - 2|p_{01}|^2 \sin \theta_{\text{opt}} \cos \theta_{\text{opt}}, \end{aligned} \quad (\text{B17})$$

whilst

$$\begin{aligned} q_{01} &= p_{00}p_{01} \sin \theta_{\text{opt}} + p_{01}p_{00} \cos \theta_{\text{opt}} + p_{01}(1 - p_{00}) \cos \theta_{\text{opt}} \\ &\quad - (1 - p_{00})p_{01} \sin \theta_{\text{opt}} \\ &= p_{01}[\cos \theta_{\text{opt}} + \sin \theta_{\text{opt}}(2p_{00} - 1)]. \end{aligned} \quad (\text{B18})$$

Inputting $\cos \theta_{\text{opt}}$ and $\sin \theta_{\text{opt}}$ into these equations then gives

$$\begin{aligned} q_{00} &= p_{00} - \frac{2(2p_{00} - 1)|p_{01}|^2}{1 + (2p_{00} - 1)^2}, \\ q_{01} &= p_{01}\sqrt{1 + (2p_{00} - 1)^2}. \end{aligned} \quad (\text{B19})$$

The output state σ , as characterized by q_{00} and q_{01} , therefore depends only on p_{00} and p_{01} from ρ . We then note that local Z rotations are always allowed, such that we can set $p_{01} \rightarrow |p_{01}|$ without loss of generality. One can now consider taking two copies of σ and repeating the same process. This would output the state

$$\omega_A = \text{tr}_B[U_{\text{opt}}(\sigma)(\sigma_A \otimes \sigma_B)U_{\text{opt}}(\sigma)^\dagger] =: \begin{bmatrix} l_{00} & l_{01} \\ l_{01}^* & 1 - l_{00} \end{bmatrix}, \quad (\text{B20})$$

where, using the same calculation as above, it can be seen that

$$\begin{aligned} l_{00} &= q_{00} - \frac{2(2q_{00} - 1)|q_{01}|^2}{1 + (2q_{00} - 1)^2}, \\ l_{01} &= |q_{01}|\sqrt{1 + (2q_{00} - 1)^2}. \end{aligned} \quad (\text{B21})$$

Again, the output state ω , as characterized by l_{00} and l_{01} , can be seen to depend on q_{00} and q_{01} from σ . This process can then be repeated an arbitrary finite number of times, where each time the parameters characterizing the output state can be formulated in terms of the parameters characterizing the input state. This, therefore, takes the form of recurrence relations, which here are coupled and nonlinear functions where the next value of the function depends on some combination of the previous value. We note that here the outputs also only depend on the previous values, and not any further back in the sequence. By labeling the input state from the m th layer as p_{00}^m and p_{01}^m , the outputs from the $m + 1$ layer can therefore be given by

$$\begin{aligned} p_{00}^{m+1} &= p_{00}^m - \frac{2(2p_{00}^m - 1)|p_{01}^m|^2}{1 + (2p_{00}^m - 1)^2}, \\ p_{01}^{m+1} &= |p_{01}^m|\sqrt{1 + (2p_{00}^m - 1)^2}, \end{aligned} \quad (\text{B22})$$

which is exactly Eq. (7) in the main text. The proof is thus completed. ■

3. Proof of Result 2

This Appendix contains a proof of Result 2, which places an upper bound on the achievable final coherence in the qubit concatenation protocol by the purity of the input state.

Result 2. Given an initial qubit state ρ , let $\sigma_{(m)}$ be the qubit state output after the m th step of the concatenation protocol. Then,

$$M^{(1)}(\sigma_m) \leq \sqrt{2\text{tr}(\rho^2) - 1}. \quad (\text{B23})$$

Proof. Firstly, if ρ is already incoherent, we must have $M^{(1)}(\sigma_{(m)}) = 0 \forall m$ and the bound holds immediately. Hence, it suffices to focus on ρ with some initial coherence.

Now, let $\mathbf{n}^0 = (n_x^0, n_y^0, n_z^0) \in \mathbb{R}^3$ be ρ 's Bloch vector, which has 2-norm $\|\mathbf{n}^0\|_2 = \sqrt{(n_x^0)^2 + (n_y^0)^2 + (n_z^0)^2}$. Let $p_{00}^0 := \langle 0|\rho|0\rangle = (1 + n_z^0)/2$ and $p_{01}^0 := \langle 0|\rho|1\rangle = (n_x^0 + in_y^0)/2$. As stated in the main text, since Z rotations are all locally allowed, we can set $n_y = 0$ without loss of generality. Also, as we assume ρ is coherent, we must have $n_x^0 \neq 0$. This means

ρ 's purity, $\text{tr}(\rho^2)$, can be expressed as

$$\text{tr}[\rho^2] = (p_{00}^0)^2 + (1 - p_{00}^0)^2 + 2|p_{01}^0|^2 = \frac{1 + \|\mathbf{n}^0\|_2^2}{2}. \quad (\text{B24})$$

Furthermore, the recurrence relations [Eq. (7) in the main text] can be rewritten in terms of the Bloch vector as

$$\begin{aligned} n_z^{m+1} &= n_z^m - \frac{n_z^m (n_x^m)^2}{1 + (n_z^m)^2}, \\ |n_x^{m+1}| &= |n_x^m| \sqrt{1 + (n_z^m)^2}, \end{aligned} \quad (\text{B25})$$

where n_z^m and n_x^m are the Z and X components of $\sigma_{(m)}$'s Bloch vector, denoted by $\mathbf{n}^m$. We note that $n_y^m = 0 \forall m$ such that $M^{(1)}(\sigma_{(m)}) = |n_x^m|$. Now, a direct calculation shows that

$$\|\mathbf{n}^{m+1}\|_2^2 = \|\mathbf{n}^m\|_2^2 + \frac{(n_z^m n_x^m)^2}{1 + (n_z^m)^2} \left[(n_z^m)^2 + \frac{(n_x^m)^2}{1 + (n_z^m)^2} - 1 \right]. \quad (\text{B26})$$

It is then noted that

$$\begin{aligned} (n_z^m)^2 + \frac{(n_x^m)^2}{1 + (n_z^m)^2} &= \frac{(n_z^m)^2 + (n_x^m)^2 + (n_z^m)^4}{1 + (n_z^m)^2} \\ &\leq \frac{1 + (n_z^m)^4}{1 + (n_z^m)^2} = 1 - (n_z^m)^2 \leq 1 \quad \forall m, \end{aligned} \quad (\text{B27})$$

as $(n_z^m)^2 + (n_x^m)^2 \leq 1$ due to normalization and $1 + (n_z^m)^4 = [1 + (n_z^m)^2][1 - (n_z^m)^2]$. As $(n_z^m n_x^m)^2 / [1 + (n_z^m)^2] \geq 0 \forall m$, Eqs. (B26) and (B27) jointly imply that

$$\|\mathbf{n}^{m+1}\|_2 \leq \|\mathbf{n}^m\|_2 \quad \forall m. \quad (\text{B28})$$

Using Eq. (B24), the definition of the 2-norm, and recalling that $\sigma_{(m)}$'s coherence reads $M^{(1)}(\sigma_{(m)}) = |n_x^m|$, we obtain

$$M^{(1)}(\sigma_{(m)}) = |n_x^m| \leq \|\mathbf{n}^m\|_2 \leq \|\mathbf{n}^0\|_2 = \sqrt{2\text{tr}(\rho^2) - 1}, \quad (\text{B29})$$

which is the desired result. ■

As a remark, we have the following observation:

If the initial qubit state ρ is coherent, then, no matter how small the initial coherence is, we must have $n_z^\infty = 0$.

Namely, a vanishing amount of coherence possessed by ρ suffices to asymptotically erase the Z component of the Bloch vector.

Proof. To see this, first, from the above proof, we observe that $n_z^{m+1} \leq n_z^m$ and $|n_x^{m+1}| \geq |n_x^m| \forall m \in \mathbb{N}$, meaning that $\{|n_x^m|\}_m$ is monotonic nondecreasing and $\{n_z^m\}_m$ is monotonic nonincreasing when $m \rightarrow \infty$. Hence, by the monotone convergence theorem (see, e.g., Ref. [26]), we must have

$$\lim_{m \rightarrow \infty} n_z^m = \inf_m n_z^m =: n_z^\infty \quad \text{and} \quad \lim_{m \rightarrow \infty} |n_x^m| = \sup_m |n_x^m| =: n_x^\infty. \quad (\text{B30})$$

Now we argue that $n_z^\infty = 0$. To see this, we apply $m \rightarrow \infty$ on both sides of $|n_x^{m+1}| = |n_x^m| \sqrt{1 + (n_z^m)^2}$ [i.e., the second line in Eq. (B25)], obtaining $(n_x^\infty)^2 = (n_x^\infty)^2 [1 + (n_z^\infty)^2]$ (which is true because both limits $\lim_{m \rightarrow \infty} n_z^m$ and $\lim_{m \rightarrow \infty} |n_x^m|$ exist).

This then implies that $|n_x^\infty|n_z^\infty = 0$. Hence, we must have either $n_z^\infty = 0$ or $|n_x^\infty| = 0$. However, if $n_x^\infty = 0$, then, since $\{|n_x^m|\}_m$ is monotonic nondecreasing, we must have $n_x^0 = 0$, meaning that the initial state ρ is already incoherent, in contradiction to our assumption. Consequently, we must have $n_z^\infty = 0$. ■

4. Proof of Result 3

This Appendix contains a proof of Result 3, which is first stated here for clarity.

Result 3. For any number of states $n \in \mathbb{N}$ and $\epsilon > 0$, there exists an initial state ρ (dependent on n, ϵ) such that

$$M^{(1)}(\rho) < \frac{1}{n} \text{ \& } \frac{M^{(1)}(\sigma_{(m)})}{M^{(1)}(\rho)} > 2^{-\epsilon} \sqrt{n} \forall m \geq \log_2(n). \quad (\text{B31})$$

Proof. For a given $N \in \mathbb{N}$, we firstly choose an initial qubit state ρ with Bloch vector $(n_x^0, n_y^0 = 0, n_z^0)$ satisfying

$$0 < |n_x^0|^2 \ll \frac{1}{2^{2N}} \times \frac{1}{2N} \times \min\{1 - |n_z^0|^2, |n_z^0|^2\}, \quad (\text{B32})$$

where the notation “ $x \ll y$ ” means that $x/y \sim O(1)$ is significantly smaller than 1. As a result, this state has a vanishing amount of coherence, as

$$M^{(1)}(\rho) = |n_x^0| \ll \frac{1}{2^N}. \quad (\text{B33})$$

For convenience, let us define, for $m \in \mathbb{N} \cup \{0\}$,

$$A_m := |n_x^m|^2 \quad \text{and} \quad B_m := |n_z^m|^2. \quad (\text{B34})$$

Then, Eq. (B32) becomes

$$0 < A_0 \ll \frac{1}{2^{2N}} \times \frac{1}{2N} \times \min\{1 - B_0, B_0\}. \quad (\text{B35})$$

Also, using Eq. (B25), we have that, for every $m \in \mathbb{N} \cup \{0\}$,

$$B_{m+1} = B_m \left(1 - \frac{A_m}{1 + B_m}\right)^2 = B_0 \prod_{k=0}^m \left(1 - \frac{A_k}{1 + B_k}\right)^2, \\ A_{m+1} = A_m(1 + B_m) = A_0 \prod_{k=0}^m (1 + B_k). \quad (\text{B36})$$

Using the fact that $0 \leq B_k \leq 1 \forall k$, we can combine Eqs. (B35) and (B36) to obtain

$$A_m \leq A_0 \prod_{k=0}^m (2) = 2^m A_0 \\ \leq 2^N A_0 \ll \frac{1}{2^N} \times \frac{1}{2N} \times \min\{1 - B_0, B_0\} \quad \forall m \leq N, \quad (\text{B37})$$

where in the first inequality we have set $B_k = 1$, and in the second we have used the fact that $m \leq N$. Finally, we have subbed in Eq. (B35).

Furthermore, using the fact that $A_0 \leq A_k \leq 1$ and $B_0 \geq B_k \geq 0 \forall k$, Eq. (B36) also implies that, for every $m \leq N$

(note, the justification details why we get to the next line),

$$B_0 \geq B_{m+1} \quad \text{as Eq. (52)} \\ = B_0 \prod_{k=0}^m \left(1 - \frac{A_k}{1 + B_k}\right)^2 \text{ as } B_k \geq 0 \forall k \\ \geq B_0 \prod_{k=0}^m (1 - A_k)^2 \text{ as } A_m \geq A_{m-1} \\ \geq B_0 \prod_{k=0}^m (1 - A_m)^2 = B_0 (1 - A_m)^{2m} \text{ as } m \leq N \\ \geq B_0 (1 - A_m)^{2N} \text{ as } (1 + x)^n \\ = \sum_{k=0}^n \binom{n}{k} x^k \text{ ignoring negative on } A_m \\ \geq B_0 \left(1 - \sum_{k=1}^{2N} (2NA_m)^k\right) \text{ as sum of a geometric series} \\ = B_0 \left(1 - 2NA_m \times \frac{1 - (2NA_m)^{2N}}{1 - 2NA_m}\right) \text{ as } (2NA_m)^{2N} \geq 0 \\ \geq B_0 \left(1 - \frac{2NA_m}{1 - 2NA_m}\right) \text{ as } 0 \\ \leq 2NA_m \ll 2^{-N} < 1/2 \text{ due to Eq. (53)} \\ \geq B_0 - 4NA_m, \quad (\text{B38})$$

giving in total $B_{m+1} \geq B_0 - 4NA_m$. This means that, as long as $m \leq N$, we have $B_{m+1} \approx B_0$, due to Eq. (B37).

Now, by setting $m = N - 1$, Eqs. (B36) and (B38) jointly suggest that (below, we define $A_{-1} := 0$, so that $B_0 \geq B_0 - 4NA_{-1}$)

$$\log_2 \frac{A_N}{A_0} = \sum_{k=0}^{N-1} \log_2(1 + B_k) \text{ as } B_{k+1} \geq B_0 - 4NA_k \forall k \\ \leq N \text{ from Eq. (54)} \\ \geq \sum_{k=0}^{N-1} \log_2(1 + B_0 - 4NA_{k-1}) \text{ as } A_k \\ \leq A_N \forall k \leq N \text{ and } A_{-1} := 0 \\ \geq \sum_{k=0}^{N-1} \log_2(1 + B_0 - 4NA_N) \\ = N \log_2(1 + B_0 - 4NA_N) \\ = N \log_2[2 - (1 - B_0 + 4NA_N)] \text{ as } 4NA_N \\ \ll (1 - B_0)/2^{N-1} \leq 1 - B_0 \text{ due to Eq. (53)} \\ \geq N \log_2[2 - 2(1 - B_0)] \\ = N + N \log_2 B_0. \quad (\text{B39})$$

Since $\{A_k\}_k$ is nondecreasing in k , and since $M^{(1)}(\sigma_{(m)}) = |n_x^m| = \sqrt{A_m}$ as well as $B_0 = |n_z^0|^2$, we thus conclude that for

$\forall m \geq N$ the following holds:

$$\begin{aligned} \frac{M^{(1)(\sigma_m)}}{M^{(1)(\rho)}} &\geq \frac{M^{(1)(\sigma_N)}}{M^{(1)(\rho)}} = \sqrt{\frac{A_N}{A_0}} \geq \sqrt{2^{N+N \log_2(|n_z^0|)}} \\ &= 2^{\frac{N}{2}} \times 2^{N \log_2(|n_z^0|)}, \end{aligned} \quad (\text{B40})$$

where we have used Eq. (B39). Note that $\log_2(|n_z^0|) \leq 0$ is in general negative. Finally, to see the exponential amplification, for any given $\epsilon > 0$, which can be as small as we want, there exists an $|n_z^0|$ satisfying $0 < |\log_2 |n_z^0|| \ll \epsilon/N$, from which we have $N \log_2 |n_z^0| \geq -\epsilon$. Putting this together, we have

$$\frac{M^{(1)(\sigma_m)}}{M^{(1)(\rho)}} \geq 2^{\frac{N}{2}-\epsilon} \quad \forall m \geq N. \quad (\text{B41})$$

To complete the proof, we note that $n = 2^N$, where n is the number of states used in N layers of the protocol. Hence, there exists an initial state that has a vanishingly small amount of coherence [Eq. (B33)], from which a state with unboundedly more coherence can be created using this concatenation protocol. ■

Numerically, it can be seen in Fig. 1 that even for states with $0 \leq |n_z^0| < 1/2$, given enough steps, an exponential increase in local coherence can be achieved. We leave a mathematical proof of this fact for future work.

5. Proof of Result 4

To prove Result 4, the following Lemma is first proved. It states that, given some operator P , the sum over the absolute value of the matrix elements of any diagonal of length k , in any basis, is upper-bounded by the k th Ky-Fan norm of P .

Lemma B1. Let P be a linear operator acting on finite-dimensional vector space, where $\{|\phi_i\rangle\}_{i=0}^m$ and $\{|\psi_j\rangle\}_{j=0}^n$ are orthonormal bases on the input and output spaces of P , respectively. Then, we have

$$\sum_n |\langle \psi_{n+j} | P | \phi_n \rangle| \leq \|P\|_{k\text{-KF}} \quad \forall j. \quad (\text{B42})$$

Proof. Firstly, we write $\langle \psi_r | P | \phi_s \rangle = |\langle \psi_r | P | \phi_s \rangle| e^{i\theta_{rs}}$ $\forall r, s$. Now, consider the operator $X := \sum_n e^{i\theta_{n+j,n}} |\psi_{n+j}\rangle \langle \phi_n|$. It can then be seen that

$$\text{tr}(X^\dagger P) = \sum_n |\langle \psi_{n+j} | P | \phi_n \rangle|. \quad (\text{B43})$$

It is noted that $\text{tr}(X^\dagger A) = |\text{tr}(X^\dagger A)|$, as it is non-negative. Now, the k th Ky-Fan norm [24] can be written as

$$\|P\|_{k\text{-KF}} = \sup_Z \{|\text{tr}(Z^\dagger P)| : \|Z\|_{(k)}^* \leq 1\}, \quad (\text{B44})$$

where $\|Z\|_{(k)}^*$ is the dual of the k th Ky-Fan norm [27] and Z is an operator on the Hilbert space. By definition, the dual of the k th Ky-Fan norm is

$$\|Q\|_{(k)}^* := \sup_Y \{|\text{tr}(Y^\dagger Q)| : \|Y\|_{k\text{-KF}} \leq 1\}, \quad (\text{B45})$$

where Y is an operator on the Hilbert space. To write the k th Ky-Fan norm in this way, we have used the fact that taking the dual of a norm's dual returns the norm [27].

Using the above definitions, it can be stated that

$$\sum_n |\langle \psi_{n+j} | P | \phi_n \rangle| = |\text{tr}(X^\dagger P)| \leq \|P\|_{k\text{-KF}} \quad \text{if} \quad \|X\|_{(k)}^* \leq 1. \quad (\text{B46})$$

To see that $\|X\|_{(k)}^* \leq 1$, we explicitly evaluate it. Firstly, one can find the eigenvalues of $\sqrt{X^\dagger X}$, as these are the singular values of X . From this, it can be seen that X has k singular values that are one, with the remaining being zero.

Now, let $\|X\|_{(k)}^* = |\text{tr}[(Y^*)^\dagger X]|$, such that Y^* is the operator satisfying Eq. (B45). The von Neumann trace inequality [28] can then be employed to state that

$$\begin{aligned} \|X\|_{(k)}^* &= |\text{tr}[(Y^*)^\dagger X]| \leq \sum_{i=0}^{\min\{n,m\}} \mu_i(X) \mu_i(Y^*) \\ &= \sum_{i=0}^{k-1} \mu_i(Y^*) =: \|Y^*\|_{k\text{-KF}} \leq 1, \end{aligned} \quad (\text{B47})$$

where, for an operator C , the set $\{\mu_i(C)\}_{i=0}^{\min\{n,m\}}$ is the singular values of C ordered in decreasing order, such that $\mu_i(C) \geq \mu_j(C) \forall i < j$. To go from the first to the second line, we have used the fact that all nonzero singular values of X are one. We have then used the definition of the k th Ky-Fan norm: it is the sum over the k largest singular values. The final inequality comes from the definition of the dual norm of the k th Ky-Fan norm, where Y was restricted to have $\|Y\|_{k\text{-KF}} \leq 1$ by definition. Hence, $\|X\|_{(k)}^* \leq 1$, meaning Eq. (B46) holds, completing the proof.

We now restate Result 4 before we begin the proof. ■

Result 4. Let $d^{(j)} := \dim(\mathcal{V}_{\text{in}}^{(j)})$. For any finite-dimensional state ρ , we have

$$\Delta M_A^{(j)} \leq \|(\rho_A \otimes \rho_B)^{(j)}\|_{d^{(j)}} - \text{KF} - \|\rho^{(j)}\|_1, \quad (\text{B48})$$

where $\|P\|_{\alpha\text{-KF}}$ is the α -Ky-Fan norm of the operator P , which is the sum of the α largest singular values of P .

Proof. The j th bipartite mode subspace, $\mathcal{B}_{AB}^{(j)}$, has $d^{(j)} := \dim(\mathcal{V}_{\text{in}}^{(j)})$ basis operators whose coefficients will contribute to the local mode upon performing the partial trace. The rough goal of coherence concentration is to maximize the value of these coefficients under conjugation by an allowed unitary. It is noted that each basis operator in $\mathcal{V}_{\text{in}}^{(j)}$ is in a distinct row and column along a diagonal of the j th bipartite mode operator.

Now, $M^{(j)}(\sigma_A) := \|\sigma_A^{(j)}\|_1$ is the sum of the absolute values of the matrix elements of σ_A in the j th mode. Each matrix element of $\sigma_A^{(j)}$ is then a sum over some subset of the coefficients of the basis states in $\mathcal{V}_{\text{in}}^{(j)}$ for the j th bipartite mode $\sigma_{AB}^{(j)}$. Hence, using the triangle inequality, it can be seen that

$$M^{(j)}(\sigma_A) \leq \sum_{n,m} |\langle n+j, m | \sigma_{AB}^{(j)} | n, m \rangle|, \quad (\text{B49})$$

where $\langle n+j, m | \sigma_{AB}^{(j)} | n, m \rangle$ are the $d^{(j)}$ coefficients of $\sigma_{AB}^{(j)}$ with respect to the basis operators in $\mathcal{V}_{\text{in}}^{(j)}$. This upper bound is then a sum over the absolute values of the matrix elements of a diagonal of the j th bipartite mode operator, $\sigma_{AB}^{(j)}$, in the eigenbasis of L_{AB} . Lemma B1 can then be employed to imply

that

$$\begin{aligned} & \sum_{n,m} |\langle n+j, m | \sigma_{AB}^{(j)} | n, m \rangle| \\ & \leq \| \sigma_{AB}^{(j)} \|_{d^{(j)}-\text{KF}} = \| (\rho_A \otimes \rho_B)^{(j)} \|_{d^{(j)}-\text{KF}}, \end{aligned} \quad (\text{B50})$$

as the Ky-Fan norm is unitarily invariant. Taking away $M^{(j)}(\rho_A)$ from both sides completes the proof. ■

6. Proof of Result 5

Here, we present a proof of Result 5, which is first stated for clarity.

Result 5. Let $d^{(j,c)} := \dim(\mathcal{V}_{\text{in}}^{(j,c)})$. For any d -dimensional state ρ , we have

$$\Delta M_A^{(j)} \leq \sum_{c=0}^{2d-2-j} \| \Pi_{c+j}(\rho \otimes \rho)^{(j)} \Pi_c \|_{d^{(j,c)}} - \text{KF} - \| \rho^{(j)} \|_1, \quad (\text{B51})$$

Proof. To prove this, one firstly needs to show that each LR-D subspace transforms independently. From here, the proof of Result 4 can then be applied to each individual space. To see the independence of each LR-D subspace, recall that an allowed unitary will apply independent unitaries within each degenerate subspace of the operator defining our chosen basis. In the concentration scenario, a global allowed unitarity V_{AB} therefore acts as an independent unitary within each degenerate space of the operator $L_{AB} = \sum_c \lambda_c \Pi_c$ with spectrum $\{\lambda_c\}_c$. More precisely, for each c , there exist unitary operators V_c such that

$$V_{AB} = \sum_{c=0}^{2d-2} V_c \Pi_c, \quad (\text{B52})$$

where the projector Π_c is given by

$$\Pi_c = \sum_{n+m=k+j=c} |nm\rangle \langle kl|. \quad (\text{B53})$$

A j th-mode operator, as seen in the main text, is some operator supported in the j th-mode subspace,

$$\sigma_{AB}^{(j)} \in \mathcal{B}_{AB}^{(j)} = \text{span}\{|n+k, m+j-k\rangle \langle n, m| : \forall n, m, k\}, \quad (\text{B54})$$

such that

$$\begin{aligned} \sigma_{AB}^{(j)} &= \sum_k \sum_{n,m} q_{n+k, m+j-k, n, m} |n+k, m+j-k\rangle \langle n, m| \\ &= \sum_{n, \alpha, \beta} \tilde{q}_{n+j, n, \alpha, \beta} |n+j, \alpha\rangle \langle n, \beta|, \end{aligned} \quad (\text{B55})$$

where we have performed a relabeling to reflect the degeneracies in L_{AB} , using the dummy indices α and β to label the degeneracies. From here, it can be seen that

$$\begin{aligned} V_{AB} \sigma_{AB}^{(j)} V_{AB}^\dagger &= \left(\sum_{c=0}^{2d-2} V_c \Pi_c \right) \sigma_{AB}^{(j)} \left(\sum_{e=0}^{2d-2} \Pi_e V_e^\dagger \right) \\ &= \sum_{c,e=0}^{2d-2} \sum_{n, \alpha, \beta} \tilde{q}_{n+j, n, \alpha, \beta} V_c \Pi_c |n+j, \alpha\rangle \langle n, \beta| \Pi_e V_e^\dagger \end{aligned}$$

$$\begin{aligned} &= \sum_{n, \alpha, \beta} \tilde{q}_{n+j, n, \alpha, \beta} V_{n+j} |n+j, \alpha\rangle \langle n, \beta| V_n^\dagger \\ &= \sum_n V_{n+j} \Pi_{n+j} \sigma_{AB}^{(j)} \Pi_n V_n, \end{aligned} \quad (\text{B56})$$

where in the first line we have used the fact that $\Pi_c |n, \alpha\rangle = \delta_{c,n} |n, \alpha\rangle \forall \alpha$, and in the second line we have used

$$\Pi_{c+j} \sigma_{AB}^{(j)} \Pi_c = \sum_{\alpha, \beta} \tilde{q}_{c+j, c, \alpha, \beta} |c+j, \alpha\rangle \langle c, \beta|. \quad (\text{B57})$$

As the unitary V_c maps states within the c th degenerate subspace back to the c th degenerate subspace, it can be concluded that each $\Pi_{c+j} \sigma_{AB}^{(j)} \Pi_c$, that is, each LR-D subspace, transforms independently under allowed unitaries V_{AB} . We note that the c th degenerate subspace of L_{AB} has dimension

$$d(c, L_{AB}) := \begin{cases} c+1 & 0 \leq c < d-1, \\ 2d-1-c & d-1 \leq c \leq 2d-2. \end{cases} \quad (\text{B58})$$

Hence, the $(c+j, c)$ -LR-D operator has its input and output dimensions as $d(c+j, L_{AB})$ and $d(c, L_{AB})$, respectively. Also, there are $2d-2-j$ LR-D operators within the j th global mode.

Returning to the original labeling, one can define the $(c+j, c)$ -LR-D subspace of the j th-mode space as

$$\mathcal{B}_{AB}^{(j,c)} := \text{span}\{|n+k, m+j-k\rangle \langle n, m| : n+m=c\} \subset \mathcal{B}_{AB}^{(j)}. \quad (\text{B59})$$

As before, this can be split into two subspaces, one where the components make a contribution to the local mode and one where they do not,

$$\begin{aligned} \mathcal{V}_{\text{in}}^{(j,c)} &:= \text{span}\{|n+j, m\rangle \langle n, m| : n+m=c\} \subseteq \mathcal{B}_{AB}^{(j,c)}, \\ \mathcal{V}_{\text{out}}^{(j,c)} &:= \text{span}\{|n+k, m+j-k\rangle \langle n, m| : n+m=c, k \neq j\} \\ &\subseteq \mathcal{B}_{AB}^{(j,c)}, \end{aligned} \quad (\text{B60})$$

where the aim of mode concentration is still to maximize the coefficients with respect to the basis operators in $\mathcal{V}_{\text{in}}^{(j,c)}$. Given that each of the LR-D subspaces evolves unitarily, the proof of Result 4 can then be applied to each LD-R subspace individually.

APPENDIX C: BOUND COMPARISON

Here, the bounds on $\Delta M_A^{(j)}$ given in Results 4 and 5 are compared for $d=3$ and $d=4$ for a sample of random states varying in rank (see Figs. 4 and 5). It is firstly noted that for $d=3$, both bounds are equal for all tested pure (i.e., rank-one) states. For states with higher ranks in $d=3$, and states in $d=4$ (including rank-one), we see that there is not a clear comparison of the two upper bounds. This can be confirmed in the insert, which shows the number of sampled states for which one bound is greater than the other. In general, it can be seen that Result 5 performs better for $d=3$, and Result 4 performs better for $d=4$. We conjecture that these bounds will not coincide for all d , and that in general one should just take whichever of the two bounds is smallest to be the upper bound on the concentrateable coherence.

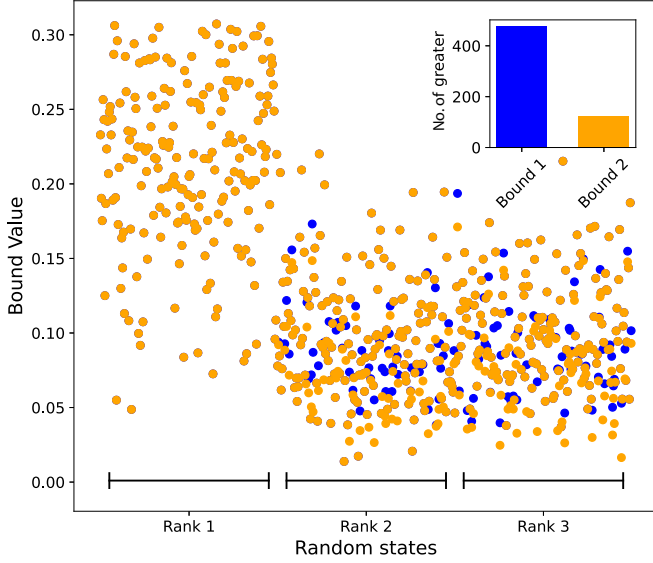


FIG. 4. The value of Result 4 (bound 1) and Result 5 (bound 2) plotted for a sample of states in $d = 3$ for all possible ranks. The insert shows the number of sampled states for which a given bound is greater than the other.

APPENDIX D: A SUFFICIENT CONDITION FOR THE EXISTENCE OF CORRELATIONS

Here, we show that if a bipartite state in AB (with equal local dimension d) has support only on the zeroth mode and modes $k \geq d$, then there must exist a correlation between A and B . Formally, we prove that:

Lemma D1. Let ρ_{AB} be a bipartite state in AB with equal local dimension $d < \infty$. Then, ρ_{AB} is not a product state, and

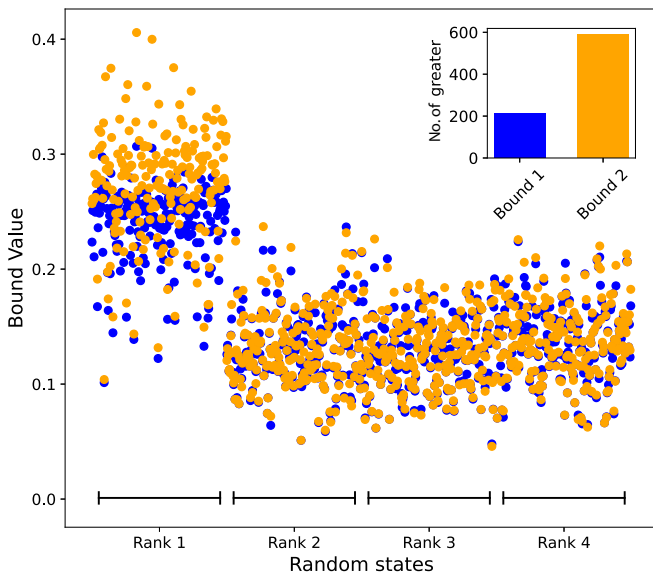


FIG. 5. The value of Result 4 (bound 1) and Result 5 (bound 2) plotted for a sample of states in $d = 4$ for all possible ranks. The insert shows the number of sampled states for which a given bound is greater than the other.

hence has correlation, if

$$\text{modes}(\rho_{AB}) \neq \{0\} \text{ but } [1, d-1] \cap \text{modes}(\rho_{AB}) = \emptyset. \quad (\text{D1})$$

Namely, ρ_{AB} satisfies

$$\rho_{AB} = \rho_{AB}^{(0)} + \sum_{i \geq d} \rho_{AB}^{(i)} \quad \text{and} \quad \sum_{i \geq d} \rho_{AB}^{(i)} \neq 0. \quad (\text{D2})$$

Proof. Suppose the opposite, that $\rho_{AB} = \rho_A \otimes \rho_B$ and

$$\rho_A \otimes \rho_B = \rho_{AB}^{(0)} + \sum_{i \geq d} \rho_{AB}^{(i)} \quad \& \quad \sum_{i \geq d} \rho_{AB}^{(i)} \neq 0. \quad (\text{D3})$$

Due to Lemma 1, it can then be concluded that $\rho_A = \text{tr}_B(\rho_{AB}^{(0)}) = \rho_A^{(0)}$ and $\rho_B = \text{tr}_A(\rho_{AB}^{(0)}) = \rho_B^{(0)}$, meaning that both ρ_A and ρ_B would need to be states supported only on the zeroth mode. However, using the fact that the bipartite modes of a tensor product of states are given by [22]

$$(\rho_A \otimes \rho_B)^{(j)} = \sum_k \rho_A^{(k)} \otimes \rho_B^{(j-k)} \quad \forall j, \quad (\text{D4})$$

it can be concluded that $\rho_{AB} = \rho_A^{(0)} \otimes \rho_B^{(0)} = \rho_{AB}^{(0)}$, meaning that $\sum_{i \geq d} \rho_{AB}^{(i)} = 0$, in contradiction with Eq. (D3). ■

APPENDIX E: QUTRIT EXAMPLE

Here, we will run through an example of mode concentration using two qutrits. We will use graphical deceptions where appropriate to make our explanations as clear as possible.

Let ρ be a qutrit state and $L = \sum_{n=0}^2 n|n\rangle\langle n|$ be the Hermitian operator defining the eigenspaces between which we aim to increase coherence. The j th mode of ρ is define as $\rho^{(j)} = \sum_n p_{n+j,n} |n+j\rangle\langle n|$, with all the modes of ρ visualizable as

$$\rho = \begin{bmatrix} p_{00} & p_{01} & p_{02} \\ p_{10} & p_{11} & p_{12} \\ p_{20} & p_{21} & p_{22} \end{bmatrix} \quad \text{where} \quad \begin{array}{c|c} \text{Mode} & \text{Colour} \\ \hline \rho^{(0)} & \text{red} \\ \rho^{(1)} & \text{yellow} \\ \rho^{(2)} & \text{orange} \\ \rho^{(-1)} & \text{blue} \\ \rho^{(-2)} & \text{green} \end{array} \quad (\text{E1})$$

such that, for example,

$$\rho^{(0)} = \begin{bmatrix} p_{00} & 0 & 0 \\ 0 & p_{11} & 0 \\ 0 & 0 & p_{22} \end{bmatrix} \quad \text{and} \quad \rho^{(2)} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ p_{20} & 0 & 0 \end{bmatrix}. \quad (\text{E2})$$

We will here aim to concentrate coherence onto the second mode, detailing the framework presented in the main text. To do so, one needs to apply an allowed unitary to the second global mode and then perform the partial trace. The second global mode is given by

$$(\rho_A \otimes \rho_B)^{(2)} = \rho_A^{(0)} \otimes \rho_B^{(2)} + \rho_A^{(1)} \otimes \rho_B^{(1)} + \rho_A^{(2)} \otimes \rho_B^{(0)}, \quad (\text{E3})$$

which, with respect to the eigenbasis of L_{AB} , is given by

$$(\rho_A \otimes \rho_B)^{(2)} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ p_{00}p_{20} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ p_{10}p_{10} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & p_{10}p_{21} & 0 & p_{11}p_{20} & 0 & 0 & 0 & 0 & 0 \\ p_{20}p_{00} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & p_{20}p_{11} & 0 & p_{21}p_{10} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & p_{20}p_{22} & 0 & p_{21}p_{21} & 0 & p_{22}p_{20} & 0 & 0 \end{bmatrix}$$

Space	Colour
$\mathcal{V}_{\text{in}}^{(2)}$	●
$\mathcal{V}_{\text{out}}^{(2)}$	●

(E4)

where we have highlighted the coefficients associated with basis operators in $\mathcal{V}_{\text{in}}^{(2)}$, i.e., the basis operators whose coefficients contribute to the local mode. In this case, the second local mode has a single component, given by the sum of the coefficients associated with the basis operators in $\mathcal{V}_{\text{in}}^{(2)}$ (the pink dots). Hence, $M^{(2)}(\sigma_A)$ is the absolute value of the sum of these components. Optimally concentrating coherence on the second mode therefore means maximizing the absolute value of the sum of the coefficients associated with the basis operators in $\mathcal{V}_{\text{in}}^{(2)}$. An upper bound on $M^{(2)}(\sigma_A)$ is then given by the sum of the maximized absolute values of each coefficient, due to the triangle inequality.

1. Qutrit example for Result 4

We now detail the logical steps that led to Result 4 in the main text. Firstly, we relax the problem by assuming that any pre- and postprocessing unitary can be applied to $(\rho_A \otimes \rho_B)^{(2)}$, not just a conjugation by an allowed unitary. From here, Lemma B1 can be employed to show that the sum over the absolute value of the matrix coefficients that give the second

local mode is upper bounded by the third Ky-Fan norm of this operator. This is because we are summing over a diagonal of length $\dim(\mathcal{V}_{\text{in}}^{(2)}) = 3$. As $M^{(2)}(\sigma_A)$ is upper bounded by the sum of the absolute values of these matrix coefficients, which is further upper bounded by the third Ky-Fan norm of this operator, one can therefore conclude that $M^{(2)}(\sigma_A) \leq \|(\rho_A \otimes \rho_B)^{(2)}\|_{3\text{-KF}}$.

2. Qutrit example for Result 5

We now consider in detail splitting $(\rho_A \otimes \rho_B)^{(2)}$ up into the individual LR-D subspaces that evolve independently under allowed unitaries, and hence go through the logical steps that led to Result 5 in the main text. Firstly, as stated in Appendix B6, any allowed unitary can be decomposed as $V_{AB} = \sum_c V_c \Pi_c$, where each V_k is an independent unitary, if considering coherence between the eigenspaces of the operator $L_{AB} = \sum_c \lambda_c \Pi_c$ with spectrum $\{\lambda_c\}_c$. Here, we consider the Hermitian operator $L_{AB} = L_A \otimes \mathbb{I}_B + \mathbb{I}_A \otimes L_B$, such that with respect to the eigenbasis of L_{AB} any allowed unitary can be decomposed as

$$V_{AB} = \begin{bmatrix} \text{yellow} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \text{pink} & 0 & \text{pink} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \text{orange} & 0 & \text{orange} & 0 & \text{orange} & 0 & 0 \\ 0 & \text{pink} & 0 & \text{pink} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \text{orange} & 0 & \text{orange} & 0 & \text{orange} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \text{blue} & 0 & \text{blue} & 0 \\ 0 & 0 & \text{orange} & 0 & \text{orange} & 0 & \text{orange} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \text{blue} & 0 & \text{blue} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \text{purple} \end{bmatrix}$$

Unitary	Colour
V_0	●
V_1	●
V_2	●
V_3	●
V_4	●

(E5)

where each matrix element associated with a given independent unitary is represented by a color. This unitary is then applied to $(\rho_A \otimes \rho_B)^{(2)}$. We now represent $(\rho_A \otimes \rho_B)^{(2)}$ while highlighting all the different LR-D subspaces with the mode,

$$(\rho_A \otimes \rho_B)^{(2)} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ p_{00}p_{20} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ p_{10}p_{10} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & p_{10}p_{21} & 0 & p_{11}p_{20} & 0 & 0 & 0 & 0 & 0 \\ p_{20}p_{00} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & p_{20}p_{11} & 0 & p_{21}p_{10} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & p_{20}p_{22} & 0 & p_{21}p_{21} & 0 & p_{22}p_{20} & 0 & 0 \end{bmatrix} \quad \begin{array}{c|c} \text{LR-D Space} & \text{Colour} \\ \hline (2,0) & \text{purple} \\ (3,1) & \text{teal} \\ (4,2) & \text{orange} \end{array} \quad (\text{E6})$$

where the $(c+j, c)$ LR-D subspace of the j th mode is defined as $\Pi_{c+j}(\rho_A \otimes \rho_B)^{(j)}\Pi_c$. As stated in the main text, the benefit of this formulation is that each LR-D subspace evolves independently and unitarily. More specifically, each of the LR-D subspaces evolve as

$$V_2 \Pi_2(\rho_A \otimes \rho_B)^{(2)} \Pi_0 V_0 = \begin{bmatrix} \text{orange} & \text{orange} & \text{orange} \\ \text{orange} & \text{orange} & \text{orange} \\ \text{orange} & \text{orange} & \text{orange} \end{bmatrix} \begin{bmatrix} p_{00}p_{20} \\ p_{10}p_{10} \\ p_{20}p_{00} \end{bmatrix} \begin{bmatrix} \text{yellow} \end{bmatrix} \quad (\text{E7})$$

$$V_3 \Pi_2(\rho_A \otimes \rho_B)^{(2)} \Pi_0 V_1 = \begin{bmatrix} \text{blue} & \text{blue} \\ \text{blue} & \text{blue} \end{bmatrix} \begin{bmatrix} p_{10}p_{21} & p_{11}p_{20} \\ p_{20}p_{11} & p_{21}p_{10} \end{bmatrix} \begin{bmatrix} \text{pink} & \text{pink} \\ \text{pink} & \text{pink} \end{bmatrix} \quad (\text{E8})$$

$$V_4 \Pi_2(\rho_A \otimes \rho_B)^{(2)} \Pi_0 V_2 = \begin{bmatrix} \text{purple} \end{bmatrix} \begin{bmatrix} p_{20}p_{22}, & p_{21}p_{21}, & p_{22}p_{20} \end{bmatrix} \begin{bmatrix} \text{orange} & \text{orange} & \text{orange} \\ \text{orange} & \text{orange} & \text{orange} \\ \text{orange} & \text{orange} & \text{orange} \end{bmatrix} \quad (\text{E9})$$

In the above, we have highlighted what coefficients within each LR-D subspace belong to basis states in $\mathcal{V}_{\text{in}}^{(2)}$ using a pink dot. The aim when performing mode concentration is still to maximize the absolute value of the sum of the coefficients associated with these basis operators. What this framework provides is a finer level of detail over how these coefficients can evolve.

To find the bound given in Result 5, one applies the same methodology as for Result 4, but this time to each LR-D subspace individually. It can be seen, as above, that an upper bound on $M^{(2)}(\sigma_A)$ is given by the sum of the absolute values of the matrix coefficients associated with basis operators in $\mathcal{V}_{\text{in}}^{(2)}$. Using Lemma B1, this can be upper bounded by the sum over the k th Ky-Fan norm of all the $(c+j, c)$ -LR-D operators, where k is the number of basis states in $\mathcal{V}_{\text{in}}^{(2)}$ upon which a given $(c+j, c)$ -LR-D operator has support.

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