

Fate of entanglement in quadratic Markovian dissipative systems

Fabio Caceffo  and Vincenzo Alba 

Dipartimento di Fisica dell'Università di Pisa and INFN, Sezione di Pisa, I-56127 Pisa, Italy



(Received 17 February 2026; accepted 24 February 2026; published 15 April 2026)

We develop a hydrodynamic description for the driven-dissipative dynamics of the entanglement negativity, which quantifies the genuine entanglement in mixed-state systems. We focus on quantum quenches in fermionic and bosonic systems subject to linear dissipation, as described by quadratic Lindblad master equations. In the spirit of hydrodynamics, we divide the system into mesoscopic cells. At early times, correlations are generated in each cell by the unitary component of the evolution. Correlations are then transported across different cells via ballistic quasiparticle propagation, while simultaneously evolving under the action of the environment. We show that in the hydrodynamic limit the negativity can be reconstructed from the correlations between the independently propagating quasiparticles. We benchmark our approach considering quenches from both homogeneous and inhomogeneous initial states in the Kitaev chain, the tight-binding chain, and the harmonic chain in the presence of gain or loss dissipation.

DOI: [10.1103/f6q2-vkyv](https://doi.org/10.1103/f6q2-vkyv)

I. INTRODUCTION

In out-of-equilibrium closed quantum many-body systems, entanglement is intricately intertwined with thermodynamics. This connection is particularly evident in integrable systems, where the complete dynamics of entanglement over time can be fully reconstructed [1–3] from the Generalized Gibbs Ensemble (GGE), which characterizes the *local* properties of the steady state. This is surprising because entanglement reflects “nonlocal” correlations within the system, which cannot be directly encoded in the GGE. Understanding the fate of entanglement and its relationship with thermodynamics in open quantum systems is of paramount importance because every realistic system interacts with an external environment. Unfortunately, the dynamics of open systems leads towards a globally mixed state, rendering popular entanglement measures for pure states, such as the von Neumann entropy, ineffective, due to contamination by environment fluctuations. The logarithmic negativity [4–7], an appropriate measure of entanglement even for mixed states, proves challenging to compute, except in very specific scenarios [8–10]. In contrast, in closed integrable systems the negativity is understood [11–13] within the quasiparticle picture for entanglement [1,2,14–18].

Here we develop a hydrodynamic framework for the logarithmic negativity between two intervals of equal length ℓ and placed at distance d [see Fig. 1(a)] in generic *quadratic* fermionic and bosonic systems following Lindblad [19]. Exact results in noninteracting systems proved to be crucial for understanding the absence of thermalization [20] in integrable

systems, provided a valuable platform to test Generalized Hydrodynamics [16] (GHD), and pioneered the application of the quasiparticle picture [1]. Free systems also facilitate the exploration of dynamics beyond the quantum quench paradigm [21]. Quadratic Lindblad dynamics are exactly solvable [22], enabling a comprehensive understanding of the dynamics of von Neumann and Rényi entropies [10,23–28].

Our framework is predicated on the idea that, at a hydrodynamic level, the system can be partitioned into “mesoscopic” cells, which are thermodynamically large relative to any microscopic length scales, yet small in comparison to the overall system size. At early stages of the dynamics, entanglement between quasiparticles generated within each cell is established by the local unitary component of the evolution. Subsequently, correlation and entanglement are “transported” through the ballistic quasiparticle propagation. Similar concepts have been previously explored for unitary dynamics [29,30]. Crucially, the quasiparticles are constructed from the *full* Liouvillian generating the dynamics. Indeed, as correlations spread across different cells, they become “dressed” by the environment, which renders them time-dependent. Generally, even with weak dissipation, entanglement is exponentially dampened [see Fig. 1(b)]. Bosonic systems experience the so-called sudden death, wherein the negativity is exactly zero at long times. Unlike in unitary dynamics, where the negativity is linked to thermodynamic entropy [11], here it depends on “off-diagonal” correlations between quasiparticles, rather than solely on their densities. This contrasts with the behavior of von Neumann and Rényi entropies in quadratic Liouvillians [24]. Thus, the dissipative dynamics profoundly reveals the breakdown of the relationship between entanglement and thermodynamics. The paper is organized as follows. In Sec. II we outline the hydrodynamic framework for the negativity in generic quadratic fermionic and bosonic systems. In Sec. III we employ the method to derive the dynamics of the fermionic negativity in

Published by the American Physical Society under the terms of the [Creative Commons Attribution 4.0 International](https://creativecommons.org/licenses/by/4.0/) license. Further distribution of this work must maintain attribution to the author(s) and the published article's title, journal citation, and DOI.

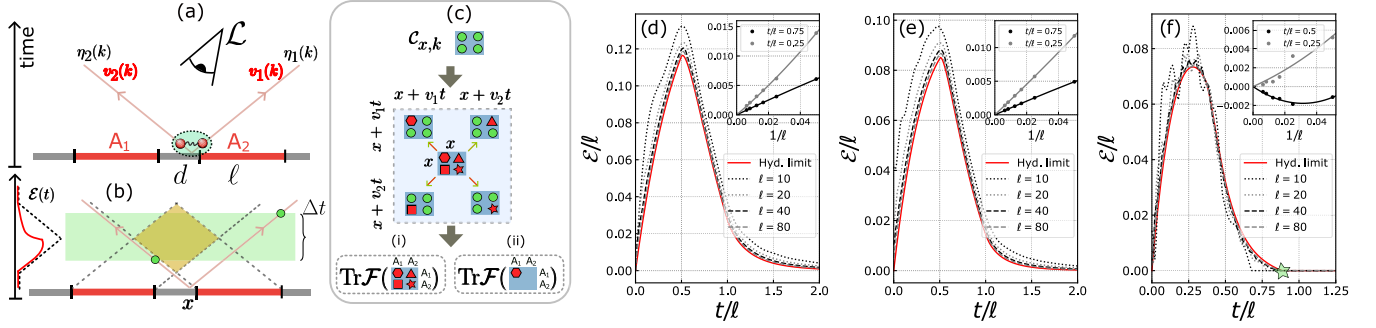


FIG. 1. Dynamics of the negativity $\mathcal{E}(t)$ between two equal intervals A_1 and A_2 of length ℓ placed at distance d under the Liouvillian $\mathcal{L}$ in dissipative noninteracting systems. (a) Two entangled quasiparticles η_1, η_2 originated in the same hydrodynamic cell propagate with velocities $v_1(k)$ and $v_2(k)$. (b) The pair entangles A_1 and A_2 only in the time interval Δt , in which it is shared between A_1 and A_2 . The number of shared pairs is the horizontal width of the shaded diamond. In the absence of dissipation, $\mathcal{E}(t)$ exhibits a linear “rise and fall” dynamics, whereas with dissipation the “rise and fall” is exponentially damped. (c) Entanglement in the hydrodynamic cell at x is encoded in the 4×4 correlator $\mathcal{C}_{x,k}$ [cf. (1) and (6)] between the quasiparticles. At time t , the “transported” correlator $\tilde{\mathcal{C}}_{x,k,t}(t)$ describes the negativity between quasiparticles in the cells at $x + v_1 t$ and $x + v_2 t \in A_2$ the contribution to the negativity between A_1 and A_2 is reported in (i), whereas (ii) shows the configuration with a quasiparticle in A_1 , which gives $\mathcal{E} = 0$. (d)–(f) Numerical benchmarks for gain or loss dissipation with rates $\gamma_{\pm}$. $\mathcal{E}(t)/\ell$ versus t/ℓ in the weakly dissipative hydrodynamic limit. (d) Quench from the dimer state in the tight-binding chain with $\gamma_+ \ell = 0.2$, $\gamma_- \ell = 0.3$. (e) Magnetic field quench with $h_0 = 0.3$, $h_1 = 3.1$, $\delta = 0.73$ in the Kitaev chain. The dissipation rates are $\gamma_+ \ell = 0.2$, $\gamma_- \ell = 0.5$. (f) Mass quench with $m_0 = 1.3$, $m_1 = 0.5$, $w = 1.73$, $\mathcal{K} = 2.1$, $\gamma_+ \ell = 0.25$, $\gamma_- \ell = 0.9$ in the harmonic chain. The full lines are analytic results. The insets show finite-size corrections at fixed t/ℓ . Curves are fits to $a/\ell + b/\ell^2$. In (f), the star marks the negativity sudden death.

paradigmatic fermion systems, such as the Kitaev chain and the tight-binding chain in the presence of gain and loss dissipation. In Sec. IV we discuss the negativity dynamics in the so-called harmonic chain. In Sec. V we discuss dynamics in the tight-binding chain starting from bipartite initial states in the presence of homogeneous gain and loss dissipation. We conclude and discuss future directions in Sec. VI. In Appendix A we discuss how the symmetries of the Hamiltonian and of the dissipation. In Appendix B we discuss the effect of symmetries on the definition of quasiparticles. In Appendix C we present a hydrodynamic derivation of our main results. In Appendix D we provide a detailed derivation of our results for the negativity in the bosonic and fermionic models that we consider.

II. HYDRODYNAMIC FRAMEWORK FOR THE NEGATIVITY

Let us consider a generic free-fermion or free-boson Hamiltonian H that is diagonalized in terms of n species of quasiparticles $\eta_j(k)$, with $j \in [1, n]$, and η_j satisfying standard commutation or anticommutation relations. The quasimomentum k lives in a reduced Brillouin zone $\mathcal{B}$. In the spirit of hydrodynamics, let us divide the system in mesoscopic cells. A mesoscopic cell is thermodynamically large compared to the microscopic length scales of the system, i.e., the lattice spacing for models defined on a lattice. At the same time each cell is small compared to the full system. After a quantum quench, each cell can be viewed as source of the same excitations $\eta_j(k)$ of the full chain. For now, to illustrate the hydrodynamic framework, let us focus on the unitary dynamics under H , starting from an initial state. Since the system is Gaussian, the correlations generated initially in the cell at position x are characterized by the $2n \times 2n$ matrix

$\mathcal{C}_{x,k}$ as

$$\mathcal{C}_{x,k} = \left\langle \begin{pmatrix} \eta_r^\dagger(k) \eta_s(k) & \eta_r^\dagger(k) \eta_s^\dagger(k) \\ \eta_r(k) \eta_s(k) & \eta_r(k) \eta_s^\dagger(k) \end{pmatrix} \right\rangle, \quad r, s \in [1, n], \quad (1)$$

with $\langle \cdot \rangle$ the initial-state expectation value. $\mathcal{C}_{x,k}$ is block diagonal in momentum space for translational-invariant H and homogeneous initial states, implying that one could drop the dependence on x in (1) because all the cells are equal. Crucially, $\mathcal{C}_{x,k}$ is not diagonal as $2n \times 2n$ matrix, reflecting that the n quasiparticles for each k form an entangled multiplet. We consider dynamics that preserve the Gaussianity of the initial states, implying that one can univocally associate with $\mathcal{C}_{x,k}$ a Gaussian density matrix $\rho_{x,k}$. The full-system correlation matrix is $\mathcal{C} = \bigoplus_{x,k} \mathcal{C}_{x,k}$, and the density matrix $\rho = \bigotimes_{x,k} \rho_{x,k}$, since different cells are decoupled. Notice that the change of basis from the lattice fermions or bosons to the $\eta_j(k)$ in each cell can be subtle and is discussed in Appendixes A and B. Quasiparticles of each multiplet spread ballistically with group velocities $v_j(k)$, which are obtained from the single-particle dispersion $\varepsilon(k)$ of the model. Correlations spread accordingly, via the mechanism highlighted in Fig. 1(c). At time t , the j th quasiparticle of the multiplet created in the cell at x is in the cell at $x + v_j(k)t$. Suggestively, we can associate with each multiplet a nonlocal correlation matrix $\mathcal{C}_{x,k,t}$ (notice the subscript t) obtained by “transporting” $\mathcal{C}_{x,k}$ [cf. (1)]. $\mathcal{C}_{x,k,t}$ describes correlations between cells at $x + v_j t$, $j = 1, \dots, n$, although its entries are simply those of $\mathcal{C}_{x,k}$. We illustrate the case with $n = 2$ in Fig. 1(c), where the triangle and the square are the 2×2 correlation blocks [cf. (1)] between η_1 in the cell at $x + v_1 t$ and η_2 in the cell at $x + v_2 t$. The hexagon and the star are the correlations $\langle \eta_j^{(t)} \eta_j^{(t)} \rangle$ inside the same cell at $x + v_j t$. Again, $\mathcal{C}_{x,k,t}$ identifies the Gaussian state $\rho_{x,k,t}$, the full-chain correlation matrix being $\mathcal{C}_t = \bigoplus_{x,k} \mathcal{C}_{x,k,t}$, whereby the full-system state is $\rho_t = \bigotimes_{x,k} \rho_{x,k,t}$.

Let us now consider the logarithmic negativity $\mathcal{E}$ between two intervals A_1 and A_2 [see Fig. 1(b)]. $\mathcal{E}$ is obtained as $f(\rho_A)$, with ρ_A the reduced density matrix of $A = A_1 \cup A_2$ and $f(x)$ a suitable function [7,31]. Gaussianity implies that $f(\rho_A(t)) = \text{Tr}_A \mathcal{F}(\mathcal{C}_t^{(A)})$, with $\mathcal{C}_t^{(A)}$ being $\mathcal{C}_t$ restricted to A , and $\mathcal{F}(x)$ a function [7,31], which is related to $f(z)$. Our hydrodynamic-limit assumptions imply

$$f(\rho_A(t)) \stackrel{\text{hydr}}{=} \sum_{x,k} \text{Tr} \mathcal{F}(\mathcal{C}_{x,k,t}^{(A)}). \quad (2)$$

In (2), $\mathcal{C}_{x,k,t}^{(A)}$ is $\mathcal{C}_{x,k,t}$, where we select the rows and columns corresponding to the quasiparticles, with labels $\mathcal{A} \subseteq \{1, 2, \dots, n\}$, that at time t are in A . The sum over x, k depends on the way quasiparticles of the same multiplet distribute between A_1 and A_2 . Equation (2) holds in the hydrodynamic limit $\ell, t, d \rightarrow \infty$ with $\ell/t, d/t$ fixed. For $n = 2$ [entangled pairs; see Figs. 1(a) and 1(b)], one has quite generically $v_1(k) > 0, v_2(k) = -v_1(k)$, with $k \in [0, \pi]$. Moreover, $\mathcal{F}(\mathcal{C}_{x,k,t}^{(A)})$ [cf. (2)] is nonzero only if the pair is shared between A_1 and A_2 [Fig. 1(c)], for a generic entanglement measure. Thus, Eq. (2) gives

$$\mathcal{E}(t) = \ell \int_0^\pi \frac{dk}{2\pi} e_2(k, t) \Theta(k, t), \quad (3)$$

where $e_2(k, t) = \text{Tr}[\mathcal{F}(\mathcal{C}_{x,k,t}^{(A)})]$, and $\Theta(k, t) = \max(2v_1 t/\ell, 2 + d/\ell) + \max(2v_1 t/\ell, d/\ell) - 2 \max(v_1/\ell, 1 + d/\ell)$ results from the sum over x in (2), and it gives the linear “rise and fall” dynamics [see Fig. 1(b)]. For unitary dynamics $\rho_{x,k,t}$ corresponds to a pure state for the pair, implying that $\mathcal{E} = 1/2 I_{A_1:A_2}^{(1/2)}$, with $I_{A_1:A_2}^{(1/2)}$ the Rényi-1/2 mutual information [11]. In the presence of dissipation $e_2(k, t)$ will depend on time, and the rise and fall curve will be exponentially damped [see Fig. 1(b)].

To deal with dissipation we employ the Lindblad master equation [19]. The evolution of a generic observable O is given as $d\langle O \rangle/dt = i\langle \hat{\mathcal{L}}^\dagger[O] \rangle$, with

$$\hat{\mathcal{L}}^\dagger[O] = [H, O] - i \sum_{j,j'=1}^{2L} K_{jj'} \left(r_j O r_{j'} - \frac{1}{2} \{r_j r_{j'}, O\} \right), \quad (4)$$

where $\langle O \rangle := \text{Tr}(O \rho_t)$ and L is the length of the chain (in our framework, we take the thermodynamic limit $L \rightarrow \infty$). The Kossakowski matrix K encodes dissipation [32], and it can be decomposed as $K = A + iB$, with $A(B)$ real symmetric (antisymmetric) matrices. For fermionic systems, $r = (w_1^1, w_1^2, \dots, w_L^1, w_L^2)$ are Majorana fermions, whereas for bosons $r = (x_1, p_1, \dots, x_L, p_L)$, with x_j, p_j satisfying $[x_j, p_{j'}] = i\delta_{jj'}$.

To proceed, we apply Eq. (4) to each hydrodynamic cell. First, we identify the quasiparticles η_j transporting the correlations as linear combinations of r_n [23,24,28]. For bosonic systems, the η_j themselves are eigenvectors of $\hat{\mathcal{L}}^\dagger$. For fermions, $\hat{\mathcal{L}}^\dagger[r_n]$ gives rise to nonlinear terms, and to define the η_j one has to consider the quadratic operators $\chi(k) = \eta_j^\dagger \eta_j, \eta_i \eta_j$ requiring that they satisfy the generalized eigenvalue equation [28] (see also the appendixes)

$$\hat{\mathcal{L}}^\dagger[\chi] = \lambda_\chi \chi(k) + \gamma_\chi, \quad (5)$$

where $\lambda_\chi(k), \gamma_\chi(k) \in \mathbb{C}$. The group velocities of the quasiparticles are extracted from $\chi = \eta_i \eta_j$ with $i \neq j$ [28]. Although Eq. (5) holds for finite dissipation, we focus on the weakly dissipative limit [23], in which $t, \ell \rightarrow \infty$, the dissipation rates γ vanish, and t/ℓ and $\gamma\ell$ are fixed. In this limit, the η_j obey canonical (anti)commutation relations (see also the appendixes), and the quasiparticle velocities $v_j(k)$ are the same as without dissipation (see Refs. [23,24]). Finally, we conjecture that Eq. (2) generalizes as

$$f(\rho_A(t)) \stackrel{\text{w.d. hydr}}{=} \sum_{x,k} \text{Tr} \mathcal{F}[\tilde{\mathcal{C}}_{x,k,t}^{(A)}(t)]. \quad (6)$$

Equation (6) can be justified within the hydrodynamic framework, and it relies on the independence of different hydrodynamic cells, and on the canonical commutation relations between the quasiparticles (see Appendix C). To build $\tilde{\mathcal{C}}_{x,k,t}(t)$, we first evolve (1) with (4), obtaining $\tilde{\mathcal{C}}_{x,k}(t)$. Then we transport ballistically $\tilde{\mathcal{C}}_{x,k}(t)$ as in the unitary case [see Fig. 1(c)], to obtain $\tilde{\mathcal{C}}_{x,k,t}(t)$. Importantly, in doing so we set to one the oscillating phases due to the unitary part of the dynamics, which in the hydrodynamic limit has already been accounted for by the quasiparticle spreading. Still, $\tilde{\mathcal{C}}_{x,k,t}(t)$ depends explicitly on time due to the dissipation, unlike in the unitary case. We expect Eq. (6) to hold also at finite dissipation rate, at least for some types of dissipations (see Ref. [28]). For finite γ the eigenmodes of the Liouvillian $\eta_j(k)$ are related to those of the Hamiltonian by a nonunitary matrix V_k (see Appendix B). The $\eta_j(k)$ satisfy the noncanonical anticommutation relations $\{\eta_j(k), \eta_j^\dagger\} = V_k^\dagger V_k$. Now, to generalize (6) one has to take into account the effect of V_k , as in Ref. [28]. Moreover, the velocities of the quasiparticles derived from the λ_χ [cf. Eq. (5)] do not coincide with the unitary case.

III. APPLICATIONS TO FERMIONIC SYSTEMS

Let us consider the periodic Kitaev chain, defined by the Hamiltonian

$$H_K = -\frac{1}{2} \sum_{j=1}^L (c_j^\dagger c_{j+1} + \delta c_j^\dagger c_{j+1}^\dagger + \text{H.c.} - 2hc_j^\dagger c_j), \quad (7)$$

where h, δ are real parameters, and we take the thermodynamic limit $L \rightarrow \infty$. The strategy to diagonalize the Hamiltonian (7) is standard, and we discuss it in Appendix D 1. The single-particle energies are

$$\varepsilon(k) = [(h - \cos k)^2 + \gamma^2 \sin^2 k]^{1/2}. \quad (8)$$

In our quench protocol, at $t < 0$, the chain is in the ground state of H_K with $h = h_0$. At $t > 0$, h is suddenly changed to $h = h_1$. Pairs of entangled quasiparticles are formed by the modes $\eta_1(k)$ and $\eta_2(k)$, with $k \in [0, \pi]$, and group velocities $v_1(k) = \partial_k \varepsilon(k) = -v_2(k)$ (see [16]). We focus on the Lindblad dynamics in (4), where $r_{2m-1} = c_m + c_m^\dagger, r_{2m} = i(c_m - c_m^\dagger)$. All the correlations are encoded in the matrix $\Gamma_{m,n}$ defined as

$$\Gamma_{m,n} := \langle [r_m, r_n]/2 \rangle. \quad (9)$$

Its Fourier transform is

$$\hat{g}_{k,a,b} = \langle r_{a,k} r_{b,-k} \rangle - \delta_{a,b}, \quad (10)$$

with $a, b \in \{e, o\}$ and $r_{o,k}, r_{e,k}$ the Fourier transforms of r_{2j+1} and r_{2j} , respectively. Equation (4) for $\hat{g}_k$ gives [24]

$$\hat{g}_k = \hat{u}_k \hat{g}_k(0) \hat{u}_{-k}^T + 4i \int_0^t du \hat{u}_k(t-u) \hat{B}_k \hat{u}_{-k}^T(t-u), \quad (11)$$

with $\hat{u}_k(t) = e^{-(4i\hat{h}_k + 2\hat{A}_k)t}$, where $\hat{h}_k, \hat{A}_k$, and $\hat{B}_k$ are the Fourier transforms of $h_{m,n}$, which is defined as $H_K = \sum_{m,n} h_{m,n} r_m r_n$, and of A, B [cf. (4)]. We consider gain and loss dissipation with rates $\gamma_{\pm}$, which corresponds to

$$\hat{A}_k = [(\gamma_+ + \gamma_-)/4] \mathbb{1}_2, \quad (12)$$

$$\hat{B}_k = i[(\gamma_+ - \gamma_-)/4] \sigma^y, \quad (13)$$

with $\sigma^{x,y,z}$ the Pauli matrices. For gain and loss the quasiparticles η_j are the same as in the unitary case (see also the appendixes). By solving (11) for $\hat{g}_k$ [25], one obtains $\tilde{C}_{x,k}(t)$ as

$$\tilde{C}_{x,k}(t) = \frac{1}{2} \begin{pmatrix} \tilde{C}_1 & \tilde{C}_2^{\dagger} \\ \tilde{C}_2 & \tilde{C}_1 \end{pmatrix}. \quad (14)$$

Here $\tilde{C}_1 := \mathbb{1}_2 - A_{k,t} \sigma^z$, $\tilde{C}_2 := iB_{k,t}(e^{2i\varepsilon(k)t} \sigma^+ + \text{H.c.})$, where

$$A_{k,t} := \lambda_t \cos(\Delta_{k,h_1} - \Delta_{k,h_0}) + (1 - \lambda_t)(1 - 2n_{\infty}) \cos \Delta_{k,h_1}, \quad (15)$$

$$B_{k,t} := \lambda_t \sin(\Delta_{k,h_1} - \Delta_{k,h_0}), \quad (16)$$

with $\lambda_t := e^{-(\gamma_+ + \gamma_-)t}$, $n_{\infty} := \gamma_+ / (\gamma_+ + \gamma_-)$, and $\Delta_{k,h}$ the Bogoliubov angle (see Appendix D 1). We now set to one the oscillatory terms $e^{\pm 2i\varepsilon(k)t}$ in $\tilde{C}_2$, and we obtain $\tilde{C}_{x,k}(t)$ as in Fig. 1(c). Finally, we compute the negativity $e_n(x, k, t)$ between the quasiparticles that at time t are shared between A_1 and A_2 , and which form a multiplet originated in the cell at x , as [33]

$$e_n(x, k, t) = \frac{1}{2} [S_{1/2}[\Gamma_{\times}(x, k, t)] - S_2(\Gamma_{A_1 \cup A_2})], \quad (17)$$

where S_{α} are the Rényi- α entropies. In (17), $\Gamma(x, k, t) = 2\tilde{C}_{x,k}(t) - \mathbb{1}$, and we defined $\Gamma_{\times} = [(\mathbb{1} + \Gamma_{A_1 \cup A_2}^2)/2]^{-1} \text{diag}(\Gamma_{A_1}, -\Gamma_{A_2})$, where A_1, A_2 contain the labels of the quasiparticles in A_1 and A_2 , respectively. In our quench we have $n = 2$, and Eq. (3) holds with

$$e_2(k, t) = \ln \left[\frac{1}{2} (1 + A_{k,t}^2 + B_{k,t}^2) + \left(\frac{1}{4} (1 + A_{k,t}^2 + B_{k,t}^2)^2 - A_{k,t}^2 \right)^{\frac{1}{2}} \right], \quad (18)$$

where $A_{k,t}, B_{k,t}$ are defined in (15) and (16).

Let us now consider the tight-binding chain, which corresponds to $\delta = h = 0$ in H_K (see Appendix D 3). The Hamiltonian is diagonalized by going to momentum space as

$$H = \int_0^{\pi} \frac{dk}{2\pi} \varepsilon(k) (c_k^{\dagger} c_k - c_{k-\pi}^{\dagger} c_{k-\pi}), \quad (19)$$

where $\varepsilon(k) = -\cos(k)$. The quasiparticles species are now $\eta_1 = c_k$ and $\eta_2 = c_{k-\pi}$. We focus on the quench from the dimer state

$$|D\rangle = 2^{-L/4} \prod_{j=1}^{L/2} (c_{2j-1}^{\dagger} + c_{2j}^{\dagger}) |0\rangle \quad (20)$$

in the presence of gain and loss dissipation, which correspond to Lindblad operators (in momentum space) $L_+ \sim \sqrt{\gamma^+} c_k^{\dagger}$ and $L_- \sim \sqrt{\gamma^-} c_k$. The correlation matrix between the species [cf. (1)] is obtained as

$$\tilde{C}_{x,k}(t) := \langle \eta_i^{\dagger} \eta_j \rangle = C_{x,k}(t) e^{-(\gamma_+ + \gamma_-)t} + (1 - e^{-(\gamma_+ + \gamma_-)t}) \begin{pmatrix} n_{\infty} & 0 & 0 & 0 \\ 0 & 1 - n_{\infty} & 0 & 0 \\ 0 & 0 & n_{\infty} & 0 \\ 0 & 0 & 0 & 1 - n_{\infty} \end{pmatrix}, \quad (21)$$

where $n_{\infty} = \gamma^+ / (\gamma^+ + \gamma^-)$, whereas $C_{x,k}(t)$ is the unitarily evolved correlator

$$C_{x,k}(t) = \frac{1}{2} \begin{pmatrix} 1 + \cos k & 0 & i \sin k e^{2i\varepsilon(k)t} & 0 \\ 0 & 1 - \cos k & 0 & i \sin k e^{-2i\varepsilon(k)t} \\ -i \sin k e^{-2i\varepsilon(k)t} & 0 & 1 - \cos k & 0 \\ 0 & -i \sin k e^{2i\varepsilon(k)t} & 0 & 1 + \cos k \end{pmatrix}. \quad (22)$$

Now, Eq. (3) should hold with $e_2(k, t) = \ln(T_1 + T_2)$, where T_1, T_2 are easily determined by applying (6) as

$$T_1^D := \lambda_t + 2g_{\times}(1 - \lambda_t)^2, \quad (23)$$

$$T_2^D := [(1 - \lambda_t)^2 [(1 - \lambda_t)g_+ + \lambda_t]^2 + \lambda_t^2 \sin^2(k)]^{\frac{1}{2}}, \quad (24)$$

with $g_{\times} := (\gamma^+ \gamma^-) / (\gamma^+ + \gamma^-)^2$, $g_+ := (\gamma^+)^2 + (\gamma^-)^2 / (\gamma^+ + \gamma^-)^2$ and λ_t as defined in (15) and (16). A similar result for the quench from the fermionic Néel state was derived *ab initio* in Ref. [8]. The derivation is cumbersome, while it is straightforward within our approach (see Appendix D 2). In Figs. 1(d)–1(f) we discuss numerical results for several

Lindblad dynamics with gain and loss dissipation. In Fig. 1(d) we consider dynamics in the tight-binding chain starting from the dimer state. The dashed and dotted curves are exact numerical data, and the continuous line is obtained by using $e_2(k, t) = \ln(T_1 + T_2)$ and (23) and (24) in (3). We plot $\mathcal{E}/\ell$ versus t/ℓ , considering the weakly dissipative limit $\gamma_{\pm} \propto 1/\ell$. The agreement between the numerical data and the quasiparticle picture is perfect in the limit $\ell \rightarrow \infty$. The inset shows fits to $\sim a/\ell + b/\ell^2$ of the finite-size corrections. In Fig. 1(e) we focus on the quench $h_0 = 0.3 \rightarrow h_1 = 3.1$ in the Kitaev chain. The agreement between the data and the analytic result [cf. (18) and (3)] is excellent.

IV. APPLICATIONS TO BOSONIC SYSTEMS

As a case study of bosonic systems, we consider the mass quench in the harmonic chain

$$H_h = \frac{1}{2} \sum_{j=1}^L \left(\frac{p_j^2}{m} + mw^2 x_j^2 + \mathcal{K}(x_{j+1} - x_j)^2 \right), \quad (25)$$

where x_j, p_j are canonically conjugated variables, and $\mathcal{K}, w$ real parameters. The initial state is the ground state of H_h with $m = m_0$, that is quenched to $m = m_1$. Bosonic gain and loss dissipation corresponds to $\hat{K}_k = \hat{A}_k + i\hat{B}_k$ [cf. (4)], with $\hat{A}_k = (\gamma_+ + \gamma_-)\mathbb{1}_2/2$ and $\hat{B}_k = i(\gamma_- - \gamma_+)\sigma^y/2$. It is straightforward to generalize Eq. (11) to the correlator $\Gamma_{m,n} := \langle \{r_m, r_n\} \rangle$ [24] (see Appendix D 4). From $\Gamma_{m,n}$ one can construct $\tilde{\mathcal{C}}_{x,k,t}$ in terms of the η_j , which for bosons are eigenvectors of $\hat{\mathcal{L}}^\dagger$. Here we consider the bosonic logarithmic negativity [4]. According to our conjecture, one has to compute the negativity between the quasiparticles $e_n(x, k, t)$, which is [31] $e_n = -1/2 \sum_j \ln[\min(1, |\mu_j|)]$. Here μ_j are the eigenvalues of $i\Omega\Gamma^{T_2}$, with

$$\Gamma^{T_2} := J_{A_2} \Gamma_{A_1 \cup A_2} J_{A_2}, \quad \Omega := (i\sigma^y)^{\oplus |A_1 \cup A_2|}, \quad (26)$$

where $\Gamma_{A_1 \cup A_2}(x, k, t)$ is $\tilde{\mathcal{C}}_{x,k,t}(t)$ transformed into the $x_j(k) := [\eta_j(k) + \eta_j(k)^\dagger]/\sqrt{2}$ and $p_j(k) := -i[\eta_j(k) - \eta_j^\dagger(k)]/\sqrt{2}$ basis and restricted to quasiparticles in $A_1 \cup A_2$. In (26), Ω is block-diagonal with equal blocks $i\sigma^y$. J_{A_2} implements the partial transposition with respect to A_2 on the correlation function Γ , and it is also block diagonal with the j th block equal to σ_z if $j \in A_2$ and to $\mathbb{1}_2$ otherwise (see Appendix C for a hydrodynamic “derivation”). For the mass quench, we have $n = 2$ [3]. The negativity is obtained from the eigenvalues of $J_{A_2} \Gamma_{A_1 \cup A_2} J_{A_2}$. They are (see Appendix D 4)

$$\lambda_1(k, t) = \frac{z_0^2}{z^2} \delta_t + \frac{z^4 + 1}{2g z^2} (1 - \delta_t), \quad (27)$$

$$\lambda_2(k, t) = \frac{z^2}{z_0^2} \delta_t + \frac{z^4 + 1}{2g z^2} (1 - \delta_t), \quad (28)$$

and $\lambda_3 = -\lambda_1, \lambda_4 = -\lambda_2$. The negativity e_2 is obtained as

$$e_2(k, t) = -\ln \left[\min \left(1, \left| \frac{\zeta_{k,m_0} \delta_t + \frac{\zeta_{k,m_1}^2 + 1}{2g \zeta_{k,m_1}} (1 - \delta_t)}{\zeta_{k,m_1}} \right| \right) \right] \\ - \ln \left[\min \left(1, \left| \frac{\zeta_{k,m_1} \delta_t + \frac{\zeta_{k,m_1}^2 + 1}{2g \zeta_{k,m_1}} (1 - \delta_t)}{\zeta_{k,m_0}} \right| \right) \right], \quad (29)$$

where $\zeta_{k,m} := [m^2 + 4 \sin^2(k/2)]^{1/2}$, $\delta_t := e^{-(\gamma_- - \gamma_+)t}$, and $g := (\gamma_- - \gamma_+)/(\gamma_- + \gamma_+)$. The derivation of (29) is cumbersome, and it is reported in Appendix D 4. Interestingly, Eq. (29) implies that, except for $g = m_1 w = 1$, $\mathcal{E}$ vanishes at a finite value of t/ℓ , which marks the so-called *sudden death of entanglement* [34]. Thus, at long times only the so-called bound entanglement [35] is present. The condition for the sudden death is derived analytically, and involves only the slowest quasiparticles, with $k \rightarrow 0$, as they are the only ones entangling A_1 and A_2 at long times (see Appendix D 5). Figure 1(f) corroborates our conjecture [cf. (29) and (3)] for the

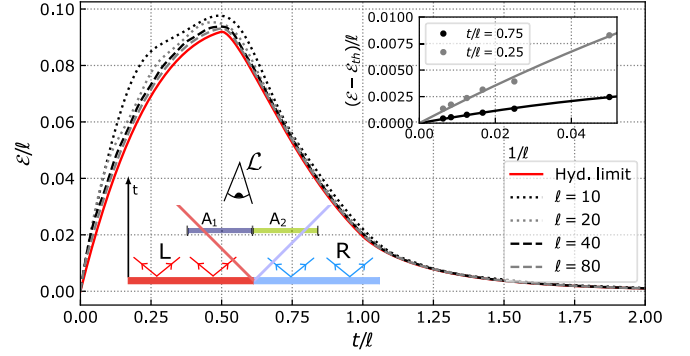


FIG. 2. Negativity dynamics in the tight-binding chain with gain and loss starting from an inhomogeneous state. At $t = 0$ the left and right parts of the chain are prepared in the dimer state and the Néel state, respectively. We plot $\mathcal{E}/\ell$ for two adjacent equal-length intervals versus t/ℓ . The continuous curve is the analytic result in the weakly dissipative hydrodynamic limit. The inset shows the finite-size corrections.

mass quench in the harmonic chain. After the “rise and fall,” the negativity exhibits the sudden death, which is correctly described by the theory.

V. INHOMOGENEOUS INITIAL STATES

We now consider the dynamics in the tight-binding chain with gain and losses starting from the inhomogeneous state $|L\rangle \otimes |R\rangle$, $|L\rangle$ and $|R\rangle$ being the states of the two halves of the chain (see Fig. 2). In the absence of dissipation the entanglement dynamics was investigated [29,36–38] using GHD [39,40]. The mechanism for correlation spreading described in Fig. 1(c) remains the same. However, now one has to “transport” the correlators $\mathcal{C}_{x,k}^{(L)}$ and $\mathcal{C}_{x,k}^{(R)}$ originating at $x < 0$ and $x > 0$, respectively. Correspondingly, one obtains two contributions $e_2^{(L)}$ and $e_2^{(R)}$. If A_1 and A_2 are placed symmetrically with respect to $x = 0$ (see Fig. 2), one obtains

$$\mathcal{E}(t) = \ell \int_0^\pi \frac{dk}{2\pi} \Theta(k, t) \frac{e_2^{(L)}(k, t) + e_2^{(R)}(k, t)}{2}, \quad (30)$$

where $\Theta(k, t)$ is the same as in (3). Despite its simplicity, an *ab initio* derivation of (30) would be a daunting task.

In Fig. 2 we consider the tight-binding chain and the quench from the state obtained by joining the Néel state and the dimer state. The continuous line in Fig. 2 is Eq. (30) with $e_2^{(L)}$ given by (3). For $e_2^{(R)}$ we used the result of [8]. The agreement with (30) is perfect in the weakly dissipative hydrodynamic limit.

VI. CONCLUSIONS AND OUTLOOK

We derived the hydrodynamic framework for the logarithmic negativity in generic bosonic and fermionic quadratic Lindblad systems. Our results open new avenues for future research. First and foremost, one could start an entanglement-based classification of dissipative dynamics in generic free-fermion and free-boson systems also beyond one dimension, generalizing results at equilibrium [41]. Second, our approach is readily generalizable to inhomogeneous dissipation or

dynamics beyond the quench paradigm, such as Floquet driving [42]. Third, it would be interesting to investigate whether the hydrodynamic framework captures *multipartite* entanglement [43] between spatially separated regions. Clearly, it is of paramount importance to investigate the effect of interactions, going beyond quadratic systems. Finally, Eq. (6) is not restricted to entanglement measures, and it could be applied to the entanglement asymmetry [44], generalizing the results of Ref. [45].

ACKNOWLEDGMENTS

This study was carried out within the National Centre on HPC, Big Data and Quantum Computing, SPOKE 10 (Quantum Computing), and received funding from the European Union Next-Generation EU, National Recovery and Resilience Plan (NRRP), MISSION 4 COMPONENT 2, INVESTMENT N. 1.4–CUP N. I53C22000690001. This work has been supported by the project “Artificially devised many-body quantum dynamics in low dimensions—ManyQLowD” funded by the MIUR Progetti di Ricerca di Rilevante Interesse Nazionale (PRIN) Bando 2022, Grant No. 2022R35ZBF.

APPENDIX A: EFFECT OF SPATIAL SYMMETRIES AND TIME REVERSAL ON QUADRATIC OPERATORS

We briefly discuss some properties of the Fourier transform of block-circulant matrices that we employ. Let our matrix M have matrix elements $[M_{m,n}]_{a,b}$, where the indices $m, n = 1, \dots, L$ denote the position of the block and a, b label the elements inside the block. Here we always consider fermionic and bosonic Hamiltonians with one-site translation invariance, meaning that $a, b = 1, 2$. Translation invariance implies that M is block-circulant, allowing us to write

$$[M_{m,n}] = [M_{m-n}] = \frac{1}{L} \sum_k \hat{m}_k e^{-ik(m-n)}, \quad (\text{A1})$$

where $k = 2\pi j/L$ and $j = 1, \dots, L$, and where $\hat{m}_k$ is defined as the Fourier transform of M :

$$\hat{m}_k := \sum_{m,n=1}^L \frac{1}{L} e^{ik(m-n)} [M_{m-n}]. \quad (\text{A2})$$

Let us now discuss how symmetries of M are implemented in $\hat{m}_k$. It is straightforward to check that

$$[M_{m-n}] = [M_{n-m}]^\dagger \Rightarrow \hat{m}_k^\dagger = \hat{m}_k, \quad (\text{A3})$$

$$[M_{m-n}] = [M_{n-m}]^T \Rightarrow \hat{m}_{-k} = \hat{m}_k^T, \quad (\text{A4})$$

$$[M_{m-n}] = -[M_{n-m}]^T \Rightarrow \hat{m}_{-k} = -\hat{m}_k^T. \quad (\text{A5})$$

Furthermore, let us now consider the operator

$$\mathcal{M} = \sum_{m,n,a,b} [M_{m,n}]_{a,b} r_{m,a} r_{n,b}, \quad (\text{A6})$$

where $r_{m,a}$ are Majorana operators for fermionic models, or the conjugated operators x_j, p_j for bosonic ones (see the main text). Let us consider the effect of the spatial inversion symmetry and of the time-reversal symmetry on $\mathcal{M}$ and $M_{m,n}$. The spatial inversion $\hat{R}$ is implemented by a unitary operator acting

on the local Majorana operators as

$$\hat{R} r_{m,a} \hat{R} = r_{-m,a}. \quad (\text{A7})$$

Thus, spatial inversion symmetry for the operator $\mathcal{M}$ implies that

$$[M_{m-n}] = [M_{n-m}] \Rightarrow \hat{m}_{-k} = \hat{m}_k. \quad (\text{A8})$$

We should notice there are several quadratic models that are *not* symmetric under the spatial inversion as defined in (A7). A prominent example is the Kitaev chain, as is easy to check.

Instead, the Kitaev chain is invariant under the time-reversal symmetry, which we now define. Time inversion $\hat{T}$ is an *antiunitary* operator leaving the local bosonic and fermionic ladder operators invariant. This implies that

$$\hat{T} r_{j,o} \hat{T} = r_{j,o}, \quad \hat{T} r_{j,e} \hat{T} = -r_{j,e}, \quad (\text{A9})$$

where $r_{j,o}$ and $r_{j,e}$ are defined in the main text, and we use that $\hat{T} i \hat{T} = -i$. Thus, invariance under time reversal implies that M satisfies

$$[M_{m-n}] = \sigma_z [M_{m-n}]^* \sigma_z \Rightarrow \hat{m}_{-k} = \sigma_z \hat{m}_k^* \sigma_z. \quad (\text{A10})$$

Here and in the following, $\sigma_{x,y,z}$ are the Pauli matrices.

In conclusion, the above results imply that if the matrix M is block-circulant, Hermitian, symmetric, and invariant under spatial inversion, its Fourier transform is of the form

$$\hat{m}_k = \alpha \mathbb{1} + \beta \sigma_x + \gamma \sigma_z, \quad \alpha, \beta, \gamma \in \mathbb{R}. \quad (\text{A11})$$

If M is block-circulant, Hermitian, antisymmetric, and invariant under spatial inversion, one has

$$\hat{m}_k = \alpha \sigma_y, \quad \alpha \in \mathbb{R}. \quad (\text{A12})$$

If M is block-circulant, Hermitian, symmetric, and time-reversal invariant, the Fourier transform satisfies

$$\hat{m}_k = \alpha \mathbb{1} + \beta \sigma_z, \quad \alpha, \beta \in \mathbb{R}. \quad (\text{A13})$$

Finally, if M is block-circulant, Hermitian, antisymmetric, and invariant under time reversal, its $\hat{m}_k$ can be written as

$$\hat{m}_k = \alpha \sigma_x + \beta \sigma_y, \quad \alpha, \beta \in \mathbb{R}. \quad (\text{A14})$$

Clearly, Eq. (C10) implies that $\hat{m}_k$ is real, which will turn out to be important to determine the quasiparticles for bosonic systems. On the other hand, both (A12) and (A13) imply that $\text{Tr}(\hat{m}_k) = 0$, which will be used in Appendix B to determine the properties of fermionic quasiparticles.

APPENDIX B: EFFECT OF THE DISSIPATION ON THE PROPAGATING MODES OF THE LIOUVILLIAN

As discussed in the main text, a key ingredient in the hydrodynamic description of the negativity is the correct identification of the quasiparticles in the presence of dissipation. Let us reiterate that the state ρ of the system evolves according to the master equation

$$\partial_t \rho = -i[H, \rho] + \sum_j \left(L_j \rho L_j^\dagger - \frac{1}{2} \{L_j^\dagger L_j, \rho\} \right) := -i\hat{\mathcal{L}}\rho, \quad (\text{B1})$$

where we have defined the Liouvillian superoperator $\hat{\mathcal{L}} = \hat{\mathcal{U}} + i\hat{\mathcal{D}}$. Here $\hat{\mathcal{U}} = [H, \cdot]$ the Hamiltonian part and $\hat{\mathcal{D}}$ is the

dissipator, acting on the space $L(\mathcal{H})$ of linear operators on the (finite) Hilbert space $\mathcal{H}$ of the theory. $L(\mathcal{H})$ is a Hilbert space itself, with the scalar product being $\langle A|B \rangle = \text{Tr}[A^\dagger B]$. From Eq. (B1), it is straightforward to derive the time evolution equation of the expected values of linear operators $\langle O \rangle = \text{Tr}[\rho O]$ as

$$\partial_t \langle O \rangle = i \langle \hat{\mathcal{L}}^\dagger O \rangle, \quad (\text{B2})$$

with

$$\hat{\mathcal{L}}^\dagger O = [H, O] - i \sum_j \left(L_j^\dagger O L_j - \frac{1}{2} \{L_j^\dagger L_j, O\} \right). \quad (\text{B3})$$

In the case of linear dissipation and quadratic Hamiltonian, we rewrite (B2), in terms of local fermionic Majorana operators or bosonic position and momentum operators x_j, p_j . We obtain

$$\begin{aligned} \hat{\mathcal{L}}^\dagger O &= \sum_{m,n=1}^{2L} h_{m,n} [r_m r_n, O] \\ &- i \sum_{m,n=1}^{2L} K_{m,n} \left(r_m O r_n - \frac{1}{2} \{r_m r_n, O\} \right). \end{aligned} \quad (\text{B4})$$

The Kossakowski matrix K is Hermitian and positive-semidefinite, and it can be decomposed as $K = A + iB$, with A real symmetric and positive-semidefinite and B real anti-symmetric. For fermionic systems, the r_j are the Majorana fermions, whereas for bosons $r = (x_1, p_1, \dots, x_L, p_L)$ (see the main text). In the following, we will always assume invariance under one-site translations.

1. Fermionic systems

Let us first consider the fermionic case, for which $h_{m,n}$ [cf. (B4)] can be chosen antisymmetric and purely imaginary. Let us also assume that the Hamiltonian is invariant under either spatial inversion [Eq. (A8)] or time reversal [Eq. (A10)]. In the former case the Fourier transform is $\hat{h}_k \propto \sigma_y$ [cf. (A12)], while in the second one it is a real linear combination of σ_x and σ_y [cf. (A14)].

For unitary dynamics, we have $\hat{\mathcal{D}} = 0$. Moreover, the Liouvillian superoperator $\hat{\mathcal{L}}$ maps local Majorana operators to linear combinations of local Majorana operators. Thus, it is natural to construct the single-particle propagating modes as eigenvectors of the Liouvillian that are linear combinations of Majorana operators. The eigenvector $\eta^\dagger$ of the adjoint Liouvillian will satisfy $[H, \eta^\dagger] = \lambda \eta^\dagger$. Let us now exploit translation invariance. In the thermodynamic limit $L \rightarrow \infty$ the Hamiltonian is rewritten as

$$H = \int_{-\pi}^{\pi} \frac{dk}{2\pi} \begin{pmatrix} r_o(-k) & r_e(-k) \end{pmatrix} \hat{h}_k \begin{pmatrix} r_o(k) \\ r_e(k) \end{pmatrix}, \quad (\text{B5})$$

where $r_o(k) := \frac{1}{\sqrt{L}} \sum_{j=1}^L e^{ikj} r_{2j-1} = c_k + c_{-k}^\dagger$ and $r_e(k) := \frac{1}{\sqrt{L}} \sum_{j=1}^L e^{ikj} r_{2j} = i(c_k - c_{-k}^\dagger)$ are the Fourier transforms of odd and even Majorana fermions, respectively. It is then apparent that for each quasimomentum k , the 2×2 subspace generated by the operators $r_o(k)$ and $r_e(k)$ is left invariant by

the dynamics. Indeed, one has

$$\begin{pmatrix} [H, r_o(k)] \\ [H, r_e(k)] \end{pmatrix} = -4\hat{h}_k \begin{pmatrix} r_o(k) \\ r_e(k) \end{pmatrix}, \quad (\text{B6})$$

where we have used Eq. (A5). From (B6) it follows that the propagating modes are linear combinations of $r_o(k)$ and $r_e(k)$. The coefficients $w(k)$ of the combination satisfy

$$w^T(k)(-4\hat{h}_k) = \lambda w^T(k). \quad (\text{B7})$$

If $w_\pm(k)$ are the two orthonormal left eigenvectors of $4\hat{h}_k$, their eigenvalues are $\lambda_\pm = \pm \varepsilon(k)$, since $\text{Tr}[\hat{h}_k] = 0$, and they are real. We can define η_k, η_{-k} as

$$\begin{pmatrix} \eta_k \\ \eta_{-k}^\dagger \end{pmatrix} = \frac{1}{\sqrt{2}} V_k \begin{pmatrix} r_o(k) \\ r_e(k) \end{pmatrix}, \quad V_k^T = \{w_+, w_-\}, \quad (\text{B8})$$

i.e., the unitary matrix V_k has rows $w_+^T(k), w_-^T(k)$. Here $k \in [0, \pi]$ and the labels $k, -k$ in the left-hand side in (B8) are consistent with the right-hand side. Indeed, by using (A3) and (A5) in (B7) one obtains that

$$w^\dagger(-4\hat{h}_{-k}) = -\lambda w^\dagger. \quad (\text{B9})$$

Thus, $w_\pm(-k) = w_\mp(k)^*$, implying that the rows of (B8) for $-k$ are the adjoints of those for k .

We can then naturally define two families of quasiparticles as $\eta_1^\dagger(k) = \eta_k^\dagger$, and $\eta_2^\dagger(k) = \eta_{-k}^\dagger$, with associated eigenvalues $\varepsilon_1(k) = \varepsilon_2(k) = \varepsilon(k)$. The anticommutation relation $\{\eta_i^\dagger(k), \eta_j(h)\} = \delta_{i,j} \delta_{k,h}$ is fulfilled because of the unitarity of V_k . In the thermodynamic limit $L \rightarrow \infty$, the Hamiltonian can be rewritten in terms of η_1, η_2 as

$$H = \int_0^\pi \frac{dk}{2\pi} \varepsilon(k) [\eta_1(k)^\dagger \eta_1(k) + \eta_2(k)^\dagger \eta_2(k) - 1]. \quad (\text{B10})$$

The propagation velocity of a species of quasiparticles is the group velocity obtained from the dispersion $\varepsilon_j(k)$ in front of $\eta_j(k)^\dagger \eta_j(k)$ after the Hamiltonian has been reduced in a form like (B10). An important caveat is that one has to consider the dispersion of the excitations with respect to the *physical* quasimomentum $k \in [-\pi, \pi]$ and not the folded one [45] $k \in [0, \pi]$, thus $v_1(k) = \varepsilon'(k)$, and $v_2(k) = -\varepsilon'(k) = -v_1(k)$. Finally, we observe that the condition $\text{Tr}[\hat{h}_k] = 0$, which is a consequence either of time reversal or of spatial inversion symmetry, ensures $v_1(k) = -v_2(k)$.

Let us now discuss the dissipative case $\hat{\mathcal{D}} \neq 0$ in (B2). As a further assumption, we require the dissipator to have the same symmetry as the Hamiltonian, that is, the Kossakowski matrix K [cf. (B2)] satisfies either (A8) or (A10). First, the subspace of operators that are linear in the fermionic local operators is not left invariant by the adjoint Liouvillian $\hat{\mathcal{L}}^\dagger$. It is easy to check that the dissipator $\hat{\mathcal{D}}$ applied to linear combinations of fermionic operators produces also cubic operators. Thus, following [24,28], we consider *quadratic* operators of the form $\chi = \eta^\dagger \eta$, where η is linear in fermion operators. We require that χ satisfies the equation

$$\hat{\mathcal{L}}^\dagger \chi = \xi(\chi + c), \quad (\text{B11})$$

with ξ and c to be determined. Thus, we identify $\eta^\dagger$ as the quasiparticle. This means that $\eta^\dagger$ is a quasiparticle if its density $\chi = \eta^\dagger \eta$ evolves according to (B11). In the following we will show that the associated eigenvalue ξ is purely imaginary.

Notice that a real part would give an oscillating behavior which is not expected for the density (in analogy with the unitary case). Moreover, ξ will have a positive imaginary part, reflecting that the quasiparticles decay.

Let us first show that in a one-site translation invariant system there are always two independent species of quasiparticles [in the sense of (B11)], $\eta_k^\dagger, \eta_{-k}^\dagger$ with $k \in [0, \pi]$. We anticipate that the η_k and η_{-k} satisfy canonical anticommutation relations only in the limit of weak dissipation. Moreover, the quadratic operator $\chi'_k = [\eta_k, \eta_{-k}]$ will also be an eigenvector of the adjoint Liouvillian as

$$\hat{\mathcal{L}}^\dagger \chi'_k = \xi'_k (\chi'_k + c'_k). \quad (\text{B12})$$

In analogy with the unitary case, the real part of ξ' , which is responsible for the oscillatory behavior of χ' , will provide the dispersion relation of the quasiparticles $\eta_j^\dagger$. Thus, ξ' is needed to identify the propagation velocities of the quasiparticles.

To proceed, let us write $\chi = \sum_{m,n=1}^L Q_{m,n} r_m r_n$, where Q can be chosen Hermitian and antisymmetric, since χ is self-adjoint. Then the eigenvalue equation (B11) becomes

$$\begin{aligned} \hat{\mathcal{L}}^\dagger \chi &= \sum_{m,n=1}^{2L} [(4h + 2iA)Q + Q(-4h + 2iA)]_{m,n} r_m r_n \\ &\quad - 4\text{Tr}[BQ] = \xi(\chi + c), \end{aligned} \quad (\text{B13})$$

where we used the decomposition of the Kossakowski matrix as $K = A + iB$, and we exploited the symmetries $h^T = -h$, $A^T = A$, $Q^T = -Q$. Moreover, if we assume translation invariance, Eq. (B13) becomes the eigenvalue equation for the 2×2 Fourier transform $\hat{q}_{k'}$ of Q as

$$(4\hat{h}_{k'} + 2i\hat{A}_{k'})\hat{q}_{k'} - \hat{q}_{k'}(4\hat{h}_{k'} + 2i\hat{A}_{k'})^\dagger = \xi \hat{q}_{k'}, \quad \forall k'. \quad (\text{B14})$$

Now let us consider the case of “small” dissipation (i.e., small $\hat{A}_k$) in which the solution of (B14) is simpler. Notice that this is not a very restrictive assumption, since our main interest is the limit of vanishing dissipation. Then the matrix $\hat{X}_k := 4\hat{h}_k + 2i\hat{A}_k$ is always diagonalizable. The symmetries of h and A [Eqs. (C10), (A12), or (A13), (A14)] imply that $\hat{x}_k$ has two distinct eigenvalues of the form $\lambda_k = \pm\omega_k + i\gamma_k$, where both ω_k and γ_k are real. We choose $\omega_k > 0$, and we have $\gamma_k = \text{Tr}[\hat{A}_k] > 0$, as expected. This means that the density of the quasiparticles decays with time, as anticipated. If $w_\pm(k)$ are the associated (column) eigenvectors of $\hat{x}_k$, we can construct a solution of (B14) as $\hat{q}_{\pm,k'}^{(k)} = w_\pm(k)w_\pm^\dagger(k)\delta_{k,k'}$ satisfying Eq. (B14), with $\xi = \xi_k^\pm := 2i\gamma_k$. Thus, if the matrix V_k with columns $w_+(k), w_-(k)$ diagonalizes $\hat{x}_k$ as $\hat{x}_k = V_k D_k V_k^{-1}$, we can define the quasiparticles as

$$\begin{pmatrix} \eta_k \\ \eta_{-k}^\dagger \end{pmatrix} = \frac{1}{\sqrt{2}} V_k^\dagger \begin{pmatrix} r_o(k) \\ r_e(k) \end{pmatrix}, \quad (\text{B15})$$

where the factor $1/\sqrt{2}$ is a normalization we will need later. Notice that in the unitary case $\hat{A}_k = 0$, and we recover Eq. (B8), as is clear from the definitions of $w_\pm(k)$.

We now define two families of quasiparticles, in analogy with the unitary case, as $\eta_1(k)^\dagger = \eta_k^\dagger$, $\eta_2(k)^\dagger = \eta_{-k}^\dagger$. They satisfy anticommutation relations

$$\begin{pmatrix} \{\eta_1(k), \eta_1(k')^\dagger\} & \{\eta_1(k), \eta_2(k')^\dagger\} \\ \{\eta_2(k)^\dagger, \eta_1(k')^\dagger\} & \{\eta_2(k)^\dagger, \eta_2(k')^\dagger\} \end{pmatrix} = \delta_{k,k'} V_k^\dagger V_k, \quad (\text{B16})$$

that in the limit of small dissipation become the canonical anticommutation relations, since the matrix V_k diagonalizing $\hat{x}_k$ becomes unitary.

Finally, we check that the operator $\chi'_k = [\eta_k, \eta_{-k}] = [\eta_1(k), \eta_2(k)]$ is also an eigenvector of $\hat{\mathcal{L}}$ in the sense of (B12). Indeed, we can write it as

$$\chi'_k = \sum_{m,n=1}^L P_{m,n}^{(k)} r_m r_n, \quad (\text{B17})$$

where $P^{(k)}$ is block circulant and has 2×2 Fourier transform

$$\hat{P}_{k'}^{(k)} = \frac{1}{2}(\delta_{k,-k'} w_+^*(k) w_-^T(k) - \delta_{k,k'} w_-(k) w_+^\dagger(k)). \quad (\text{B18})$$

Now, the eigenvalue equation (B12) for χ'_k becomes

$$\begin{aligned} \hat{\mathcal{L}} \chi'_k &= \xi'_k (\chi'_k + c'_k) = \sum_{m,n=1}^{2L} (4[h, P^{(k)}] \\ &\quad + i\{A, P^{(k)} - (P^{(k)})^T\})_{m,n} r_m r_n - 4\text{Tr}[BP^{(k)}], \end{aligned} \quad (\text{B19})$$

which implies, exploiting the fact that $(\hat{P}_{-k'}^{(k)})^T = -\hat{P}_{k'}^{(k)}$, that $\hat{P}_{k'}^{(k)}$ must satisfy

$$\hat{x}_{k'} \hat{P}_{k'}^{(k)} - \hat{P}_{k'}^{(k)} \hat{x}_{k'}^\dagger = \xi'_k \hat{P}_{k'}^{(k)}, \quad (\text{B20})$$

which is true for $\xi'_k = -2\omega_k + 2i\gamma_k$. In the unitary case, i.e., without dissipation, one has $\hat{\mathcal{L}} \chi'_k = -2\varepsilon(k) \chi'_k$, with $\varepsilon(k)$ the dispersion of the model. This means that the eigenvalue equation for χ' gives access to the propagation velocities of the two quasiparticles as $v_1(k) = -v_2(k) = \partial_k \omega_k$. A similar idea holds in the presence of dissipation [28], where the real part of ξ'_k gives access to the velocities of the quasiparticles. Moreover, in the limit of weak dissipation [24] the velocities of the quasiparticles become the same as in the unitary case, i.e., $\pm\varepsilon'(k)$. Incidentally, we notice that the symmetry under spatial reflection and time reversal was crucial to derive the results above. Indeed, dropping the symmetry requirements on the dissipation would produce different dissipation rates for the densities of the two quasiparticles of the pair, distinguishing between right and left movers. Indeed, x_k would have eigenvalues $\lambda_k = \pm\omega_k \pm i\delta_k + i\gamma_k$, with $\delta_k \in \mathbb{R}$, implying $\xi_k^\pm = 2i(\gamma_k \pm \delta_k)$ (it is quite cumbersome but straightforward to prove that the imaginary part of $\xi_k^\pm$ would still be positive). For the time-reversal symmetric Kitaev chain, the dissipations without time-reversal symmetry turn out to be the “odd dissipations” of [25].

2. Bosonic systems

Let us now consider the bosonic case, for which the matrix h in Eq. (B4) can be always chosen real and symmetric, and is usually positive-definite. We also assume invariance under spatial inversion [Eq. (A8)] for both the Hamiltonian and the dissipation. The bosonic case is easier to deal with than the fermionic one, since the subspace of linear operators is always invariant under the action of $\hat{\mathcal{L}}^\dagger$. Thus, we can construct the quasiparticles as linear combinations of r_j . If we write $\eta = \sum_j w_j r_j$, the vector w must satisfy the equation

$$i(2h - B)\Omega w = \lambda w, \quad (\text{B21})$$

with λ the eigenvalue, and the Hamiltonian matrix h and B as in (B4). The symplectic form Ω encodes the commutation relations between the bosonic operators and reads $[\Omega_{m,n}] = i\sigma_y \delta_{m,n}$. Since we have translation invariance, we can look for eigenvectors belonging to the 2×2 invariant subspace generated by $x_k := \frac{1}{\sqrt{L}} \sum_{j=1}^L e^{ikj} x_j$ and $p_k := \frac{1}{\sqrt{L}} \sum_{j=1}^L e^{ikj} p_j$, which are the Fourier transforms of position and momentum operators. In that subspace, the eigenvalue problem in (B21) becomes

$$i(2\hat{h}_{-k} - \hat{B}_{-k})(i\sigma_y)w_k = \lambda_k w_k. \quad (\text{B22})$$

The invariance under spatial inversion, as we have shown in Appendix A, implies that $i\hat{B}_k = b_k \sigma_y$, $b_k \in \mathbb{R}$. In turn, this implies $-i\hat{B}_{-k}(i\sigma_y) = -ib_{-k} \mathbb{1}_{2 \times 2}$. In conclusion, spatial inversion symmetry ensures that the quasiparticles are the same quasiparticles of the Hamiltonian. Since h is positive-definite, $\hat{h}_{-k}$ is also positive. Then for Williamson's theorem it always satisfies

$$\begin{aligned} i\hat{h}_{-k}(i\sigma_y) &= s_k^{-1} \begin{pmatrix} 0 & i\omega_k \\ -i\omega_k & 0 \end{pmatrix} s_k \\ &= \frac{1}{2} s_k^{-1} \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix} \begin{pmatrix} -\omega_k & 0 \\ 0 & \omega_k \end{pmatrix} \begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix} s_k, \end{aligned} \quad (\text{B23})$$

where the real matrix s_k is symplectic and satisfies $s_k \hat{h}_{-k} s_k^T = \text{diag}(\omega_k, \omega_k)$, $\omega_k \in \mathbb{R}^+$ [46]. Thus, we can define the quasiparticles as

$$\begin{pmatrix} \eta_k \\ \eta_{-k}^\dagger \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix} (s_k^{-1})^T \begin{pmatrix} x_k \\ p_k \end{pmatrix}, \quad (\text{B24})$$

where the rows of the matrix are the transposed of the two eigenvectors $w_{\mp}(k)$ satisfying (B22), with eigenvalues $\mp 2\omega_k - ib_{-k}$. As in the fermionic case, $k \in [0, \pi]$, since in Eq. (B24) for $-k$ we have the adjoint operators η_{-k} , $\eta_k^\dagger$. This follows from the fact that $s_{-k} = s_k$, because the spatial inversion symmetry implies $\hat{h}_{-k} = \hat{h}_k$. We identify our two families of quasiparticles as $\eta_1^\dagger(k) = \eta_k^\dagger$, $\eta_2^\dagger(k) = \eta_{-k}^\dagger$. The fact that s_k is symplectic ensures that quasiparticles have the correct commutation relations. In terms of these two families, the Hamiltonian can be rewritten (in the thermodynamic limit $L \rightarrow \infty$) as

$$H = \int_0^\pi \frac{dk}{2\pi} 2\omega_k [\eta_1(k)^\dagger \eta_1(k) + \eta_2(k)^\dagger \eta_2(k) + 1]. \quad (\text{B25})$$

The velocities of the two families of quasiparticles are given by the group velocity obtained from their energy dispersion, with the same caveat as in the fermionic case that they have to be extracted from the dispersion before the folding. One has $v_1(k) = -v_2(k) = 2\partial_k \omega_k$.

3. Summary and generalization

The construction above shows that for fermions the effect of the dissipation is simply a mixing between the quasiparticles $\eta_1(k)$, $\eta_2(k)^\dagger$ of the Hamiltonian, as is clear by comparing Eqs. (B8) and (B15). Thus, if the initial state is translationally invariant, the two-point function is block-diagonal in the quasimomentum space with respect to the Hamiltonian quasiparticles. The time-dependent correlator $\hat{C}_{x,k,t}(t)$ can be

reconstructed by solving the time evolution equation of the Fourier transform $\hat{g}_k(t)$ of the two-point function, which contains all the nonzero correlations allowed by the translational symmetry.

For bosons, typical dissipations have no effect on the structure of the quasiparticles of the Hamiltonian. A case of dissipation that does not respect spatial inversion symmetry, implying that the quasiparticles are modified by the dissipation, was studied in [24]. In the setup of Ref. [24] the net effect of the dissipation is, again, a mixing between $\eta_1(k)^\dagger$ and $\eta_2(k)$. Thus, also in the bosonic case if the initial state is translationally invariant, the two-point function is block-diagonal in the basis η_1 , η_2 , and the time dependence of $\hat{C}_{x,k,t}(t)$ can be reconstructed from the Fourier transform of the two-point function $\hat{g}(k, t)$.

When the translational symmetry is weakened (for example, becomes a two-site translation invariance), to preserve the block-diagonality one has to write the two-point function in quasimomentum space in terms of larger number n of families of quasiparticles. In the unitary case, this is equivalent to require that we can rewrite the Hamiltonian in the thermodynamic limit $L \rightarrow \infty$ as [45]

$$H = \int_0^{\frac{2\pi}{n}} \frac{dk}{2\pi} \sum_{j=1}^n \epsilon_j(k) \eta_j(k)^\dagger \eta_j(k) + C, \quad (\text{B26})$$

where the quasiparticles η_j satisfy canonical commutation or anticommutation relations, and the velocities of the species are identified as $v_j(k) = \pm \epsilon_j'(k)$, where the sign, as discussed above, is the one obtained from the energy dispersion before folding the quasimomentum. We expect that the effect on this multiplet structure of a dissipation with at least the same spatial symmetry as the Hamiltonian should be simply to mix the η_i , $\eta_j^\dagger$ with fixed quasimomentum k . Thus, one could expect that the procedure to construct the quasiparticles we presented can be generalized to arbitrary multiplets.

APPENDIX C: HYDRODYNAMIC DERIVATION OF EQS. (2) AND (6)

In this appendix we give a more detailed justification of our conjectures (2) and (6) in the main text. We show how they rely on the fact that quasiparticles obey canonical (anti)commutation relations, and on the hydrodynamic assumptions of independent mesoscopic cells and ballistic propagation of the correlations [cf. Fig. 1(c)]. We also show that, for bosons, the way we construct the position-momentum correlator $\Gamma(x, k, t)$ associated with a multiplet [see Eq. (26)] is the correct one, even though it does not correspond to the correlator of any set of Fourier-transformed position-momentum operators. In the following, it will be useful to employ the notation $m^\oplus$ to indicate a block-diagonal matrix with the proper number (depending on the context) of equal blocks m on the diagonal. Moreover, with $m \sim n$ we mean that the matrices m and n are similar (thus, they have the same spectrum).

Let us consider any local observable $f(\rho_A(t))$ that on Gaussian states can be evaluated as $f(\rho_A(t)) = \text{Tr}[\mathcal{F}(C_A(t))]$, where $C_A(t)$ is the two-point function in real space restricted to a subsystem A and $\mathcal{F}$ a suitable function related to f . We

assume that in the hydrodynamic limit we can instead calculate the trace of the same function $\mathcal{F}$ of the correlator $\mathcal{C}_t = \bigoplus_{x,k} \tilde{\mathcal{C}}_{x,k,t}(t)$, where the restriction on a subsystem A becomes the restriction on the quasiparticles in that subsystem. Thus, in our conjecture, we assume that in the hydrodynamic limit $\mathcal{F}(C_A(t))$ has the same spectrum as $\mathcal{F}(\mathcal{C}_t^{(A)})$. Provided that the quasiparticles satisfy canonical (anti)commutation relations, this is simply a consequence of the fact that the mesoscopic cells are independent and the dynamics in each cell produces the same quasiparticles of the full chain. This is justified because the cell is “large.” Indeed, within this approximation, the transformation between $C_A(t)$ and $\mathcal{C}_t^{(A)}$ is of the form

$$C_A(t) = u^{\oplus} \mathcal{C}_t^{(A)} (u^{\dagger})^{\oplus}, \quad (\text{C1})$$

where each copy of u acts on a cell contained in A . Now it is useful to distinguish between the fermionic case and the bosonic one. The fermionic case is much easier. Indeed, if the quasiparticles obey canonical anticommutation relations, then u is unitary and $C_A(t) \sim \mathcal{C}_t^{(A)}$ for any subsystem A , thus one simply has to check that the required $\mathcal{F}$ is well behaved, i.e., it preserves the similarity. In this work, we are interested in the case of the fermionic negativity. One has [33]

$$\mathcal{E}(t) = \frac{1}{2} [S_{1/2}(\Gamma_{\times}(t)) - S_2(\Gamma_{A_1 \cup A_2}(t))], \quad (\text{C2})$$

where we define

$$\Gamma_{\times} := [(\mathbb{1} + \Gamma_{A_1 \cup A_2}^2)/2]^{-1} \text{diag}(\Gamma_{A_1}, -\Gamma_{A_2}) \quad (\text{C3})$$

and $\Gamma_A := 2C_A(t) - \mathbb{1}$. In (C2), we also defined

$$S_{\alpha} = \frac{1}{2(1-\alpha)} \ln \left[\left(\frac{1+x}{2} \right)^{\alpha} + \left(\frac{1-x}{2} \right)^{\alpha} \right]. \quad (\text{C4})$$

$S_{\alpha}(\Gamma_A)$ is the Rényi- α entropy of a subsystem A [47]. With these definitions, it is apparent that $\mathcal{F}(C_A(t)) \sim \mathcal{F}(\mathcal{C}_t^{(A)})$. We thus obtain Eq. (17). Notice that the generalization to other quantities, such as the charged moments of the entanglement asymmetry [45], is straightforward. An important caveat is that in the presence of dissipation the quasiparticles might not satisfy the canonical anticommutation relations for finite dissipation rates, in contrast with the weak dissipation limit [see Eq. (B16)]. This means that for finite dissipation Eq. (6) has to be modified. To be precise, Eq. (6) will depend on the matrix V_k in Eq. (B16). This strategy has been employed in [28] for the von Neumann entropy.

In the bosonic case, the preservation of the commutation relations implies (notice that we are working in the ladder operators’ basis)

$$u \sigma_z^{\oplus} u^{\dagger} = \sigma_z^{\oplus} = u^{\dagger} \sigma_z^{\oplus} u, \quad (\text{C5})$$

where the second equality follows from the fact that $\sigma_z^{-1} = \sigma_z$. Then $C_A(t)$ and $\mathcal{C}_t^{(A)}$ have different spectra. This is not very surprising, since for bosonic correlators the relevant spectrum is usually the *symplectic* one. Indeed, the issue can be solved by the form of $\mathcal{F}$, which can ensure that $\mathcal{F}(C_A(t)) \sim \mathcal{F}(\mathcal{C}_t^{(A)})$. This is exactly what happens for our case of interest, the bosonic logarithmic negativity, as we are going to show. For bosons the negativity is obtained as [31]

$$\mathcal{E}(t) = -1/2 \sum_j \ln[\min(1, |\mu_j|)], \quad (\text{C6})$$

where $\mu_j(t)$ are the eigenvalues of $i\Omega J_{A_2} \Gamma_A(t) J_{A_2}$, with Γ_A the two-point function restricted to $A := A_1 \cup A_2$, $\Omega = i\sigma_y^{\oplus}$, and J_{A_2} is a matrix that represents the effect of the partial transposition of the state with respect to A_2 on Γ_A , reading $J_{A_2} = \mathbb{1}_2^{\oplus |A_1|} \oplus \sigma_z^{\oplus |A_2|}$, thus acting as $\mathbb{1}$ on each site of A_1 and as σ_z on each site of A_2 . Notice that Eq. (C6) does not provide explicitly the form of $\mathcal{F}$ for the bosonic negativity. Now the change of basis from the position and momentum x_j , p_j to the bosonic ladder operators b_j is implemented by a unitary matrix R as

$$\begin{pmatrix} x_j \\ p_j \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix} \begin{pmatrix} b_j^{\dagger} \\ b_j \end{pmatrix} := R \begin{pmatrix} b_j^{\dagger} \\ b_j \end{pmatrix}. \quad (\text{C7})$$

Next, let us observe that (with J_{A_2} from now on called simply J for brevity)

$$i\Omega J \Gamma_A(t) J = -R^{\oplus} \sigma_z^{\oplus} J' C_A(t) J' (R^{\dagger})^{\oplus}, \quad (\text{C8})$$

with

$$J' := \mathbb{1}_2^{\oplus} \oplus \sigma_x^{\oplus}. \quad (\text{C9})$$

Equation (C8) implies that

$$i\Omega J \Gamma_A(t) J \sim -\sigma_z^{\oplus} J' C_A(t) J' = -J'' \sigma_z^{\oplus} C_A(t) J', \quad (\text{C10})$$

with

$$J'' := \mathbb{1}_2^{\oplus} \oplus (-\sigma_x)^{\oplus}, \quad (\text{C11})$$

providing the definition of the bosonic negativity as $\text{Tr} \mathcal{F}(C_A) := -\text{Tr}(J'' \sigma_z^{\oplus} C_A J')$. We now apply (C1) and (C5), obtaining

$$\begin{aligned} -J'' \sigma_z^{\oplus} C_A(t) J' &= -J'' \sigma_z^{\oplus} u^{\oplus} \mathcal{C}_t^{(A)} (u^{\dagger})^{\oplus} J' \\ &= -J'' (u^{-1})^{\dagger \oplus} \sigma_z^{\oplus} \mathcal{C}_t^{(A)} (u^{\dagger})^{\oplus} J'. \end{aligned} \quad (\text{C12})$$

Then, since $J'^2 = J''^2 = \mathbb{1}$, we have

$$\begin{aligned} -J'' \sigma_z^{\oplus} C_A(t) J' &= -J'' (u^{-1})^{\dagger \oplus} J'' J'' \sigma_z^{\oplus} \mathcal{C}_t^{(A)} J' J' (u^{\dagger})^{\oplus} \\ &\sim -J'' \sigma_z^{\oplus} \mathcal{C}_t^{(A)} J', \end{aligned} \quad (\text{C13})$$

as we wanted to show, where the last step follows from the fact that $J'' (u^{-1})^{\dagger \oplus} J'' = [J' (u^{\dagger})^{\oplus} J']^{-1}$. By using (C8), (C10), and (C13), we obtain that

$$i\Omega J \Gamma_A(t) J \sim i\Omega J \bar{\Gamma}_t^{(A)} J, \quad \text{with } \bar{\Gamma}_t^{(A)} := R^{\oplus} \mathcal{C}_t^{(A)} (R^{\dagger})^{\oplus}. \quad (\text{C14})$$

From (C7) one has that, as anticipated, $\bar{\Gamma}_t^{(A)}$ is the two-point correlators of the position-momentum operators $x^{(k)}$ and $p^{(k)}$ defined from the quasiparticles operators η_j as

$$x^{(k)} = \frac{\eta_j(k) + \eta_j^{\dagger}(k)}{\sqrt{2}}, \quad p_j^{(k)} = -i \frac{\eta_j(k) - \eta_j^{\dagger}(k)}{\sqrt{2}}. \quad (\text{C15})$$

Thus, the correlator $\Gamma_{A_1 \cup A_2}(x, k, t)$ of the generic n -quasiparticle multiplet with quasimomentum k and originated at x is

$$\Gamma_{A_1 \cup A_2}(x, k, t) := R^{\oplus |A_1 \cup A_2|} \tilde{\mathcal{C}}_{x,k,t}^{(A_1 \cup A_2)}(t) (R^{\dagger})^{\oplus |A_1 \cup A_2|}, \quad (\text{C16})$$

and we have $\bar{\Gamma}_t^{(A)} = \bigoplus_{x,k} \Gamma_{A_1 \cup A_2}(x, k, t)$. In terms of $x^{(k)}$ and $p^{(k)}$ the negativity of the multiplet resembles the standard definition of the negativity (see Ref. [33]). We stress again that the operators (C15) satisfy canonical commutation relations and

are *not* Fourier transforms of any set of position-momentum operators, since, for example, they are self-adjoint, while Fourier transforms of position-momentum operators are not.

There is an alternative way to justify Eqs. (2) and (6). One can notice that the identities of the form $f(\rho_A(t)) = \text{Tr}[\mathcal{F}(C_A(t))]$ holding for Gaussian states usually rest only upon the assumption that the local operators satisfy canonical (anti)commutation relations. Thus, choosing another basis implies simply that $\mathcal{F}$ has to be applied to the corresponding two-point function, which in our case is exactly $C_t^{(A)}$.

APPENDIX D: DETAILED CALCULATIONS FOR OUR MODELS

1. Quench of the magnetic field in the Kitaev chain with gain and loss dissipation

First, let us focus on the key fermionic model we presented in the main text. Namely, let us consider a quench of the magnetic field in the Kitaev chain. The Hamiltonian reads as

$$H_K = -\frac{1}{2} \sum_{j=1}^L (c_j^\dagger c_{j+1} + \delta c_j^\dagger c_{j+1}^\dagger + \text{H.c.} - 2h c_j^\dagger c_j). \quad (\text{D1})$$

We assume periodic boundary conditions, i.e., $c_{L+1} \equiv c_1$. At $t < 0$, the system is in the ground state of H_K with $h = h_0$. At $t = 0$, the field is instantly quenched to $h = h_1$. Let us first consider the unitary case. H_K can be diagonalized by means of a Fourier transform $c_k = \frac{1}{\sqrt{L}} \sum_{j=1}^L e^{ikj} c_j$ followed by a Bogoliubov transformation, i.e.,

$$\begin{pmatrix} \eta_k \\ \eta_{-k}^\dagger \end{pmatrix} = \begin{pmatrix} \cos(\Delta_{k,h}/2) & -i \sin(\Delta_{k,h}/2) \\ -i \sin(\Delta_{k,h}/2) & \cos(\Delta_{k,h}/2) \end{pmatrix} \begin{pmatrix} c_k \\ c_{-k}^\dagger \end{pmatrix}, \quad (\text{D2})$$

with the Bogoliubov angle $\Delta_{k,h}$ satisfying

$$\cos \Delta_{k,h} = \frac{h - \cos k}{\sqrt{(h - \cos k)^2 + \delta^2 \sin^2 k}}, \quad (\text{D3})$$

$$\sin \Delta_{k,h} = \frac{\delta \sin k}{\sqrt{(h - \cos k)^2 + \delta^2 \sin^2 k}}. \quad (\text{D4})$$

In the thermodynamic limit $L \rightarrow \infty$, we can rewrite

$$H_K = \int_0^\pi \frac{dk}{2\pi} \varepsilon(k) (\eta_k^\dagger \eta_k + \eta_{-k}^\dagger \eta_{-k} - 1), \quad (\text{D5})$$

with $\varepsilon(k)$ given as

$$\varepsilon(k) := \sqrt{(h - \cos k)^2 + \delta^2 \sin^2 k}. \quad (\text{D6})$$

The initial state is the ground state of H_K with the field $h = h_0$. Let us call $\eta_k^{(0)}$ the eigenmodes for $h = h_0$. The ground state then reads $|\Omega\rangle \propto \prod_{k=-\pi}^\pi \eta_k^{(0)} |0\rangle$. From Eq. (D2), we see that the postquench eigenmodes η_k satisfy

$$\begin{pmatrix} \eta_k \\ \eta_{-k}^\dagger \end{pmatrix} = M \begin{pmatrix} \eta_k^{(0)} \\ \eta_{-k}^{(0)\dagger} \end{pmatrix}, \quad (\text{D7})$$

where we defined M as

$$M = \begin{pmatrix} \cos(\theta_{h_1,h_0}(k)/2) & -i \sin(\theta_{h_1,h_0}(k)/2) \\ -i \sin(\theta_{h_1,h_0}(k)/2) & \cos(\theta_{h_1,h_0}(k)/2) \end{pmatrix} \quad (\text{D8})$$

with $\theta_{h',h}(k) := \Delta_{k,h'} - \Delta_{k,h}$. In the following, we drop the subscripts h_0, h_1 for the sake of simplicity, and write θ_k for $\theta_{h_1,h_0}(k)$. As explained above, we identify the pairs of entangled excitations, which are responsible of entanglement spreading, as $\eta_1^\dagger(k) = \eta_k^\dagger, \eta_2^\dagger(k) = \eta_{-k}^\dagger, k \in [0, \pi]$, with the dispersions $\varepsilon_1(k) = \varepsilon_2(k) = \varepsilon(k)$ that we obtain from (D5). The group velocities of the quasiparticles are then $v_1(k) = -v_2(k) = \varepsilon'(k)$ [cf. (D6)].

By using (D5) and (D7), one obtains that the two-point function is block-diagonal in the quasimomentum space, with the block encoding the correlation between the species [see Eq. (1)] reading

$$C_{x,k}(t) = \frac{1}{2} \begin{pmatrix} 1 - \cos \theta_k & 0 & 0 & i \sin \theta_k e^{2i\varepsilon(k)t} \\ 0 & 1 + \cos \theta_k & i \sin \theta_k e^{-2i\varepsilon(k)t} & 0 \\ 0 & -i \sin \theta_k e^{2i\varepsilon(k)t} & 1 - \cos \theta_k & 0 \\ -i \sin \theta_k e^{-2i\varepsilon(k)t} & 0 & 0 & 1 + \cos \theta_k \end{pmatrix}. \quad (\text{D9})$$

Notice that in (D9) we keep the phases $e^{2i\varepsilon(k)t}$ associated with the unitary evolution, thus it represents the time evolution of $C_{x,k}$ in Eq. (1). We then add gain and loss dissipation, defined by the two families of Lindblad operators $L_{+,j} = \sqrt{\gamma_+} c_j^\dagger, L_{-,j} = \sqrt{\gamma_-} c_j$ in Eq. (B1), where $\gamma_\pm$ are the gain and loss rates. We straightforwardly obtain that the Fourier transforms of A and B [cf. (B4)] read

$$\hat{A}_k = \frac{\gamma_+ + \gamma_-}{4} \mathbb{1}_2, \quad \hat{B}_k = i \frac{\gamma_+ - \gamma_-}{4} \sigma_y. \quad (\text{D10})$$

In the presence of a homogeneous dissipation, the two-point correlation function remains block-diagonal with respect of the two families $\eta_1^\dagger(k), \eta_2^\dagger(k)$. Indeed, a homogeneous dissipation can at most mix the modes $\eta_1, \eta_2^\dagger$. However, this does not

happen for gain and loss dissipation, for which the modes that propagate entanglement remain the same as the eigenmodes of the Hamiltonian. This is due to the fact that $\hat{A}_k \propto \mathbb{1}_2$ [cf. (B14)]. To derive the time-dependent correlator $\tilde{C}_{x,k,t}(t)$, it is easier to first go to the Majorana basis, by defining the Majorana operators r_j as $r_{2j-1} = c_j + c_j^\dagger, r_{2j} = i(c_j - c_j^\dagger)$. The correlation matrix $\Gamma_{m,n} := \langle [r_m, r_n]/2 \rangle$, which contains the same information as $\tilde{C}_{x,k,t}(t)$, satisfies the evolution equation [24]

$$\Gamma(t) = U(t) \Gamma(0) U^T(t) + 4i \int_0^t du U(t-u) B U^T(t-u), \quad (\text{D11})$$

where $U(t) = e^{-(4ih+2A)t}$, with A , B , and h defined as in (B2). Translational invariance allows us to rewrite Eq. (D11) into a time-evolution equation for the Fourier transform $\hat{g}_k$ of the correlation matrix as

$$\hat{g}_k = \hat{u}_k \hat{g}_k(0) \hat{u}_{-k}^T + 4i \int_0^t du \hat{u}_k(t-u) \hat{B}_k \hat{u}_{-k}^T(t-u), \quad (\text{D12})$$

where $\hat{u}_k(t) = e^{-(4i\hat{h}_k+2\hat{A}_k)t}$, with $\hat{h}_k$, $\hat{A}_k$, and $\hat{B}_k$ the Fourier transforms of h , A , B , respectively. The Fourier transform $\hat{g}_k$

of the correlation matrix reads

$$\hat{g}_k = \begin{pmatrix} \langle r_{o,k} r_{o,-k} \rangle - 1 & \langle r_{o,k} r_{e,-k} \rangle \\ \langle r_{e,k} r_{o,-k} \rangle & \langle r_{e,k} r_{e,-k} \rangle - 1 \end{pmatrix}. \quad (\text{D13})$$

Here $r_{o,k} := 1/\sqrt{L} \sum_{j=1}^L e^{ikj} r_{2j-1} = c_k + c_{-k}^\dagger$ and $r_{e,k} := 1/\sqrt{L} \sum_{j=1}^L e^{ikj} r_{2j} = i(c_k - c_{-k}^\dagger)$ the Fourier transforms of odd and even Majorana fermions r_{2j-1} and r_{2j} , respectively. We obtain $\tilde{C}_{x,k}(t)$ by solving Eq. (D12) in the thermodynamic limit $L \rightarrow \infty$, and then changing basis as

$$\tilde{C}_{x,k}(t) = \frac{1}{4} \begin{pmatrix} 0 & 0 & m_k & im_k^* \\ m_k^* & m_k^* & 0 & 0 \\ -im_k & im_k & 0 & 0 \\ 0 & 0 & m_k & -im_k^* \end{pmatrix} \begin{pmatrix} \hat{g}_k(t) + \mathbb{1}_{2 \times 2} & \mathbf{0}_{2 \times 2} \\ \mathbf{0}_{2 \times 2} & \hat{g}_{-k}(t) + \mathbb{1}_{2 \times 2} \end{pmatrix} \begin{pmatrix} 0 & m_k & im_k^* & 0 \\ 0 & m_k & -im_k^* & 0 \\ m_k^* & 0 & 0 & m_k^* \\ -im_k & 0 & 0 & im_k \end{pmatrix}, \quad m_k := e^{i\frac{\Delta_{k,h_1}}{2}}. \quad (\text{D14})$$

In (D14), translation invariance ensures that $\langle r_{o,k} r_{e,k} \rangle = 0$, and Δ_{k,h_1} is the Bogoliubov angle defined in (D3). The time evolution of $\hat{g}_k(t)$ was computed in [25] for generic homogeneous linear dissipation in the weakly dissipative hydrodynamic limit $t \rightarrow \infty$, $\gamma \rightarrow 0$ with constant γt , where γ is the strength of the dissipation. In particular, for gain and loss dissipation one has

$$\hat{g}_k(t) = (\cos \theta_k \sigma_y^{(k)} - \sin \theta_k \sigma_x^{(k)} e^{2i\varepsilon(k)t} \sigma_y) \lambda_t + (1 - \lambda_t)(1 - 2n_\infty) \cos \Delta_{k,h_1} \sigma_y^{(k)}, \quad (\text{D15})$$

where σ_a , $a = x, y, z$ are the Pauli matrices, and we define $\sigma_a^{(k)} := e^{i\Delta_{k,h_1} \sigma_z/2} \sigma_a e^{-i\Delta_{k,h_1} \sigma_z/2}$, $\lambda_t := e^{-(\gamma_+ + \gamma_-)t}$, $n_\infty := \gamma_+ / (\gamma_+ + \gamma_-)$. Plugging (D15) into (D14), we obtain

$$\tilde{C}_{x,k}(t) = \frac{1}{2} \begin{pmatrix} 1 - A_{k,t} & 0 & 0 & -iB_{k,t} e^{2i\varepsilon(k)t} \\ 0 & 1 + A_{k,t} & -iB_{k,t} e^{-2i\varepsilon(k)t} & 0 \\ 0 & iB_{k,t} e^{2i\varepsilon(k)t} & 1 - A_{k,t} & 0 \\ iB_{k,t} e^{-2i\varepsilon(k)t} & 0 & 0 & 1 - A_{k,t} \end{pmatrix}, \quad (\text{D16})$$

where

$$A_{k,t} := \lambda_t \cos(\Delta_{k,h_1} - \Delta_{k,h_0}) + (1 - \lambda_t)(1 - 2n_\infty) \cos \Delta_{k,h_1} \quad (\text{D17})$$

and

$$B_{k,t} := \lambda_t \sin(\Delta_{k,h_1} - \Delta_{k,h_0}), \quad (\text{D18})$$

which is Eq. (14). It is now straightforward to determine the negativity between the two quasiparticles forming an entangled pair.

2. Quench from the Néel state in the tight-binding chain with gain and loss dissipation

As a second example, we consider a quench in the tight-binding chain with the gain and loss dissipation (D10). The initial state we choose is the fermionic Néel state

$$|N\rangle = \prod_{j=1}^{L/2} c_{2j-1}^\dagger |0\rangle. \quad (\text{D19})$$

This setting is particularly interesting because the exact expression for the negativity in the weakly dissipative hydrodynamic limit is already known, as was calculated in [8]. We now show that our quasiparticle framework reproduces correctly and in a simpler way this result, whose *ab initio* analytical derivation is long and involved. The tight-binding chain is obtained as the limiting case $h = \delta = 0$ of the Kitaev

chain (D1). The tight-binding Hamiltonian thus reads

$$H = -\frac{1}{2} \sum_{j=1}^L (c_j^\dagger c_{j+1} + c_{j+1}^\dagger c_j), \quad (\text{D20})$$

and we assume periodic boundary conditions. Now the initial state has a weaker translational symmetry than the Kitaev chain, as it presents only a two-site translation invariance. In principle, as suggested by Eq. (D12), one should thus expect a 4×4 correlation matrix $\hat{g}_k$ encoding the correlations between the Fourier modes $c_k^\dagger$, $c_{-k}^\dagger$, $c_{k-\pi}^\dagger$, $c_{-k+\pi}^\dagger$, with $k \in [0, \pi/2]$. However, the simple structure of the tight-binding chain and of the Néel state ensures that some of these correlations are zero. This allows us to describe the dynamics of the negativity in terms of pairs. Indeed, the tight-binding chain can be diagonalized in terms of the Fourier modes, in the thermodynamic limit $L \rightarrow \infty$, as

$$H = \int_0^\pi \varepsilon(k) (c_k^\dagger c_k - c_{k-\pi}^\dagger c_{k-\pi}), \quad (\text{D21})$$

with $\varepsilon(k) = -\cos k$. In (D21) $c_k^\dagger$, $c_{k-\pi}^\dagger$ are the two species of entangled quasiparticles. To be precise, we define $\eta_1^\dagger(k) = c_k^\dagger$, $\eta_2^\dagger(k) = c_{k-\pi}^\dagger$. The modes propagate with opposite velocities $v_1(k) = -v_2(k) = \varepsilon'(k) = \sin k$. Notice that $v_j(k) = \varepsilon'_j(k)$, with $\varepsilon_j(k)$ the dispersions of the eigenmodes, unlike the case of the Kitaev chain, in which $v_2(k) = -\varepsilon'_2(k)$, due to the different folding of the Brillouin zone. The correlation function

in quasimomentum space is block-diagonal and reads

$$C_{x,k}(t) = \frac{1}{2} \begin{pmatrix} 1 & 0 & -e^{2i\varepsilon(k)t} & 0 \\ 0 & 1 & 0 & e^{-2i\varepsilon(k)t} \\ -e^{-2i\varepsilon(k)t} & 0 & 1 & 0 \\ 0 & e^{2i\varepsilon(k)t} & 0 & 1 \end{pmatrix}. \quad (\text{D22})$$

In the presence of gain and loss dissipation, the block structure of (D22) remains the same. Indeed, for gain and loss dissipation the Lindblad equation (B2) can be rewritten, for $L \rightarrow \infty$, as

$$\begin{aligned} \hat{\mathcal{L}}[O] = & i[H, O] + \int_{-\pi}^{\pi} \frac{dk}{2\pi} \left[\gamma_+ \left(c_k O c_k^\dagger - \frac{1}{2} \{c_k c_k^\dagger, O\} \right) \right. \\ & \left. + \gamma_- \left(c_k^\dagger O c_k - \frac{1}{2} \{c_k^\dagger c_k, O\} \right) \right]. \end{aligned} \quad (\text{D23})$$

It is easy to derive the time evolution of the two-point correlators as

$$\begin{aligned} \frac{d}{dt} \langle c_k^\dagger c_h \rangle &= \{i[\varepsilon(k) - \varepsilon(h)] - (\gamma_+ + \gamma_-)\} \langle c_k^\dagger c_h \rangle + \gamma_+ \delta_{k,h}, \\ \frac{d}{dt} \langle c_k^\dagger c_h^\dagger \rangle &= \{i[\varepsilon(k) + \varepsilon(h)] - (\gamma_+ + \gamma_-)\} \langle c_k^\dagger c_h^\dagger \rangle. \end{aligned} \quad (\text{D24})$$

From (D24), it is apparent that the two bilinears $c_k^\dagger c_k$ and $c_{k-\pi}^\dagger c_{k-\pi}$ are eigenfunctions of the Liouvillian in the sense of (B11). This also means that the propagating modes remain the same as in the unitary case. In contrast with the case of systems with one-site translation invariance (see Appendix D1), the eigenmodes are not of the form $c_k^\dagger, c_{-k}^\dagger$. However, comparing with the unitary case, it is apparent that the other bilinear we have to consider to extract the dispersion is $[c_k^\dagger, c_{k-\pi}]$. The velocities of the modes η_1, η_2 are readily obtained from the dynamics of $[c_k^\dagger, c_{k-\pi}]$. Alternatively, considering $[c_k, c_{-k}]$ and $[c_{k-\pi}, c_{\pi-k}]$ would work as well, since, as suggested by the symmetry, this quench actually produces *quadruplets* that are split into two superposed and uncorrelated pairs each. One obtains that the velocities are the same as in the unitary

case, i.e., $v_1(k) = -v_2(k) = \varepsilon'(k)$. The time-evolved correlation matrix reads

$$\begin{aligned} \tilde{C}_{x,k}(t) := \langle \eta_i^\dagger \eta_j \rangle &= C_{x,k}(t) e^{-(\gamma_+ + \gamma_-)t} + (1 - e^{-(\gamma_+ + \gamma_-)t}) \\ &\times \begin{pmatrix} n_\infty & 0 & 0 & 0 \\ 0 & 1 - n_\infty & 0 & 0 \\ 0 & 0 & n_\infty & 0 \\ 0 & 0 & 0 & 1 - n_\infty \end{pmatrix}, \end{aligned} \quad (\text{D25})$$

where $C_{x,k}(t)$ is the correlation matrix in the case without dissipation [cf. Eq. (D22)], and $n_\infty := \gamma_+ / (\gamma_+ + \gamma_-)$ as above. After removing the phases $e^{\pm 2i\varepsilon(k)t}$ from $\tilde{C}_{x,k}(t)$ in (D25) and passing to $\tilde{C}_{x,k,t}(t)$, we use Eq. (11), and we obtain the negativity of a pair of shared quasiparticles as

$$\begin{aligned} e_2(k, t) = & \ln(\lambda_t + 2g_\times(1 - \lambda_t^2)) \\ & + \sqrt{(1 - \lambda_t)^2[(1 - \lambda_t)g_+ + \lambda_t]^2 + \lambda_t^2}, \end{aligned} \quad (\text{D26})$$

where we define $g_\times := (\gamma^+ \gamma^-) / [(\gamma^+ + \gamma^-)^2]$, $g_+ := [(\gamma^+)^2 + (\gamma^-)^2] / [(\gamma^+ + \gamma^-)^2]$, and $\lambda_t := e^{-(\gamma_+ + \gamma_-)t}$ as above. It is straightforward to check that Eq. (D26) coincides with the result of the *ab initio* derivation in [8]. Notice that the analytical derivation required very long and involved calculations, therefore our method represents a remarkable simplification.

3. Quench from the dimer state in the tight-binding chain with gain and loss dissipation

Let us consider a different initial state in the tight-binding chain (D20) with the gain and loss dissipation (D10), specifically, the dimer state

$$|D\rangle = \prod_{j=1}^{L/2} \frac{1}{\sqrt{2}} (c_{2j-1}^\dagger + c_{2j}^\dagger) |0\rangle. \quad (\text{D27})$$

It is easy to verify that the eigenmodes are the same as for the Néel state, and in the unitary case the correlation matrix is given as

$$C_{x,k}(t) = \frac{1}{2} \begin{pmatrix} 1 + \cos k & 0 & i \sin k e^{2i\varepsilon(k)t} & 0 \\ 0 & 1 - \cos k & 0 & i \sin k e^{-2i\varepsilon(k)t} \\ -i \sin k e^{-2i\varepsilon(k)t} & 0 & 1 - \cos k & 0 \\ 0 & -i \sin k e^{2i\varepsilon(k)t} & 0 & 1 + \cos k \end{pmatrix}. \quad (\text{D28})$$

Similarly to the Néel case, the velocities are the same as in the unitary case and are left unmodified by the gain and loss dissipation (D10). In the dissipative case, we obtain

$$\tilde{C}_{x,k}(t) = C_{x,k}(t) e^{-(\gamma_+ + \gamma_-)t} + (1 - e^{-(\gamma_+ + \gamma_-)t}) \begin{pmatrix} n_\infty & 0 & 0 & 0 \\ 0 & 1 - n_\infty & 0 & 0 \\ 0 & 0 & n_\infty & 0 \\ 0 & 0 & 0 & 1 - n_\infty \end{pmatrix}, \quad (\text{D29})$$

where $C_{x,k}(t)$ is the correlator in the unitary case, Eq. (D28). We remove the phases associated with the unitary part of the dynamics and pass to $\tilde{C}_{x,k,t}(t)$, and then we apply Eq. (6) in the case of entangled pairs. We finally obtain

$$e_2(k, t) = \ln(\lambda_t + 2g_\times(1 - \lambda_t^2) + \sqrt{(1 - \lambda_t)^2[(1 - \lambda_t)g_+ + \lambda_t]^2 + \lambda_t^2 \sin^2 k}), \quad (\text{D30})$$

where $g_\times$ and g_+ are the same as in (D26). Equation (D30) is the same formula as discussed in the main text.

4. Quench of the mass in the harmonic chain with gain and loss dissipation

As a case study of bosonic systems, we consider a mass quench in the harmonic chain

$$H_h = \frac{1}{2} \sum_{j=1}^L \left(\frac{p_j^2}{m} + mw^2 x_j^2 + \mathcal{K}(x_{j+1} - x_j)^2 \right) \quad (\text{D31})$$

with periodic boundary conditions. The chain is prepared in the ground state of (D31) for $m = m_0$. At $t = 0$, the mass is suddenly changed to the value $m = m_1$. The local canonical operators x_j, p_j play the same role as the Majorana operators for fermionic systems in (B4), by identifying $r_{2j-1} = x_j$, $r_{2j} = p_j$. It is useful to summarize their canonical commutation relations as

$$[r_m, r_n] = i\Omega_{m,n}, \quad \text{with} \quad [\Omega_{i,j}] = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \delta_{i,j} = i\sigma_y \delta_{i,j}. \quad (\text{D32})$$

Notice that in the bosonic case the commutation relations are preserved by the symplectic transformations S , i.e., real linear transformations such that $S\Omega S^T = \Omega$. The canonical operators can be combined to form the local ladder operators $b_j = (x_j + ip_j)/\sqrt{2}$, $b_j^\dagger = (x_j - ip_j)/\sqrt{2}$, satisfying $[b_i, b_j^\dagger] = \delta_{i,j}$. For a Gaussian state, all the information is contained in the two-point function $\Gamma_{m,n} := \langle \{r_m, r_n\} \rangle$. From Eqs. (B2) and (B4), it follows that Γ satisfies the time-evolution equation (see [24])

$$\Gamma(t) = W(t)\Gamma(0)W(t)^T + 2 \int_0^t du W(t-u)\Omega A \Omega^T W(t-u)^T. \quad (\text{D33})$$

Here $W(t) = e^{\Omega(2h+B)t}$, and A, B, h are defined as in (B4). By exploiting the translational invariance, we can rewrite Eq. (D33) as an evolution equation for the Fourier transform $\hat{g}_k$ of the two-point function, reading

$$\hat{g}_k = 2 \begin{pmatrix} \langle x_k x_{-k} \rangle & \langle x_k p_{-k} \rangle \\ \langle p_k x_{-k} \rangle & \langle p_k p_{-k} \rangle \end{pmatrix} + \sigma_y. \quad (\text{D34})$$

Here x_k and p_k are the Fourier transforms of the canonical operators, defined as $x_k := 1/\sqrt{L} \sum_j e^{ikj} x_j = (b_k + b_{-k}^\dagger)/\sqrt{2}$ and $p_k := 1/\sqrt{L} \sum_j e^{ikj} p_j = -i(b_k - b_{-k}^\dagger)/\sqrt{2}$, where $b_k, b_k^\dagger$ are the Fourier modes associated with the ladder operators. In terms of $\hat{g}_k$, Eq. (D33) becomes [24]

$$\begin{aligned} \hat{g}_k(t) &= \hat{W}_k(t) \hat{g}_k(0) \hat{W}_k^\dagger(t) \\ &+ 2 \int_0^t du \hat{W}_k(t-u) \sigma_y \hat{A}_k \sigma_y \hat{W}_k^\dagger(t-u), \end{aligned} \quad (\text{D35})$$

$$C_{x,k}(t) = \frac{1}{4} \begin{pmatrix} \frac{(z(k)^2 - z_0(k)^2)^2}{z(k)^2 z_0(k)^2} & 0 & 0 & \frac{z(k)^4 - z_0(k)^4}{z(k)^2 z_0(k)^2} e^{2i\omega(k)t} \\ 0 & \frac{(z(k)^2 + z_0(k)^2)^2}{z(k)^2 z_0(k)^2} & \frac{z(k)^4 - z_0(k)^4}{z(k)^2 z_0(k)^2} e^{-2i\omega(k)t} & 0 \\ 0 & \frac{z(k)^4 - z_0(k)^4}{z(k)^2 z_0(k)^2} e^{2i\omega(k)t} & \frac{(z(k)^2 - z_0(k)^2)^2}{z(k)^2 z_0(k)^2} & 0 \\ \frac{z(k)^4 - z_0(k)^4}{z(k)^2 z_0(k)^2} e^{-2i\omega(k)t} & 0 & 0 & \frac{(z(k)^2 + z_0(k)^2)^2}{z(k)^2 z_0(k)^2} \end{pmatrix}. \quad (\text{D43})$$

We then add a gain and loss dissipation. It is defined as in the fermionic case by two families of Lindblad operators

where $\hat{W}_k(t) = e^{i\sigma_y(2\hat{h}_k + \hat{B}_k)t}$. In the thermodynamic limit $L \rightarrow \infty$, the Fourier transform of the Hamiltonian reads

$$\hat{h}_k = \begin{pmatrix} 2a(k) & 0 \\ 0 & 2b \end{pmatrix}, \quad (\text{D36})$$

where we defined $a(k) := m_1 w^2 / 4 + \mathcal{K} \sin^2(k/2)$ and $b := 1/(4m_1)$. The harmonic chain (D31) can be diagonalized, in the thermodynamic limit $L \rightarrow \infty$, as

$$\begin{aligned} H_h &= \int_{-\pi}^{\pi} \frac{dk}{2\pi} \omega(k) \left(\eta_k^\dagger \eta_k + \frac{1}{2} \right) \\ &= \int_0^{\pi} \frac{dk}{2\pi} \omega(k) (\eta_k^\dagger \eta_k + \eta_{-k}^\dagger \eta_{-k} + 1), \end{aligned} \quad (\text{D37})$$

with

$$\omega(k) := \sqrt{w^2 + \frac{4\mathcal{K}}{m} \sin^2(k/2)}. \quad (\text{D38})$$

This can be done by means of a Fourier transform followed by the transformation

$$\begin{pmatrix} \eta_k \\ \eta_{-k}^\dagger \end{pmatrix} = \frac{1}{2} \begin{pmatrix} z_m(k) + \frac{1}{z_m(k)} & z_m(k) - \frac{1}{z_m(k)} \\ z_m(k) - \frac{1}{z_m(k)} & z_m(k) + \frac{1}{z_m(k)} \end{pmatrix} \begin{pmatrix} b_k \\ b_{-k}^\dagger \end{pmatrix}, \quad (\text{D39})$$

where

$$z_m(k) := \sqrt{m^2 w^2 + 4m\mathcal{K} \sin^2(k/2)}, \quad (\text{D40})$$

that can be seen as a bosonic version of the Bogoliubov transformation (D2). The modes η_k satisfy the same commutation relations as the b_k . Indeed, the latter can be summarized as

$$\left[\begin{pmatrix} b_k \\ b_{-k}^\dagger \end{pmatrix}, (b_{-h} \quad b_h^\dagger) \right] = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \delta_{k,h} = i\sigma_y \delta_{k,h}, \quad (\text{D41})$$

thus they are preserved by the transformation (D39), which is symplectic and even in k . Our initial state is the ground state $|\Omega\rangle$ of H_h with $m = m_0$. Let us call $\eta_k^{(0)}$ the diagonal modes of H_h for $m = m_0$. They annihilate the initial state. Equation (D39) ensures that the postquench modes η_k satisfy

$$\begin{pmatrix} \eta_k \\ \eta_{-k}^\dagger \end{pmatrix} = \frac{1}{2} \begin{pmatrix} \frac{z(k)^2 + z_0(k)^2}{z(k)z_0(k)} & \frac{z(k)^2 - z_0(k)^2}{z(k)z_0(k)} \\ \frac{z(k)^2 - z_0(k)^2}{z(k)z_0(k)} & \frac{z(k)^2 + z_0(k)^2}{z(k)z_0(k)} \end{pmatrix} \begin{pmatrix} \eta_k^{(0)} \\ \eta_{-k}^{(0)\dagger} \end{pmatrix}, \quad (\text{D42})$$

where we call $z_0(k) := z_{m_0}(k)$ and $z(k) := z_{m_1}(k)$ [see (D39)]. Now we can identify the two families of entangled quasiparticles as $\eta_1^\dagger(k) = \eta_k^\dagger$, $\eta_2^\dagger(k) = \eta_{-k}^\dagger$. In terms of these quasiparticles the two-point function is block-diagonal in the quasimomentum space, as for the fermionic models. The associated velocities are $v_1(k) = -v_2(k) = \omega'(k)$. In the unitary case, we easily find

$L_{+,j} = \sqrt{\gamma_+} b_j^\dagger$, $L_{-,j} = \sqrt{\gamma_-} b_j$ in Eq. (B1), where $\gamma_\pm$ are the gain and loss rates. The different definition of the r_j in the

bosonic case yields Fourier transforms of A and B [cf. (B4)] that are slightly different from the fermionic case, reading

$$\hat{A}_k = \frac{\gamma_+ + \gamma_-}{2} \mathbb{1}_2, \quad \hat{B}_k = i \frac{\gamma_- - \gamma_+}{2} \sigma_y. \quad (\text{D44})$$

As in the fermionic case, the block structure in (D43) is preserved when a linear, homogeneous dissipation is added. Indeed, the translation invariance ensures that the only nonzero correlations are those contained in (D34). Thus, dissipation can at most mix the propagating modes $\eta_1(k)$, $\eta_2^\dagger(k)$. This, however, does not happen for the gain and loss dissipation, which preserves the eigenmodes of the Hamiltonian as propagating modes, since $\hat{B}_k \propto \sigma_y$, and the propagation velocities (see the discussion in Appendix B above). One can obtain the time-dependent correlator $\tilde{C}_{x,k}(t)$ by solving the time-evolution Eq. (D35) for the Fourier transform (D34) and then changing basis as

$$\tilde{C}_{x,k}(t) = \frac{1}{2} M(k) \begin{pmatrix} \hat{g}_k(t) - \sigma_y & 0_{2 \times 2} \\ 0_{2 \times 2} & \hat{g}_{-k}(t) - \sigma_y \end{pmatrix} M(k)^\dagger, \quad (\text{D45})$$

with

$$M(k) := \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & z(k) & -\frac{i}{z(k)} \\ z(k) & \frac{i}{z(k)} & 0 & 0 \\ z(k) & -\frac{i}{z(k)} & 0 & 0 \\ 0 & 0 & z(k) & \frac{i}{z(k)} \end{pmatrix}. \quad (\text{D46})$$

In Eq. (D45) the translation invariance ensures that $\langle x_k x_k \rangle = \langle p_k p_k \rangle = \langle x_k p_k \rangle = 0$, and $z(k)$ is defined in (D42). At $t = 0$ the Fourier transform of the correlator is given as

$$\begin{aligned} & \frac{1}{2} \begin{pmatrix} \hat{g}_k(0) - \sigma_y & 0_{2 \times 2} \\ 0_{2 \times 2} & \hat{g}_{-k}(0) - \sigma_y \end{pmatrix} \\ &= M_0^{-1}(k) \frac{1}{2} \begin{pmatrix} \mathbb{1}_{2 \times 2} - \sigma_z & 0_{2 \times 2} \\ 0_{2 \times 2} & \mathbb{1}_{2 \times 2} - \sigma_z \end{pmatrix} (M_0^\dagger(k))^{-1}, \end{aligned} \quad (\text{D47})$$

where $M_0(k)$ is the same matrix as $M(k)$ with $z_0(k)$ instead of $z(k)$, and we have used that the $\eta_j^{(0)}$ annihilate the initial state. By using Eq. (D35) with the initial condition (D47), we obtain

$$\begin{aligned} \hat{g}_k(t) = & \begin{pmatrix} \left(\frac{\delta_t}{z_0^2} + \frac{1-\delta_t}{g} \right) \cos^2(\omega t) + \frac{1}{z^4} \left(z_0^2 \delta_t + \frac{1-\delta_t}{g} \right) \sin^2(\omega t) & \frac{2}{\omega} \left(\frac{b-a}{g} + \delta_t \left(b z_0^2 + \frac{a-b}{g} - \frac{a}{z_0^2} \right) \right) \sin(2\omega t) \\ \frac{2}{\omega} \left(\frac{b-a}{g} + \delta_t \left(b z_0^2 + \frac{a-b}{g} - \frac{a}{z_0^2} \right) \right) \sin(2\omega t) & \left(z_0^2 \delta_t + \frac{1-\delta_t}{g} \right) \cos^2(\omega t) + z^4 \left(\frac{\delta_t}{z_0^2} + \frac{1-\delta_t}{g} \right) \sin^2(\omega t) \end{pmatrix} \delta_t \\ & + \begin{pmatrix} -\frac{z^4+1}{2gz^4} + \frac{2\omega^2(z^4+1)/z^4+g^2}{g(4\omega^2+g^2)\delta_t} + \frac{(z^4-1)(2\omega \sin(2\omega t)-g_- \cos(2\omega t))}{2z^4(4\omega^2+g_-^2)} & \frac{(z^4-1)(2\omega \cos(2\omega t)+g_- \sin(2\omega t)-2\omega \delta_{-t})}{2z^2(4\omega^2+g_-^2)} \\ \frac{(z^4-1)(2\omega \cos(2\omega t)+g_- \sin(2\omega t)-2\omega \delta_{-t})}{2z^2(4\omega^2+g_-^2)} & -\frac{z^4+1}{2g} + \frac{2\omega^2(z^4+1)+g_-^2}{g(4\omega^2+g_-^2)\delta_t} + \frac{(z^4-1)(g_- \cos(2\omega t)-2\omega \sin(2\omega t))}{2(4\omega^2+g_-^2)} \end{pmatrix} \delta_t. \end{aligned} \quad (\text{D48})$$

Here we define $g_- := \gamma_- - \gamma_+$, $g := (\gamma_- - \gamma_+)/(\gamma_- + \gamma_+)$, $\delta_t := e^{-g_- t}$, and $\omega(k) = 4\sqrt{a(k)b}$ is defined in (D37). We also omit the dependence on k of z , z_0 , ω , a for the sake of brevity, and we use the fact that $z(k) = \sqrt[4]{a(k)/b}$ [see (D39)]. We are interested in the weakly dissipative hydrodynamic limit $\gamma_+, \gamma_- \rightarrow 0$, $t \rightarrow \infty$ with fixed $\gamma_+ t$, $\gamma_- t$. In this limit, we can neglect some terms in (D48), and the Fourier-transformed correlator $\hat{g}_k(t)$ becomes

$$\begin{aligned} \hat{g}_k(t) = & \begin{pmatrix} \left(\frac{\delta_t}{z_0^2} + \frac{1-\delta_t}{g} \right) \cos^2(\omega t) + \frac{1}{z^4} \left(z_0^2 \delta_t + \frac{1-\delta_t}{g} \right) \sin^2(\omega t) & \frac{2}{\omega} \left(\frac{b-a}{g} + \delta_t \left(b z_0^2 + \frac{a-b}{g} - \frac{a}{z_0^2} \right) \right) \sin(2\omega t) \\ \frac{2}{\omega} \left(\frac{b-a}{g} + \delta_t \left(b z_0^2 + \frac{a-b}{g} - \frac{a}{z_0^2} \right) \right) \sin(2\omega t) & \left(z_0^2 \delta_t + \frac{1-\delta_t}{g} \right) \cos^2(\omega t) + z^4 \left(\frac{\delta_t}{z_0^2} + \frac{1-\delta_t}{g} \right) \sin^2(\omega t) \end{pmatrix} \delta_t \\ & + \begin{pmatrix} \frac{z^4+1}{2gz^4} (1 - \delta_t) + \frac{(z^4-1) \sin(2\omega t) \delta_t}{4\omega z^4} & \frac{(z^4-1) (\cos(2\omega t) \delta_t - 1)}{4z^2 \omega} \\ \frac{(z^4-1) (\cos(2\omega t) \delta_t - 1)}{4z^2 \omega} & \frac{z^4+1}{2g} (1 - \delta_t) - \frac{(z^4-1) \sin(2\omega t) \delta_t}{4\omega} \end{pmatrix}. \end{aligned} \quad (\text{D49})$$

Plugging Eq. (D49) into (D45), we finally obtain

$$\tilde{C}_{x,k}(t) = C_{x,k}(t) \delta_t - (1 - \delta_t) \times \begin{pmatrix} \frac{(z^4+1)\delta_t}{4gz^2} + \frac{1}{2} & 0 & 0 & 0 \\ 0 & \frac{(z^4+1)\delta_t}{4gz^2} - \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{(z^4+1)\delta_t}{4gz^2} + \frac{1}{2} & 0 \\ 0 & 0 & 0 & \frac{(z^4+1)\delta_t}{4gz^2} - \frac{1}{2} \end{pmatrix}, \quad (\text{D50})$$

where $C_{x,k}(t)$ is the time-dependent correlator in the absence of dissipation [cf. Eq. (D43)]. To compute the negativity carried by a shared pair, we first remove the oscillating phases in $\tilde{C}_{x,k}(t)$ and pass to $\tilde{C}_{x,k,t}(t)$. Next, we change basis to obtain the position-momentum correlator $\Gamma(x, k, t)_{m,n} := \langle \{r_m^{(k)}, r_n^{(k)}\} \rangle$, where $r_{2j-1}^{(k)} := x_j^{(k)} := (\eta_j(k) + \eta_j^\dagger(k))/\sqrt{2}$, $r_{2j}^{(k)} := p_j^{(k)} :=$

$-i(\eta_j(k) - \eta_j^\dagger(k))/\sqrt{2}$, as explained in Appendix C. After having determined $\Gamma(x, k, t)$, we perform the partial transposition with respect to one of the two particles. For a system of two bosons the partial transposition of the density matrix is identified by the correlation matrix $J_{A_2} \Gamma J_{A_2}$ (see, for instance, Ref. [31]), where for a shared pair J_{A_2} is the matrix

$J_{A_2} = \mathbb{1}_2 \oplus \sigma_z$ [cf. Eq. (26)]. Thus, from (D50), we obtain

$$J_{A_2} \Gamma(x, k, t) J_{A_2} = \begin{pmatrix} \frac{z^4 + z_0^4}{2z^2 z_0^2} - \frac{(z^4 + 1)(1 - \delta_t)}{2gz^2} & 0 & \frac{z^4 - z_0^4}{2z^2 z_0^2} & 0 \\ 0 & \frac{z^4 + z_0^4}{2z^2 z_0^2} - \frac{(z^4 + 1)(1 - \delta_t)}{2gz^2} & 0 & \frac{z^4 - z_0^4}{2z^2 z_0^2} \\ \frac{z^4 - z_0^4}{2z^2 z_0^2} & 0 & \frac{z^4 + z_0^4}{2z^2 z_0^2} - \frac{(z^4 + 1)(1 - \delta_t)}{2gz^2} & 0 \\ 0 & \frac{z^4 - z_0^4}{2z^2 z_0^2} & 0 & \frac{z^4 + z_0^4}{2z^2 z_0^2} - \frac{(z^4 + 1)(1 - \delta_t)}{2gz^2} \end{pmatrix} \delta_t. \quad (\text{D51})$$

The logarithmic negativity is obtained from the symplectic eigenvalues of $J_{A_2} \Gamma(x, k, t) J_{A_2}$, i.e., the two positive eigenvalues

$$\lambda_1(k, t) = \frac{z_0^2}{z^2} \delta_t + \frac{z^4 + 1}{2gz^2} (1 - \delta_t), \quad (\text{D52})$$

$$\lambda_2(k, t) = \frac{z^2}{z_0^2} \delta_t + \frac{z^4 + 1}{2gz^2} (1 - \delta_t) \quad (\text{D53})$$

of $i(i\sigma_y \oplus i\sigma_y)J\Gamma(x, k, t)J$, with the other two eigenvalues that are the opposite of (D52) and (D53). Precisely, the negativity shared between the two quasiparticles of a pair is given as [31] $e_2(k, t) = -\ln[\min(1, \lambda_1(k, t))] - \ln[\min(1, \lambda_2(k, t))]$, which is exactly Eq. (16).

5. Sudden death of the negativity in the harmonic chain with gain and loss dissipation

Here we discuss the sudden death of negativity for arbitrary mass quenches in the harmonic chain in the presence of gain and loss dissipation. Let us first observe that the two eigenvalues of $i(i\sigma_y)^\oplus J_{A_2} \Gamma(x, k, t) J_{A_2}$ in (D52) and (D53) can become *both* larger than one at long times. This is in contrast with the case of unitary dynamics, i.e., without dissipation [11]. This means that the negativity $e_2(k)$ of a pair with quasimomentum k can vanish at finite time, and the so-called sudden death of the negativity can occur. Interestingly, within the hydrodynamic framework, the slow entangled pairs with $k \rightarrow 0$ play a crucial role to determine whether the sudden death can happen. Indeed, any pair with finite k can entangle two finite intervals only for a finite time. At times larger than $(2\ell + d)/(2v_1(k))$, with ℓ the subsystems' size, and d their

distance, no pair of quasiparticles with quasimomentum k can be shared between the two intervals. Thus, the only entangled pairs that can entangle the two subsystems at arbitrarily long times are the pairs with $k \rightarrow 0$. This means that if the negativity of the pairs with small quasimomentum $k \rightarrow 0$ vanishes at finite time, the sudden death of the negativity occurs. Notice that for times larger than the sudden death time, the two subsystems may still be entangled, although their entanglement is the so-called bound entanglement [35], which cannot be revealed by the negativity.

From (D52) and (D53), we obtain that the condition for the sudden death to take place if $\lambda_1 > \lambda_2$ (or, equivalently, $m_0 > m_1$) is

$$|g_-|t \geq \max_{k \in [0, \epsilon]} \left| \ln \left(\frac{2gz^4 - z_0^2 z^4 - z_0^2}{2gz^2 z_0^2 - z_0^2 z^4 - z_0^2} \right) \right|, \quad (\text{D54})$$

where $\epsilon \rightarrow 0$, and we put $z_0 = z_{m_0}$, $z = z_{m_1}$, with z_m defined in (D39). Conversely, for $\lambda_1 < \lambda_2$, one has

$$|g_-|t \geq \max_{k \in [0, \epsilon]} \left| \ln \left(\frac{2gz^2 - z^4 - 1}{2gz_0^2 - z^4 - 1} \right) \right|. \quad (\text{D55})$$

The suitable one (depending on the order between m_0 and m_1) among Eqs. (D54) and (D55) is always true for finite times, except when the logarithm diverges for $k = 0$. In that case, the sudden death does not occur. Notice that the system can escape the sudden death only for $m_1 w = z(0)^2 = g = 1$, which means $\gamma^+ = 0$. The case $g = 0$, i.e., $\gamma_+ = \gamma_-$, is not included in the conditions (D54) and (D55). However, it is straightforward to check in (D52) and (D53) that both λ_1 and λ_2 become greater than one at finite time for any k , implying the presence of the sudden death in this case as well.

- [1] M. Fagotti and P. Calabrese, Evolution of entanglement entropy following a quantum quench: Analytic results for the XY chain in a transverse magnetic field, *Phys. Rev. A* **78**, 010306(R) (2008).
- [2] V. Alba and P. Calabrese, Entanglement and thermodynamics after a quantum quench in integrable systems, *Proc. Natl. Acad. Sci. USA* **114**, 7947 (2017).
- [3] V. Alba and P. Calabrese, Entanglement dynamics after quantum quenches in generic integrable systems, *SciPost Phys.* **4**, 017 (2018).
- [4] G. Vidal and R. F. Werner, Computable measure of entanglement, *Phys. Rev. A* **65**, 032314 (2002).
- [5] M. B. Plenio, Logarithmic negativity: A full entanglement monotone that is not convex, *Phys. Rev. Lett.* **95**, 090503 (2005).
- [6] P. Calabrese, J. Cardy, and E. Tonni, Entanglement negativity in quantum field theory, *Phys. Rev. Lett.* **109**, 130502 (2012).
- [7] H. Shapourian and S. Ryu, Entanglement negativity of fermions: Monotonicity, separability criterion, and classification of few-mode states, *Phys. Rev. A* **99**, 022310 (2019).
- [8] V. Alba and F. Carollo, Logarithmic negativity in out-of-equilibrium open free-fermion chains: An exactly solvable case, *SciPost Phys.* **15**, 124 (2023).
- [9] S. Murciano, P. Calabrese, and V. Alba, Symmetry-resolved entanglement in fermionic systems with dissipation, *J. Stat. Mech.* (2023) 113102.
- [10] F. Caceffo and V. Alba, Entanglement negativity in a fermionic chain with dissipative defects: Exact results, *J. Stat. Mech.* (2023) 023102.

- [11] V. Alba and P. Calabrese, Quantum information dynamics in multipartite integrable systems, *Europhys. Lett.* **126**, 60001 (2019).
- [12] S. Murciano, V. Alba, and P. Calabrese, Quench dynamics of Rényi negativities and the quasiparticle picture, in *Entanglement in Spin Chains*, Quantum Science and Technology edited by A. Bayat, S. Bose, and H. Johannesson (Springer, Cham, 2022).
- [13] J. Kudler-Flam, Y. Kusuki, and S. Ryu, Correlation measures and the entanglement wedge cross-section after quantum quenches in two-dimensional conformal field theories, *J. High Energy Phys.* **04** (2020) 074.
- [14] P. Calabrese and J. Cardy, Evolution of entanglement entropy in one-dimensional systems, *J. Stat. Mech.* (2005) P04010.
- [15] P. Calabrese, Entanglement and thermodynamics in non-equilibrium isolated quantum systems, *Physica A* **504**, 31 (2018).
- [16] V. Alba, B. Bertini, M. Fagotti, L. Piroli, and P. Ruggiero, Generalized-hydrodynamic approach to inhomogeneous quenches: Correlations, entanglement and quantum effects, *J. Stat. Mech.* (2021) 114004.
- [17] K. Klobas, B. Bertini, and L. Piroli, Exact thermalization dynamics in the “Rule 54” Quantum cellular automaton, *Phys. Rev. Lett.* **126**, 160602 (2021).
- [18] B. Bertini, K. Klobas, V. Alba, G. Lagnese, and P. Calabrese, Growth of Rényi entropies in interacting integrable models and the breakdown of the quasiparticle picture, *Phys. Rev. X* **12**, 031016 (2022).
- [19] H. P. Breuer and F. Petruccione, *The Theory of Open Quantum Systems* (Oxford University Press, Oxford, 2002).
- [20] P. Calabrese, F. H. L. Essler, and M. Fagotti, Quantum quench in the transverse-field Ising chain, *Phys. Rev. Lett.* **106**, 227203 (2011).
- [21] T. Prosen and E. Ilievski, Nonequilibrium phase transition in a periodically driven XY spin chain, *Phys. Rev. Lett.* **107**, 060403 (2011).
- [22] T. Prosen, Third quantization: A general method to solve master equations for quadratic open Fermi systems, *New J. Phys.* **10**, 043026 (2008).
- [23] V. Alba and F. Carollo, Spreading of correlations in Markovian open quantum systems, *Phys. Rev. B* **103**, L020302 (2021).
- [24] F. Carollo and V. Alba, Dissipative quasiparticle picture for quadratic Markovian open quantum systems, *Phys. Rev. B* **105**, 144305 (2022).
- [25] V. Alba and F. Carollo, Hydrodynamics of quantum entropies in Ising chains with linear dissipation, *J. Phys. A: Math. Theor.* **55**, 074002 (2022).
- [26] V. Alba and F. Carollo, Noninteracting fermionic systems with localized losses: Exact results in the hydrodynamic limit, *Phys. Rev. B* **105**, 054303 (2022).
- [27] V. Alba, Unbounded entanglement production via a dissipative impurity, *SciPost Phys.* **12**, 011 (2022).
- [28] E. Starchl and L. M. Sieberer, Relaxation to a parity-time symmetric generalized Gibbs ensemble after a quantum quench in a driven-dissipative Kitaev chain, *Phys. Rev. Lett.* **129**, 220602 (2022).
- [29] B. Bertini, M. Fagotti, L. Piroli, and P. Calabrese, Entanglement evolution and generalised hydrodynamics: Noninteracting systems, *J. Phys. A: Math. Theor.* **51**, 39LT01 (2018).
- [30] A. Bastianello and P. Calabrese, Spreading of entanglement and correlations after a quench with intertwined quasiparticles, *SciPost Phys.* **5**, 033 (2018).
- [31] K. Audenaert, J. Eisert, M. B. Plenio, and R. F. Werner, Entanglement properties of the harmonic chain, *Phys. Rev. A* **66**, 042327 (2002).
- [32] G. Lindblad, On the generators of quantum dynamical semi-groups, *Commun. Math. Phys.* **48**, 119 (1976).
- [33] J. Eisert, V. Eisler, and Z. Zimboras, Entanglement negativity bounds for fermionic Gaussian states, *Phys. Rev. B* **97**, 165123 (2018).
- [34] T. Yu and J. H. Eberly, Sudden death of entanglement, *Science* **323**, 598 (2009).
- [35] M. Horodecki, P. Horodecki, and R. Horodecki, Mixed-state entanglement and distillation: Is there a “bound” entanglement in nature? *Phys. Rev. Lett.* **80**, 5239 (1998).
- [36] V. Alba, Entanglement and quantum transport in integrable systems, *Phys. Rev. B* **97**, 245135 (2018).
- [37] V. Alba, B. Bertini, and M. Fagotti, Entanglement evolution and generalised hydrodynamics: Interacting integrable systems, *SciPost Phys.* **7**, 005 (2019).
- [38] V. Alba, Towards a generalized hydrodynamics description of Rényi entropies in integrable systems, *Phys. Rev. B* **99**, 045150 (2019).
- [39] B. Bertini, M. Collura, J. De Nardis, and M. Fagotti, Transport in out-of-equilibrium XXZ chains: Exact profiles of charges and currents, *Phys. Rev. Lett.* **117**, 207201 (2016).
- [40] O. A. Castro-Alvaredo, B. Doyon, and T. Yoshimura, Emergent hydrodynamics in integrable quantum systems out of equilibrium, *Phys. Rev. X* **6**, 041065 (2016).
- [41] Y. Zhang and T. Barthel, Criticality and phase classification for quadratic open quantum many-body systems, *Phys. Rev. Lett.* **129**, 120401 (2022).
- [42] M. van Caspel, S. E. T. Arze, and I. P. Castillo, Dynamical signatures of topological order in the driven-dissipative Kitaev chain, *SciPost Phys.* **6**, 026 (2019).
- [43] G. Perez and W. Witczak-Krempa, The fate of entanglement, *SciPost Phys.* **20**, 002 (2026).
- [44] F. Ares, S. Murciano, and P. Calabrese, Entanglement asymmetry as a probe of symmetry breaking, *Nat. Commun.* **14**, 2036 (2023).
- [45] F. Caceffo, S. Murciano, and V. Alba, Entangled multiplets, asymmetry, and quantum Mpemba effect in dissipative systems, *J. Stat. Mech.* (2024) 063103.
- [46] V. I. Arnold, *Mathematical Methods of Classical Mechanics* (Springer, New York, 1989).
- [47] I. Peschel and V. Eisler, Reduced density matrices and entanglement entropy in free lattice models, *J. Phys. A: Math. Theor.* **42**, 504003 (2009).