

Role of quantum dynamics in coherent and incoherent radiation during tunneling ionization

Michael Klaiber^{✉,*}, Karen Z. Hatsagortsyan^{✉,†} and Christoph H. Keitel
Max-Planck-Institut für Kernphysik, Saupfercheckweg 1, 69117 Heidelberg, Germany

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Radiation during strong-field tunneling ionization is investigated. The spontaneous as well as the coherent components of the radiation are calculated describing the ionization dynamics via the strong-field approximation, and the role of the quantum dynamics at tunneling is analyzed. The competition between different mechanisms in different spectral regions is examined. Transitionlike radiation (Brunel radiation) is dominant at low frequencies, Thomson scattering at the laser frequency, and radiative recombination via the three-step process at high-order harmonics. To distinguish the role of the quantum dynamics, simple man Drude models are developed for coherent as well as for spontaneous radiation; the models are based on the electron trajectory out of the tunneling barrier. The quantum dynamics is shown to modify the coherent Brunel radiation for near-zero frequencies in asymmetric laser pulses. The significant role of free-free transitions is demonstrated for spontaneous radiation in the low-frequency region.

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I. INTRODUCTION

It is well known that tunneling ionization is accompanied by high-order harmonic generation (HHG) via the three-step model [1], when the recombination of the ionized electron wave packet to the bound state at the recollision after the excursion in the laser field and gaining energy results in the emission of a high-energy photon [2–4]. At the recollision, radiation may emerge also due to electron scattering, rather than recombination, yielding free-free HHG [5]. In the macroscopic picture, the ionization of a gas target results in the creation of an electron current after the interaction with a laser pulse, which can be a source of low-frequency THz radiation, with the emitted frequency determined by the interaction time [6–19]. The same mechanism can work also in the case of a single atom producing THz radiation in asymmetric laser pulses, e.g., in a two-color laser field [20,21]. The origin of this radiation is similar to the transition radiation, when the electron velocity [22] (or dielectric constant of a medium where the electron propagates [23]) abruptly changes. A similar effect exists in plasma physics—Brunel radiation [24,25], when an electron in the plasma is dragged out of the plasma. In the case of tunneling ionization, the electron is initially in a stationary bound state, while abruptly starting to move after tunneling.

In the tunneling regime of strong-field ionization, the bound electron tunnels through the barrier created by the laser

and the Coulomb field of the atomic core. However, this common simple dynamics can continue to evolve in a nontrivial way. The electron traversing the classically forbidden region may reflect from the potential barrier surface, propagate back and scatter off the core, and then finally tunnel out of the barrier. Although this nontrivial path has no large probability, it has consequences, inducing the tunneling time delay when interfering with the direct ionization path [26,27], and it can be responsible for driving Freeman resonances in the strong-field regime [28], which has recently been confirmed experimentally [29]. The question arises whether the signatures of the quantum dynamics at tunneling, in particular, those of the under-the-barrier recollisions, can be observed in the radiation accompanying tunneling ionization.

In contrast to the coherent radiation discussed above, emission of spontaneous low-order harmonic radiation during the bound-continuum transition with the origin similar to Brunel radiation has been recently shown in [30], with the typical frequency determined by the Keldysh tunneling time [31]. Finally, harmonic radiation at tunneling ionization arises also from Thomson scattering of the laser wave off the ionized free electron wave packet, with higher harmonics than the fundamental one suppressed in the nonrelativistic regime.

Coherent radiation should be distinguished from incoherent radiation of the same origin. For instance, the spontaneous recombination of a high-energy electron to an atomic bound state results in a photon emission of a random phase, which is an incoherent process [32]. However, when the recombining electron is released from the bound state and recombines into the same bound state driven by a laser field, the HHG is coherent with the phase determined by the coherent superposition of the ground state wave function with the ionization continuum wave packet. Similarly, the coherent part of radiation due to Compton scattering of a strong laser wave from an ionized electron wave packet [33,34] is phase matched in the forward direction, while the incoherent radiation has a broad angular distribution [35–39]. For the incoherent process, in contrast

*Contact author: klaiber@mpi-hd.mpg.de

†Contact author: k.hatsagortsyan@mpi-hd.mpg.de

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to the coherent one, the final state of the electron does not exist initially. The coherent radiation intensity scales as $\sim \mathcal{N}^2$ with the number of emitters $\mathcal{N}$, due to the phase matching of the radiation, while for the incoherent radiation interference is suppressed and the total intensity scaling is $\sim \mathcal{N}$ [40]. The coherent radiation can be calculated using the quantum mechanical expectation value of the current density within the classical radiation theory [35].

In this paper we investigate the role of the under-the-barrier dynamics for the spontaneous and coherent Brunel radiation during tunneling ionization. The ionization quantum dynamics is described by the strong-field approximation (SFA) which implicitly includes the under-the-barrier path. Additionally, we develop simple man Drude models for the coherent as well as for spontaneous radiation, which include only the electron trajectory out of the tunneling barrier. Comparison of the results of the Drude model with those of SFA allows us to single out the contribution of the under-the-barrier path in the radiation spectra. We analyze how the different mechanisms of the radiation during the strong-field ionization are scaled in different spectral regions of emission and the feasibility of their observation in a coherent or incoherent form.

The harmonics emitted from a gas target are the result of a multiatom contribution, and therefore, are dominated by the coherent radiation. For the observation of the incoherent radiation either a single atom experiment should be carried out (when each time a single atom existed in the interaction region), or the emission should be measured at a stray direction far from the phase-matching one [38].

The structure of the paper is the following. In Sec. II the coherent radiation during tunneling ionization is calculated using the SFA. Spontaneous emission at the tunneling ionization is discussed in Sec. III. Relation of our calculations to an experiment is considered in Sec. IV. We give an order of magnitude estimation of radiation intensities in Sec. V. Our conclusions are given in Sec. VI.

II. COHERENT RADIATION DURING TUNNELING IONIZATION

A. Radiation via SFA current

We calculate coherent radiation emitted during tunneling ionization of a laser-driven hydrogenlike atom in the non-relativistic regime. The dynamics of the tunneling ionization is treated by the SFA, which provides the current density of the ionizing electron. The latter is employed in Maxwell equations to calculate the intensity of coherent radiation.

The strong-field ionization dynamics is described by the Schrödinger equation,

$$i\partial_t |\Psi(t)\rangle = \hat{H} |\Psi(t)\rangle, \quad (1)$$

with the Hamiltonian,

$$\hat{H} = \hat{\mathbf{p}}^2/2 + \hat{V}(r) + \mathbf{r} \cdot \mathbf{E}(t), \quad (2)$$

where $\hat{V}(\mathbf{r}) = 2\pi/\kappa \delta(\mathbf{r})(1 + \partial_r)$ is the regularized atomic zero-range potential, and $\mathbf{E}(t)$ the laser electric field. The interaction with the laser field is described in the length gauge which is the most suitable physically in the SFA [41,42]. Atomic units are used throughout.

We seek the solution for the electron wave function Ψ via the iterative SFA expansion [43],

$$|\Psi(t)\rangle = |\psi^{(0)}(t)\rangle + |\psi^{(1)}(t)\rangle + |\psi^{(2)}(t)\rangle, \quad (3)$$

where the zeroth-order term $|\psi^{(0)}\rangle$ coincides with the unperturbed bound state $|\psi^{(a)}\rangle$ in the atomic potential, with the ionization energy $I_p = 0.5$ a.u. The first-order SFA wave function describes the direct tunneling path,

$$|\psi^{(1)}(t)\rangle = -i \int^t dt' \int d^3\mathbf{p} |\psi_{\mathbf{p}}^{(f)}(t)\rangle \times \langle \psi_{\mathbf{p}}^{(f)}(t') | \mathbf{r} \cdot \mathbf{E}(t') | \psi^{(a)}(t') \rangle, \quad (4)$$

with the continuum Volkov state $|\psi_{\mathbf{p}}^{(f)}(t)\rangle$ in the laser field [44], and in the second-order SFA [43],

$$|\psi^{(2)}(t)\rangle = - \int^t dt'' \int^{t'} dt' \int d^3\mathbf{p} \int d^3\mathbf{q} |\psi_{\mathbf{p}}^{(f)}(t)\rangle \times \langle \psi_{\mathbf{p}}^{(f)}(t'') | V | \psi_{\mathbf{q}}^{(f)}(t'') \rangle \langle \psi_{\mathbf{q}}^{(f)}(t') | \mathbf{r} \cdot \mathbf{E}(t') | \psi^{(a)}(t') \rangle. \quad (5)$$

The states $|\psi^{(1)}\rangle$ and $|\psi^{(2)}\rangle$ describe the electron dynamics in the continuum and at recollisions, including the under-the-barrier dynamics, and the motion in the laser field after leaving the barrier.

From previous studies it is known [35] that the coherent radiation can be calculated interpreting the quantum current density as a classical one within the classical radiation theory. We calculate the quantum current density for the tunneling electron $\mathbf{j}(\mathbf{r}, t) = \frac{1}{2} \{ \Psi(\mathbf{r}, t)^* \hat{\mathbf{p}} \Psi(\mathbf{r}, t) + \text{c.c.} \}$, interpret it as a classical current, and use it to calculate the emitted *coherent* radiation [40] within the standard gauge invariant representation of Liénard-Wiechert radiation theory. From the latter the photon emission probability reads [45]

$$dw = \frac{\alpha}{(2\pi)^2} \frac{d^3\mathbf{k}}{k^3} \left| \int dt \int d^3\mathbf{r} \mathbf{k} \times \mathbf{j}(\mathbf{r}, t) e^{i\omega t - i\mathbf{k} \cdot \mathbf{r}} \right|^2, \quad (6)$$

where α is the fine structure constant; ω and $\mathbf{k}$ are the emitted photon energy and momentum, respectively; and $k = |\mathbf{k}|$. In the nonrelativistic regime the dipole approximation for the emission is relevant, $\mathbf{k} \cdot \mathbf{r} \ll 1$. Then the angular integration of the emitted photon probability can be carried out, yielding

$$\frac{dw}{d\omega} = \frac{2\alpha\omega}{3\pi c^2} \left| \int dt \mathbf{J}(t) \exp(i\omega t) \right|^2, \quad (7)$$

where $\mathbf{J}(t) = \int d^3\mathbf{r} \mathbf{j}(\mathbf{r}, t) = \langle \Psi(t) | \hat{\mathbf{p}} | \Psi(t) \rangle$ is the expectation value of the electron probability current, and c is the speed of light. We can identify different probability currents $\mathbf{J}(t) = \mathbf{J}_{01}(t) + \mathbf{J}_{10}(t) + \mathbf{J}_{11}(t)$, where

$$\begin{aligned} \mathbf{J}_{01}(t) &= \langle \psi^{(0)}(t) | \hat{\mathbf{p}} | \psi^{(1)}(t) \rangle, \\ \mathbf{J}_{10}(t) &= \langle \psi^{(1)}(t) | \hat{\mathbf{p}} | \psi^{(0)}(t) \rangle, \\ \mathbf{J}_{11}(t) &= \langle \psi^{(1)}(t) | \hat{\mathbf{p}} | \psi^{(1)}(t) \rangle. \end{aligned} \quad (8)$$

The radiation via the combination of $\mathbf{J}_{01}$ and $\mathbf{J}_{10}$ emerges due to the recombination to the bound state and, therefore,

describes three-step HHG, including below-threshold harmonics. The radiation via $\mathbf{J}_{11}$ term includes the effect of the electron transition to the continuum and the continuum dynamics, which will be the main point of our study. The second-order terms $\mathbf{J}_{02}$, $\mathbf{J}_{20}$ describe the contribution to the radiation due to the under-the-barrier recollisions, which are usually small in the net radiation, and the observation of its signatures requires more subtle analysis; see Sec. III C.

The expression for the photon emission probability of Eq. (7) can be equivalently derived via the quantum perturbation theory, treating the interaction of the electron with the photon field in the velocity gauge $H_{\text{rad}} = \hat{\mathbf{A}}(t) \cdot \hat{\mathbf{p}}$ by perturbation:

$$dw = |m_{\mathbf{k}}|^2 d^3\mathbf{k},$$

$$m_{\mathbf{k}} = -i \int dt \langle \Psi(t), 1_{\mathbf{k}} | \hat{\mathbf{A}}(t) \cdot \hat{\mathbf{p}} | \Psi(t), 0_{\mathbf{k}} \rangle, \quad (9)$$

where $\hat{\mathbf{A}}(\mathbf{r}, t)$ is the vector potential operator for the emitted photon, and $\hat{\mathbf{E}}(\mathbf{r}, t) = -\partial_t \hat{\mathbf{A}}(\mathbf{r}, t) = \sum_{\mathbf{k}} \sqrt{\omega}/(2\pi) \hat{\mathbf{e}}(a_{\mathbf{k}} e^{i\mathbf{k} \cdot \mathbf{r} - i\omega t} + a_{\mathbf{k}}^\dagger e^{-i\mathbf{k} \cdot \mathbf{r} + i\omega t})$ is the field of the emitted photons in second quantization, with the creation and annihilation operators $a_{\mathbf{k}}^\dagger$ and $a_{\mathbf{k}}$. As the classical radiation theory is gauge invariant, we take its result as a benchmark to choose the gauge of the quantum perturbation theory with respect to the radiation Hamiltonian. Thus we use different gauges in description of different interactions: the length gauge to describe the electron interaction with the laser field, justified by the SFA experience, and the velocity gauge for the perturbative treatment of the interaction with the emitted photon field, justified by the matching with the Liénard-Wiechert theory.

B. Simple man Drude model for Brunel radiation

The radiation via the current $\mathbf{J}_{11}$ in Eq. (7) describes radiation during tunneling ionization due to free-free transitions. This includes, first of all, Brunel radiation, which arises due to the bound electron abruptly starting to move after tunneling, and, secondly, Thomson scattering of the laser wave on the ionized free electron, which contributes mostly at the laser frequency. The bremsstrahlung in the applied short-range atomic potential is suppressed. We are mostly interested in Brunel radiation. We will see that the current $\mathbf{J}_{11}$ contains contributions from the continuum dynamics after the tunneling, as well as from the under-the-barrier path. To clearly single out the effect of the under-the-barrier dynamics for the Brunel radiation, we introduce a simple man model dubbed as "Drude model" in Ref. [20], and simulate the radiation only due to the continuum dynamics excluding the under-the-barrier contribution.

In this model we evaluate the ionized electron probability current in the continuum $\mathbf{J}_D(t)$ at the time instant t , consisting of the tunnel ionized electron probability contribution at previous times $t' < t$, multiplied by the laser-driven velocity after the tunneling. We use the ionization probability via the Perelomov-Popov-Terent'ev (PPT) theory [46,47], including nonadiabatic corrections:

$$W(t') = \frac{\kappa^2}{2} \frac{|\mathbf{E}(t')|}{E_a} \exp \left[-\frac{2E_a(1 - \frac{\gamma^2}{10})}{3|\mathbf{E}(t')|} \right], \quad (10)$$

where $E_a = \kappa^3$ is the atomic field, I_p is the atomic ionization energy, and $\kappa = \sqrt{2I_p}$, $\gamma = \kappa\omega_0/E_0$ are the Keldysh parameter. The laser driven velocity after the tunneling is

$$\mathbf{v}(t, t') = \mathbf{A}(t) - \mathbf{A}(t'). \quad (11)$$

Thus the ionized electron probability current in the continuum reads

$$\mathbf{J}_D(t) = \int_{-\infty}^t dt' \mathbf{v}(t, t') W(t'). \quad (12)$$

Using $\mathbf{J}_D(t)$, the radiation spectra with the Drude model is calculated using classical radiation theory via Eq. (7).

Within this model, the radiation has a spike in the near-zero-frequency domain in the case of an asymmetric pulse, which has a simple explanation. Initially, the electron is at rest at the core, then it tunnels out during the current formation time τ , and it is accelerated in the continuum due to the laser field up to its final velocity v . The current associated with this motion can be parametrized as

$$J(t) \sim v [1 + \text{erf}(t/\tau)], \quad (13)$$

with the Fourier component of the current $J_\omega \sim v \exp(-\omega^2 \tau^2/4)/\omega$. Thus, from Eq. (7),

$$\frac{dw}{d\omega} \sim \frac{v^2}{\omega} e^{-\omega^2 \tau^2/2}, \quad (14)$$

and the spectrum has a singularity at zero frequency. Note that the Drude model exaggerates the singularity because it does not account accurately for the formation time τ of the current, because the error function of Eq. (13) is replaced by a Heaviside-theta function.

C. Coherent Brunel radiation

1. Calculation of the current

We have carried out the calculation of the coherent radiation spectra in the case of a symmetric and an asymmetric few-cycle laser pulse defined by their displacement,

$$\alpha(t) = -\hat{\mathbf{e}}_x \frac{E_0}{\omega_0^2} [\cos(\omega_0 t) + \mathcal{A} \sin(2\omega_0 t)] \cos(\omega_0 t/N)^4, \quad (15)$$

where $\alpha''(t) = A'(t) = -E(t)$; E_0 and ω_0 are the laser field amplitude and frequency, respectively; $N = 12$; $\mathcal{A}$ is the asymmetry parameter of the laser pulse; $\mathcal{A} = 0$ for the symmetric monochromatic pulse; and $\mathcal{A} = 0.1$ for the asymmetric two-color one. In the calculation of currents, the time integrations are performed numerically, while all other integrals are evaluated analytically in the applied three-dimensional (3D) zero-range potential case.

The current component $\mathbf{J}_{01}(t)$ reads

$$\mathbf{J}_{01}(t) = -i \int_{-\infty}^t dt' \int d^3\mathbf{p} \langle \psi^{(0)}(t) | \hat{\mathbf{p}} | \psi_{\mathbf{p}}^{(f)}(t) \rangle \times \langle \psi_{\mathbf{p}}^{(f)}(t') | \mathbf{r} \cdot \mathbf{E}(t') | \psi^{(a)}(t') \rangle,$$

which after the saddle-point integration over the momentum is expressed as follows:

$$\mathbf{J}_{01}(t) = -i \int^t dt' \left(\frac{2\pi i}{t - t' - i\epsilon} \right)^{3/2} \mathbf{m}_p^*(\mathbf{p}_s, t) m_x(\mathbf{p}_s, t'), \quad (16)$$

where $\mathbf{p}_s = -[\boldsymbol{\alpha}(t) - \boldsymbol{\alpha}(t')]/(t - t')$ is the saddle-point momentum, and the singularity is handled using $\epsilon = 0.4/\kappa^2$ [48]. Here, the matrix elements are defined as

$$\begin{aligned} \mathbf{m}_p(\mathbf{p}, t) &= \int d^3\mathbf{r} \psi_p^{(f)}(\mathbf{r}, t)^* \hat{\mathbf{p}} \psi^{(a)}(\mathbf{r}, t) \\ &= \frac{\sqrt{\kappa}(\mathbf{p} + \mathbf{A}(t))}{\pi[\kappa^2 + (\mathbf{p} + \mathbf{A}(t))^2]^2} \\ &\quad \times \exp[ip^2/2t + i\mathbf{p} \cdot \boldsymbol{\alpha}(t) + i\beta(t) + i\kappa^2/2t], \\ m_x(\mathbf{p}, t) &= \int d^3\mathbf{r} \psi_p^{(f)}(\mathbf{r}, t)^* \hat{\mathbf{r}} \cdot \mathbf{E}(t) \psi^{(a)}(\mathbf{r}, t) \\ &= -2i \mathbf{m}_p(\mathbf{p}, t) \cdot \mathbf{E}(t), \end{aligned} \quad (17)$$

with $\beta = \int dt A^2(t)/2$.

The current component $\mathbf{J}_{11}(t)$ reads

$$\begin{aligned} \mathbf{J}_{11}(t) &= - \int^t dt' \int^t dt'' \int d^3\mathbf{p} \int d^3\mathbf{p}' \\ &\quad \times \langle \psi^{(a)}(t') | \mathbf{r} \cdot \mathbf{E}(t') | \psi_p^{(f)}(t') \rangle \\ &\quad \times \langle \psi_p^{(f)}(t) | \hat{\mathbf{p}} | \psi_p^{(f)}(t) \rangle \langle \psi_p^{(f)}(t'') | \mathbf{r} \cdot \mathbf{E}(t'') | \psi^{(a)}(t'') \rangle, \\ &= \int d^3\mathbf{p} (\mathbf{p} + \mathbf{A}(t)) \left| \int^t dt' m_x(\mathbf{p}, t') \right|^2. \end{aligned} \quad (18)$$

The t integration in Eq. (16), the t and p integrations in Eq. (18), and the Fourier transform in Eq. (7) are calculated numerically.

Further, because of a finite duration of the laser pulse with sharp time boundaries, unphysical contributions arise [49] in the temporal Fourier transformation of the currents, which are removed by the following procedure. In the interval (t_i, t_f) the time integration for the Fourier transform is carried out numerically, while the integrations in the intervals $(-\infty, t_i)$ and (t_f, ∞) are calculated analytically,

$$\int_{-\infty}^{\infty} dt f(t) = \frac{f(t_i)^2}{f'(t_i)} + \int_{t_i}^{t_f} dt f(t) - \frac{f(t_f)^2}{f'(t_f)}, \quad (19)$$

where t_i and t_f are the switching-on and -off times of the laser pulse, respectively, using

$$\int_{-\infty}^{t_i} dt f(t) \approx \int_{-\infty}^{t_i} dt \exp(\ln[f(t_i) + f'(t_i)(t - t_i)]) = \frac{f(t_i)^2}{f'(t_i)},$$

with the function f being the integrand of the Fourier transformation $f(t) = J(t) \exp[i\omega t - \epsilon t]$, where the parameter $\epsilon = 2.4 \times 10^{-5}$ a.u. is introduced corresponding to the interaction time of 1 ps (frequency resolution of 1 THz).

2. Radiation spectra

The spectra in symmetric and asymmetric few-cycle laser pulses with $E_0 = 0.1$, $\omega_0 = 0.05$, and the number of cycles $N = 12$ in the 3D zero-range potential, with $\kappa = 1$, are

shown in Fig. 1. In both cases the three-step HHG dominates after the third harmonic, $\omega > 3\omega_0 \approx 4$ eV, while the continuum contribution via the J_{11} term dominates at low frequencies $\omega < 4$ eV; see zoom of spectra for the case of the asymmetric laser pulse in Fig. 2. The first and second harmonics are mostly due to Thomson scattering, and the SFA calculation via J_{11} are well approximated by the Drude model in the interval $0.5 < \omega < 4$ eV. In particular, in the asymmetric pulse (Fig. 2), the radiation has a spike in the near-zero-frequency domain dominated by the continuum contribution (J_{11}), characteristic to the Drude model. Thus we conclude that J_{11} at low frequencies describes the Brunel radiation.

While Fig. 1 makes the impression that the J_{11} induced radiation spectrum in the near-zero-frequency domain coincides with that via the Drude model (with the solely classical continuum contribution), the zoomed-in view of Fig. 2 demonstrates differences from the Drude model, especially at the peak of the near-zero-frequency region. The reason for the difference is that J_{11} includes additionally the quantum free-free transition effect. The latter modifies the emission in most of the near-zero-frequency spectral region. The comparison of the cases $\gamma = 0.5$ and $\gamma = 0.7$ in Figs. 2(c) and 2(d) shows that the quantum dynamics leads to small enhancement of the THz spectrum with respect to the classical Drude model in the nonadiabatic regime and vice versa in the adiabatic regime.

The width of the near-zero-frequency spike is about $\Delta\omega \sim 0.2$ eV, which is related to the laser pulse duration. In fact, the FWHM of the pulse in intensity is approximately $\tau \sim 200$ a.u. which corresponds to an energy of $1/\tau \sim 0.15$ eV, roughly fitting to $\Delta\omega \sim 1/\tau$.

III. SPONTANEOUS RADIATION DURING TUNNELING IONIZATION

A. Spontaneous radiation spectra

The spontaneous radiation during tunneling ionization is calculated as a photon emission during the transition between the state of the tunneling electron $\Psi(t)$ and the continuum state in the laser field:

$$dw = |m(\mathbf{k}, \mathbf{p})|^2 d^3\mathbf{k} d^3\mathbf{p}, \quad (20)$$

where the emission amplitude for the photon momentum $\mathbf{k}$, and the electron final momentum $\mathbf{p}$ reads

$$m(\mathbf{k}, \mathbf{p}) = \int dt \langle \psi_p^{(f)}(t), 1 | \hat{\mathbf{p}} \cdot \hat{\mathbf{A}}(t) | \Psi(t), 0 \rangle, \quad (21)$$

with the SFA wave function of the tunneling electron $\Psi(t)$. Integrating over the final electron momenta we obtain

$$\frac{dw}{d\omega} = \frac{4\pi\omega^2}{c^3} \int d^3\mathbf{p} \left| \int dt \langle \psi_p^{(f)}(t), 1 | \hat{\mathbf{p}} \cdot \hat{\mathbf{A}}(t) | \Psi(t), 0 \rangle \right|^2. \quad (22)$$

In zeroth-order SFA $\Psi = \psi^{(0)}(t)$ and we obtain the photon emission probability which coincides with the result of Ref. [30] in velocity gauge. In the first-order SFA calculation, the electron initial state during the photon emission process is $\Psi(t) = \psi^{(0)}(t) + \psi^{(1)}(t)$, which includes the continuum

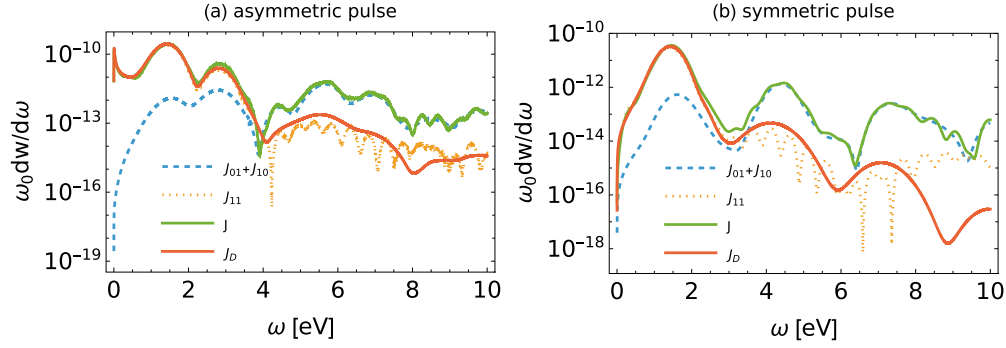


FIG. 1. Coherent radiation spectra during tunneling ionization with $E_0 = 0.1$, $I_p = 0.5$ a.u., $\omega_0 = 0.05$, and $N = 12$: (a) in the asymmetric laser pulse $\mathcal{A} = 0.1$ [Eq. (15)], (b) in the symmetric laser pulse $\mathcal{A} = 0$. Comparison of the Brunel radiation via J_{11} with the three-step HHG ($J_{01} + J_{10}$) and with the Drude model (J_D). The total radiation with the full current $J = J_{01} + J_{10} + J_{11}$ is given for comparison.

dynamics in the tunneling ionization process via $\psi^{(1)}(t)$. Comparing the zeroth-order probability with the first-order one, we learn about the role of the continuum dynamics for the spontaneous radiation. We define the zero-order amplitude as

$$m_0(\mathbf{k}, \mathbf{p}) = \int dt \langle \psi_{\mathbf{p}}^{(f)}(t), 1 | \hat{\mathbf{p}} \cdot \hat{\mathbf{A}}(t) | \psi^{(0)}(t), 0 \rangle, \quad (23)$$

$$(24)$$

and the first-order one,

$$m_1(\mathbf{k}, \mathbf{p}) = \int dt \langle \psi_{\mathbf{p}}^{(f)}(t), 1 | \hat{\mathbf{p}} \cdot \hat{\mathbf{A}}(t) | \psi^{(1)}(t), 0 \rangle, \quad (25)$$

as well as the total amplitude,

$$m(\mathbf{k}, \mathbf{p}) = m_0(\mathbf{k}, \mathbf{p}) + m_1(\mathbf{k}, \mathbf{p}). \quad (26)$$

The amplitudes are expressed in the following form:

$$m_0(\mathbf{k}, \mathbf{p}) = \int dt \langle \hat{\mathbf{A}}(t) \rangle \cdot \mathbf{m}_p^*(\mathbf{p}, t),$$

$$m_1(\mathbf{k}, \mathbf{p}) = -i \int dt' [\mathbf{p} + \mathbf{A}(t')] \cdot \langle \hat{\mathbf{A}}(t') \rangle \int^{t'} dt m_x(\mathbf{p}, t), \quad (27)$$

where $\langle \hat{\mathbf{A}}(t) \rangle = \langle 1 | \hat{\mathbf{A}}(t) | 0 \rangle$. The amplitude m_1 has a structure of a nested integral and is calculated via two partial integrations, see Appendix A, leading to the following expression:

$$m_1 = i \int dt m_x(\mathbf{p}, t) [\mathbf{p} + \mathbf{A}(t)] \cdot \int^t dt'' \langle \hat{\mathbf{A}}(t'') \rangle$$

$$+ i \int dt m_x(\mathbf{p}, t) \int^t dt' \mathbf{E}(t') \cdot \int^{t'} dt'' \langle \hat{\mathbf{A}}(t'') \rangle$$

$$- i \int dt m_x(\mathbf{p}, t) \int dt' \mathbf{E}(t') \cdot \int^{t'} dt'' \langle \hat{\mathbf{A}}(t'') \rangle. \quad (28)$$

The inner time integrals are analytical and the outer integration is performed numerically. The final momentum integration is also performed numerically.

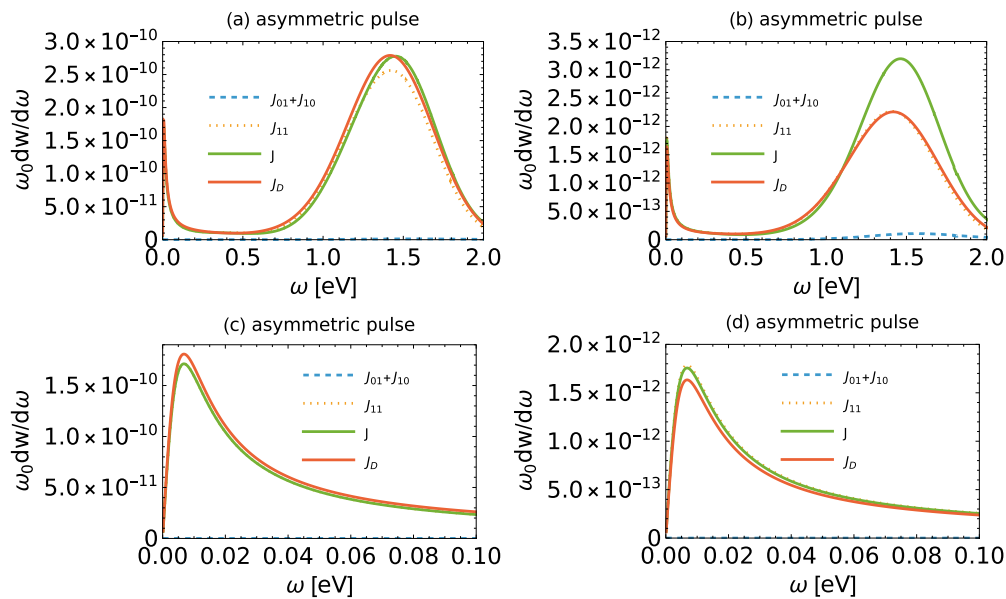


FIG. 2. Coherent radiation spectra during tunneling ionization in the asymmetric laser pulse: (first column) $E_0 = 0.1$ [zoomed-in view Fig. 1(a)]; (second column) $E_0 = 0.075$. Comparison of the Brunel radiation via J_{11} with the three-step HHG ($J_{01} + J_{10}$) and with the Drude model (J_D).

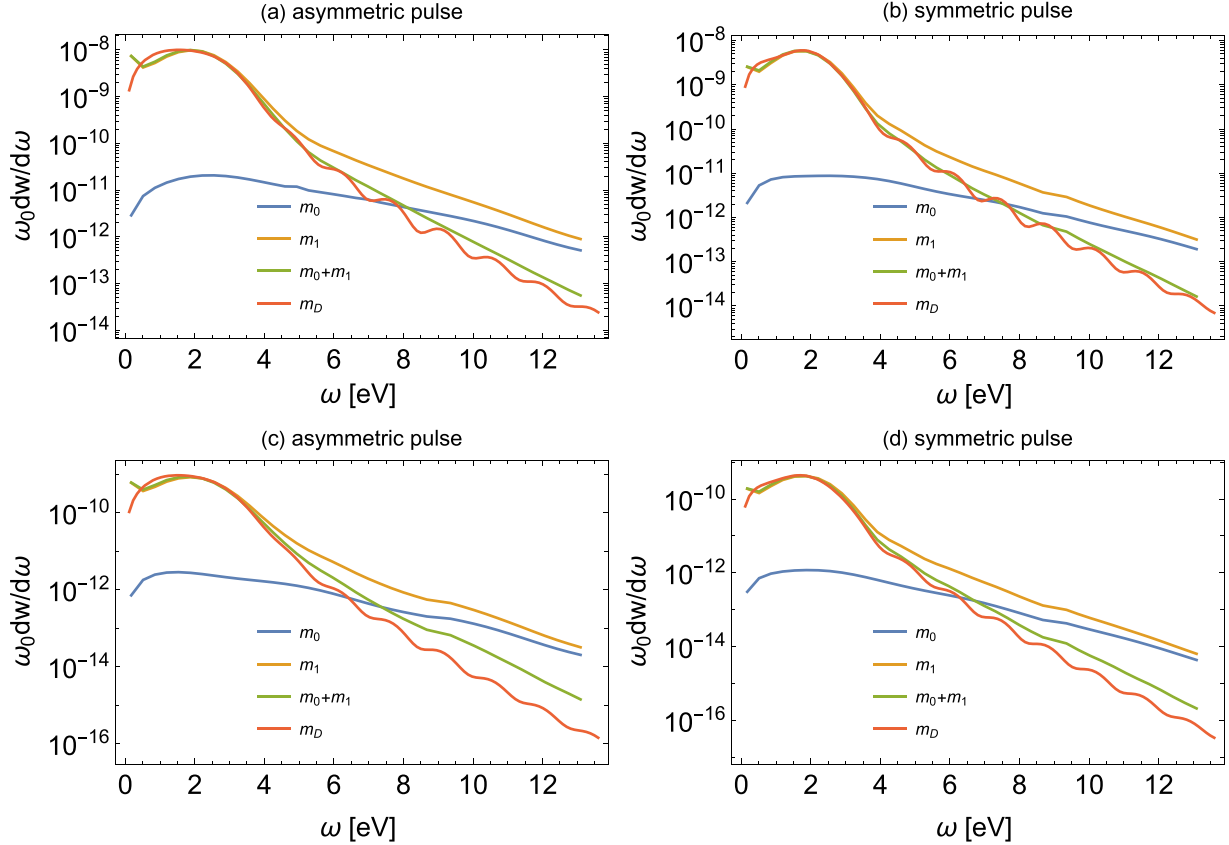


FIG. 3. Spontaneous radiation spectrum during tunneling ionization: (left column) asymmetric few-cycle laser pulse, (right column) symmetric laser pulse, in both cases $N = 4$; (a,b) for $E_0 = 0.1$, (c,d) for $E_0 = 0.075$. The contributions of amplitudes in different approximations m_0 , m_1 , and $m_0 + m_1$ are specified. The results of the simple man Drude model are indicated by m_D .

We have carried out the calculation of the incoherent radiation spectra in the case of symmetric and asymmetric few-cycle laser pulses defined above with $E_0 = 0.1$ ($\gamma = 0.5$) and $E_0 = 0.075$ ($\gamma = 0.7$), $\omega_0 = 0.05$, and $N = 4$. The spectra for the spontaneous radiation are shown in Fig. 3. The spectrum via $m_0(\mathbf{k}, \mathbf{p})$ is similar to the calculation of Ref. [30], with the spectral width $\Delta\omega \sim 1/\tau_K$, determined by the Keldysh time $\tau_K = \kappa/E_0$. Up to $\omega < 6$ eV it is much smaller with respect to $m_1(\mathbf{k}, \mathbf{p})$, which includes the continuum motion. In the near-zero-frequency region, $m_1(\mathbf{k}, \mathbf{p})$ has a spike similar to that in the coherent radiation. At larger frequencies $\omega > 6$ eV, $m_0(\mathbf{k}, \mathbf{p})$ and $m_1(\mathbf{k}, \mathbf{p})$ compensate each other. Thus the amplitude $m_0(\mathbf{k}, \mathbf{p})$ does not describe the observable spectrum either at low frequencies $\omega \rightarrow 0$, or at high ones $\omega \rightarrow \infty$, but only near the cutoff $\omega \sim 8$ eV. Our remarkable observation is that there is no significant difference in spontaneous radiation in symmetric or asymmetric pulses. The qualitative behavior of the spectra does not change with increasing the Keldysh parameter, but the intensity of radiation decreases with larger γ as the driving electric field decreases. The spectral width of the radiation is related to the Keldysh time $\Delta\omega \sim 1/\tau_K = \gamma/\omega_0$.

When considering the correction to the ground state in the photon emission amplitude at strong-field ionization of Eq. (21), a question will arise as to why the correction to the final continuum state is neglected. The latter correction describes free-free harmonic generation at a recollision due

to bremsstrahlung, which is negligible compared with the discussed Brunel radiation; see Appendix B.

B. Simple man Drude model for spontaneous radiation

For an intuitive proof of the spontaneous radiation spectra, we have generalized the simple man Drude model of Eq. (12) for the spontaneous radiation described by Eq. (22). Taking into account that in the strong-field ionization three-step model [1] there is a mapping between the ionization time and the momentum, and substituting $\langle \psi_p^{(f)}(t) | \hat{\mathbf{p}} | \Psi(t) \rangle \rightarrow \mathbf{v}(t, t')$, as well as $\langle 1 | \hat{\mathbf{A}}(t) | 0 \rangle \rightarrow \langle \hat{\mathbf{A}}(t) \rangle \equiv i/(2\pi\sqrt{\omega})\hat{\mathbf{e}}e^{i\omega t}$, we arrive at the simple man Drude model representation for the spontaneous radiation of Eq. (22):

$$\frac{dw_D}{d\omega} = \frac{4\pi\omega^2}{c^3} \int dt' \left| \int_{t'+i\gamma/\omega_0}^{t_f+i\gamma/\omega_0} dt \mathbf{v}(t, t') \cdot \langle \hat{\mathbf{A}}(t) \rangle \right|^2 W(t'). \quad (29)$$

The results via the Drude model (via the current J_D) for the spontaneous radiation spectra are also presented in Fig. 3. The Drude model is qualitatively in accordance with m_1 in the low-frequency region and confirms the enhancement of spectra in this frequency region in contrast to the behavior of the spectra based on the direct ionization m_0 amplitude. At large frequencies $\omega > 6$ eV, when the radiation is dominated

by interference of m_0 and m_1 , the Drude model still surprisingly agrees quite well with the SFA calculation.

C. Correlated electron and photon measurement

The under-the-barrier recollisions are described by the m_2 amplitude. Our calculations show that its contribution to the coherent and spontaneous radiation spectral distribution is negligible; i.e., the radiation spectra do not carry information about under-the-barrier recollisions. Is there any other way to see this signature in the radiation properties? To answer this question, we discuss the correlated electron and photon measurement.

The spontaneous emission spectra in Fig. 3 have been calculated integrating over the final electron momentum. However, when the correlated electron and photon measurement is carried out in an experiment, one can access the electron momentum resolved photon spectra. The electron momentum resolved total photon yield in a half-cycle laser pulse in the case of the correlated measurement is defined as follows:

$$\frac{dw}{d^3\mathbf{p}} = \frac{4\pi}{c^3} \int \omega^2 d\omega |m_0(\mathbf{k}, \mathbf{p}) + m_1(\mathbf{k}, \mathbf{p})|^2. \quad (30)$$

The zeroth-order amplitude,

$$m_0(\mathbf{k}, \mathbf{p}) = \int dt \hat{\mathbf{A}}(t) \cdot \mathbf{m}_p(\mathbf{p}, t), \quad (31)$$

is calculated via the saddle-point integration in time t , where the saddle points are found numerically in a half-cycle sinusoidal laser pulse. For the calculation of the first-order amplitude, the Drude model for spontaneous emission is applied:

$$m_1(\mathbf{k}, p_x, p_\perp = 0) = \int_{t'+iy/\omega_0}^{t_f+iy/\omega_0} dt \mathbf{v}(t, t') \cdot \langle \hat{\mathbf{A}}(t) \rangle \sqrt{W(t', p_\perp = 0)}, \quad (32)$$

with the time-to-momentum mapping $p_x = -A(t')$, and $W(t, p_\perp = 0) = W(t)\kappa/(\pi|\mathbf{E}(t)|)$. The t integration in m_1 , and ω integration in Eq. (30) are carried out numerically.

We include the second-order correction to the emission amplitude m_2 heuristically, taking into account the correction in the ionization rate, where we replace the binding energy of the initial state $-\kappa^2/2$ by $-(\kappa^2/2)\{1 + i \exp(-(2/3)[\kappa^3/|\mathbf{E}(t)|])\}$ [50].

The correlated photon yield is presented in Fig. 4. The higher-order corrections to the spectrum via m_0 and m_1 induce a shift of the position of the maximum. This is a result of the continuum motion. While in the given half-cycle laser pulse the photoelectron momentum peak is shifted from the zero momentum, the momentum shift δp in the photoelectron momentum resolved photon yield is smaller, because it is additionally multiplied in Eq. (32) by the Fourier component of the ionizing electron velocity. The momentum shift approximately scales as $\delta p \sim (E_0/\kappa^3)^{2/3}(E_0/\omega_0)$ according to the fit to the numerical calculations.

The small contribution of the under-the-barrier recollisions via m_2 can be distinguished near the peak. However, this does not lead to any qualitative change of the spectrum.

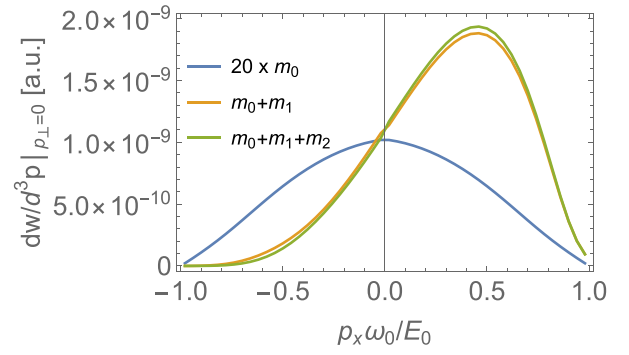


FIG. 4. Total probability of photon emission vs electron final momenta in the case of the correlated electron and photon measurement, for $E_0 = 0.2$ a.u. and a half-cycle pulse; the value for the m_0 contribution is multiplied by 20.

IV. COMPARISON WITH EXPERIMENTAL RESULTS

The main distinguishing feature of the Brunel radiation is a large spike at near-zero frequencies. To validate our theory based on SFA, we compare its result with an example of the experimental spectrum of Ref. [15]. In the experiment THz radiation is observed from a gas target exposed to a two-color strong-field. In such a many-atom situation the coherent radiation is relevant and we have calculated the spectrum using results of Sec. II via Eqs. (7), (16), and (18). The laser parameters employed in this experiment are as follows: the wavelength $\lambda = 800$ nm, pulse duration $\tau = 60$ fs, and pulse energy $\varepsilon = 400$ μ J. Then the average intensity can be estimated as $I_0 \approx 3.3 \times 10^{14}$ W/cm², or the laser field is $E_0 \approx 0.1$ a.u. The second harmonic energy is approximately 5%; then the asymmetry $\mathcal{A} = \sqrt{0.05} \approx 0.2$. The radiation spectrum in the nitrogen gas with the laser parameters approximately deduced from the provided information is shown in Fig. 5 in the low-frequency region. We see an order of magnitude decrease of the radiation intensity in the spectral region up to 20 THz. A similar spike existed in the experimental spectrum of Fig. 3 in Ref. [15]. Thus our simple SFA theory qualitatively reproduces the main feature of the experimental spectrum. There exist deviations in details from the experimental result, for which more accurate description is necessary, in particular, the inclusion of explicit atomic potential, rather than the short-range one as in our SFA theory.

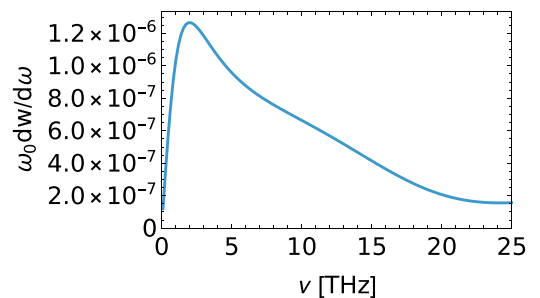


FIG. 5. The radiation spectrum at strong-field ionization near the low-frequency region for the approximate parameters of the experiment of Ref. [15]: $E_0 = 0.1$, $\mathcal{A} = 0.2$.

V. ESTIMATION OF RADIATION INTENSITIES

Finally, let us give estimations, by an order of magnitude, of the coherent and incoherent Brunel radiation. We proceed with the dipole radiation power [45]:

$$I = \frac{2\ddot{d}^2}{3c^3}. \quad (33)$$

We estimate the current from Eq. (12),

$$\dot{d}_{\text{coh}} \sim (E_0/\omega_0)w_iT, \quad (34)$$

where $w_i \sim (\kappa^2/2)(E_0/E_a)\exp[-2E_a/(3E_0)]$ is the laser ionization rate, and $T = 2\pi/\omega_0$ the laser period. The probability of the coherent Brunel photon emission per laser period with the energy ω is

$$w_{\text{coh}} \sim \frac{I_{\text{coh}}T}{\omega} \sim \frac{\alpha E_0^2}{c^2\omega} \left(\frac{\omega}{\omega_0}\right)^2 w_i T^3 \sim \alpha a_0^2 \gamma^{-1} (w_i T)^2, \quad (35)$$

where $a_0 = E_0/c\omega_0$ is the relativistic invariant field parameter.

Now let us estimate the probability of the spontaneous Brunel radiation again via the dipole radiation theory. For spontaneous radiation, we estimate the dipole moment derivative as

$$\dot{d}_{\text{sp}} \sim \omega_c \ell_c, \quad (36)$$

with the characteristic frequency $\omega_c \sim 1/\tau_K$, and $\ell_c \sim \kappa \tau_K$, with the Keldysh time $\tau_K = \kappa/E_0$, and the atomic velocity κ . Then the spontaneous radiation power is

$$I_{\text{sp}} \sim \frac{\alpha \omega_c^4 \ell_c^2}{c^2} w_i T, \quad (37)$$

where we multiplied by the ionization probability because the dipole moment d_{sp} is created during tunneling. Then the probability of the spontaneous photon emission during tunneling ionization is

$$w_{\text{sp}} \sim \frac{I_{\text{sp}}T}{\omega_c} w_i T \sim \frac{\alpha \omega_c^3 \ell_c^2}{c^2} w_i T^2 \sim \alpha \frac{\kappa^2}{c^2} \gamma^{-1} w_i T, \quad (38)$$

with the Keldysh parameter $\gamma = \omega_0 \tau_K$.

Thus the ratio of the probabilities is

$$\frac{w_{\text{sp}}}{w_{\text{coh}}} \sim \frac{\kappa^2}{c^2} \frac{1}{a_0^2 w_i T}. \quad (39)$$

For $E_0 = 0.1$ (the laser intensity 3.5×10^{14} W/cm²), $E_0/E_a \sim 0.1$, and $a_0 \sim 1.5 \times 10^{-2}$, $\frac{I_p}{\omega_0} \sim 10$, $e^{-\frac{2E_a}{3E_0}} \sim 1.3 \times 10^{-3}$, $w_i T \sim 10^{-2}$, we obtain $w_{\text{sp}}/w_{\text{coh}} \sim 10^2$. The latter ratio is roughly in accordance with spectra in Figs. 1 and 3.

We may estimate also the spontaneous radiation per electron: $w_{\text{sp}}^{(e)} \sim w_{\text{sp}}/w_i T \sim (\alpha/\gamma)(\kappa/c)^2 \sim 10^{-6}$, which indicates complications for the electron and photon coincidence measurement. The Brunel radiation scales as $1/\gamma$ with the Keldysh parameter.

When the Brunel radiation is observed with a gas target, the coherent Brunel radiation will be dominating over the spontaneous one, if the target density is large enough: $N = nV > w_{\text{sp}}/w_{\text{coh}}$, with the particle number density n , the focal volume $V \sim (\pi w^2/\lambda)(\pi w^2)$, and the laser beam waist size w . With $w \sim 3\lambda$, $\lambda \sim 1 \mu$, this will take place at a gas density $n > 10^{11}$ cm⁻³.

VI. CONCLUSION

We have investigated the coherent and incoherent radiation during tunneling ionization. The coherent radiation in the near-zero-frequency THz domain ($\omega \sim 0.2$ eV) is dominated by the continuum dynamics via the quantum free-free transition J_{11} term in the current. We have shown that due to the quantum dynamics the radiation is modified especially in the adiabatic limit.

The spontaneous radiation is not affected by the asymmetry of the driving laser pulse. The characteristic width of the spontaneous radiation is determined by the Keldysh time $\Delta\omega \sim 1/\tau_K$. There are two contributions into the spontaneous radiation: due to the direct transition during tunneling (with the amplitude m_0) investigated in Ref. [30], and due to additional motion in the continuum (with the amplitude m_1). At larger frequencies $\omega \gtrsim 1/\tau_K$, the m_0 and m_1 terms compensate each other and the total radiation is suppressed. At low frequencies $\omega \lesssim 1/\tau_K$, the radiation is dominated by the continuum dynamics via the amplitude m_1 . Thus the contribution of the direct transition in the radiation essentially describes the spontaneous radiation only near the cutoff.

DATA AVAILABILITY

The data that support the findings of this article are not publicly available upon publication because it is not technically feasible and/or the cost of preparing, depositing, and hosting the data would be prohibitive within the terms of this research project. The data are available from the authors upon reasonable request.

APPENDIX A: DERIVATION OF THE MATRIX ELEMENT m_1

The amplitude m_1 represented by Eq. (25) is transformed as follows:

$$m_1(\mathbf{k}, \mathbf{p}) = -i \int d^3\mathbf{q} \int dt \langle \psi_{\mathbf{p}}^{(f)}(t), 1 | \hat{\mathbf{p}} \cdot \hat{\mathbf{A}}(t) | \psi_{\mathbf{q}}^{(f)}(t), 0 \rangle \int^t dt' \langle \psi_{\mathbf{q}}^{(f)}(t') | \mathbf{r} \cdot \mathbf{E}(t') | \psi^{(a)}(t') \rangle \quad (A1)$$

$$= -i \int d^3\mathbf{q} \int dt [\mathbf{q} + \mathbf{A}(t)] \cdot \langle \hat{\mathbf{A}}(t) \rangle \delta(\mathbf{p} - \mathbf{q}) \int^t dt' \langle \psi_{\mathbf{q}}^{(f)}(t') | \mathbf{r} \cdot \mathbf{E}(t') | \psi^{(a)}(t') \rangle \quad (A2)$$

$$= -i \int dt [\mathbf{p} + \mathbf{A}(t)] \cdot \langle \hat{\mathbf{A}}(t) \rangle \int^t dt' \langle \psi_{\mathbf{p}}^{(f)}(t') | \mathbf{r} \cdot \mathbf{E}(t') | \psi^{(a)}(t') \rangle \quad (A3)$$

$$= -i \int dt [\mathbf{p} + \mathbf{A}(t)] \cdot \langle \hat{\mathbf{A}}(t) \rangle \int^t dt' m_x(\mathbf{p}, t') \quad (A4)$$

$$= i \int dt \left[m_x(\mathbf{p}, t) [\mathbf{p} + \mathbf{A}(t)] - \int^t dt' m_x(\mathbf{p}, t') \mathbf{E}(t) \right] \cdot \int^t dt'' \langle \hat{\mathbf{A}}(t'') \rangle - \left\{ i \int dt' m_x(\mathbf{p}, t') \mathbf{p} \cdot \int dt'' \langle \hat{\mathbf{A}}(t'') \rangle \right\} \quad (\text{A5})$$

$$= i \int dt \left[m_x(\mathbf{p}, t) [\mathbf{p} + \mathbf{A}(t)] - \int^t dt' m_x(\mathbf{p}, t') \mathbf{E}(t) \right] \cdot \int^t dt'' \langle \hat{\mathbf{A}}(t'') \rangle + \left\{ \int dt' m_x(\mathbf{p}, t') \mathbf{p} \cdot \hat{\mathbf{e}} \delta(\omega) / \sqrt{\omega} \right\} \quad (\text{A6})$$

$$= i \int dt m_x(\mathbf{p}, t) [\mathbf{p} + \mathbf{A}(t)] \cdot \int^t dt'' \langle \hat{\mathbf{A}}(t'') \rangle - i \int dt \int^t dt' m_x(\mathbf{p}, t') \mathbf{E}(t) \cdot \int^t dt'' \langle \hat{\mathbf{A}}(t'') \rangle \quad (\text{A7})$$

$$= i \int dt m_x(\mathbf{p}, t) [\mathbf{p} + \mathbf{A}(t)] \cdot \int^t dt'' \langle \hat{\mathbf{A}}(t'') \rangle + i \int dt m_x(\mathbf{p}, t) \int^t dt' \mathbf{E}(t') \cdot \int^{t'} dt'' \langle \hat{\mathbf{A}}(t'') \rangle \quad (\text{A8})$$

$$- \left\{ i \int dt m_x(\mathbf{p}, t) \int dt' \mathbf{E}(t') \cdot \int^{t'} dt'' \langle \hat{\mathbf{A}}(t'') \rangle \right\}, \quad (\text{A9})$$

where $\langle \hat{\mathbf{A}}(t) \rangle \equiv i/(2\pi\sqrt{\omega})\hat{\mathbf{e}}e^{i\omega t}$. The first partial integration is carried out in Eq. (A4), and the second one in the second term of Eq. (A7). The surface term in the curly brackets $\{\dots\}$ in Eq. (A6) is zero because $\omega > 0$, and the surface term in Eq. (A9) gives a not-significant contribution to the radiation at laser frequencies ω_0 and $2\omega_0$ due to Thomson scattering.

All integrand functions in Eqs. (A8) and (A9) are well behaved and the time integrals converge. The inner time integrals are analytical and the outer integration is performed numerically. The boundaries of the finite time integration introduce not-physically-relevant time diffraction oscillations in the integration result. To avoid it, we shift the outer time integration contour in the complex plane passing over the saddle point $t_s = i\gamma/\omega_0$:

$$\int dt f(t) = \int_{t_i + i\gamma/\omega_0}^{t_f + i\gamma/\omega_0} dt f(t), \quad (\text{A10})$$

and omit the contribution of the vertical contours, which are responsible for the time diffraction; see a similar procedure in Ref. [49].

APPENDIX B: FREE-FREE HARMONICS

When considering the correction to the continuum wave function, an additional amplitude of the process arises:

$$m_{\text{FFH}}(\mathbf{k}, \mathbf{p}) = i \int d^3\mathbf{q} \int dt \int^t dt' \langle \psi_{\mathbf{p}}^f(t) | V | \psi_{\mathbf{q}}^f(t) \rangle \langle \psi_{\mathbf{q}}^f(t') | 1 | \hat{\mathbf{p}} \cdot \hat{\mathbf{A}}(t') | \psi^{(0)}(t') + \psi^{(1)}(t'), 0 \rangle, \quad (\text{B1})$$

$$= i \int d^3\mathbf{q} \int dt \int^t dt' \langle \psi_{\mathbf{p}}^f(t) | V | \psi_{\mathbf{q}}^f(t) \rangle m_0(\mathbf{k}, \mathbf{q}, t) + i \int d^3\mathbf{q} \int dt \int^t dt' \langle \psi_{\mathbf{p}}^f(t) | V | \psi_{\mathbf{q}}^f(t) \rangle m_1(\mathbf{k}, \mathbf{q}, t), \quad (\text{B2})$$

where $m_0(\mathbf{k}, \mathbf{p}, t) = \int^t dt' \langle \psi_{\mathbf{p}}^{(f)}(t') | 1 | \hat{\mathbf{p}} \cdot \hat{\mathbf{A}}(t') | \psi^{(0)}(t'), 0 \rangle$, and $m_1(\mathbf{k}, \mathbf{p}, t) = \int^t dt' \langle \psi_{\mathbf{p}}^{(f)}(t') | 1 | \hat{\mathbf{p}} \cdot \hat{\mathbf{A}}(t') | \psi^{(1)}(t'), 0 \rangle$. The first term of Eq. (B2) describes rescattering after the Brunel radiation, and leads to merely electron momentum distribution change after the photon emission. The second term of Eq. (B2) describes the ionized wave packet recollision and bremsstrahlung photon emission at rescattering.

Let us estimate the free-free harmonics via the bremsstrahlung mechanism. We use the cross section of bremsstrahlung in the nonrelativistic limit for fast electrons [51]: $\sigma_{\text{BS}} \sim \alpha r_e^2 c^2 / v^2$, with the electron classical radius r_e , and the electron velocity v . The probability of bremsstrahlung is

$$w_{\text{BS}} \sim \sigma_{\text{BS}} \rho v T \sim \alpha r_e^2 \frac{c^2}{v^2} \frac{v}{\omega_0} \frac{w_i/\omega_0}{r_{\perp}^2 l_{\parallel}}, \quad (\text{B3})$$

where $\rho \sim (w_i/\omega_0)/(r_{\perp}^2 l_{\parallel})$ is the density of the recolliding electron wave packet, which is spread during the rescattering in the transverse direction $r_{\perp} \sim \sqrt{E_0/\kappa}/\omega_0$, and in the longitudinal direction $l_{\parallel} \sim E_0/\omega_0^2$. Then the ratio of the probabilities of the bremsstrahlung to the spontaneous Brunel radiation [Eq. (38)] at ionization is

$$\frac{w_{\text{BS}}}{w_{\text{sp}}} \sim \alpha^2 c^2 \gamma^4 \frac{\omega_0}{\kappa^4} \sim 10^{-3}, \quad (\text{B4})$$

where we estimated $v \sim E_0/\omega_0$, $\gamma \sim 0.3$, $\kappa \sim 1$, and $\omega_0 \sim 0.1$. Thus the bremsstrahlung at the recollision is negligible with respect to the Brunel radiation, and consequently it is justified to neglect the contribution of Eq. (B2) in the radiation amplitude.

- [1] P. B. Corkum, Plasma perspective on strong field multiphoton ionization, *Phys. Rev. Lett.* **71**, 1994 (1993).
- [2] M. Ferray, A. L'Huillier, X. F. Li, L. A. Lompre, G. Mainfray, and C. Manus, Multiple-harmonic conversion of 1064 nm radiation in rare gases, *J. Phys. B: At., Mol. Opt. Phys.* **21**, L31 (1988).
- [3] M. Lewenstein, P. Balcou, M. Y. Ivanov, A. L'Huillier, and P. B. Corkum, Theory of high-harmonic generation by low-frequency laser fields, *Phys. Rev. A* **49**, 2117 (1994).
- [4] P. Agostini and L. F. DiMauro, The physics of attosecond light pulses, *Rep. Prog. Phys.* **67**, 813 (2004).
- [5] M. C. Kohler, C. Ott, P. Raith, R. Heck, I. Schlegel, C. H. Keitel, and T. Pfeifer, High harmonic generation via continuum wavepacket interference, *Phys. Rev. Lett.* **105**, 203902 (2010).
- [6] P. Sprangle, J. R. Peñano, B. Hafizi, and C. A. Kapetanakis, Ultrashort laser pulses and electromagnetic pulse generation in air and on dielectric surfaces, *Phys. Rev. E* **69**, 066415 (2004).
- [7] K. Kim, A. Taylor, J. Glowina, and G. Rodriguez, Coherent control of terahertz supercontinuum generation in ultrafast laser-gas interactions, *Nat. Photonics* **2**, 605 (2008).
- [8] K.-Y. Kim, Generation of coherent terahertz radiation in ultrafast laser-gas interactions, *Phys. Plasmas* **16**, 056706 (2009).
- [9] J. Dai, N. Karpowicz, and X.-C. Zhang, Coherent polarization control of terahertz waves generated from two-color laser-induced gas plasma, *Phys. Rev. Lett.* **103**, 023001 (2009).
- [10] H. Wen and A. M. Lindenberg, Coherent terahertz polarization control through manipulation of electron trajectories, *Phys. Rev. Lett.* **103**, 023902 (2009).
- [11] I. Babushkin, S. Skupin, and J. Herrmann, Generation of terahertz radiation from ionizing two-color laser pulses in Ar filled metallic hollow waveguides, *Opt. Express* **18**, 9658 (2010).
- [12] I. Babushkin, W. Kuehn, C. Köhler, S. Skupin, L. Bergé, K. Reimann, M. Woerner, J. Herrmann, and T. Elsaesser, Ultrafast spatiotemporal dynamics of terahertz generation by ionizing two-color femtosecond pulses in gases, *Phys. Rev. Lett.* **105**, 053903 (2010).
- [13] I. Babushkin, S. Skupin, A. Husakou, C. Köhler, E. Cabrera-Granado, L. Bergé, and J. Herrmann, Tailoring terahertz radiation by controlling tunnel photoionization events in gases, *New J. Phys.* **13**, 123029 (2011).
- [14] Y. S. You, T. I. Oh, and K. Y. Kim, Off-axis phase-matched terahertz emission from two-color laser-induced plasma filaments, *Phys. Rev. Lett.* **109**, 183902 (2012).
- [15] M. Clerici, M. Peccianti, B. E. Schmidt, L. Caspani, M. Shalaby, M. Giguère, A. Lotti, A. Couairon, F. Légaré, T. Ozaki, D. Faccio, and R. Morandotti, Wavelength scaling of terahertz generation by gas ionization, *Phys. Rev. Lett.* **110**, 253901 (2013).
- [16] T. Balciunas, A. Verhoef, A. Mitrofanov, G. Fan, E. Serebryannikov, M. Ivanov, A. Zheltikov, and A. Baltuska, Optical and thz signatures of sub-cycle tunneling dynamics, *Chem. Phys.* **414**, 92 (2013).
- [17] T. I. Oh, Y. J. Yoo, Y. S. You, and K. Y. Kim, Generation of strong terahertz fields exceeding 8 MV/cm at 1 kHz and real-time beam profiling, *Appl. Phys. Lett.* **105**, 041103 (2014).
- [18] N. V. Vvedenskii, A. I. Korytin, V. A. Kostin, A. A. Murzanev, A. A. Silaev, and A. N. Stepanov, Two-color laser-plasma generation of terahertz radiation using a frequency-tunable half harmonic of a femtosecond pulse, *Phys. Rev. Lett.* **112**, 055004 (2014).
- [19] V. A. Tulsy, M. Baghery, U. Saalmann, and S. V. Popruzhenko, Boosting terahertz-radiation power with two-color circularly polarized midinfrared laser pulses, *Phys. Rev. A* **98**, 053415 (2018).
- [20] I. Babushkin, C. Bree, C. M. Dietrich, A. Demircan, U. Morgner, and A. Husakou, Terahertz and higher-order bruno harmonics: From tunnel to multiphoton ionization regime in tailored fields, *J. Mod. Opt.* **64**, 1078 (2017).
- [21] I. Babushkin, Á. J. Galán, J. R. C. de Andrade, A. Husakou, F. Morales, M. Kretschmar, T. Nagy, V. Vaičaitis, L. Shi, D. Zuber, L. Bergé, S. Skupin, I. A. Nikolaeva, N. A. Panov, D. E. Shipilo, O. G. Kosareva, A. N. Pfeiffer, A. Demircan, M. J. J. Vrakking, U. Morgner, *et al.*, All-optical attoclock for imaging tunnelling wavepackets, *Nat. Phys.* **18**, 417 (2022).
- [22] I. I. Abbasov, B. M. Bolotovskii, and V. A. Davydov, High-frequency asymptotic behavior of radiation spectra of moving charges in classical electrodynamics, *Usp. Fiz. Nauk.* **149**, 709 (1986) [*Sov. Phys. Usp.* **29**, 788 (1986)].
- [23] V. L. Ginzburg, *Theoretical Physics and Astrophysics* (Pergamon Press, New York, 1979).
- [24] F. Brunel, Not-so-resonant, resonant absorption, *Phys. Rev. Lett.* **59**, 52 (1987).
- [25] F. Brunel, Harmonic generation due to plasma effects in a gas undergoing multiphoton ionization in the high-intensity limit, *J. Opt. Soc. Am. B* **7**, 521 (1990).
- [26] M. Klaiber, K. Z. Hatsagortsyan, and C. H. Keitel, Under-the-tunneling-barrier recollisions in strong-field ionization, *Phys. Rev. Lett.* **120**, 013201 (2018).
- [27] M. Klaiber, Q. Z. Lv, S. Sukiasyan, D. Bakucz Canário, K. Z. Hatsagortsyan, and C. H. Keitel, Reconciling conflicting approaches for the tunneling time delay in strong field ionization, *Phys. Rev. Lett.* **129**, 203201 (2022).
- [28] M. Klaiber, K. Z. Hatsagortsyan, and C. H. Keitel, Sub-barrier pathways to freeman resonances, *Phys. Rev. A* **102**, 053105 (2020).
- [29] T. Khurelbaatar, M. Klaiber, S. Sukiasyan, K. Z. Hatsagortsyan, C. H. Keitel, and D. E. Kim, Unveiling under-the-barrier electron dynamics in strong field tunneling, *Phys. Rev. Lett.* **134**, 213201 (2025).
- [30] D. B. Milošević, Quantum theory of photon emission during strong-laser-field-induced ionization, *Phys. Rev. A* **108**, 033110 (2023).
- [31] L. V. Keldysh, Ionization in the field of a strong electromagnetic wave, *Zh. Eksp. Teor. Fiz.* **37**, 1945 (1964) [*Sov. Phys. JETP* **20**, 1307 (1965)].
- [32] B. Sundaram and P. W. Milonni, High-order harmonic generation: Simplified model and relevance of single-atom theories to experiment, *Phys. Rev. A* **41**, 6571 (1990).
- [33] P. Krekora, R. E. Wagner, Q. Su, and R. Grobe, Retardation and kinematic relativistic effects in scattered light, *Laser Phys.* **12**, 455 (2002).
- [34] E. A. Chowdhury, I. Ghebregziabihier, and B. C. Walker, Larmor radiation from the ultra-intense field ionization of atoms, *J. Phys. B: At., Mol. Opt. Phys.* **38**, 517 (2005).
- [35] J. Peatross, C. Müller, K. Z. Hatsagortsyan, and C. H. Keitel, Photoemission of a single-electron wave packet in a strong laser field, *Phys. Rev. Lett.* **100**, 153601 (2008).
- [36] J. P. Corson and J. Peatross, Quantum-electrodynamic treatment of photoemission by a single-electron wave packet, *Phys. Rev. A* **84**, 053832 (2011).

- [37] G. Tarbox, E. Cunningham, R. Sandberg, J. Peatross, and M. Ware, Radiation from free electrons in a laser focus at 10^{18} W/cm²: modeling of photon yields and required focal conditions, *J. Opt. Soc. Am. B* **32**, 743 (2015).
- [38] M. Ware, E. Cunningham, C. Coburn, and J. Peatross, Measured photoemission from electron wave packets in a strong laser field, *Opt. Lett.* **41**, 689 (2016).
- [39] Y. Pan and A. Gover, Spontaneous and stimulated emissions of a preformed quantum free-electron wave function, *Phys. Rev. A* **99**, 052107 (2019).
- [40] D. Marcuse, Emission of radiation from a modulated electron beam, *J. Appl. Phys.* **42**, 2255 (1971).
- [41] D. B. Milošević, G. G. Paulus, D. Bauer, and W. Becker, Above-threshold ionization by few-cycle pulses, *J. Phys. B* **39**, R203 (2006).
- [42] M. O. Scully and M. S. Zubairy, *Quantum Optics* (Cambridge University Press, Cambridge, 1997).
- [43] W. Becker, F. Grasbon, R. Kopold, D. B. Milošević, G. G. Paulus, and H. Walther, Above-threshold ionization: From classical features to quantum effects, *Adv. At., Mol., Opt. Phys.* **48**, 35 (2002).
- [44] D. M. Wolkow, Über eine Klasse von Lösungen der Diracschen Gleichung, *Z. Phys.* **94**, 250 (1935).
- [45] L. D. Landau and E. M. Lifshitz, *The Classical Theory of Fields* (Elsevier, Oxford, 1975).
- [46] A. M. Perelomov, V. S. Popov, and V. M. Terent'ev, Ionization of atoms in an alternating electric field. II, *Zh. Exp. Theor. Fiz.* **51**, 309 (1966).
- [47] V. S. Popov, Tunneling and multiphoton ionization of atoms and ions in a strong laser field (Keldysh theory), *Phys. Usp.* **47**, 855 (2004).
- [48] N. Suárez, A. Chacón, M. F. Ciappina, J. Biegert, and M. Lewenstein, Above-threshold ionization and photoelectron spectra in atomic systems driven by strong laser fields, *Phys. Rev. A* **92**, 063421 (2015).
- [49] M. Klaiber, Q. Z. Lv, K. Z. Hatsagortsyan, and C. H. Keitel, Tunneling ionization in ultrashort laser pulses: Edge effect and remedy, *Phys. Rev. A* **105**, 063109 (2022).
- [50] M. Klaiber, K. Z. Hatsagortsyan, and C. H. Keitel, Signatures of under-the-barrier dynamics in a tunneling electron wavepacket, *Commun. Phys.* **7**, 384 (2024).
- [51] V. B. Berestetskii, E. M. Lifshitz, and L. P. Pitevskii, *Quantum Electrodynamics* (Pergamon, Oxford, 1982).