

Ancilla-train quantum algorithm for simulating non-Markovian open quantum systems

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(Received 14 November 2025; accepted 19 August 2026; published 13 October 2026)

We present a quantum algorithm for simulating open quantum systems coupled to Gaussian environments valid for any configuration and coupling strength. The algorithm is applicable to problems with strongly coupled, or non-Markovian, environments; problems with multiple environments out of mutual equilibrium; and problems with time-dependent Hamiltonians. We show that the algorithm can reproduce the true dynamics of such problems at arbitrary accuracy and, for a broad range of problems, only adds a minor resource cost relative to Trotterized time evolution; the cost is low-degree polynomial in the inverse target accuracy. The algorithm is based on the insight that any Gaussian environment can be represented as a train of ancillary qubits that sequentially interact with the system through a time-local coupling, given by the convolution square root of the bath correlation function; this is a secondary result of our work. In numerical simulations of a spin-boson model, we find accurate simulations can be achieved with moderate (10^1 – 10^3) ancilla counts in the intermediate to strong-coupling regimes. Our results open up applications of quantum computers for efficient simulation of nonequilibrium and open quantum systems.

DOI: [10.1103/PhysRevA.114.042422](https://doi.org/10.1103/PhysRevA.114.042422)

I. INTRODUCTION

A key motivation for the development of quantum computers is their ability to dramatically speed up simulation of physical processes governed by quantum mechanics [1]. To harness this potential capability, a diverse range of quantum algorithms have been developed to simulate quantum systems in nature [2–11]. Much effort has focused on isolated and equilibrium quantum systems, which can be characterized in terms of ground states, excitation spectra, or thermal ensembles [5,9,12–22]. At the same time, many important processes in nature involve *open* and *nonequilibrium* quantum systems, which feature dynamics, structured environments, time-dependent forces, or multiple baths out of mutual equilibrium. Nonequilibrium and open quantum systems emerge, e.g., in the contexts of biology [23], chemistry [24–26], or light-harvesting [27,28]. While a variety of classical methods have been developed for these problems [29–43], relatively simple systems remain challenging to simulate on classical computers, e.g., due to non-Markovian effects or the breakdown of thermodynamic principles [35,36,44]. This raises the question: *How can we efficiently harness quantum computers for simulating open and nonequilibrium quantum systems?*

In this work, we address the question above by introducing a quantum algorithm that efficiently simulates the dynamics and steady states of any open quantum system coupled to Gaussian environments. The algorithm, termed the ancilla-train algorithm (ATA), is, for instance, applicable to problems with strongly coupled, or non-Markovian, environ-

ments; problems with multiple environments out of mutual equilibrium; and problems with time-dependent Hamiltonians. The ATA consists of a standard Trotterized evolution where each iteration is augmented with a gate sequence coupling system observables with a finite array—or *train*—of ancillary qubits (ancillas); see Fig. 1. With a suitable but simple system–ancilla-train coupling, we show that the ATA efficiently simulates any open quantum system with Gaussian environments to *arbitrary accuracy* with respect to the true dynamics resulting from the underlying microscopic model.

We provide rigorous error bounds and resource estimates for the ATA. In particular, we demonstrate that the ATA incurs only a minor cost relative to Trotterized Hamiltonian simulation for a broad class of problems: namely a strictly or quasilocal system–ancilla coupling circuit (depending on the class of model), along with a finite ancilla register. As another advantage, we argue that ATA can tolerate a finite hardware error rate because its output—fixed-point trajectories or steady states of open quantum systems—is perturbatively robust against generic noise.

We construct the ATA based on a single key insight: *Any Gaussian environment can be represented by a train of qubits—or quantum binary white noise signal—sequentially interacting with the system through an appropriate time-local interaction.* Specifically, we show that such a representation is accurate when the time-local interaction is proportional to the convolution square root of the bath correlation function, or *jump correlator*, identified in Ref. [39]. We refer to this representation as the *ancilla-train representation* of Gaussian quantum noise (ATR). The ATR offers a systematic time-domain, or collision, model, with qubit representation of any Gaussian quantum environment, complementary to frequency-domain, or pseudomode, bosonic representations [31,32,35,42]. As a key secondary result, we obtain a rigorous bound on the error incurred by truncating the ancilla

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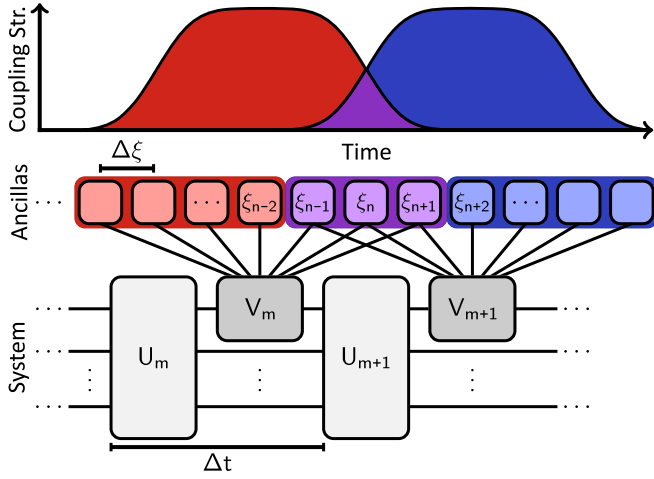


FIG. 1. Ancilla-train algorithm (ATA): The ATA simulates the evolution of any open quantum system with Gaussian environments to arbitrary accuracy via Trotterized evolution. Each iteration—here the m th—propagate the system through a time step Δt via two gate sequences, U_m and V_m , with U_m the time evolution generated by the system Hamiltonian, while V_m connects one or more system observables to a register of ancilla qubits (red, purple, and blue boxes; black lines indicate couplings). After the iteration, a subset of ancillas are carried over to the next $[(m+1)\text{th}]$ iteration (purple); the remainder are discarded (red), and an identical number of new ancillas are (re)introduced in the $|0\rangle$ state (blue). With a suitable system-ancilla coupling structure obtained from the jump correlator of the bath [39], $g(t)$, we show that the ATA can reproduce the true dynamics of the system to arbitrary accuracy with a finite ancilla register.

train of the ATR, enabling a systematic approximate representation of any Gaussian environment with a finite qubit register. Exploiting the natural qubit representation of Gaussian environments offered by the ATR, we obtain the ATA simply by Trotterizing the resulting time evolution.

Interestingly, a construction related to the ATR emerged recently in the open quantum systems literature [43], involving a train of harmonic oscillators, rather than qubits, and a construction based on orthogonal polynomials that does not have the discrete time-translation symmetry of the ATR.

An illustrative special case of the ATR emerges for symmetric power spectral densities, where the environment is equivalent to a classical noise signal acting on the system [44]. Here the ATR implies that any classical noise signal with Gaussian correlations—i.e., colored noise—can be generated as a sufficiently dense sequence of identical pulses with random signs; see Fig. 3. Thus the ATR can be viewed as a quantum noise generalization of the well-known result that a colored noise signal can be generated by filtering a (binary) white noise signal [45]; here the jump correlator provides the filtering kernel.

The ATA complements two other approaches that have recently emerged for systematic quantum simulation of open quantum systems: quantum Gibbs sampling (QGS) algorithms [12,17–22] and Markovian collision-model algorithms based on systematic, nonsecular Lindblad equations that have recently been derived rigorously [39–41,46], here termed systematic Lindblad (SL) algorithms; see Sec V A for a detailed

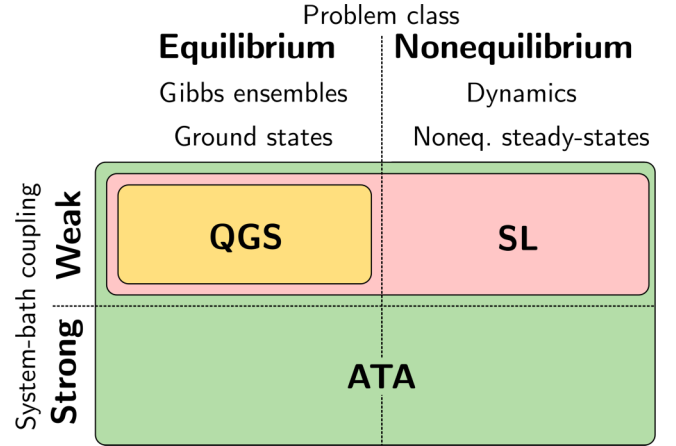


FIG. 2. Regimes of applicability of the ATA, in relation to quantum Gibbs samplers (QGS) [12,17–20], and non-Markovian collision models based on systematic, nonsecular, Lindblad equations (SL algorithms) [39–41,46]; see Secs. I and V A for details.

discussion (see also Ref. [47] for a proposal preceding these developments). We summarize the regimes of applicability and advantages of the ATA, QGS, and SL algorithms in Fig. 2 and Table I. Compared to QGS and SL algorithms, which are generally only accurate in the limit of weak system-environment coupling [48,49], and require compilation of quasilocal jump operators and Lamb shifts, the ATA offers key advantages: (1) the ATA is accurate to all orders in system-environment coupling and can thus efficiently capture non-Markovian effects. (2) the ATA does not require compilation of quasilocal jump-operators or Lamb shifts, i.e., renormalizations of the system Hamiltonian, as required in QGS or SL algorithms, and can thus potentially be implemented with the addition of a low-depth unitary circuit to each Trotter iteration. (3) Like SL but unlike QGS algorithms, the ATA can be applied to nonequilibrium problems and can yield dynamics, fixed-point trajectories, along with steady states. The cost of the ATA relative to QGS and SL algorithms is a register of a finite number of ancilla qubits that scales linearly with system-bath coupling for a given, fixed target accuracy. Thus, the ATA offers both a broad range of applicability *and* an efficient implementation in cases where circuit depth is the limiting resource.

We expect that the advantages of the ATA outlined above make it a promising approach for thermal state preparation and quantum simulation in a broad range of problems involving equilibrium and nonequilibrium quantum systems and non-Markovian dynamics. Moreover, we expect that its potential robustness against finite error rates makes it a good candidate for implementation on near-term quantum hardware.

The rest of the paper is structured as follows: In Sec. II we introduce the problem that we study and present our first main result: the ancilla-train representation of Gaussian environments. In Sec. III we use this result to present our second main result: the ancilla-train quantum algorithm, based on the ATR. We conclude with a discussion, including a more detailed analysis of the relationship among the ATA, QGS, and SL

TABLE I. Comparison of quantum algorithms for systematic open quantum system simulation. We compare the features of ATA, quantum Gibbs sampling (QGS), and systematic Lindblad (SL) algorithms; see the text for definition of these. Here ε denotes the target accuracy of the algorithm, while Γ , τ , and Λ denote the characteristic scales for system-bath coupling, correlation time, and ultraviolet cutoff for the environment defined in Eqs. (17) and (20).

Algorithm	QGS	SL	ATA
Can be realized with			
Quasiloca system-ancilla coupling	✓	✓	✓
Strictly local system-ancilla coupling ^a	✗ ^b	✗	✓
No Lamb shift compilation ^c	✗	✗	✓
Can return			
Thermal state ^d	✓	(✓)	(✓)
Nonequilibrium steady state	✗	✓	✓
Dynamics	✗	✓	✓
Can simulate			
Nonequilibrium baths	✗	✓	✓
Time-dependent Hamiltonians	✗	✓	✓
Non-Markovian regimes	✗	✗	✓
Non-Gaussian environments	✗	✗	✗
No. of ancillas required	1	1 ^e	$\sim \frac{\Gamma\tau}{\varepsilon^2}, \frac{\Lambda\tau}{\varepsilon}$

^aThe ATA involves a strictly local ancilla-system coupling in each Trotter iteration if observables in the simulation coupled to environments can be represented as a local observable in the system register.

^bReferences [19–21] found approximate QGS-like algorithms with strictly local system-ancilla coupling and multiple ancillas, which could yield approximate (not exact) Gibbs states.

^cHere Lamb shift refers to a quasiloca renormalization of the Hamiltonian, required for the Lindblad processes corresponding to QGS and SL algorithms.

^dPhysical systems only converge to Gibbs states in the limit of vanishing system-bath coupling [48]; (✓) indicates that the steady state contains corrections accounting for this fact, while ✓ indicates that the algorithms return the exact Gibbs state.

^eFor example, for collision models based on the Lindblad equations from Refs. [39,41].

algorithms, in Sec. V. Detailed error bounds, estimates, and their proofs are found in the appendices.

II. ANCILLA-TRAIN REPRESENTATION OF GAUSSIAN ENVIRONMENTS

Here we introduce the class of models for open quantum systems that can be simulated with the ATA (Sec. II A) and provide our first main result, namely that any Gaussian environment can be represented by a train of ancilla qubits that sequentially act with the system (Sec. II B). We then demonstrate that this train can be truncated to have a finite number of qubits at the cost of a controllable error that vanishes in the limit of many qubits. This result underlies the ATA. As a special case, we finally demonstrate that classical noise signals with Gaussian correlations can be represented by a discrete binary noise signal convolved with the jump correlator of the bath power spectral density (PSD)—defined as the inverse Fourier transform of the square root of the bath PSD.

A. Introduction of problem

We consider an arbitrary quantum system, $\mathcal{S}$, coupled to one or more Gaussian environments, henceforth referred to as the bath, $\mathcal{B}$. See below and Appendix A for a definition of Gaussian environments. We allow for the system Hamiltonian, $H_S(t)$, to be time dependent. The Hamiltonian of the combined system we consider thus reads

$$H_{SB}(t) = H_S(t) + H_B + H_{\text{int}}, \quad (1)$$

where H_B is the Hamiltonian of $\mathcal{B}$ and the system-bath interaction H_{int} contains all terms of $H_{SB}(t)$ that act nontrivially on both $\mathcal{S}$ and $\mathcal{B}$. In general, H_{int} can be written in the form

$$H_{\text{int}} = \sqrt{\gamma} \sum_{\alpha} S_{\alpha} \otimes B_{\alpha}, \quad (2)$$

with S_{α}, B_{α} Hermitian operators (observables) of the system and bath, respectively, while γ is an arbitrary energy scale parametrizing the system-bath coupling. We refer to each term in the sum above as a *noise channel*. We assume S_{α} to be bounded. Without loss of generality, we may therefore set $\|S_{\alpha}\| = 1$ [50] and assume $\langle B_{\alpha}(t) \rangle = 0$ for each α ; any nonzero expectation value can be removed by modifying $H_S(t)$. For simplicity, our discussion mainly considers the case where the sum above holds a single term, and hence we suppress the subscript α below. In Sec. III C we present our results for the general case.

Furthermore, we assume that the system is decoupled from the bath at the beginning of the simulation, i.e., that the system-bath state is separable at some initial time t_0 ,

$$\rho_{SB}(t_0) = \rho(t_0) \otimes \rho_B. \quad (3)$$

Our goal is to obtain the resulting evolution of the system, fully described by its reduced density matrix,

$$\rho(t) := \text{Tr}_B[\rho_{SB}(t)]. \quad (4)$$

with $\rho_{SB}(t)$ denoting the time evolution of $\rho_{SB}(t_0)$ generated by $H_{SB}(t)$, and Tr_B denoting partial trace over the environment Hilbert space.

The Gaussian property of $\mathcal{B}$ implies that the correlation functions of B with respect to H_B and ρ_B are fully determined from the two-point correlation function, and structured according to Wick's theorem [44]; see Appendix A for details. This property emerges, e.g., when $\mathcal{B}$ consists of a set or continuum of bosonic modes, with B a linear combination of mode creation and annihilation operators and ρ_B a Gaussian state with respect to these modes [44]. More broadly, Gaussianity emerges in the limit where the bath consists of many effectively decoupled subsystems, no matter their internal structure. For this reason, Gaussian environments are ubiquitous, emerging in a broad range of settings, including electromagnetic, phononic, electronic, vibronic, and spin environments [44,51]. The universality of Gaussian environments means that they can be represented in many equivalent ways, e.g., in terms of configurations of discrete dissipative bosonic modes [31,36]. In the next section, we will identify one such way of efficiently representing Gaussian environments which is particularly useful for quantum simulation.

B. Ancilla-train representation

We now show how the bath can be represented as a train of ancilla qubits sequentially interacting with the system. We first show that the bath can be represented as a train of harmonic oscillators sequentially interacting with the system, all initialized in the ground state. A bosonic construction for quantum collision models, similarly to ours, was recently proposed in Ref. [43] for Gaussian baths with a hard ultraviolet (UV) cutoff. Subsequently, we show that the probability for excitation to the second excited state can be neglected, implying that the harmonic oscillators can be replaced by qubits initialized in their $|0\rangle$ states.

First, we transform to the interaction picture with respect to H_B [52], introducing $B(t) := e^{iH_B t} B e^{-iH_B t}$. Note that the reduced density matrices of $\mathcal{S}$ are identical in the interaction and Schrödinger pictures. In the interaction picture, when restricting to a single term in Eq. (2), the Hamiltonian of the combined system becomes

$$H_{SB}^{(1)}(t) = H_S(t) + \sqrt{\gamma} S \otimes B(t). \quad (5)$$

The observable $B(t)$ can be viewed as a quantum generalization of a classical noise signal [51]; we henceforth refer to it as the *quantum noise signal* generated by the bath.

We next exploit that $\mathcal{B}$ is Gaussian. This means that, in the state ρ_B , all correlation functions of $B(t)$ are determined by the two-point correlation function, according to Wick's theorem [44]. Here the two-point correlation function is given by

$$J(t - t') := \text{Tr}[B(t)B(t')\rho_B], \quad (6)$$

The above fact means that the system evolution $\rho(t)$ can be reproduced by substituting the quantum noise signal $B(t)$ in Eq. (5) with *any* observable $A(t)$ whose correlations are structured according to Wick's theorem and produce the two-point correlation function $J(t)$. This defines a broad equivalence class of observables that emulate the same environment as $B(t)$.

We now identify *two* observables, which are equivalent to $B(t)$ in the sense above and offer efficient representations of the environment for quantum simulation. The first, $A_{\text{osc}}(t)$, is a time-dependent linear combination of creation and annihilation operators in an array, or *train*, of bosonic modes. We use this to obtain the second, $A_q(t)$, which is analogously constructed from a train of *qubits*, and forms the basis for the ATA in the next section.

We construct $A_q(t)$ and $A_{\text{osc}}(t)$ using the *jump correlator* from Ref. [39], $g(t)$, defined as

$$g(t) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} d\omega \sqrt{\tilde{J}(\omega)} e^{-i\omega t} \quad (7)$$

with $\tilde{J}(\omega) := \frac{1}{2\pi} \int_{-\infty}^{\infty} dt e^{i\omega t} J(t)$ the power spectral density of $B(t)$. Note that $\tilde{J}(\omega)$ is always non-negative so the square root is non-negative as well. The jump correlator g is the unique positive-semidefinite *convolution square root* of J , i.e., $J = g * g$ with $*$ denoting convolution [53]:

$$J(t - t') = \int_{-\infty}^{\infty} ds g(t - s) g(s - t'). \quad (8)$$

We construct $A_{\text{osc}}(t)$ by introducing an ancillary system consisting of infinitely many independent harmonic oscillators, or free bosonic modes, $\mathcal{A}_{\text{osc}}$, with bosonic annihilation operators $\{a_n | n \in \mathbb{Z}\}$. We initialize $\mathcal{A}_{\text{osc}}$ in the vacuum state of all ancilla modes, $\rho_{\text{osc}}^0 := |\mathbf{0}\rangle\langle\mathbf{0}|$ with $|\mathbf{0}\rangle := \bigotimes_{n=-\infty}^{\infty} |0\rangle_n$, and $|0\rangle_n$ the vacuum state of mode n : $a_n|0\rangle_n = 0$. We construct $A_{\text{osc}}(t)$ as a sliding linear combination of creation and annihilation operators:

$$A_{\text{osc}}(t) := \sqrt{\Delta\xi} \sum_{n=-\infty}^{\infty} (g^*(t - \xi_n) a_n^\dagger + \text{H.c.}), \quad (9)$$

where $\xi_n = n\Delta\xi$, and $\Delta\xi$ is a timescale of our choice. It is straightforward to verify that, in the state ρ_{osc}^0 , $A_{\text{osc}}(t)$ satisfies Wick's theorem, and has two-point correlation function

$$\text{Tr}[A_{\text{osc}}(t)A_{\text{osc}}(t')\rho_{\text{osc}}^0] = \sum_{n=-\infty}^{\infty} \Delta\xi g(t - \xi_n) g(\xi_n - t'). \quad (10)$$

We identify the right-hand side above with that of Eq. (8) with a discretized integral. Thus the above expectation value converges to $J(t)$ in the limit $\Delta\xi \rightarrow 0$. Since the correlation functions of $A_{\text{osc}}(t)$ satisfy Wick's theorem, and have two-point correlation function $J(t)$, $A_{\text{osc}}(t)$ reproduces the same evolution as $B(t)$ in the limit $\Delta\xi \rightarrow 0$. In Appendix B 2 we provide a rigorous proof for this fact, along with a bound for the error incurred by choosing $\Delta\xi$ nonzero.

We now identify the second observable, $A_q(t)$, that represents the bath in terms of *qubits*. First, note that leakage to the second excited state of any ancilla oscillator in $\mathcal{A}_{\text{osc}}$ is negligible in the limit $\Delta\xi \rightarrow 0$. Even for nonzero $\Delta\xi$, the leakage error can be controlled (see Appendix B 3). This allows us to project the ancilla system into the subspace of the ground and first-excited state of each oscillator at a vanishing error cost. This effectively replaces the harmonic oscillators in $\mathcal{A}_{\text{osc}}$ with *qubits*. Building on this result, we introduce the *qubit* ancilla system, $\mathcal{A}$, consisting of infinitely many qubits, with Pauli operators $\{(\sigma_n^x, \sigma_n^y, \sigma_n^z) | n \in \mathbb{Z}\}$. We initialize the ancilla system in the mutual $|0\rangle$ state of all qubits $\rho_A^0 := \bigotimes_{n=-\infty}^{\infty} |0\rangle_n \langle 0|_n$ and introduce the energy-truncated ancilla-train operator,

$$A_q(t) := \sqrt{\Delta\xi} \sum_{n=-\infty}^{\infty} (g^*(t - \xi_n) \sigma_n^+ + \text{H.c.}), \quad (11)$$

with $\sigma_n^\pm := \frac{1}{2}(\sigma_n^x \pm i\sigma_n^y)$ the usual ladder operators of qubit n . By the arguments above, in the limit $\Delta\xi \rightarrow 0$, this signal satisfies Wick's theorem and has the correct two-point correlation function; $J(t)$. Thus, $B(t)$ and $A_q(t)$ result in the same system dynamics in this limit.

The above result shows that any Gaussian environment, or quantum noise signal $B(t)$, can be represented by an array, or *train* of qubits that sequentially interact with the system in a time-dependent manner controlled by the jump correlator. This ATR of the quantum noise signal converges to the *exact* quantum noise signal as we take the limit $\Delta\xi \rightarrow 0$.

ATR for classical noise

An illuminating example of the ATR emerges in the special case when the bath PSD is *even*, or, equivalently, the correlation function is real valued [51]. In this case, $B(t)$ can

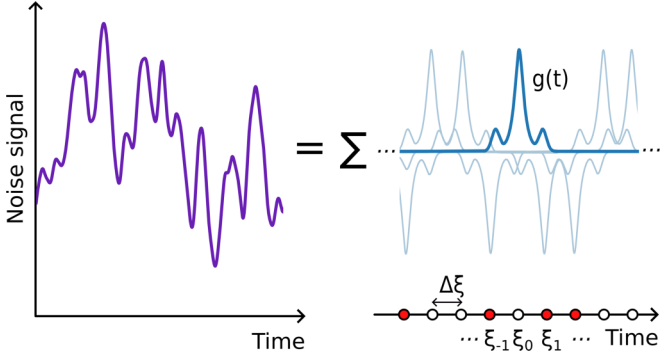


FIG. 3. Ancilla-train representation of classical colored noise: Environments with even power spectral densities are equivalent to classical noise signals acting on the system [44]. Here our results imply that any classical noise signal with Gaussian correlations—or colored noise signals—can be generated by summing a sequence of identical pulses with random signs (red and white dots) and spacing $\Delta\xi$, and taking the limit $\Delta\xi \rightarrow 0$. Here the pulse is given by $g(t)\sqrt{\Delta\xi}$ [45].

be represented as a *classical*, scalar-valued, noise signal with Gaussian correlations—or colored noise signal—acting on the system through S . In this case, $g(t)$ is real valued, and Eq. (15) becomes $A_q(t) = \sqrt{\Delta\xi} \sum_n g(t - \xi_n) \sigma_n^x$. Note that σ_n^x is now an integral of motion of the total time evolution. Since each ancilla is initialized in an even-weighted superposition of the two eigenstates of σ_n^x , tracing out the ancilla is equivalent to replacing σ_n^x with 1 and -1 at probability 1/2 each. We may therefore replace each σ_n^x with an independent random variable x_n uniformly sampled from $\{-1, 1\}$. This replaces $A_q(t)$ with the following stochastic (scalar-valued) function:

$$A_{cl}(t) = \sqrt{\Delta\xi} \sum_n g(t - \xi_n) x_n. \quad (12)$$

Since the right-hand side above is simply a scalar-valued stochastic function, we identify it with a classical noise signal. This special case of the ancilla-train representation thus shows that *a classical Gaussian noise signal can always be generated by convolving its jump correlator with a white noise signal*. We provide an illustration of this result in Fig. 3.

The above results show how the ATR can be viewed as a quantum noise generalization of the known result from signal processing that a colored noise signal can be generated by filtering a white noise signal [45]; here we identify the jump correlator as the filtering kernel.

C. Truncating ATR to finite ancilla number

We now demonstrate that the infinite ancilla train of the ATR can be well approximated by an ancilla-train quantum noise signal with a finite register: We show that a quantum noise signal $B(t)$ can be substituted with an emulating noise signal of the form in Eq. (15), with the sum truncated to a finite number of terms by picking a *finite* step size $\Delta\xi$ and discarding terms in the sum with $|t - \xi_n| \leq \tau_c$, for some cutoff time τ_c .

We prove in Appendix B 2 that the ATR can be truncated to a finite ancilla register as above at the cost of an arbitrary

small error, if we simultaneously replace $g(t)$ with a *filtered* jump correlator [54], defined as

$$g_f(t) := \int_{-\infty}^{\infty} ds g(s) \varphi(t - s) \quad (13)$$

with $\varphi(t) = \frac{1}{2\pi} \int d\omega \tilde{\varphi}(\omega) e^{-i\omega t}$ a low-pass filtering kernel satisfying $\tilde{\varphi}(\omega) = 0$ for $|\omega| \geq \pi/\Delta\xi$ and $\tilde{\varphi}(\omega) = 1$ for $|\omega| < \pi/2\Delta\xi$. In between these two ranges, $\tilde{\varphi}$ should have uniformly bounded first and second derivatives (almost everywhere), implying that $g_f(t)$ decays on approximately the same timescale as $g(t)$ —in Appendix B 2 we provide a particular construction of $\tilde{\varphi}(\omega)$ via polynomial interpolation. Below, we present three operators, which faithfully reproduce the evolution generated by $B(t)$ when choosing $\Delta\xi$ sufficiently small and choosing a sufficiently large cutoff time, τ_c :

$$A_{osc,f}(t) := \sqrt{\Delta\xi} \sum_{n=-\infty}^{\infty} (g_f^*(t - \xi_n) a_n^\dagger + \text{H.c.}), \quad (14)$$

$$A_{q,f}(t) := \sqrt{\Delta\xi} \sum_{n=-\infty}^{\infty} (g_f^*(t - \xi_n) \sigma_n^+ + \text{H.c.}), \quad (15)$$

$$A_T(t) := \sqrt{\Delta\xi} \sum_{n=n_-(t)}^{n_+(t)} (g_f^*(t - \xi_n) \sigma_n^+ + \text{H.c.}), \quad (16)$$

with $n_{\pm}(t) = \text{round}([t \pm \tau_c]/\Delta\xi)$. We will use the first two observables as intermediate steps to bound the error for substituting $B(t)$ with $A_T(t)$, which only requires access to a finite ancilla register of size $\sim \tau_c/\Delta\xi$, and serves as a basis for the ATA. In the following, we define this error as the trace norm distance between the evolution, $\rho_T(t)$, generated with the quantum noise signal $A_T(t)$ with positive $\Delta\xi$ and finite τ_c , and the true evolution, $\rho(t)$, resulting from $B(t)$.

Our error bound depends on two quantities of the bath: a characteristic system-bath interaction rate, Γ , and the characteristic correlation time of the bath, τ , defined as

$$\Gamma = 4\gamma \left[\int_{-\infty}^{\infty} dt |g(t)| \right]^2 \quad \text{and} \quad \tau = \frac{\int_{-\infty}^{\infty} dt |g(t)|}{\int_{-\infty}^{\infty} dt |g(t)|}. \quad (17)$$

Importantly, Γ defines an upper bound for the rate of bath-induced evolution [39].

Below, we identify values of $\Delta\xi$ and τ_c such that the error induced by the truncation is smaller than some given target relative accuracy, ε . Specifically, we require that

$$\|\rho(T) - \rho_T(T)\|_{tr} = O(\varepsilon \Gamma T), \quad (18)$$

where $\|\cdot\|_{tr}$ denotes the trace norm. We employ a triangle inequality argument to bound this error by a sum of three contributions: The first error, ε_1 , arises from substituting $B(t)$ with $A_{osc,f}(t)$. The second error, ε_2 , arises from substituting the oscillator signal $A_{osc,f}(t)$ with $A_{q,f}(t)$. The final error, ε_3 arises from discarding terms involving $|\xi_n - t|$ beyond the cutoff time τ_c , resulting in $A_T(t)$. We discuss each of these three errors separately below.

The discretization error ε_1 can be bounded by a quantity depending on the effective UV cutoff of the bath PSD, i.e., an energy beyond which $\tilde{J}(\omega)$ effectively vanishes. Specifically, we find in Appendix B 2, that the substitution of $B(t)$ with $A_{osc,f}(t)$ induces a correction to $\rho(T)$ with trace norm of order

at most $O(\epsilon(\Omega)\Gamma T)$, when choosing $\Delta\xi \leq \pi/\Omega$, with

$$\epsilon(\Omega) = \frac{4\sqrt{(\int_{|\omega|>\Omega} d\omega \tilde{J}(\omega))(\int_{|\omega|>\Omega} d\omega |\tilde{J}''(\omega)|)}}{[\int_{-\infty}^{\infty} dt |g(t)|]^2}. \quad (19)$$

In particular, $\epsilon(\Omega) \rightarrow 0$ as $\Omega \rightarrow \infty$ when $\tilde{J}$ and $\tilde{J}''$ are integrable. For such power spectral densities, we reach the target accuracy by choosing the following UV cutoff:

$$\Lambda(\epsilon) := \inf\{\Omega \geq 1/\tau | \epsilon(\Omega) < \epsilon\}. \quad (20)$$

The requirement that $\Omega \geq 1/\tau$ ensures that the frequency filtering does not significantly alter Γ and τ (see Appendix C for details). We expect that $\Lambda(\epsilon)$ can effectively be chosen to be given by the effective spectral width of the observable coupled to the bath, which is equivalent to the Trotter step in the algorithm, and hence $\Delta\xi$ can be chosen to be of the same order as the Trotter step.

We now consider the error ϵ_2 from using qubits instead of harmonic oscillators. This error is small whenever leakage to the second excited states of the harmonic oscillators is negligible. The leakage vanishes when $\Delta\xi$ goes to zero, as the strength of the interaction with each ancillary qubit becomes vanishingly small. In Appendix B 3 we make this intuition precise and show that the error stays within the target accuracy when choosing $\Delta\xi \leq \frac{\epsilon}{\Gamma}$.

Finally, we consider the error ϵ_3 from discarding ancilla qubits whose distance from a given time $|\xi_n - t|$ exceeds the cutoff τ_c . In Appendix B 4 we show that ϵ_3 is within the target accuracy when choosing $\tau_c \geq \frac{\tau}{\epsilon}$.

Together, the above results imply that $A_T(t)$ reproduces the correct dynamics up to a relative target accuracy ϵ when choosing

$$\Delta\xi \leq \min\left(\frac{\epsilon}{\Gamma}, \frac{\pi}{\Lambda(\epsilon)}\right), \quad \tau_c \geq \frac{\tau}{\epsilon}. \quad (21)$$

With these parameters, $A_T(t)$ only involves a *limited* number of ancilla qubits at any given point in time, given by

$$N = \max\left(\frac{2\Gamma\tau}{\epsilon^2}, \frac{2\Lambda(\epsilon)\tau}{\pi\epsilon}\right). \quad (22)$$

Importantly, we note that the number of ancillas required scales as a low-degree polynomial in ϵ^{-1} when the bath PSD decays sufficiently fast.

Our results above show that any Gaussian environment can accurately be represented by $A_T(t)$, even in the case of an infinite number of unbounded degrees of freedom. This is significant since $A_T(t)$ is constructed from a finite ancillary register of N qubits. Importantly, $A_T(t)$ is *bounded*; with the choice for $\Delta\xi$ above, $\sqrt{\gamma}\|A_T(t)\| \sim \max\{\Gamma/\sqrt{\epsilon}, \sqrt{\Lambda\Gamma}\}$ [55]. This fact enables efficient Trotterization of the system evolution with the ancilla-train representation. In the next section, we exploit this fact to construct an efficient quantum algorithm for simulating dynamics of open quantum systems connected to arbitrary Gaussian environments at arbitrary coupling strength.

III. ANCILLA-TRAIN ALGORITHM

We now use the ATR to obtain our main result: a quantum algorithm for systematically simulating open quantum sys-

Ancilla-train algorithm (2nd-order Trotterized)

Input $\rho(t_0)$, $\tilde{J}(\omega)$, H_S , S , $\gamma > 0$, $T > 0$, $\epsilon > 0$
Set $\Lambda \geq 1/\tau$ with $\epsilon(\Lambda) < \epsilon$ $\triangleright$ See Eq. (19);
 $\tau_c = \frac{\tau}{\epsilon}$; $\Delta\xi = \min(\frac{\epsilon}{\Gamma}, \frac{\pi}{\Lambda})$;
 $\Delta t = \frac{\epsilon}{\sqrt{\Gamma\|H_S, S\|}}$; $M = \text{round}(\frac{T}{\Delta t})$
Prepare $|\Psi\rangle \sim \rho(t_0)$ $\triangleright$ Sample from initial state
for $m \in \{1, 2, 3, \dots, M\}$ **do**
 Load new ancillas used in step m
 Evolve $|\Psi\rangle \rightarrow U_m^{1/2} V_m U_m^{1/2} |\Psi\rangle$ $\triangleright$ Trotter step
 Discard/reset ancillas not used in step $m + 1$.
end for
Discard remaining ancillas
Return $|\Psi\rangle|_S \sim \rho(M\Delta t)$ to accuracy $\mathcal{O}(\epsilon\Gamma T)$

FIG. 4. Pseudocode for the ancilla-train algorithm. Input parameters denote the initial system state $\rho(t_0)$, bath PSD $\tilde{J}(\omega)$, system Hamiltonian H_S , system coupling operator S , system-bath coupling γ , total simulation time T , and (relative) target accuracy ϵ . U_m , given in (26), denotes a Trotter interaction generated by the system Hamiltonian and V_m , given in (27) and (29), denotes a Trotter iteration in the time evolution generated by the ancilla(s)-system coupling. We write $|\Psi\rangle \sim \rho$ to indicate that the expected value of $|\Psi\rangle\langle\Psi|$ is ρ .

tems coupled to any configuration of Gaussian environments at any coupling strength. The ATA is obtained by Trotterizing the combined system-ancilla (SA) evolution in the ATR, using the truncated ancilla-train quantum noise signal $A_T(t)$. We provide a compact pseudocode of the ATA in Fig. 4.

The section is structured as follows: For the sake of simplicity, we first introduce the ATA for a single noise channel (Sec. III A) [56] and provide a resource estimate (Sec. III B). We then extend these results to the most general (multichannel) configuration of Gaussian environments in Sec. III C.

A. Construction of ATA

The ATA simulates the evolution of the open quantum system of the class defined in Sec. II A: i.e., an arbitrary physical system S with (possibly time-dependent Hamiltonian) $H_S(t)$, connected to any configuration of Gaussian environments $\mathcal{B}$, through an arbitrary system-bath coupling H_{int} . For simplicity, we first consider the case where H_{int} factorizes into a product of a system observable S and a bath observable B : $H_{\text{int}} = \sqrt{\gamma}SB$, with $\|S\| = 1$. Here γ parametrizes the system-bath coupling and can be arbitrarily large. We consider the more general case in Sec. III C.

The ATA operates on a system qubit register $\mathcal{R}_S$, encoding the state of the system S , and an ancilla register, $\mathcal{A}$, used to emulate the bath $\mathcal{B}$ through the ATR, as described in Sec. II. The system can be encoded in $\mathcal{R}_S$ through standard Hamiltonian simulation techniques [2, 57–62]. Henceforth, we simply use S in place of $\mathcal{R}_S$, while S and $H_S(t)$ refer to the observable and Hamiltonian in $\mathcal{R}_S$ representing their counterparts in the simulated system.

The ATA takes as input the initial state of S in the simulation, $|\psi_0\rangle_S$ [63]. The ATA then simulates the evolution of $\rho(t)$

resulting from $|\psi_0\rangle$ by iteratively propagating the system forward in time through the Trotterized evolution: Specifically, the m th iteration of the ATA returns a state of $\mathcal{SA}$, $|\Psi_m\rangle$, whose corresponding reduced density matrix,

$$\rho_m := \text{Tr}_{\mathcal{A}}[|\Psi_m\rangle\langle\Psi_m|], \quad (23)$$

provides a controlled approximations of $\rho(t_m)$ (see below). Here $t_m := m\Delta t$ and Δt , termed the *Trotter step*, is an input parameter of the simulation that controls the accuracy and output resolution.

The first step of the ATA prepares the full system-ancilla register $\mathcal{SA}$ in the initial state $|\Psi_0\rangle := |\psi_0\rangle_{\mathcal{S}} \otimes |\mathbf{0}\rangle_{\mathcal{A}}$ with $|\mathbf{0}\rangle := \bigotimes_n |0\rangle_n$ the tensor product of $|0\rangle$ states in $\mathcal{A}$. The ATA then iteratively generates the trajectory $\{|\Psi_m\rangle\}$ through Trotterized evolution with $H_{\mathcal{SA}}(t) = H_{\mathcal{S}}(t) + \sqrt{\gamma}SA_{\mathcal{T}}(t)$. Specifically, $|\Psi_m\rangle$ is iteratively obtained through a sequence of gate sequences—*Trotter unitaries*— $G_1, G_2, \dots$, where the m th Trotter unitary, G_m , propagates the system forward from time t_{m-1} to t_m :

$$|\Psi_m\rangle = G_m|\Psi_{m-1}\rangle. \quad (24)$$

We use a second-order Trotterization scheme, such that G_m is given by

$$G_m = U_m^{1/2} V_m U_m^{1/2}, \quad (25)$$

where U_m and V_m are the Trotter unitaries generated by $H_{\mathcal{S}}(t)$ and $\sqrt{\gamma}SA_{\mathcal{T}}(t)$:

$$U_m := \mathcal{T} \exp \left[-i \int_{t_{m-1}}^{t_m} ds H_{\mathcal{S}}(s) \right], \quad (26)$$

$$V_m := \mathcal{T} \exp \left[-i \int_{t_{m-1}}^{t_m} ds \sqrt{\gamma} S \otimes A_{\mathcal{T}}(s) \right], \quad (27)$$

with $\mathcal{T}$ denoting the time-ordering symbol and $A_{\mathcal{T}}(t)$ the emulated bath signal from Eq. (15)

$$A_{\mathcal{T}}(t) := \sqrt{\Delta\xi} \sum_{n=n_-(t)}^{n_+(t)} (g_f^*(t - \xi_n) \sigma_n^+ + \text{H.c.}). \quad (28)$$

Here $\xi_n := n\Delta\xi$, $n_{\pm}(t) = \text{round}([t \pm \tau_c]/\Delta\xi)$, with $\Delta\xi$ and τ_c input parameters of the simulation that, together with Δt , control the accuracy of the simulation; see below for more details.

The first gate sequence of the Trotter unitary, $U_m^{1/2}$, can be implemented with existing Hamiltonian simulation techniques [2, 57–62]. The bath propagator, V_m , is itself a product of commuting unitaries that each are generated by coupling S to a single qubit in $\mathcal{A}$: with

$$V_m = \prod_{n=n_-(t_{m-1})}^{n_+(t_m)} V_{mn}, \quad (29)$$

$$V_{mn} := \mathcal{T} \exp \left[-i \sqrt{\gamma} \Delta\xi \int_{t_{mn}^{(0)}}^{t_{mn}^{(1)}} ds S \otimes (g_f^*(s - \xi_n) \sigma_n^+ + \text{H.c.}) \right],$$

where $t_{mn}^{(0)} := \max(t_{m-1}, \xi_n - \tau_c)$ and $t_{mn}^{(1)} := \min(t_m, \xi_n + \tau_c)$; $t_{mn}^{(0/1)}$ denote the boundaries of the integration domain, accounting both for the restriction from a finite τ_c and from the bounds of the time interval corresponding to the Trotter

step. Note that for local or quasilocal S , V_{mn} can be efficiently compiled to a finite sequence of quantum gates and may even be classically tractable. Thus, we expect the compilation of V_{mn} to generically be low cost.

In Appendix B we show that the ATA returns the true system evolution to any nonzero target accuracy, ε , for $\Delta\xi$, $1/\tau_c$, and Δt chosen sufficiently small. We establish this result by combining the discussion in Sec. II with an error bound for second-order Trotterization of the time evolution [64]. Our bound assumes a time-independent system Hamiltonian, $H_{\mathcal{S}}(t) = H_{\mathcal{S}}$ [65], but we expect generalization to time-dependent Hamiltonians is straightforward. Specifically, we show that the output trajectory of the ATA is guaranteed to reproduce the true system evolution to a given (relative) target accuracy ε ,

$$\|\rho_m - \rho(t_m)\|_{\text{Tr}} \leq O(\varepsilon \Gamma t_m), \quad (30)$$

provided

$$\Delta\xi \leq \min\left(\frac{\varepsilon}{\Gamma}, \frac{\pi}{\Lambda(\varepsilon)}\right), \quad \tau_c \geq \frac{\tau}{\varepsilon}, \quad \Delta t \leq \frac{\varepsilon}{\sqrt{\Gamma s}}. \quad (31)$$

where $s := \|H_{\mathcal{S}}, S\|$, while Γ , τ , and Λ are defined in Eqs. (17) and (20). A more detailed requirement on Δt for the bound for nonvanishing ε can be found in Appendix B 5. The results are obtained by combining the error bounds in Sec. II with an error bound for the Trotterization of the evolution, ε_4 ; see Appendix B for details.

We finally emphasize that the ancilla qubits in $\mathcal{A}$ need only be prepared at the first Trotter step where they are used: i.e., ancilla n needs only be prepared at Trotter step $m_n^{(i)} := \inf\{m : n \leq n_+(m\Delta t)\}$. Likewise, ancilla n may be discarded after the last Trotter step where it affects the system evolution: $m_n^{(f)} := \sup\{m : n \geq n_-(m\Delta t)\}$. Depending on the hardware, in particular, the resource cost of qubits, discarded ancillas may be reset and reintroduced in the register. In this scheme, $\mathcal{A}$ only needs to hold $n_+(t_m) - n_-(t_{m-1}) = \lceil \frac{2\tau_c + \Delta t}{\Delta\xi} \rceil$ qubits; among these, $N_{\text{reset}} := \lceil \Delta t / \Delta\xi \rceil$ are reset and reintroduced in the register, and the remaining qubits are preserved, at every Trotter step. It is still possible to implement the ancilla register as a continuous stream of incoming qubits in hardware, where this is more resource efficient [66, 67].

B. Resource cost

We now estimate the additional resource cost of the ATA compared to the standard unitary time evolution generated by U_m .

We first consider the number of extra qubits required to implement the ATA—i.e., the minimal number of qubits in the ancilla register, $\mathcal{A}$. Recall from the discussion below Eq. (31) that the ATA only requires access to $N_{\mathcal{A}} = \lceil \frac{\Delta t + 2\tau_c}{\Delta\xi} \rceil$ qubits in the ancilla register. In terms of error scaling, the number of required ancillas is thus upper bounded by $\max(\sqrt{\frac{\Gamma}{s}} + \frac{2\Gamma\tau}{\varepsilon^2}, \frac{\Lambda\varepsilon}{\pi\sqrt{\Gamma s}} + \frac{2\Lambda\tau}{\pi\varepsilon})$. In the limit $\varepsilon \rightarrow 0$, the required number of ancillary qubits thus scales as

$$\text{No. of ancillas} = O\left(\max\left[\frac{\Gamma\tau}{\varepsilon^2}, \frac{\Lambda(\varepsilon)\tau}{\varepsilon}\right]\right). \quad (32)$$

Next, we consider the gate complexity of implementing ATA relative to standard unitary evolution. Each Trotter step V_m involves a product of $N_A = \lceil \frac{2\tau_c + \Delta t}{\Delta \xi} \rceil$ unitaries V_{mn} . As described in Sec. III A, each V_{mn} is generated by a coupling of S with observables of the n th qubit in $\mathcal{A}$. Therefore, we expect V_{mn} can be efficiently compiled to a finite sequence of quantum gates. The additional gate count of the ATA compared to the standard Trotterized Hamiltonian simulation thus scales as $N_A T / \Delta t$, which leads to

$$\text{No. of ancilla gates} = O\left(\max\left[\frac{\tau T \sqrt{\Gamma^3 \dot{s}}}{\varepsilon^3}, \frac{\Lambda(\varepsilon) \tau T \sqrt{\Gamma \dot{s}}}{\varepsilon^2}\right]\right). \quad (33)$$

If we pick $\Delta \xi$ commensurate with Δt , i.e., $\Delta \xi / \Delta t = a/b$ for integers a, b , then the gate V_m is identical to $V_{m+\text{lcm}(a,b)}$ up to a shift of the ancilla register, where "lcm" denotes the least common multiple. In this case, the gates $\{V_m\}$ may be efficiently precompiled. Given a set of unitary gates for simulating the isolated system evolution $\{U_m^{1/2}\}$, the compilation of G_m follows easily. For time-independent H_S , all $\{U_m^{1/2}\}$ are identical, and thus, the number of precompiled G_m is independent of the simulation time T .

Our resource estimates for the ATA in Eqs. (32) and (33) show that the extra resource cost of the ATA, relative to the standard Trotterized evolution, is a low-degree polynomial of the inverse target accuracy when the bath PSD decaying sufficiently fast. Importantly, the extra cost is *independent* of the complexity of the Hamiltonian simulation problem. Thus, the extra cost of replacing standard Trotterized evolution with the ATA may be negligible for a broad class of problems.

C. Definition for multichannel environments

We finally demonstrate how the ATA can be straightforwardly extended to arbitrary Gaussian environments and arbitrary system-bath coupling, H_{int} .

Without loss of generality [39], in this case we may express H_{int} as

$$H_{\text{int}} = \sqrt{\gamma} \mathbf{S} \cdot \mathbf{B}. \quad (34)$$

Here $\mathbf{S}$ and $\mathbf{B}$ represent the column vectors $(S_1, \dots, S_{N_C})$ and $\mathbf{B} = (B_1, \dots, B_{N_C})$, respectively, with S_α, B_α denoting observables on S and B , respectively, and N_C denoting the total number of noise channels. We employ this vector notation to emphasize the similarity to the single-channel case. The bath correlation function now becomes matrix-valued, $\mathbf{J}(t)$, with (α, β) element given by

$$J_{\alpha\beta}(t-s) := \text{Tr}[B_\alpha(t)B_\beta(s)\rho_B], \quad (35)$$

We extend the definition of the jump correlator, $\mathbf{g}(t)$, to be the unique convolution matrix square root of $\mathbf{J}$ of positive type, i.e., $\mathbf{g}(t) := \int d\omega e^{-i\omega t} \sqrt{\mathbf{J}(\omega)} / 2\pi$ with $\sqrt{\cdot}$ denoting the matrix square root with branch cut along the negative real axis [68].

With these definitions in place, we extend the ATA from the single- to the multichannel case as follows: First, we extending the ancilla register $\mathcal{A}$ from one to N_C ancilla trains, i.e., an array of qubits with Pauli operators $\{\sigma_{\lambda n}^i | \lambda = 1, \dots, N_C, n \in \mathbb{Z}, i \in \{x, y, z\}\}$ [69]. From these ancilla trains, we can replace

each $B_\alpha(t)$ with the truncated, emulating noise signal $A_{T,\alpha}(t)$, where

$$A_{T,\alpha}(t) := \sum_{\lambda=1}^{N_C} \sum_{n=n_-(t)}^{n_+(t)} \sqrt{\Delta \xi} (g_{f,\alpha\lambda}^*(t - \xi_n) \sigma_{\lambda n}^+ + \text{H.c.}), \quad (36)$$

where $g_{f,\alpha\lambda}(t)$ denotes the (α, λ) entry of the filtered jump correlator $\mathbf{g}_f(t) := \int ds \varphi(t-s) \mathbf{g}(s)$. Note that each ancilla train does not necessarily emulate a single channel; instead, each channel is generally generated from combinations of all ancilla trains.

The ATA is defined for the multichannel case as for the single-channel case, with the replacement of the system-ancilla coupling V_m to account for the multiple channels, i.e.,

$$V_m := \mathcal{T} \exp \left[-i \int_{t_{m-1}}^{t_m} ds \sqrt{\gamma} \mathbf{S} \cdot \mathbf{A}_T(s) \right]. \quad (37)$$

This unitary can, with care, be further Trotterized by utilizing commutativity between different ancillas (see Appendix D); if $\{S_\alpha\}$ mutually commute, then V_m directly factorizes into operators acting on individual ancillas.

Except for the substitution above, the ATA for the multichannel case is unchanged from the single-channel case. Indeed, our error bound in Appendix B was derived for the multichannel case; it shows that the ATA for multichannel baths is guaranteed to be ε -accurate in the sense of Eq. (30) if $\Delta \xi$, τ_c , and Δt are chosen to satisfy Eq. (31), with Γ , τ , and $\Lambda(\varepsilon)$ now defined by appropriate substitution of $|J(\omega)|$, $|J''(\omega)|$, and $|g(t)|$ with $\|\mathbf{J}(\omega)\|_{1,1}$, $\|\mathbf{J}''(\omega)\|_{1,1}$, and $\|\mathbf{g}(t)\|_{2,1}$, respectively. Here, for any matrix $\mathbf{M}$, $\|\mathbf{M}\|_{p,q} := (\sum_\beta [\sum_\alpha |\mathbf{M}_{\alpha\beta}|^p]^{1/p})^{1/q}$ denotes the (p, q) matrix norm.

IV. NUMERICAL SIMULATION

In this section, we simulate the ATA for a spin-boson model consisting of a two-level system coupled to an environment with Ohmic spectral density. Our goal is to demonstrate the operating principle of the algorithm and to probe its resource requirements, which in practice may be significantly smaller than our rigorous bounds in Sec. III B, which cover worst-case scenarios.

The model we simulate has Hamiltonian

$$H_{\text{SB}} = -\Delta \sigma_z + H_B + \sqrt{\gamma} \sigma_x \otimes B, \quad (38)$$

where B is an observable of a Gaussian environment with an Ohmic cutoff. The power spectral density is given as

$$\tilde{J}_\beta(\omega) = \frac{\omega e^{-(\frac{\omega}{\omega_s})^2}}{1 - e^{-\beta\omega}}, \quad (39)$$

where β is the inverse temperature of the bath and ω_s is the Gaussian cutoff. In all of our simulations, we set $\Delta = 1$, $\gamma = 0.01$, and $\omega_s = 10$. We consider three different temperatures: $\beta = 0.1$, $\beta = 1$, and $\beta = 10$.

We consider the dynamics that results from initializing the system in $|1\rangle$. Through direct time evolution, we simulate the evolution of the system under the ATA, using up to 13 ancillas; see Fig. 5 for the pseudocode. In all simulations, we set $\Delta t = \Delta \xi$ to minimize numerical costs [70]. From the expression for the ancilla number N_A in Sec. III B and Eq. (31) we find the

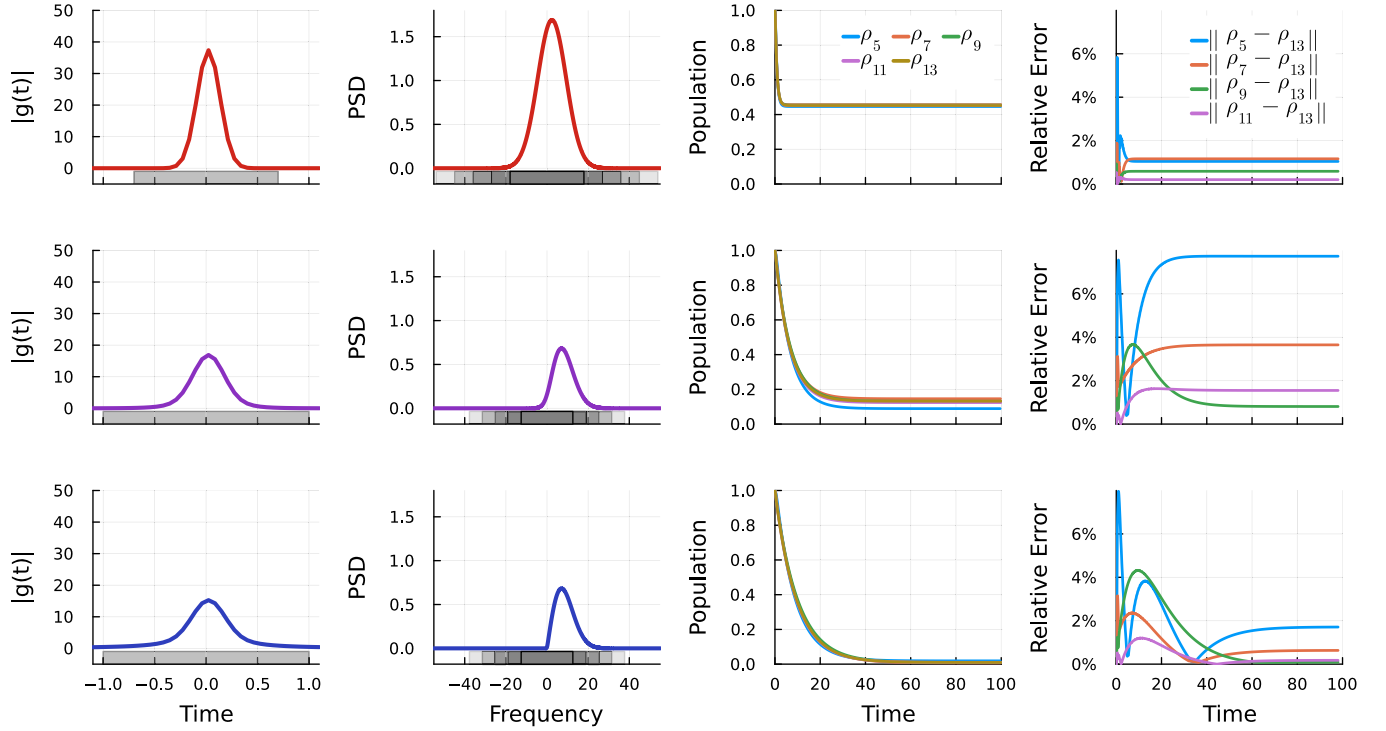


FIG. 5. Simulation of the ATA for Ohmic spin-boson model. (see main text for details). In all the plots, we set $\Delta = 1$, $\gamma = 0.01$, and $\omega_s = 10$. The three rows result from different inverse temperatures of the baths: $\beta = 0.1$ (top), $\beta = 1$ (middle), and $\beta = 10$ (bottom). *Column 1*: Absolute value of the jump correlator. The gray bars show the extent of the cutoff time τ_c chosen for the simulation. *Column 2*: Bath power spectral densities for chosen parameters. Gray bars show the implemented UV cutoff for different numbers of ancillas. The innermost region bar is for 5 ancillas and the outermost region is for 13 ancillas. *Column 3*: Population in state $|1\rangle$ of the spin as a function of time for simulations using different numbers of ancillas; ρ_j indicates simulation with j ancillas. *Column 4*: Trace distance between the simulation using 13 ancillas and the simulations using fewer ancillas. We see that the simulations converge with respect to each other to within 2% when using 13 ancillas.

following inequality:

$$\Lambda \tau_c \leq \frac{\pi}{2} (N_A + 1), \quad (40)$$

where τ_c is the cutoff time and Λ is the UV cutoff. In all simulations, we choose the largest possible value of $\Lambda \tau_c$. The first two columns of Fig. 5 show how the value of τ_c and Λ are chosen such that the jump correlator and power spectral densities have little weight outside of their respective cutoffs. In all simulations, we choose the smallest possible ancilla spacing, $\Delta \xi$, given the number of ancillas, N_A , and the cutoff time τ_c through $\Delta \xi = \frac{2\tau_c}{N_A+1} + \delta$. Here $\delta = 10^{-5} \log 2 / \Delta$ is a small irrational correction included to avoid Floquet resonances from the Trotterized evolution [71]. For numerical tractability, we finally restrict the Hilbert space of the ancilla register to the subspace with up to four simultaneous excitations.

In the third column of Fig. 5 we plot the evolution of the population of the $|1\rangle$ state from our simulation of the ATA. In the fourth column, we plot the trace distance between the evolution of the system state to the evolution with 13 ancillas. This serves as an indicator for convergence with respect to the ancilla number. As seen in Fig. 5, the simulations with the ATA converge to within 2% when comparing simulations using 11 and 13 ancillas. There is a clear trend that the distance decreases with increasing ancilla number—the trend is most pronounced in the transients. This suggests convergence

of the algorithm with respect to ancilla number, and an error from the finite ancilla number of a few percentages for the 13-ancilla simulation.

It is illuminating to compare our results with the error bounds from Sec. III B. From Eq. (32) and Eq. (30), we see that the number of ancillas required to ensure a total error of less than $\mathcal{E} = O(\epsilon \Gamma T)$ is

$$\text{No. of ancillas} = O\left(\frac{T^2 \Gamma^3 \tau}{\mathcal{E}^2}\right), \quad (41)$$

where T is the total simulation time. For the parameters we probe, we observe the model relaxes to a stationary state within a time $T \sim 40$; see Fig. 5. In order to ensure an error below 10% within this time span, our analytical bounds informs us that we require $O(10^5)$ ancillas as evident in the third column of Table II. In comparison, our simulations indicate

TABLE II. Parameters and analytical upper bound on the ancilla count for the Ohmic spin-boson model. The estimate is based on an error threshold of 10% at $T = 40$.

β	$\Gamma \tau$	Γ	Ancilla bound	$\max_r \ \rho_{11} - \rho_{13}\ $
0.1	0.2	2.7	2.5×10^5	0.5%
1	0.2	1.2	3.6×10^4	1.6%
10	0.7	1.9	3.6×10^5	1.2%

convergence within a few percentages for ~ 13 ancillas. These results thus suggest that, in practice, numerical convergence occurs for much fewer ancillas than predicted by our bound.

V. DISCUSSION

In this paper, we introduced a quantum algorithm for systematically simulating quantum systems interacting with arbitrary Gaussian environments at any coupling strength: the ancilla-train algorithm (ATA). The ATA can be used to simulate dynamics and steady states of equilibrium and nonequilibrium problems and is applicable to non-Markovian regimes. We have provided rigorous bounds on the accuracy of the algorithm and shown that the simulation error can be made arbitrarily small with an appropriate choice of simulation resolution parameters.

The resource cost for the ATA is polynomial in the inverse target accuracy, provided modest decay properties of the bath PSD, and is controlled by a combination of four physical energy scales of the simulated model: A characteristic Heisenberg picture velocity of the system observables coupled to the environment, along with the bath correlation time, system-bath coupling strength, and effective ultraviolet cutoff of the environment. Importantly, resource costs are *independent* of the complexity of the system. Thus, for a broad range of problems, the ATA only requires a small amount of additional resource relative to the cost of simulating the time evolution in the absence of external environments.

The ATR of quantum noise provides a secondary result of our work that underlies the ATA: Any Gaussian quantum environment can be emulated by convolving a quantum binary white noise signal—an ancilla train—with a time-local convolution kernel. A corollary is that a classical noise signal with Gaussian correlations can be generated by convolving a white noise signal with the jump correlator. This result is consistent with the known result in classical signal processing that a colored noise signal can be obtained from filtering a white noise signal [45]; here the filtering kernel is given by the jump correlator. The ATR thus provides a generalization of this result to quantum noise, through a non-Markovian collision model. For quantum noise, the filtering kernel (jump correlator) is simply generalized from a real to a complex-valued function, and the classical (binary) white noise signal—a bit stream—is replaced with a stream of qubits.

A. Relation to quantum Gibbs sampling and systematic Lindblad equation algorithms

We now comment on the relationship between the ATA and other algorithms for quantum simulation of open quantum systems that have recently emerged in the literature: QGS [12,16–20] and collision models based on systematic and rigorously derived Lindblad equations (SL) [39–41].

QGS algorithms yield the exact Gibbs state for a given model Hamiltonian. This is achieved through collision models [46] based on Lindblad processes with quasilocal jump operators that exhibit KMS detailed balance [72]. Physical systems connected to environments in thermal equilibrium are known to converge to such exact Gibbs states in the limit of weak system-bath coupling. However, for nonzero finite

couplings—as emerge in physical systems—the steady state deviates from the Gibbs state by a correction controlled by the system-bath coupling due to the ambiguity in defining an appropriate subsystem Hamiltonian for a partitioned quantum system [48]. Thus, from a quantum simulation perspective, QGS algorithms yield accurate physical steady states of open quantum systems only in the limit of vanishing system-bath coupling.

In parallel with the emergence of QGS algorithms, a family of systematic rigorously derived Lindblad equations has emerged that circumvent the rotating-wave, or secular, approximation. These Lindblad equations have been successfully employed in classical simulation of dissipative quantum many-body systems [73–78]. Similarly to QGS Lindblad processes, these systematic Lindblad equations feature quasilocal jump operators and can thus be efficiently implemented as quantum algorithms using collision-model-type approaches [46]. We term such quantum algorithms *SL algorithms*. Contrasting QGS algorithms, SL algorithms are systematically derived from underlying microscopic models and extend to nonequilibrium problems, e.g., featuring driving or nonequilibrium reservoirs. These SL approaches yield dynamics and steady states—equilibrium or nonequilibrium—that are accurate up to corrections proportional to the system-environment coupling (relative to an intrinsic correlation timescale for the environment [49]). Thus, from a quantum simulation perspective, SL algorithms yield steady states that are accurate on the *same level of approximation* as those returned by QGS algorithms—namely both QGS and SL algorithms are accurate in the limit of vanishing system-bath coupling.

The ATA complements the QGS and SL algorithms above, offering important advantages: (1) The accuracy of the ATA simulation is limited only by the available resources; it can yield the exact steady state and dynamics of the simulated microscopic model to arbitrary accuracy. (2) The ATA involves only a simple, strictly local system-ancilla coupling and thus does not necessarily require compilation of a quasilocal system-ancilla coupling. In particular, if the coupling operators in the system can be represented by strictly local operators, S_α , then the unitary system-ancilla coupling operators, V_m , are also strictly local. This contrasts many quantum algorithms based on Markovian collision models [17,39]. (3) The ATA does not require compilation of a quasilocal Lamb shift or generalizations thereof. For a given, fixed target accuracy, the additional cost of the ATA is a register of ancilla qubits of size scaling linearly with system-bath coupling. Thus, the ATA offers both a broader range of applicability *and* a simpler implementation than the QGS and SL algorithms in cases where the depth of the circuit is the limiting resource.

During the preparation of this manuscript, ancilla-train-like constructions appeared in the QGS literature in Refs. [19–21]. Here the goal was to simulate a Lindblad process with a thermal (Gibbs) state as an approximate steady state. References [19–21] provides rigorous performance guarantees for this construction, demonstrating that ancilla trains are broadly applicable for QGS, where they, moreover, bring the advantage of a strictly local system-ancilla coupling.

B. Noise resilience

A final potential advantage of the ATA, shared by other algorithms for open quantum systems simulation [12,17–20,39,40], is a potential *intrinsic robustness against hardware errors* due to its entropy-extracting nature [79]. Specifically, noise in hardware executing the ATA will map to a noise channel in the simulated microscopic model. Since the simulation already involves a mechanism for dissipation, this noise channel will compete with other channels for dissipation and noise and thus only generate a continuous perturbation of the output (simulated steady state) if weak enough: Hence for open quantum system simulators such as the ATA the *noise-induced error is generically a finite-slope function of the noise strength*. Equivalently, since the dissipative channels in an ATA simulation provide a finite relaxation time for perturbations of the steady state, extrinsic noise will generate a negligible error if the characteristic rate of errors is small relative to the relaxation time. The extent of the inherent error resilience in simulations of open quantum dynamics is still an open area of research. However, several recent theoretical and numerical results corroborate the intuition that simulations of open quantum systems dynamics is significantly more resilient to errors than simulations of closed quantum systems [80–83].

C. Outlook

In summary, the ATA offers a systematic and accurate approach for simulating open quantum systems at arbitrary coupling strength, and in arbitrary nonequilibrium or equilibrium configurations of Gaussian environments. The restriction of ATA to Gaussian environments can be seen as a second-order cumulant expansion of the multitime correlation functions of the environment. A natural extension of ATA beyond Gaussian environments would be to include higher-order cumulants. We leave such an investigation for future work. We expect the algorithm has intrinsic resilience against hardware error and is relatively simple to implement. We expect these results make the ATA a promising candidate for quantum simulation of nonequilibrium quantum systems and for implementation on near-term quantum hardware.

ACKNOWLEDGMENTS

We gratefully acknowledge insightful discussions with M. Kastoryano, C.-F. Chen, M. S. Teynor, G. C. Solomon, and D. Malz. This work is supported by the Novo Nordisk Foundation (Grant No. NNF22SA0081175), NNF Quantum Computing Programme, and the Danish E-Infrastructure Consortium (Grant No. 4317-00014B).

DATA AVAILABILITY

There are no publicly available research data or software supporting this manuscript. Requests for further information or data should be sent to the authors.

APPENDIXES

In the appendixes, we provide proofs of our claims from the main text. In Appendix A we demonstrate that the evolution

TABLE III. Overview of the relationship between bath correlation function, power spectral density, square root of the power spectral density, and the jump correlator.

Name	Notation	Definition
Bath correlation function	$\mathbf{J}(t-s)$	$\text{Tr}[\mathbf{B}(t)\mathbf{B}(s)^\dagger \rho_B]$
Power spectral density (PSD)	$\tilde{\mathbf{J}}(\omega)$	$\frac{1}{2\pi} \int_{-\infty}^{\infty} dt \mathbf{J}(t) e^{i\omega t}$
Square-root PSD	$\tilde{\mathbf{g}}(\omega)$	$\sqrt{\tilde{\mathbf{J}}(\omega)}/2\pi$
Jump correlator	$\mathbf{g}(t)$	$\int_{-\infty}^{\infty} d\omega \tilde{\mathbf{g}}(\omega) e^{-i\omega t}$

of a system coupled to a Gaussian environment only depends on the environment through its bath correlation function. In Appendix B we give rigorous error bounds for our ATR as well as for the Trotter approximation required to obtain the ATA. In Appendix C, we prove that the jump correlator can be passed through a suitable low-pass filter without significantly increasing its characteristic interaction rate and correlation timescales. In Appendix D, we give a formula for V_m from Eq. (27) in the case where $\{S_\alpha\}$ are mutually commuting operators. Finally, in Appendix E we derive some important relations that are frequently used throughout the appendixes.

Throughout these appendixes, we consider the general case of multiple quantum noise channels [i.e., with multiple terms in the sum in Eq. (2)], where the bath correlation function $\mathbf{J}(t)$ is matrix valued. The special case considered in the main text of the article is equivalent to the case where $\mathbf{J}(t)$ is a 1×1 matrix. As a help to the reader, in Table III we provide an overview of the relevant functions of the bath for multiple quantum noise channels.

APPENDIX A: PROPERTIES OF GAUSSIAN BATHS

In this Appendix, we show that the effect on a given quantum system from a Gaussian environment solely depends on the two-point correlation function of the environment.

To recap, we consider the evolution of the reduced density matrix of the system, $\rho(t)$, generated by the full system-bath Hamiltonian $H_{SB}(t) = H_S(t) + H_B + \sqrt{\gamma} \sum_\alpha S_\alpha \otimes B_\alpha$ [see also Eq. (1) of the main text].

We first go to the interaction picture with respect to the system-bath interaction, $\sqrt{\gamma} \mathbf{S} \cdot \mathbf{B}$, obtaining

$$\rho(t) = U_S(t) \text{Tr}_B[U_{SB}^{(C)}(t) \rho(t_0) \otimes \rho_B U_{SB}^{(C)\dagger}(t)] U_S^\dagger(t), \quad (\text{A1})$$

with $U_S(t) = \mathcal{T} e^{-i \int_{t_0}^t ds H_S(s)}$, $U_{SB}^{(C)}(t) = \mathcal{T} e^{-i \int_{t_0}^t dt' \sqrt{\gamma} \mathbf{S}(t') \cdot \mathbf{B}(t')}$, and $\mathbf{S}(t) = U_S(t) \mathbf{S} U_S^\dagger(t)$. Here we use the dot-product notation that was introduced in Eq. (34) of the main text. Next we expand $U_{SB}^{(C)}$ in terms of its Dyson series:

$$U_{SB}^{(C)}(t, t_0) = 1 + \sum_{n=1}^{\infty} (-i)^n \int_{t_0}^t dt_1 \int_{t_0}^{t_1} dt_2 \cdots \int_{t_0}^{t_{n-1}} dt_n \prod_{i=1}^n (\sqrt{\gamma} \mathbf{S}(t_i) \cdot \mathbf{B}(t_i)). \quad (\text{A2})$$

We refer to the series expansion of $\rho(t)$ obtained by inserting the above expression into Eq. (A1) as the *Dyson series* for $\rho(t)$.

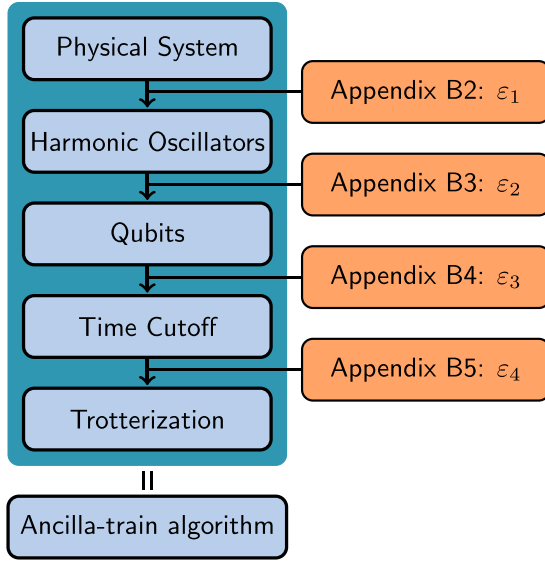


FIG. 6. Overview of approximations leading from the physical system to the ATA. Orange boxes indicate the subsections of the appendixes where the approximations are introduced and bounded.

Importantly, each term in the Dyson series for $\rho(t)$ depends solely on $H_S(t)$, $\{S_\alpha\}$, and the N -point correlation functions of the bath (for $N \in \mathbb{N}$),

$$\langle B_1 \cdots B_N \rangle := \text{Tr}[B_1 \cdots B_N \rho_B], \quad (\text{A3})$$

where B_j is shorthand for $B_{\alpha_j}(t_j) := e^{iH_B t_j} B_{\alpha_j} e^{-iH_B t_j}$. A defining fact about Gaussian baths used in this work is that they obey Wick's theorem [84],

$$\langle B_1 \cdots B_N \rangle = \sum_{j=2}^N \langle B_1 B_j \rangle \langle B_2 \cdots B_{j-1} B_{j+1} \cdots B_N \rangle. \quad (\text{A4})$$

Repeated use of Wick's theorem together with our assumption (without loss of generality) that $\langle B(t) \rangle = 0$ shows that all terms in the Dyson series for $\rho(t)$ are *completely determined* by the system Hamiltonian $H_S(t)$, the local coupling operators $\{S_\alpha\}$, and the matrix-valued two-point correlation function, defined as the matrix, $\mathbf{J}(t)$, whose (α, β) entry is given by

$$J_{\alpha\beta}(t - t') := \langle B_\alpha(t) B_\beta(t') \rangle. \quad (\text{A5})$$

We now briefly argue that the above bath properties ensure that the dynamics are fixed by the two-point correlation function of the environment. Assuming $\Gamma < \infty$, $\|\mathbf{J}\|_{1,1}$ is integrable [85]. This implies that the Dyson series for $\rho(t)$ is absolutely convergent. Therefore, the effects of Gaussian baths on the system dynamics are also *completely determined* by $H_S(t)$, $\{S_\alpha\}$, and $\mathbf{J}(t)$ [30,44]. For a more detailed proof of this fact, see Refs. [86,87].

APPENDIX B: ERROR BOUNDS OF THE ATR AND ATA

In this Appendix, we bound the error of the ATA, i.e., the deviation of the evolution generated by the ATA and the true evolution of the simulated microscopic model.

We obtain the ATA through four successive approximations, outlined in Fig. 6, and described in Appendixes B 2–B 5

below. We establish a rigorous bound on the error induced by each of these approximations; the error of the ATA is bounded by the sum of these bounds. Likewise, the error of the ATR—i.e., the deviation of the system dynamics obtained from the ATR with respect to the true dynamics—is bounded by the sum of the errors induced by the first three of these approximations.

Throughout this appendix, we make use of superoperator notation. We provide an introduction to this notation in Appendix B 1. Throughout this appendix, we make frequent reference to a set of relations associated with matrix norms or Dyson expansions; we provide proofs for these relations in Appendix E

1. Superoperator notation

We first introduce a convenient notation for superoperators that will be used extensively in this and in the following appendixes. Superoperators are defined as linear operators acting on the set of linear operators on a given Hilbert space $\mathcal{H}$ [88]. Importantly, this set of operators itself forms a Hilbert space, termed *operator space*; superoperators can be represented as matrices acting on this operator space. To highlight their Hilbert space nature, we will, in the following, use double brackets, $|\cdot\rangle\rangle$, to indicate operators acting on $\mathcal{H}$ and use $\langle\langle\cdot|\cdot\rangle\rangle$ to indicate the Hilbert-Schmidt inner product between two such operators [89]:

$$\langle\langle A|B \rangle\rangle := \text{Tr}[A^\dagger B]. \quad (\text{B1})$$

A (bounded) operator on $\mathcal{H}$, O , induces two natural superoperators, namely left- and right-multiplication by O . We denote these induced superoperators by

$$O^L|A\rangle\rangle := |OA\rangle\rangle \text{ and } O^R|A\rangle\rangle := |AO\rangle\rangle. \quad (\text{B2})$$

Note that $[A^L, B^R] = 0$ for any operators A, B . As an important example, the von-Neumann equation, $\frac{\partial}{\partial t} \rho(t) = -i[H, \rho(t)]$, may be written in the superoperator notation as

$$\frac{\partial}{\partial t} |\rho(t)\rangle\rangle = -i\nu_d H^d |\rho(t)\rangle\rangle, \quad (\text{B3})$$

where $\nu_L = -\nu_R = 1$, and we use the Einstein summation convention

$$\nu_d H^d := \sum_{d \in \{L, R\}} \nu_d H^d. \quad (\text{B4})$$

We use the above summation convention for the indices $\{L, R\}$ when they appear in the following. We will moreover in general use calligraphic symbols to indicate superoperators, when they are *not* given by left or right multiplication by bounded operators on $\mathcal{H}$. In particular, we reserve $\mathcal{U}$ for time-evolution superoperators. Given a Hamiltonian, $H(t)$, we define

$$\mathcal{U}(t, t_0) := \mathcal{T} \exp \left(-i\nu_d \int_{t_0}^t ds H^d(s) \right). \quad (\text{B5})$$

We refer to $\mathcal{U}(t, t_0)$ as *the time-evolution superoperator generated by $H(t)$* .

2. Approximating the bath with a discrete train of harmonic oscillators

We now proceed to the proofs of our error bounds. We first bound the error that comes from representing the bath as a

discrete train of harmonic oscillators of the form described in Eq. (14) of the main text [90].

In the following, we let $\rho(t)$ denote the exact time-evolved system state as in Eq. (4) and by $\rho_{\text{osc}}(t)$ the time-evolved system state resulting from an interaction with a discrete train of harmonic oscillators in place of the bath, defined via Eqs. (B29) and (B32) below. Our goal is to bound the trace norm distance between these two evolutions,

$$\varepsilon_1 := \|\rho_{\text{osc}}(t) - \rho(t)\|_{\text{tr}}. \quad (\text{B6})$$

We call this error the *discretization error*, since the system of harmonic oscillators can be seen as a discretization of the bath. Our derivation proceeds as follows: We first show that the bath $\mathcal{B}$ is equivalent to a combination of *two* specific environments, $\mathcal{B}_1$ and $\mathcal{B}_2$, in the sense that they produce the same evolution of the reduced density matrix of the system, $\rho(t)$. Here $\mathcal{B}_1$ can be taken to have a compactly supported PSD and $\mathcal{B}_2$ can be neglected at the cost of a small error (Appendix B 2 a). Subsequently, we show, via Poisson's summation formula that $\mathcal{B}_1$ is equivalent to the harmonic oscillator train Hamiltonian that generates the evolution of $\rho_{\text{osc}}(t)$ (Appendix B 2 b).

a. Dividing the bath in two parts

To construct $\mathcal{B}_1$ and $\mathcal{B}_2$, we express the bath PSD $\tilde{\mathcal{J}}(\omega)$ as a sum of two power spectral densities, $\tilde{\mathcal{J}}(\omega) = \tilde{\mathcal{J}}_1(\omega) + \tilde{\mathcal{J}}_2(\omega)$, where $\tilde{\mathcal{J}}_1(\omega)$ and $\tilde{\mathcal{J}}_2(\omega)$ are positive semidefinite and have support only for $|\omega| < \Omega$ and $|\omega| > \Omega/2$, respectively, with Ω some arbitrary effective frequency cutoff we introduce as a free parameter. In the following, we show that, when choosing Ω large enough, $\tilde{\mathcal{J}}_2(\omega)$ can be neglected at the cost of a small bounded error [we give an explicit expression below in Eq. (B21)], resulting in a bath whose PSD $[\tilde{\mathcal{J}}_1(\omega)]$ has compact support.

We construct the bath PSDs $\tilde{\mathcal{J}}_1(\omega)$ and $\tilde{\mathcal{J}}_2(\omega)$ as follows:

$$\begin{aligned} \tilde{\mathcal{J}}_1(\omega) &:= (\tilde{\varphi}(\omega))^2 \tilde{\mathcal{J}}(\omega), \\ \tilde{\mathcal{J}}_2(\omega) &:= (1 - (\tilde{\varphi}(\omega))^2) \tilde{\mathcal{J}}(\omega), \end{aligned} \quad (\text{B7})$$

Here φ can be any function satisfying $0 \leq \tilde{\varphi}(\omega) \leq 1$ for all $\omega \in \mathbb{R}$, $\tilde{\varphi}(\omega) = 1$ for $|\omega| \leq \Omega/2$ and $\tilde{\varphi}(\omega) = 0$ for $|\omega| > \Omega$. To avoid introducing long time correlations in the bath described by $\tilde{\mathcal{J}}_1(\omega)$, we shall be careful in choosing $\varphi(\omega)$ to be sufficiently smooth. To this end, we pick a second-order polynomial interpolation for $\tilde{\varphi}(\omega)$, ensuring that it is twice differentiable almost everywhere:

$$\tilde{\varphi}(\omega) = \begin{cases} 1 & \text{for } |\omega| < \frac{\Omega}{2} \\ 1 - \frac{1}{2} \left(\frac{4}{\Omega} \right)^2 (|\omega| - \Omega/2)^2 & \text{for } \frac{\Omega}{2} \leq |\omega| < \frac{3\Omega}{4} \\ \frac{1}{2} \left(\frac{4}{\Omega} \right)^2 (|\omega| - \Omega)^2 & \text{for } \frac{3\Omega}{4} \leq |\omega| < \Omega \\ 0 & \text{for } |\omega| \geq \Omega \end{cases}. \quad (\text{B8})$$

By this choice, the characteristic timescales Γ_f and τ_f of $\mathcal{B}_1$ are of similar magnitude as the corresponding timescales Γ and τ of the original bath $\mathcal{B}$ (see Appendix C).

Importantly, since $\tilde{\mathcal{J}}(\omega) = \tilde{\mathcal{J}}_1(\omega) + \tilde{\mathcal{J}}_2(\omega)$, and the effects of a Gaussian environment is completely determined by its two-point correlation function (see Appendix A and Ref. [30]), the physical environment we wish to simulate can be exactly represented as a combination of two uncorrelated environments, $\mathcal{B}_1$ and $\mathcal{B}_2$, with PSDs $\tilde{\mathcal{J}}_1(\omega)$ and $\tilde{\mathcal{J}}_2(\omega)$ such that the system-bath Hamiltonian takes the form [30]

$$H_{12}(t) = H_S(t) + H_{\mathcal{B}_1} + H_{\mathcal{B}_2} + \sqrt{\gamma} \mathbf{S} \cdot (\mathbf{B}_1 + \mathbf{B}_2), \quad (\text{B9})$$

with $\mathbf{B}_{1,2}$ observables of Gaussian environments with PSDs

$$\mathcal{J}_1(t) = \langle \mathbf{B}_1(t) (\mathbf{B}_1(0))^\dagger \rangle_{\mathcal{B}_1}, \quad (\text{B10})$$

$$\mathcal{J}_2(t) = \langle \mathbf{B}_2(t) (\mathbf{B}_2(0))^\dagger \rangle_{\mathcal{B}_2}, \quad (\text{B11})$$

$$\langle \mathbf{B}_1(t) (\mathbf{B}_2(0))^\dagger \rangle_{\mathcal{B}} = \langle \mathbf{B}_2(t) (\mathbf{B}_1(0))^\dagger \rangle_{\mathcal{B}} = 0. \quad (\text{B12})$$

Specifically, the corresponding equation of motion for the reduced density of the system described by $H_{12}(t)$ is the same as that arising from the system-bath Hamiltonian given in Eq. (1) of the main text:

$$\begin{aligned} |\rho(t)\rangle\rangle &:= \langle\langle I |_{\mathcal{B}} \mathcal{U}(t, t_0) | \rho_0 \rangle\rangle | \rho_{\mathcal{B}} \rangle\rangle \\ &= \langle\langle I |_{\mathcal{B}} \mathcal{U}_{12}(t, t_0) | \rho_0 \rangle\rangle | \rho_{\mathcal{B}} \rangle\rangle. \end{aligned} \quad (\text{B13})$$

Here $\mathcal{U}$ denotes the time-evolution superoperator of H_{SB} from Eq. (1) and $\mathcal{U}_{12}$ the time-evolution superoperator generated by H_{12} . As the combination of the two baths $\mathcal{B}_1 \otimes \mathcal{B}_2$ is equivalent to the original bath $\mathcal{B}$, we use $\mathcal{B}$ also to denote the combined baths $\mathcal{B}_1 \otimes \mathcal{B}_2$.

We now show that the bath $\mathcal{B}_2$ can be neglected at the cost of a small error when Ω is sufficiently large. To formalize this statement, we define $H_1(t) = H_S(t) + H_{\mathcal{B}_1} + H_{\mathcal{B}_2} + \sqrt{\gamma} \mathbf{S} \cdot \mathbf{B}_1$ and the time-evolution superoperator of H_1 by $\mathcal{U}_1$. We also denote the time-evolved reduced density matrix of the system arising from $\mathcal{U}_1$ by

$$|\rho_1(t)\rangle\rangle := \langle\langle I |_{\mathcal{B}} \mathcal{U}_1(t, t_0) | \rho_0 \rangle\rangle | \rho_{\mathcal{B}} \rangle\rangle. \quad (\text{B14})$$

Since the time evolution of $\rho(t)$ generated by $H_{\text{SB}}(t)$ and $H_{12}(t)$ are identical, bounding the distance $\|\rho_1(t) - \rho(t)\|_{\text{tr}}$ is reduced to bounding the effect of decoupling the split system from $\mathcal{B}_2$.

To proceed, we transform the system described by $H_{12}(t)$ to the interaction picture with respect to $\sqrt{\gamma} \mathbf{S} \cdot \mathbf{B}_2$, finding

$$|\rho(t)\rangle\rangle = \langle\langle I |_{\mathcal{B}} \mathcal{U}_1(t, t_0) \mathcal{U}_2(t, t_0) | \rho_0 \rangle\rangle | \rho_{\mathcal{B}} \rangle\rangle \quad (\text{B15})$$

with $\mathcal{U}_2(t, t_0) = \mathcal{T} e^{-i \int_{t_0}^t ds \mathcal{H}_2(s)}$ and $\mathcal{H}_2(s) = \sum_{d \in \{L, R\}} \nu_d \sqrt{\gamma} \mathbf{S}^d(s) \cdot \mathbf{B}_2^d(s)$. Here we defined the interaction picture operators; $\mathbf{S}^d(s) = e^{iH_1 s} \mathbf{S}^d e^{-iH_1 s}$ and $\mathbf{B}_2(s) = e^{iH_{\mathcal{B}_2} s} \mathbf{B}_2 e^{-iH_{\mathcal{B}_2} s}$. Using a simple Dyson expansion on $\mathcal{U}_2$ [see Eq. (E6)], along with Eq. (B14), we conclude

$$|\rho(t)\rangle\rangle = |\rho_1(t)\rangle\rangle - i \int_{t_0}^t ds \langle\langle I |_{\mathcal{B}} \mathcal{U}_1(t, t_0) \mathcal{H}_2(s) \mathcal{U}_2(s, t_0) | \rho_0 \rangle\rangle | \rho_{\mathcal{B}} \rangle\rangle. \quad (\text{B16})$$

Next, we write $\mathcal{U}_2$ as a Trotterized product with infinitesimal Trotter step:

$$\mathcal{U}_2(s, t_0) = \lim_{M \rightarrow \infty} \prod_{m=0}^M \left[1 - i \frac{s - t_0}{M} \sum_{d \in \{L, R\}} v_d \sqrt{\gamma} \mathbf{S}^d \left(\frac{m(s - t_0)}{M} + t_0 \right) \cdot \mathbf{B}^d \left(\frac{m(s - t_0)}{M} + t_0 \right) \right]. \quad (\text{B17})$$

Upon direct substitution of the above into Eq. (B16), and using Wick's theorem (A4) for $\mathcal{B}_2$, it follows that

$$|\rho(t)\rangle\rangle = |\rho_1(t)\rangle\rangle - i\gamma \sum_{\alpha, \beta} v_d v_e \int_{t_0}^t ds \int_{t_0}^s ds' J_{2, \alpha\beta}^{de}(s - s') \langle\langle I | \mathcal{B} \mathcal{U}_1(t, t_0) \mathbf{S}_\alpha^d(s) \mathcal{U}_2(s, s') \mathbf{S}_\beta^e(s') \mathcal{U}_2(s', t_0) | \rho_0 \rangle\rangle | \rho_B \rangle\rangle, \quad (\text{B18})$$

where, for $d = L, R$, $J_{2, \alpha\beta}^{de}$ denotes the (α, β) entry of the matrix $\mathbf{J}_2$ when $e = L$ and the (α, β) entry of the matrix $\mathbf{J}_2^\dagger$ for $e = R$. Furthermore, we used the Einstein summation convention from (B4). To bound the integrand above we use that $|\text{Tr}[X\rho Y]| \leq \|X\| \|Y\|$; this auxiliary result is established in Appendix E [Eq. (E11)]. Inserting this above implies

$$\|\langle\langle I | \mathcal{B} \mathcal{U}_1(t, t_0) \mathbf{S}_\alpha^d(s) \mathcal{U}_2(s, s') \mathbf{S}_\beta^e(s') \mathcal{U}_2(s', t_0) | \rho_0 \rangle\rangle | \rho_B \rangle\rangle\|_{\text{Tr}} \leq 1. \quad (\text{B19})$$

Thus,

$$\|\rho_1(t) - \rho(t)\|_{\text{Tr}} \leq 4\gamma \int_{t_0}^t ds \int_{t_0}^s ds' \|\mathbf{J}_2(s - s')\|_{1,1}. \quad (\text{B20})$$

Splitting the integration domain in $s' > s_0$ and $s' < s_0$, where s_0 is some free parameter to be chosen below, we find

$$\|\rho_1(t) - \rho(t)\|_{\text{Tr}} \leq 4\gamma \int_{t_0}^t ds \left(\int_{-s_0}^{s_0} ds' \|\mathbf{J}_2(s')\|_{1,1} + 2 \int_{s_0}^\infty ds' \frac{1}{(s')^2} \|(s')^2 \mathbf{J}_2(s')\|_{1,1} \right). \quad (\text{B21})$$

Using pointwise bounds $|J_{2, \alpha\beta}(s)| \leq \int_{-\infty}^\infty d\omega \tilde{J}_{2, \alpha\beta}(\omega)$ and $s^2 |J_{2, \alpha\beta}(s)| \leq \int_{-\infty}^\infty d\omega |\tilde{J}_{2, \alpha\beta}''(\omega)|$, we find

$$\|\rho_1(t) - \rho(t)\|_{\text{Tr}} \leq 4\gamma \left(s_0 \epsilon_1(\Omega) + \frac{1}{s_0} \epsilon_2(\Omega) \right) T, \quad (\text{B22})$$

with

$$\epsilon_1(\Omega) = 2 \int_{|\omega| > \Omega} d\omega \|\tilde{\mathbf{J}}(\omega)\|_{1,1} \quad (\text{B23})$$

$$\epsilon_2(\Omega) = 2 \int_{|\omega| > \Omega} d\omega \|\tilde{\mathbf{J}}''(\omega)\|_{1,1}. \quad (\text{B24})$$

It follows that, for $t < T$,

$$\|\rho_1(t) - \rho(t)\|_{\text{Tr}} \leq \epsilon(\Omega) \Gamma T, \quad (\text{B25})$$

with

$$\epsilon(\Omega) := \frac{2\sqrt{\epsilon_1(\Omega)\epsilon_2(\Omega)}}{\left[\int_{-\infty}^\infty dt \|\mathbf{g}(t)\|_{2,1} \right]^2}. \quad (\text{B26})$$

For $\|\tilde{\mathbf{J}}\|_{1,1}, \|\tilde{\mathbf{J}}''\|_{1,1} \in L^1(\mathbb{R})$ we note that $\epsilon(\Omega) \rightarrow 0$ as $\Omega \rightarrow \infty$. Hence, the full bath can be closely approximated by the interaction with a bath where the PSD has compact support. In particular, fixing $\Omega = \Lambda(\varepsilon)$, where

$$\Lambda(\varepsilon) := \inf\{\Omega \geq 1/\tau | \epsilon(\Omega) < \varepsilon\}, \quad (\text{B27})$$

the trace distance between $\rho(t)$ and $\rho_1(t)$ stays within the target accuracy, i.e.,

$$\|\rho_1(t) - \rho(t)\|_{\text{Tr}} \leq \varepsilon \Gamma \tau. \quad (\text{B28})$$

b. Equivalence between the bath $\mathcal{B}_1$ and a train of harmonic oscillators

We now show that the dynamics of a system coupled to the bath $\mathcal{B}_1$ are identical to those generated by the a train of

harmonic oscillators, as in Eq. (14). Specifically, exploiting the compact support of $\tilde{\mathbf{J}}_1(\omega)$, we show that the evolution of $|\rho_1(t)\rangle\rangle$ is equivalent to the reduced density matrix evolution generated by the harmonic oscillator train Hamiltonian

$$H_{\text{osc}}(t) = H_S(t) + \sqrt{\gamma} \mathbf{S} \cdot \mathbf{A}_{\text{osc}}(t), \quad (\text{B29})$$

where $\mathbf{A}_{\text{osc}}(t)$ is the column vector whose α th entry is given by

$$A_{\text{osc}, \alpha}(t) = \sum_{n=-\infty}^\infty \sum_{\lambda=1}^{N_C} \sqrt{\Delta\xi} (g_{f; \alpha\lambda}^*(t - \xi_n) a_{\lambda n}^\dagger + \text{H.c.}). \quad (\text{B30})$$

Here $\Delta\xi$ can take any value smaller than, or equal to, $2\pi/\Omega$, $\{a_{\lambda n} | \lambda = 1, \dots, N_C, n \in \mathbb{Z}\}$ are a set of distinct bosonic annihilation operators, and $\mathbf{g}_f(t)$ denotes the jump correlator of the filtered environment:

$$\mathbf{g}_f(t) = \int d\omega e^{-i\omega t} \sqrt{\tilde{\mathbf{J}}_1(\omega)/2\pi}, \quad (\text{B31})$$

with $\sqrt{\cdot}$ denoting the positive-semidefinite matrix square root. We define the harmonic oscillator train evolution of the system as

$$\rho_{\text{osc}}(t) := \langle\langle I_{\text{osc}} | \mathcal{U}_{\text{osc}}(t, t_0) | \rho_0 \rangle\rangle | \mathbf{0} \rangle \quad (\text{B32})$$

with $\mathcal{U}_{\text{osc}}(t, t_0)$ the unitary time-evolution superoperator generated by $H_{\text{osc}}(t)$, and $|\mathbf{0}\rangle = |0, 0, \dots, 0\rangle$ the mutual vacuum states of all bosonic modes corresponding to $\{a_{\lambda n}\}$. We note that $|\mathbf{0}\rangle\langle\mathbf{0}|$ is a Gaussian state; hence the n -point correlation functions of $\mathbf{A}_{\text{osc}}$ in this state satisfy Wick's theorem. Thus, due to the equivalence among Gaussian environments described in Appendix A, the evolution of $\rho_{\text{osc}}(t)$ is identical to that of any open quantum system coupled to a Gaussian environment with system Hamiltonian $H_S(t)$, system-bath coupling $\sqrt{\gamma} \mathbf{S} \cdot \mathbf{B}'$, with $\mathbf{B}'$ a bath observable with two-point

correlation function given by

$$\langle \mathbf{0} | \mathbf{A}_{\text{osc}}(t) (\mathbf{A}_{\text{osc}}(s))^\dagger | \mathbf{0} \rangle = \sum_n \mathbf{g}_f(t - \xi_n) \mathbf{g}_f(\xi_n - s) \Delta \xi. \quad (\text{B33})$$

We now show that the right-hand side above is equal to $\mathbf{J}_1(t - s)$, implying that $\rho_{\text{osc}}(t) = \rho_1(t)$.

To establish that the right-hand-side of Eq. (B33) is identical to $\mathbf{J}_1(t - s)$, we use Poisson's summation formula [91] which states

$$\begin{aligned} \sum_{n=-\infty}^{\infty} \mathbf{g}_f(t - \xi_n) \mathbf{g}_f(\xi_n - s) \Delta \xi \\ = \sum_{k=-\infty}^{\infty} \int_{-\infty}^{\infty} d\xi \mathbf{g}_f(t - \xi) \mathbf{g}_f(\xi - s) e^{-i \frac{2\pi \xi k}{\Delta \xi}}. \end{aligned} \quad (\text{B34})$$

Inserting $\mathbf{g}_f(t) = \int_{-\infty}^{\infty} d\omega \sqrt{\mathbf{J}_1(\omega)/2\pi} e^{-i\omega t}$, it is straightforward to show that

$$\begin{aligned} \int_{-\infty}^{\infty} d\xi \mathbf{g}_f(t - \xi) \mathbf{g}_f(\xi - s) e^{-i \frac{2\pi \xi k}{\Delta \xi}} \\ = \int_{-\infty}^{\infty} d\omega \sqrt{\mathbf{J}_1(2\pi k/\Delta \xi + \omega)} \sqrt{\mathbf{J}_1(\omega)} e^{-i\omega(t-s)} e^{-i \frac{2\pi k t}{\Delta \xi}}. \end{aligned} \quad (\text{B35})$$

Thus, since $\sqrt{\mathbf{J}_1(\omega)} = 0$ for $|\omega| \geq \pi/\Delta \xi$, we find, for $k \neq 0$,

$$\int_{-\infty}^{\infty} d\xi \mathbf{g}_f(t - \xi) \mathbf{g}_f(\xi - s) e^{-i \frac{2\pi \xi k}{\Delta \xi}} = 0. \quad (\text{B36})$$

It follows that

$$\begin{aligned} \sum_{n=-\infty}^{\infty} \mathbf{g}_f(t - \xi_n) \mathbf{g}_f(\xi_n - s) \Delta \xi \\ = \int_{-\infty}^{\infty} d\xi \mathbf{g}_f(t - \xi) \mathbf{g}_f(\xi - s) = \mathbf{J}_1(t - s). \end{aligned} \quad (\text{B37})$$

This is what we wanted to show; thus

$$\rho_1(t) = \rho_{\text{osc}}(t). \quad (\text{B38})$$

Combining the above result with the bound on $\|\rho_1(t) - \rho(t)\|_{\text{tr}}$ in Eq. (B28), and recalling that $\Delta \xi = 2\pi/\Omega$, we conclude that

$$\varepsilon_1 := \|\rho_{\text{osc}}(t) - \rho(t)\|_{\text{tr}} \leq \varepsilon \Gamma T, \quad (\text{B39})$$

when ρ_{osc} is constructed with $\Delta \xi = 2\pi/\Omega$ and $\Omega = \Lambda(\varepsilon)$. This establishes our bound on ε_1 .

3. Error from using qubits in place of harmonic oscillators in the ancilla train

In this section we bound the error incurred by using qubits in place of harmonic oscillators in the ancilla train. Specifically, we let $\rho_q(t)$ denote the time evolution of the reduced density matrix of the system generated by $H_q(t)$, where

$$H_q(t) = H_S(t) + \sqrt{\gamma} \mathbf{S} \cdot \mathbf{A}_q(t). \quad (\text{B40})$$

The qubit error is defined as the trace norm distance between these states:

$$\varepsilon_2 := \|\rho_{\text{osc}}(t) - \rho_q(t)\|_{\text{tr}}, \quad (\text{B41})$$

In this subsection, we obtain the following bound for this error:

$$\varepsilon_2 \leq 25.96 \Delta \xi (T + \Delta \xi) \Gamma^2, \quad (\text{B42})$$

where T is the total simulation time.

To establish Eq. (B42), it is convenient to consider the quantum channels for the two time evolutions ρ_q and ρ_{osc} ; $\mathcal{C}_q$ and $\mathcal{C}_{\text{osc}}$, respectively, such that $\rho_{\text{osc}}(t) = \mathcal{C}_{\text{osc}}(t, t_0)[\rho(t_0)]$ and $\rho_q(t) = \mathcal{C}_q(t, t_0)[\rho(t_0)]$. We quantify the error in terms of the difference between the two quantum channels, defining

$$\mathcal{E}_q(t, t_0) := \mathcal{C}_q(t, t_0) - \mathcal{C}_{\text{osc}}(t, t_0). \quad (\text{B43})$$

We term $\mathcal{E}_q(t, t_0)$ the error map. Importantly, ε_2 is upper bounded by the superoperator norm of the error map

$$\varepsilon_2 \leq \|\mathcal{E}_q(t, t_0)\|_s, \quad (\text{B44})$$

where $\|\cdot\|_s$ denotes the superoperator norm induced by the trace norm:

$$\|\mathcal{E}_q(t, t_0)\|_s := \sup_{\rho: \|\rho\|_{\text{tr}} \leq 1} \|\mathcal{E}_q(t, t_0)[\rho]\|_{\text{tr}}. \quad (\text{B45})$$

The rest of this appendix is dedicated to bounding $\|\mathcal{E}_q(t, t_0)[\rho]\|_{\text{tr}}$. To this end, we first obtain a convenient explicit expression for $\mathcal{E}_q$ in Appendix B 3 a. Subsequently, in Appendix B 3 b, we expand this expression in a collection of terms that we bound individually through a Dyson expansion. In Appendix B 3 c, we finally combine these bounds through triangle inequality to obtain a bound on $\|\mathcal{E}_q(t, t_0)\|_s$ and thereby ε_2 .

a. Constructing the error map

To obtain an explicit expression for the error map $\mathcal{E}_q(t, t_0)$ we seek to express the quantum channel $\mathcal{C}_q$ in terms of $\mathcal{C}_{\text{osc}}$. Letting $\mathcal{U}_{\text{osc}}(t, t_0)$ denote the time-evolution superoperator generated by $H_{\text{osc}}(t)$ [Eq. (B29)], the quantum channel, $\mathcal{C}_{\text{osc}}$, can be explicitly written as

$$\mathcal{C}_{\text{osc}}(t, t_0) = \langle \langle I_A | \mathcal{U}_{\text{osc}}(t, t_0) | \mathbf{0} \rangle \rangle, \quad (\text{B46})$$

with $|I_A\rangle\rangle$ the identity operator on the Hilbert space of the ancilla oscillators. Since the harmonic oscillators are initiated in the vacuum state and since $H_{\text{osc}}(t)$ is linear in creation operators acting on the harmonic oscillators, we can obtain $\mathcal{C}_q$ by removing terms in H_{osc} that couple the first and second excited level in each ancillary oscillator. We introduce the following notation: Given an operator defined on the Hilbert space of a single harmonic oscillator, X , we denote by $(X)_{\lambda n}$ the operator which acts with X on oscillator (λ, n) and acts as the identity on all other oscillators [92]. This allows us to arrive at an equivalent expression for H_q in terms of H_{osc} by removing the terms in the individual oscillators which mix the first and second excited states [93]:

$$H_q(t) = H_{\text{osc}}(t) - \sum_{\lambda} \sum_{n=-\infty}^{\infty} K_{\lambda n; 2}(t) \quad (\text{B47})$$

with

$$K_{\lambda n; j}(t) := \sum_{\alpha} \sqrt{j\gamma \Delta \xi} S_{\alpha} [g_{f; \alpha \lambda}^*(t - \xi_n) (|j\rangle\langle j-1|)_{\lambda n} + \text{H.c.}], \quad (\text{B48})$$

where $|j\rangle$ denotes the j th harmonic oscillator Fock state. Specifically, denoting the time-evolution superoperator generated by $H_q(t)$ by $\mathcal{U}_q(t, t_0)$, we can express $\mathcal{C}_q$ in terms of $\mathcal{U}_q$ via

$$\mathcal{C}_q(t, t_0) = \langle \langle I_A | \mathcal{U}_q(t, t_0) | \mathbf{0} \rangle \rangle. \quad (\text{B49})$$

The error map can now be found as the difference between the two quantum channels. Through Eq. (B47), we can now express $\mathcal{U}_q$ in terms of $\mathcal{U}_{\text{osc}}$ via a Dyson expansion allowing us to obtain a compact expression for $\mathcal{E}_q$. To this end, we make extensive use of the following two Dyson-type identities in the remainder of this appendix: Given a superoperator $\mathcal{U}(t, t_0)$ generated by a Hamiltonian of the form $H(t) = H_1(t) + H_2(t)$, the following two relations hold:

$$\mathcal{U}(t, t_0) = \mathcal{U}_1(t, t_0) - i v_d \int_{t_0}^t ds \mathcal{U}_1(t, s) H_2^d(s) \mathcal{U}(s, t_0), \quad (\text{B50})$$

$$\mathcal{U}(t, t_0) = \mathcal{U}_1(t, t_0) - i v_d \int_{t_0}^t ds \mathcal{U}(t, s) H_2^d(s) \mathcal{U}_1(s, t_0), \quad (\text{B51})$$

where $\mathcal{U}_1(t, t_0)$ is the time-evolution superoperator generated by $H_1(t)$. Due to our extensive use of these identities, for shorthand we refer to the substitutions in Eqs. (B50) and (B51) as a left and right $H_2(t)$ -Dyson substitution of $\mathcal{U}(t, t_0)$ respectively. We moreover refer to $\mathcal{U}_1(t, t_0)$ as the leading-order term, and the second terms above as the residual term. The effect of a $H_2(t)$ -Dyson substitution is to isolate the dynamics generated by $H_2(t)$ to the residual term. Using suitable choices of H_2 , this allows us to eliminate leading-order terms in our analysis below. We provide proofs of the Dyson substitutions above in Appendix E 1.

To express $\mathcal{U}_q$ in terms of $\mathcal{U}_{\text{osc}}$ via the above Dyson substitutions, we first perform a $\sum_{\lambda} \sum_{n=-\infty}^{\infty} K_{\lambda n;2}(t)$ -Dyson substitution of $\mathcal{U}_{\text{osc}}(t, t_0)$, obtaining

$$\begin{aligned} \mathcal{U}_{\text{osc}}(t, t_0) &= \mathcal{U}_q(t, t_0) - i \sum_{\lambda, n} v_d \int_{t_0}^t ds \mathcal{U}_{\text{osc}}(t, s) K_{\lambda n;2}^d(s) \mathcal{U}_q(s, t_0). \end{aligned} \quad (\text{B52})$$

From the expressions for the quantum channels [Eqs. (B49) and (B46)], we thus identify the error map as

$$\mathcal{E}_q(t, t_0) = \sum_{\lambda, n} \left\langle \left\langle I_A \left| i v_d \int_{t_0}^t ds \mathcal{U}_{\text{osc}}(t, s) K_{\lambda n;2}^d(s) \mathcal{U}_q(s, t_0) \right| \mathbf{0} \right\rangle \right\rangle. \quad (\text{B53})$$

This expression serves as a starting point for obtaining our bound on the error map.

b. Expanding the error map

In this section, we use Dyson substitution to expand the error map in Eq. (B53) into several maps whose superoperator norm can be bounded individually. To this end, we seek to

bound each individual term in the sum in Eq. (B53)

$$\mathcal{E}_{q;\lambda n}(t, t_0) := \langle \langle I_A \left| i v_d \int_{t_0}^t ds \mathcal{U}_{\text{osc}}(t, s) K_{\lambda n;2}^d(s) \mathcal{U}_q(s, t_0) \right| \mathbf{0} \rangle \rangle. \quad (\text{B54})$$

Our strategy is to further expand $\mathcal{E}_{q;\lambda n}(t, t_0)$ in powers of $K_{\lambda n;j}(t)$ using Dyson substitutions in such a way that the leading-order terms vanish when we trace out the harmonic oscillators. Each time we perform such a Dyson substitution, the resulting expression gets suppressed by a factor $\sqrt{\gamma \Delta \xi}$. We are able to perform four such Dyson substitutions, allowing us to eventually achieve a bound for the norm of $\mathcal{E}_{q;\lambda n}(t, t_0)$ which is of order $(\gamma \Delta \xi)^2$. To establish that the leading-order terms are indeed vanishing, we make extensive use of arguments based on the support of quantities in the operator spaces of the individual ancillas. For simplicity, we introduce the following terminology: Below, we say that $|A\rangle\rangle$ has its (λ, n) support confined to the subspace spanned by $|i_1\rangle\langle j_1|, \dots, |i_m\rangle\langle j_m|$ if $\text{Tr}_{\lambda n}[A(|\phi\rangle\langle\psi|)_{\lambda n}] = 0$ for any $(|\phi\rangle\langle\psi|)_{\lambda n}$ orthogonal to the subspace spanned by $(|i_1\rangle\langle j_1|)_{\lambda n}, \dots, (|i_m\rangle\langle j_m|)_{\lambda n}$. Here $\text{Tr}_{\lambda n}$ denotes the partial trace over oscillator (λ, n) , and orthogonality is defined with respect to the Hilbert-Schmidt inner product.

To identify the first vanishing leading-order term in $\mathcal{E}_{q;\lambda n}(t, t_0)$, we perform a $K_{\lambda n;1}$ -Dyson substitution of $\mathcal{U}_q$ in Eq. (B54) to obtain

$$\mathcal{U}_q(s, t_0) = \mathcal{U}_q^{\lambda n;1}(s, t_0) + i v_e \int_{t_0}^s du \mathcal{U}_q(s, u) K_{\lambda n;1}^e(u) \mathcal{U}_q^{\lambda n;1}(u, t_0), \quad (\text{B55})$$

where we use $\mathcal{U}_q^{\lambda n;j}$ to denote the time-evolution superoperator generated by $H_q - K_{\lambda n;j}$. Crucially, $\mathcal{U}_q^{\lambda n;1}$ cannot couple the 0th Fock state of oscillator (λ, n) to any other Fock state (note that the first Fock state is already decoupled from all higher Fock states, by the very definition of H_q). Hence, the operator $\mathcal{U}_q^{\lambda n;1}|\mathbf{0}\rangle\rangle$ has its (λ, n) support confined to the subspace spanned by $|\mathbf{0}\rangle\langle\mathbf{0}|$. Reinserting the leading-order term of the above Dyson substitution into Eq. (B54) results in a map, which is proportional to the operator $K_{\lambda n;2} \mathcal{U}_q^{\lambda n;1}|\mathbf{0}\rangle\rangle$. Recall from the definition in Eq. (B48) that $K_{\lambda n;2}$ annihilates any operator with (λ, n) support confined to the subspace spanned by $|\mathbf{0}\rangle\langle\mathbf{0}|$. $K_{\lambda n;2} \mathcal{U}_q^{\lambda n;1}|\mathbf{0}\rangle\rangle$ is therefore the zero operator, which in turn means that the map resulting from the leading order of the Dyson substitution is the zero map. We can therefore replace $\mathcal{U}_q$ in Eq. (B54) with the residual of the above Dyson substitution, resulting in

$$\begin{aligned} \mathcal{E}_{q;\lambda n}(t, t_0) &= \left\langle \left\langle I_A \left| v_d v_e \int_{t_0}^t ds \int_{t_0}^s du \mathcal{U}_{\text{osc}}(t, s) \right. \right. \right. \\ &\quad \times K_{\lambda n;2}^d(s) \mathcal{U}_q(s, u) K_{\lambda n;1}^e(u) \mathcal{U}_q^{\lambda n;1}(u, t_0) \left. \left. \left. \right| \mathbf{0} \right\rangle \right\rangle. \end{aligned} \quad (\text{B56})$$

We now further subdivide the error map by performing a $K_{\lambda n;1}$ -Dyson substitution of $\mathcal{U}_q(s, u)$ in Eq. (B56)

$$\mathcal{E}_{q;\lambda n}(t, t_0) = \mathcal{E}_{q;\lambda n}^{(1)}(t, t_0) + \mathcal{E}_{q;\lambda n}^{(2)}(t, t_0), \quad (\text{B57})$$

where $\mathcal{E}_{q;\lambda n}^{(1)}(t, t_0)$ and $\mathcal{E}_{q;\lambda n}^{(2)}(t, t_0)$ result from the leading-order term and residual term of the Dyson substitution respectively:

$$\begin{aligned} \mathcal{E}_{q;\lambda n}^{(1)}(t, t_0) = & \left\langle \left\langle I_A \left| v_d v_e \int_{t_0}^t ds \int_{t_0}^s du \mathcal{U}_{\text{osc}}(t, s) \right. \right. \right. \\ & \times K_{\lambda n;2}^d(s) \mathcal{U}_q^{\lambda n;1}(s, u) \\ & \times K_{\lambda n;1}^e(u) \mathcal{U}_q^{\lambda n;1}(u, t_0) \left. \left. \left. \right| \mathbf{0} \right\rangle \right\rangle \end{aligned} \quad (\text{B58})$$

and

$$\begin{aligned} \mathcal{E}_{q;\lambda n}^{(2)}(t, t_0) = & -i \left\langle \left\langle I_A \left| v_d v_e v_f \int_{t_0}^t ds \int_{t_0}^s du \int_u^s dv \right. \right. \right. \\ & \times \mathcal{U}_{\text{osc}}(t, s) K_{\lambda n;2}^d(s) \mathcal{U}_q(s, v) \\ & \times K_{\lambda n;1}^f(v) \mathcal{U}_q^{\lambda n;1}(v, u) \\ & \times K_{\lambda n;1}^e(u) \mathcal{U}_q^{\lambda n;1}(u, t_0) \left. \left. \left. \right| \mathbf{0} \right\rangle \right\rangle. \end{aligned} \quad (\text{B59})$$

We bound the norm of these error maps individually below.

To bound the norm of $\mathcal{E}_{q;\lambda n}^{(2)}(t, t_0)$, we analyze the structure of the (λ, n) support of the operators making up the integrand in $\mathcal{E}_{q;\lambda n}^{(2)}(t, t_0)$. For brevity, the time dependencies are implicit in the rest of this paragraph. We first note that $|\mathbf{0}\rangle\rangle$ has its (λ, n) support confined to the span of $|0\rangle\langle 0|$. Since $\mathcal{U}_q^{\lambda n;1}$ cannot couple the vacuum state of oscillator (λ, n) with higher Fock states, the operator $|O_1^{(2)}\rangle\rangle := \mathcal{U}_q^{\lambda n;1}|\mathbf{0}\rangle\rangle$ also has its (λ, n) support confined to the span of $|0\rangle\langle 0|$. Recall from the definition in Eq. (B48) that $K_{\lambda n;1}^e$ couples the vacuum and first excited state of oscillator (λ, n) . Thus, the operator $|O_2^{(2);e}\rangle\rangle := K_{\lambda n;1}^e|O_1^{(2)}\rangle\rangle$ has its (λ, n) support confined to the subspace spanned by $|0\rangle\langle 1|, |1\rangle\langle 0|$. We observe that $\mathcal{U}_q^{\lambda n;1}$, $K_{\lambda n;1}^f$, and $\mathcal{U}_q$ at most couple the vacuum and first excited state of oscillator (λ, n) . Therefore, the operator $|O_5^{(2);ef}\rangle\rangle := \mathcal{U}_q K_{\lambda n;1}^f \mathcal{U}_q^{\lambda n;1} |O_2^{(2);e}\rangle\rangle$ has its (λ, n) support confined to the subspace spanned by $|0\rangle\langle 0|, |0\rangle\langle 1|, |1\rangle\langle 0|, |1\rangle\langle 1|$. Finally, since $K_{\lambda n;2}^d$ couples the first and second Fock state of oscillator (λ, n) and annihilates any other Fock state, we conclude that the operator to the right of $\mathcal{U}_{\text{osc}}$ in Eq. (B59), $|O_6^{(2);def}\rangle\rangle := K_{\lambda n;2}^d |O_5^{(2);ef}\rangle\rangle$, has its (λ, n) support confined to the subspace spanned by $|0\rangle\langle 2|, |2\rangle\langle 0|, |1\rangle\langle 2|, |2\rangle\langle 1|$.

We now perform a $K_{\lambda n;2}$ Dyson substitution on $\mathcal{U}_{\text{osc}}$ in Eq. (B59). The leading-order term resulting from this consists of an integral with integrand $\langle\langle I_A | O_7^{(2);def} \rangle\rangle$ with $|O_7^{(2);def}\rangle\rangle := \mathcal{U}_{\text{osc}}^{\lambda n;2} |O_6^{(2);def}\rangle\rangle$ and $\mathcal{U}_{\text{osc}}^{\lambda n;j}$ denoting the time-evolution super-operator generated by $H_{\text{osc}} - K_{\lambda n;j}$. Crucially, $\mathcal{U}_{\text{osc}}^{\lambda n;j}$ cannot couple Fock states of oscillator (λ, n) with index below j to Fock states with index j or above. Hence $\mathcal{U}_{\text{osc}}^{\lambda n;2}$ is unable to couple the second Fock state of oscillator (λ, n) with the zeroth and first Fock state. Since we found above that $|O_6^{(2);def}\rangle\rangle$ has its (λ, n) support confined to the subspace spanned by $|0\rangle\langle 2|, |2\rangle\langle 0|, |1\rangle\langle 2|, |2\rangle\langle 1|$, we conclude that $|O_7^{(2);def}\rangle\rangle$ has its (λ, n) support confined to the span of operators on the form $|0\rangle\langle j|, |j\rangle\langle 0|, |1\rangle\langle j|, |j\rangle\langle 1|$ with $j \geq 2$. Since the (λ, n) support of $|O_7^{(2);def}\rangle\rangle$ is fully off-diagonal, we thus conclude that $\langle\langle I_A | O_7^{(2);def} \rangle\rangle = 0$. This leaves us with the map resulting

from the residual term of the Dyson substitution

$$\begin{aligned} \mathcal{E}_{q;\lambda n}^{(2)}(t, t_0) = & - \left\langle \left\langle I_A \left| v_d v_e v_f v_g \int_{t_0}^t ds \int_{t_0}^s du \int_u^s dv \int_s^t dw \right. \right. \right. \\ & \times \mathcal{U}_{\text{osc}}(t, w) K_{\lambda n;2}^g(w) \mathcal{U}_{\text{osc}}^{\lambda n;2}(w, s) \\ & \times K_{\lambda n;2}^d(s) \mathcal{U}_q(s, v) K_{\lambda n;1}^f(v) \mathcal{U}_q^{\lambda n;1}(v, u) \\ & \times K_{\lambda n;1}^e(u) \mathcal{U}_q^{\lambda n;1}(u, t_0) \left. \left. \left. \right| \mathbf{0} \right\rangle \right\rangle. \end{aligned} \quad (\text{B60})$$

To bound the right-hand side above, we now use that, for ρ trace class and X and Y bounded operators, we have

$$\|\text{Tr}_A[X\rho Y]\| \leq \|X\| \|Y\| \|\rho\|_{\text{Tr}}, \quad (\text{B61})$$

where Tr_A is a partial trace, i.e., $\text{Tr}_A[O] = \langle\langle I_A | O \rangle\rangle$ [see Appendix E2 for the proof]. Using the above result, along with the triangle inequality and the submultiplicative property of the operator norm, we arrive at the following bound:

$$\begin{aligned} \|\mathcal{E}_{q;\lambda n}^{(2)}(t, t_0)\| \leq & 16 \int_{t_0}^t ds \int_{t_0}^s du \int_u^s dv \int_s^t dw \\ & \times \|K_{\lambda n;2}(s)\| \|K_{\lambda n;1}(u)\| \\ & \times \|K_{\lambda n;1}(v)\| \|K_{\lambda n;2}(w)\|. \end{aligned} \quad (\text{B62})$$

We next note that

$$\|K_{\lambda n;j}\| \leq \sqrt{j\gamma\Delta\xi} \sum_{\alpha} |g_{f;\alpha\lambda}(t - \xi_n)|. \quad (\text{B63})$$

Using this bound in Eq. (B62), and exploiting the symmetry of the resulting integral under exchange of integration variables, we find

$$\|\mathcal{E}_{q;\lambda n}^{(2)}(t, t_0)\| \leq \frac{32(\gamma\Delta\xi)^2}{4!} \left[\int_{t_0}^t ds \sum_{\alpha} |g_{f;\alpha\lambda}(s - \xi_n)| \right]^4. \quad (\text{B64})$$

Here the factor of 32 arises from counting over the indices governing right and left multiplication and the two factors of $\sqrt{2}$ from $K_{\lambda n;2}$.

We now bound the norm of $\mathcal{E}_{q;\lambda n}^{(1)}(t, t_0)$. To this end, we observe in Eq. (B58) that $|O_3^{(1);e}\rangle\rangle := \mathcal{U}_q^{\lambda n;1} K_{\lambda n;1}^e \mathcal{U}_q^{\lambda n;1} |\mathbf{0}\rangle\rangle$ has its (λ, n) support confined to the subspace spanned by $|0\rangle\langle 1|, |1\rangle\langle 0|$. Recall that $K_{\lambda n;2}^d$ couples the first and second Fock state and annihilates the vacuum state of oscillator (λ, n) . Therefore, the operator $|O_4^{(1);de}\rangle\rangle := K_{\lambda n;2}^d |O_3^{(1);e}\rangle\rangle$ has its (λ, n) support confined to the subspace spanned by $|0\rangle\langle 2|, |2\rangle\langle 0|$. We further note that $|O_4^{(1);de}\rangle\rangle$ is nonzero only when $d = e$. Performing a $(K_{\lambda n;1} + K_{\lambda n;2})$ -Dyson substitution on $\mathcal{U}_{\text{osc}}$ in Eq. (B58) results in a leading-order term, $\mathcal{U}_{\text{osc}}^{\lambda n;1,2}$ generated by $H_{\text{osc}} - K_{\lambda n;1} - K_{\lambda n;2}$. $\mathcal{U}_{\text{osc}}^{\lambda n;1,2}$ is unable to couple the vacuum, first, and second Fock states of oscillator (λ, n) . The operator, $|O_5^{(1);de}\rangle\rangle := \mathcal{U}_{\text{osc}}^{\lambda n;1,2} |O_4^{(1);de}\rangle\rangle$, therefore has its (λ, n) support confined to the span of operators on the form $|0\rangle\langle j|, |j\rangle\langle 0|$ with $j \geq 2$. The map resulting from the leading-order term of the Dyson substitution consists of sums and integrals of terms on the form $\langle\langle I_A | O_5^{(1);de} \rangle\rangle$. Since the (λ, n) support of $|O_5^{(1);de}\rangle\rangle$ is purely off diagonal, this map vanishes in the partial trace over the oscillators, leaving us with two separate maps resulting from the individual Hamiltonians in

the residual term of the Dyson expansion

$$\mathcal{E}_{q;\lambda n}^{(1)}(t, t_0) = \mathcal{E}_{q;\lambda n}^{(1a)}(t, t_0) + \mathcal{E}_{q;\lambda n}^{(1b)}(t, t_0), \quad (\text{B65})$$

where

$$\begin{aligned} \mathcal{E}_{q;\lambda n}^{(1a)}(t, t_0) = & -i \left\langle \left\langle I_A \left| v_d v_e v_f \int_{t_0}^t ds \int_{t_0}^s du \int_s^t dv \right. \right. \right. \\ & \times \mathcal{U}_{\text{osc}}(t, v) K_{\lambda n;1}^f(v) \mathcal{U}_{\text{osc}}^{\lambda n;1,2}(v, s) \\ & \times K_{\lambda n;2}^d(s) \mathcal{U}_q^{\lambda n;1}(s, u) \\ & \left. \left. \left. \times K_{\lambda n;1}^e(u) \mathcal{U}_q^{\lambda n;1}(u, t_0) \right| \mathbf{0} \right\rangle \right\rangle \end{aligned} \quad (\text{B66})$$

and

$$\begin{aligned} \mathcal{E}_{q;\lambda n}^{(1b)}(t, t_0) = & -i \left\langle \left\langle I_A \left| v_d v_e v_f \int_{t_0}^t ds \int_{t_0}^s du \int_s^t dv \right. \right. \right. \\ & \times \mathcal{U}_{\text{osc}}(t, v) K_{\lambda n;2}^f(v) \mathcal{U}_{\text{osc}}^{\lambda n;1,2}(v, s) \\ & \times K_{\lambda n;2}^d(s) \mathcal{U}_q^{\lambda n;1}(s, u) \\ & \left. \left. \left. \times K_{\lambda n;1}^e(u) \mathcal{U}_q^{\lambda n;1}(u, t_0) \right| \mathbf{0} \right\rangle \right\rangle. \end{aligned} \quad (\text{B67})$$

We consider the two residual terms individually, starting with $\mathcal{E}_{q;\lambda n}^{(1a)}$. To this end, we note that the operator $|O_6^{(1a);def}\rangle := K_{\lambda n;1}^f |O_5^{(1);de}\rangle$ has its (λ, n) support confined to the span of operators of the form $|1\rangle\langle j|, |j\rangle\langle 1|$ with $j \geq 2$. We further note that $|O_6^{(1a);def}\rangle$ is nonzero only when $d = e \neq f$. We now perform a $K_{\lambda n;2}$ -Dyson substitution on $\mathcal{U}_{\text{osc}}$ in Eq. (B66) with leading-order term $\mathcal{U}_{\text{osc}}^{\lambda n;2}$. The (λ, n) support of the operator $|O_7^{(1a);def}\rangle := \mathcal{U}_{\text{osc}}^{\lambda n;2} |O_6^{(1a);def}\rangle$ is confined to the span of $|0\rangle\langle j|, |j\rangle\langle 0|, |1\rangle\langle j|, |j\rangle\langle 1|$ with $j \geq 2$. The map resulting from the leading-order term of the Dyson substitution consists of sums and integrals of terms of the form $\langle\langle I_A | O_7^{(1a);def} \rangle\rangle$. Once again this map vanishes in the partial trace, leaving us with the map resulting from the residual

$$\begin{aligned} \mathcal{E}_{q;\lambda n}^{(1a)}(t, t_0) = & - \left\langle \left\langle I_A \left| v_d v_e v_f v_g \int_{t_0}^t ds \int_{t_0}^s du \int_s^t dv \int_v^t dw \right. \right. \right. \\ & \times \mathcal{U}_{\text{osc}}(t, w) K_{\lambda n;2}^g(w) \mathcal{U}_{\text{osc}}^{\lambda n;2}(w, v) \\ & \times K_{\lambda n;1}^f(v) \mathcal{U}_{\text{osc}}^{\lambda n;1,2}(v, s) K_{\lambda n;2}^d(s) \mathcal{U}_q^{\lambda n;1}(s, u) \\ & \left. \left. \left. \times K_{\lambda n;1}^e(u) \mathcal{U}_q^{\lambda n;1}(u, t_0) \right| \mathbf{0} \right\rangle \right\rangle. \end{aligned} \quad (\text{B68})$$

By the same arguments as for the bound on the norm of $\mathcal{E}_{q;\lambda n}^{(2)}$, we arrive at the following bound:

$$\|\mathcal{E}_{q;\lambda n}^{(1a)}(t, t_0)\| \leq \frac{8(\gamma \Delta \xi)^2}{4!} \left[\int_{t_0}^t ds \sum_{\alpha} |g_{f;\alpha\lambda}(s - \xi_n)| \right]^4. \quad (\text{B69})$$

The prefactor is smaller than before since we showed that only terms where $d = e \neq f$ provide nonzero contributions to the bound.

By a line of arguments similar to the ones we made below Eq. (B67) we conclude that the (λ, n) support of $|O_6^{(1b);def}\rangle := K_{\lambda n;2}^f |O_5^{(1);de}\rangle$ is confined to the span of $|0\rangle\langle 1|, |1\rangle\langle 0|$ and is moreover only nonzero when $d = e = f$.

Performing a $K_{\lambda n;1}$ -Dyson substitution on $\mathcal{U}_{\text{osc}}$ in Eq. (B67), we thus see that the leading-order term vanishes, leaving us with

$$\begin{aligned} \mathcal{E}_{q;\lambda n}^{(1b)}(t, t_0) = & -i \left\langle \left\langle I_A \left| v_d v_e v_f v_g \int_{t_0}^t ds \int_{t_0}^s du \int_s^t dv \int_v^t dw \right. \right. \right. \\ & \times \mathcal{U}_{\text{osc}}(t, w) K_{\lambda n;1}^g(w) \mathcal{U}_{\text{osc}}^{\lambda n;1}(w, v) \\ & \times K_{\lambda n;2}^f(v) \mathcal{U}_{\text{osc}}^{\lambda n;1,2}(v, s) K_{\lambda n;2}^d(s) \mathcal{U}_q^{\lambda n;1}(s, u) \\ & \left. \left. \left. \times K_{\lambda n;1}^e(u) \mathcal{U}_q^{\lambda n;1}(u, t_0) \right| \mathbf{0} \right\rangle \right\rangle. \end{aligned} \quad (\text{B70})$$

By the same arguments as for the bound on the norm of $\mathcal{E}_{q;\lambda n}^{(2)}$, we arrive at the following bound:

$$\|\mathcal{E}_{q;\lambda n}^{(1b)}(t, t_0)\| \leq \frac{8(\gamma \Delta \xi)^2}{4!} \left[\int_{t_0}^t ds \sum_{\alpha} |g_{f;\alpha\lambda}(s - \xi_n)| \right]^4. \quad (\text{B71})$$

The small prefactor results from the fact that only terms where $d = e = f$ provide nonzero contributions to the bound. We have now bounded the norm of all terms that make up the error map.

c. Combining the error bounds

We finally combine the bounds on the norm of the subdivisions of the error map in order to arrive at a bound on the norm of the entire error map. We start by combining the three bounds from the last section. Using the triangle inequality, we find

$$\|\mathcal{E}_{q;\lambda n}(t, t_0)\| \leq \|\mathcal{E}_{q;\lambda n}^{(2)}(t, t_0)\| + \|\mathcal{E}_{q;\lambda n}^{(1a)}(t, t_0)\| + \|\mathcal{E}_{q;\lambda n}^{(1b)}(t, t_0)\|. \quad (\text{B72})$$

Substituting in the bounds for the terms from Eqs. (B64), (B69), and (B71) and changing the integration variable results in the bound

$$\|\mathcal{E}_{q;\lambda n}(t, t_0)\| \leq 2(\gamma \Delta \xi)^2 \left[\int_{t_0 - \xi_n}^{t - \xi_n} ds \sum_{\alpha} |g_{f;\alpha\lambda}(s)| \right]^4. \quad (\text{B73})$$

Using the triangle inequality once more, we can bound the norm of the entire error map by summing up the bounds for the individual oscillators

$$\|\mathcal{E}_q(t, t_0)\| \leq \sum_{\lambda, n} 2(\gamma \Delta \xi)^2 \left[\int_{t_0 - \xi_n}^{t - \xi_n} ds \sum_{\alpha} |g_{f;\alpha\lambda}(s)| \right]^4. \quad (\text{B74})$$

To simplify this bound, we use the inequality, which states that for any function $f(t)$,

$$\sum_{n=-\infty}^{\infty} \int_{t_0 - \xi_n}^{t - \xi_n} dt |f(t)| \leq \left(\frac{T}{\Delta \xi} + 1 \right) \int_{-\infty}^{\infty} dt |f(t)|, \quad (\text{B75})$$

where $T = t - t_0$ and $\xi_n = n\Delta\xi$ (see Appendix E 3 for proof). Extending the domain of integration to the entire real line for three of the integrals in Eq. (B74) and using the above inequality on the fourth integral allows us to get rid of the

sum over n and provides the following bound:

$$\|\mathcal{E}_q(t, t_0)\| \leq 2(\gamma \Delta \xi)^2 \left(\frac{T}{\Delta \xi} + 1 \right) \times \sum_{\lambda} \left[\int_{-\infty}^{\infty} ds \sum_{\alpha} |g_{f;\alpha\lambda}(s)| \right]^4. \quad (\text{B76})$$

since $\sum_n |a_n|^2 \leq (\sum_n |a_n|)^2$ for any set of numbers $\{a_n\}$, we obtain

$$\|\mathcal{E}_q(t, t_0)\| \leq 2\Delta \xi (T + \Delta \xi) \gamma^2 \left(\sum_{\lambda} \left[\int_{-\infty}^{\infty} ds \sum_{\alpha} |g_{f;\alpha\lambda}(s)| \right]^2 \right)^2. \quad (\text{B77})$$

Minkowski's inequality, $\sum_{\lambda} |\int_{-\infty}^{\infty} ds \sum_{\alpha} f_{\alpha\lambda}(s)|^p \leq [\int_{-\infty}^{\infty} ds \sum_{\alpha} (\sum_{\lambda} |f_{\alpha\lambda}(s)|^p)^{1/p}]^p$, allows us to further bound the error by

$$\|\mathcal{E}_q(t, t_0)\| \leq 2\Delta \xi (T + \Delta \xi) \gamma^2 \left(\int_{-\infty}^{\infty} ds \sum_{\alpha} \sqrt{\sum_{\lambda} |g_{f;\alpha\lambda}(s)|^2} \right)^4. \quad (\text{B78})$$

Using $|g_{\alpha\lambda}(s)| = |g_{\lambda\alpha}(-s)|$, and relabeling the integration variable s to $-s$, we obtain

$$\|\mathcal{E}_q(t, t_0)\| \leq 2\Delta \xi (T + \Delta \xi) \gamma^2 \left(\int_{-\infty}^{\infty} ds \sum_{\lambda} \sqrt{\sum_{\alpha} |g_{f;\alpha\lambda}(s)|^2} \right)^4. \quad (\text{B79})$$

We recognize the integrand above as the 2,1-norm of $\mathbf{g}$. Defining $\Gamma_f := 4\gamma [\int_{-\infty}^{\infty} dt \|\mathbf{g}_f(t)\|_{2,1}]^2$ analogously to Γ , we arrive at the following error bound:

$$\|\mathcal{E}_q(t, t_0)\| \leq \frac{\Delta \xi}{8} (T + \Delta \xi) \Gamma_f^2. \quad (\text{B80})$$

If we choose $\Delta \xi \leq \frac{\varepsilon}{\Gamma}$ and $\varepsilon \leq \Gamma T$, then we get the following error scaling:

$$\|\mathcal{E}_q(t, t_0)\| = O(\varepsilon \Gamma_f^2 T / \Gamma). \quad (\text{B81})$$

Importantly, with the choice of filtering function $\tilde{\varphi}$ made in Eq. (B8), we have $\Gamma_f \leq 14.41\Gamma$ —see Eq. (C7) for the proof. Thereby, we obtain

$$\varepsilon_2 \leq 25.96 \Delta \xi (T + \Delta \xi) \Gamma^2. \quad (\text{B82})$$

This tells us that a train of qubits can emulate the effect of a train of harmonic oscillators in the limit of a small discretization step.

4. Error from restricting interactions to a finite number of qubits

In this section, we bound the error from truncating the qubit register connected to the system at each point in time. Specifically, we introduce a cutoff time, τ_c , such that the

(λ, n) th ancilla qubit only interacts with the system at times t for which $|t - \xi_n| \leq \tau_c$,

$$H_T = H_S + \sqrt{\gamma} \sum_{\alpha} S_{\alpha} \otimes A_{T;\alpha}(t), \quad (\text{B83})$$

with $A_{T;\alpha}(t)$ given in Eq. (36). We define $\rho_T(t)$ to be the reduced density matrix of the system time evolved under H_T from the initial state $\rho(t_0) \otimes |\mathbf{0}\rangle\langle\mathbf{0}|_{\mathcal{A}}$. Similarly to before, our goal is to quantify the error by considering the trace distance between the time evolution with and without a time cutoff

$$\varepsilon_3 := \|\rho_T(t) - \rho_q(t)\|_{\text{Tr}}. \quad (\text{B84})$$

In this Appendix, we show that this error can be upper bounded by

$$\varepsilon_3 \leq \Gamma_f (T + \Delta \xi) \frac{\tau_f}{\tau_c}, \quad (\text{B85})$$

where τ_f and Γ_f are related to the characteristic interaction rate and correlation timescale of the bath.

Similarly to the previous subsection, we arrive at the bound above by defining a quantum channel for the time truncated dynamics, $\mathcal{C}_T$, such that $\rho_T(t) = \mathcal{C}_T(t, t_0)[\rho_0]$. As in the last Appendix, we define the error map as the difference between the two quantum channels and arrive at the bound by bounding the superoperator norm of the following error map:

$$\mathcal{E}_T(t, t_0) := \mathcal{C}_T(t, t_0)[\rho(t_0)] - \mathcal{C}_q(t, t_0)[\rho(t_0)], \quad (\text{B86})$$

where $\mathcal{C}_q(t, t_0)[\rho(t_0)]$ is the quantum channel for the non-truncated evolution. The rest of this Appendix is dedicated to constructing the error map in Appendix B 4 a and subsequently splitting and bounding the norm of the terms in the error map in Appendix B 4 b.

a. Constructing the error map

To obtain an expression for the error map $\mathcal{E}_T$, we define $\chi_n(t)$ to be the indicator function of an interval of width $2\tau_c$ centered at ξ_n :

$$\chi_n(t) := \begin{cases} 1, & |t - \xi_n| \leq \tau_c \\ 0, & |t - \xi_n| > \tau_c \end{cases}. \quad (\text{B87})$$

We can then construct a Hamiltonian which is equivalent to H_T by explicitly subtracting interactions with ancillary qubits, which fall outside of the cutoff interval

$$H_T(t) = H_q(t) - \sum_{\lambda} \sum_{n=-\infty}^{\infty} K_{\lambda n}(t) [1 - \chi_n(t)], \quad (\text{B88})$$

where

$$K_{\lambda n}(t) = \sum_{\alpha} S_{\alpha} \sqrt{\gamma \Delta \xi} (g_{f;\alpha\lambda}^*(t - \xi_n) \sigma_{\lambda n}^+ + \text{H.c.}) \quad (\text{B89})$$

is the time-dependent coupling to qubit (λ, n) . Note that $K_{\lambda n}$ is simply a restriction of $K_{\lambda n,1}$ from the previous appendix to the subspace spanned by the zeroth and first Fock states, which is equivalent to the Hilbert space of the ancilla qubits. We can now construct the quantum channel for the time truncated dynamics $\mathcal{C}_T(t, t_0) = \langle \langle I_{\mathcal{A}} | \mathcal{U}_T(t, t_0) | \mathbf{0} \rangle \rangle$, where $\mathcal{U}_T$ is the time-evolution superoperator generated by H_T . Recall that the quantum channel for the qubit evolution is

$\mathcal{C}_q(t, t_0) = \langle \langle I_A | \mathcal{U}_q(t, t_0) | \mathbf{0} \rangle \rangle$, where $\mathcal{U}_q$ is the time-evolution superoperator generated by H_q . The error map can now be found as the difference between the two channels. Similarly to before, we relate the two time-evolution superoperators using a Dyson substitution to obtain a compact expression for the error map [see paragraph above Eq. (B50)]. Specifically, we perform a $\sum_\lambda \sum_{n=-\infty}^{\infty} K_{\lambda n}(t)[1 - \chi_n(t)]$ -Dyson substitution on $\mathcal{U}_q$, allowing us to write

$$\mathcal{U}_q(t, t_0) = \mathcal{U}_T(t, t_0) - i \sum_\lambda \sum_{n=-\infty}^{\infty} \left[v_d \int_{t_0}^t ds \mathcal{U}_q(t, s) K_{\lambda n}^d(s) \times [1 - \chi_n(t)] \mathcal{U}_T(s, t_0) \right]. \quad (\text{B90})$$

This allows us to identify

$$\mathcal{E}_T(t, t_0) = \sum_{\lambda, n} i v_d \int_{t_0}^t ds \langle \langle I_A | \mathcal{U}_q(t, s) K_{\lambda n}^d(s) \times [1 - \chi_n(s)] \mathcal{U}_T(s, t_0) | \mathbf{0} \rangle \rangle. \quad (\text{B91})$$

We now set out to expand and bound the error map by methods similar to those used in the previous subsection.

b. Expanding and bounding the error map

In this section, we expand the error map into a sum of terms that can be conveniently bounded individually. We first define the error map associated with each individual ancilla qubit (λ, n) as

$$\mathcal{E}_{T;\lambda n}(t, t_0) = i v_d \int_{t_0}^t ds \langle \langle I_A | \mathcal{U}_q(t, s) K_{\lambda n}^d(s) \times [1 - \chi_n(s)] \mathcal{U}_T(s, t_0) | \mathbf{0} \rangle \rangle. \quad (\text{B92})$$

The total error map is the sum over the error maps above. We now use Dyson substitutions on the unitary superoperators above to express $\mathcal{E}_{T;\lambda n}(t, t_0)$ in such a way that it can be conveniently bounded. As in the previous subsection, our strategy is to choose Dyson substitutions where the leading-order terms vanish based on considerations about (λ, n) support (see the second paragraph under Eq. (B54) for a definition of this terminology). To aid this analysis, we introduce the following notation: We let $\mathcal{U}_q^{\lambda n}(t)$ denote the time-evolution superoperator generated by $H_q(t) - K_{\lambda n}(t)$ and let $\mathcal{U}_T^{\lambda n}(t)$ denote the time-evolution superoperator generated by $H_T(t) - K_{\lambda n}(t)\chi_n(t)$. Crucially, both these superoperators act trivially on the operator space of ancilla qubit (λ, n) , implying that they do not modify the (λ, n) support of the operator they act on.

Having laid out the strategy and notation, we are ready to expand and bound $\mathcal{E}_{T;\lambda n}(t, t_0)$. As a first step, we perform a $K_{\lambda n}\chi_n$ -Dyson substitution on $\mathcal{U}_T$ in Eq. (B92), resulting in

$$\mathcal{E}_{T;\lambda n} = \mathcal{E}_{T;\lambda n}^{(1)} + \mathcal{E}_{T;\lambda n}^{(2)}, \quad (\text{B93})$$

where the two terms above arise from the leading-order and residual terms of the Dyson substitution, respectively:

$$\mathcal{E}_{T;\lambda n}^{(1)}(t, t_0) = i \left\langle \left\langle I_A \left| v_d \int_{t_0}^t ds \mathcal{U}_q(t, s) K_{\lambda n}^d(s) [1 - \chi_n(s)] \mathcal{U}_T^{\lambda n}(s, t_0) \right| \mathbf{0} \right\rangle \right\rangle \quad (\text{B94})$$

and

$$\mathcal{E}_{T;\lambda n}^{(2)}(t, t_0) = - \left\langle \left\langle I_A \left| v_d v_e \int_{t_0}^t ds \int_{t_0}^s du \times \mathcal{U}_q(t, s) K_{\lambda n}^d(s) [1 - \chi_n(s)] \mathcal{U}_T(s, u) \times K_{\lambda n}^e(u) \chi_n(u) \mathcal{U}_T^{\lambda n}(u, t_0) \right| \mathbf{0} \right\rangle \right\rangle. \quad (\text{B95})$$

We bound the error of the maps individually, starting with $\mathcal{E}_{T;\lambda n}^{(2)}$. In Eq. (B61) [or (E11)] we saw that $\|\text{Tr}_A[X\rho Y]\| \leq \|X\| \|Y\| \|\rho\|_{\text{Tr}}$, where Tr_A denotes the partial trace over the ancilla qubits. Using this result together with the triangle inequality and the submultiplicativity of the operator norm, we obtain

$$\|\mathcal{E}_{T;\lambda n}^{(2)}(t, t_0)\| \leq 4 \int_{t_0}^t ds \int_{t_0}^s du [1 - \chi_n(s)] \chi_n(u) \times \|K_{\lambda n}(s)\| \|K_{\lambda n}(u)\|, \quad (\text{B96})$$

where the factor of 4 comes from counting over the left and right multiplication indices, d and e . Recalling that $K_{\lambda n}$ is the restriction of $K_{\lambda n,1}$ to the two lowest Fock states of the oscillator, we use the bound on $K_{\lambda n,1}$ from Eq. (B63) and the fact that $\chi_n \leq 1$ to obtain

$$\|\mathcal{E}_{T;\lambda n}^{(2)}(t, t_0)\| \leq 4\gamma \Delta \xi \int_{t_0}^t ds \int_{t_0}^s du [1 - \chi_n(s)] \times \sum_{\alpha, \alpha'} |g_{f;\alpha\lambda}(s - \xi_n)| |g_{f;\alpha'\lambda}(u - \xi_n)|, \quad (\text{B97})$$

We now bound the term $\mathcal{E}_{T;\lambda n}^{(1)}$ in Eq. (B94). We first note that the operator to the right of $\mathcal{U}_q$ in Eq. (B94) has its (λ, n) support confined to the subspace spanned by $|0\rangle\langle 1|, |1\rangle\langle 0|$. Performing a $K_{\lambda n}$ -Dyson substitution on $\mathcal{U}_q$ in Eq. (B94) results in a leading-order term $\mathcal{U}_{\lambda n}^q$, which preserves the confinement of the (λ, n) support. Since this support is purely off-diagonal, the map resulting from the leading-order term of the Dyson substitution vanishes in the partial trace over the ancilla qubits. This leaves us with the following map resulting from the residual term of the Dyson substitution:

$$\mathcal{E}_{T;\lambda n}^{(1)}(t, t_0) = i \left\langle \left\langle I_A \left| v_d v_e \int_{t_0}^t ds \int_{t_0}^s du \times \mathcal{U}_q(t, u) K_{\lambda n}^e(u) \mathcal{U}_q^{\lambda n}(u, s) \times K_{\lambda n}^d[1 - \chi_n(s)](s) \mathcal{U}_{\lambda n}^T(s, t_0) \right| \mathbf{0} \right\rangle \right\rangle. \quad (\text{B98})$$

By the same arguments used above for bounding the norm of $\mathcal{E}_{T;\lambda n}^{(2)}$, we obtain the bound

$$\begin{aligned} \|\mathcal{E}_{T;\lambda n}^{(1)}(t, t_0)\| &\leq 4\gamma \Delta\xi \int_{t_0}^t ds \int_s^t du [1 - \chi_n(s)] \\ &\quad \times \sum_{\alpha, \alpha'} |g_{f;\alpha\lambda}(s - \xi_n)| |g_{f;\alpha'\lambda}(u - \xi_n)|. \end{aligned} \quad (\text{B99})$$

We next combine the two bounds in Eqs. (B97) and (B99) to obtain a bound for $\|\mathcal{E}_{T;\lambda n}(t, t_0)\|$. Using the triangle inequality and the fact that the integration domains in Eq. (B97) and Eq. (B99) line up, we find the following bound:

$$\begin{aligned} \|\mathcal{E}_{T;\lambda n}(t, t_0)\| &\leq 4\gamma \Delta\xi \int_{t_0}^t ds \int_{t_0}^t du [1 - \chi_n(s)] \\ &\quad \times \sum_{\alpha, \alpha'} |g_{f;\alpha\lambda}(s - \xi_n)| |g_{f;\alpha'\lambda}(u - \xi_n)|. \end{aligned} \quad (\text{B100})$$

We finally obtain a bound for $\|\mathcal{E}_T(t, t_0)\|$ by summing the above bound over all ancilla qubits

$$\begin{aligned} \|\mathcal{E}_T(t, t_0)\| &\leq \sum_{\lambda, n} 4\gamma \Delta\xi \int_{t_0 - \xi_n}^{t - \xi_n} ds \int_{t_0 - \xi_n}^{t - \xi_n} du [1 - \chi_0(s)] \\ &\quad \times \sum_{\alpha, \alpha'} |g_{f;\alpha\lambda}(s)| |g_{f;\alpha'\lambda}(u)|. \end{aligned} \quad (\text{B101})$$

By extending the integration over s to the entire real line and then applying the same argument used below Eq. (B75), we arrive at the bound

$$\begin{aligned} \|\mathcal{E}_T(t, t_0)\| &\leq 4\gamma (T + \Delta\xi) \int_{-\infty}^{\infty} du \int_{-\infty}^{\infty} ds [1 - \chi_0(s)] \\ &\quad \times \sum_{\lambda, \alpha, \alpha'} |g_{f;\alpha\lambda}(s)| |g_{f;\alpha'\lambda}(u)|, \end{aligned} \quad (\text{B102})$$

where $T = t - t_0$ is the full simulation time. Using the inequality (see Appendix E 4 for the proof)

$$\sum_{\lambda, \alpha, \alpha'} |g_{\alpha\lambda}(s)| |g_{\alpha'\lambda}(u)| \leq \|\mathbf{g}(-s)\|_{2,1} \|\mathbf{g}(-u)\|_{2,1}, \quad (\text{B103})$$

we obtain, relabeling integration variables

$$\begin{aligned} \|\mathcal{E}_T(t, t_0)\| &\leq 4\gamma (T + \Delta\xi) \int_{-\infty}^{\infty} du \int_{-\infty}^{\infty} ds [1 - \chi_0(s)] \\ &\quad \times \|\mathbf{g}_f(s)\|_{2,1} \|\mathbf{g}_f(u)\|_{2,1}. \end{aligned} \quad (\text{B104})$$

Recalling the definition of Γ_f [see also below Eq. (C1)], we find

$$\varepsilon_3 \leq 2(T + \Delta\xi) \sqrt{\gamma \Gamma_f} \int_{-\infty}^{\infty} ds [1 - \chi_0(s)] \|\mathbf{g}_f(s)\|_{2,1}. \quad (\text{B105})$$

Since $1 - \chi_0(t)$ vanishes for $|t| \leq \tau_c$, we see that the norm of the error map is small if $\|\mathbf{g}_f\|_{2,1}$ has negligible weight in the regions where $|t| \geq \tau_c$. The above constitutes our best bound for ε_3 .

While the above bound is generally valid, we can obtain a simple expression for the right-hand side in terms of the bath correlation time τ_f and Γ_f . Since the integrand in the above

integrand is nonzero only when $\frac{|s|}{\tau_c} \geq 1$, we find

$$\|\mathcal{E}_T(t, t_0)\| \leq 2\sqrt{\gamma \Gamma_f} (T + \Delta\xi) \frac{1}{\tau_c} \int_{-\infty}^{\infty} ds |s| \|\mathbf{g}_f(s)\|_{2,1}. \quad (\text{B106})$$

Defining $\tau_f := \int_{-\infty}^{\infty} ds |s| \|\mathbf{g}_f(s)\|_{2,1} / \int_{-\infty}^{\infty} ds \|\mathbf{g}_f(s)\|_{2,1}$ analogously to τ , we find

$$\|\mathcal{E}_T(t, t_0)\| \leq \Gamma_f (T + \Delta\xi) \frac{\tau_f}{\tau_c}. \quad (\text{B107})$$

In Appendix C we show that $\Gamma_f \tau_f \leq 55.16 \Gamma \tau$ with the choice of filtering function φ made in Eq. (B8). Thus, we obtain the following bound on the error

$$\varepsilon_3 \leq 55.16 \Gamma \frac{\tau}{\tau_c} (T + \Delta\xi). \quad (\text{B108})$$

This tells us that we can emulate the effects of the bath to precision $O(\varepsilon \Gamma T)$ by choosing $\tau_c \sim \tau/\varepsilon$.

5. Trotter error

In this section, we bound the error from Trotterizing the evolution of the truncated Hamiltonian $H_T = H_S + \sqrt{\gamma} \sum_{\alpha} S_{\alpha} \otimes A_{T;\alpha}(t)$ first defined in Eq. (B83) with $A_{T;\alpha}(t)$ given in Eq. (36) and Δt a finite Trotter step. Such a Trotterized evolution is directly implemented in the ATA; thereby this bound completes the establishment of an error bound for the ATA, see, e.g., Fig. 6. For simplicity, we focus on the case where H_S is time independent. Trotterization techniques for time-ordered exponentials are analogous to time-independent Trotterization, and both first- and second-order formulas exist, see, e.g., Ref. [64]. The case of time-dependent $H_S(t)$ can thus be analyzed with similar techniques as below but is more cumbersome; we therefore leave such an investigation for future studies.

Our construction of the ATA employs a second-order Trotterization scheme. Denoting $t_{\ell} := t_0 + \ell \Delta t$ and defining $M := \text{round}(T/\Delta t)$ as the total number of Trotter steps, this scheme approximates the time-evolution operator generated by H_T ,

$$U_T(t_M, t_0) = \mathcal{T} e^{-i \int_{t_0}^{t_M} dt H_T(t)}, \quad (\text{B109})$$

with the Trotterized system-ancilla ATR evolution operator given by $\prod_{\ell=1}^M G_{\ell}$, where

$$G_{\ell} = e^{-\frac{iH_S \Delta t}{2}} \left[\mathcal{T} e^{-i \int_{t_{\ell-1}}^{t_{\ell}} dt' \sqrt{\gamma} \sum_{\alpha} S_{\alpha} \otimes A_{T;\alpha}(t')} \right] e^{-\frac{iH_S \Delta t}{2}}, \quad (\text{B110})$$

as in Eq. (25) in the main text. Letting and $\rho_T(t_M)$ denote the system reduced density matrix generated by the system-ancilla Hamiltonian H_T and $\rho_{\text{Trotter}}(t_M)$ the system reduced density matrix generated by the Trotterized system-ancilla time-evolution operator, we define the Trotter error:

$$\varepsilon_4 := \|\rho_T(t_M) - \rho_{\text{Trotter}}(t_M)\|_{\text{Tr}}. \quad (\text{B111})$$

In Appendix E 5 we prove that the sum of the distances between the Trotterized and original time-evolution increments provides an upper bound for the deviation. With the notation $U_{\ell} := U_T(t_{\ell}, t_{\ell-1})$, we have the bound

$$\|\rho_T(t_M) - \rho_{\text{Trotter}}(t_M)\|_{\text{Tr}} \leq 2 \sum_{\ell=1}^M \|U_{\ell} - G_{\ell}\|. \quad (\text{B112})$$

Therefore,

$$\varepsilon_4 \leq 2 \sum_{\ell=1}^M \|U_\ell - G_\ell\|. \quad (\text{B113})$$

We next bound the terms on the right-hand side of (B113). To this end, for brevity, we introduce the notation $X = H_S$ and $Y(t) = \sqrt{\gamma} \sum_{\alpha} S_{\alpha} \otimes A_{T;\alpha}(t)$. The evolution we wish to Trotterize is of the form $\mathcal{T}e^{-i \int_s^{s+z} X(u)+Y(u)du}$; our goal hence is to bound the distance of this operator to $e^{-\frac{iXz}{2}} [\mathcal{T}e^{-i \int_s^{s+z} Y(u)du}] e^{-\frac{iXz}{2}}$. Here we included an argument, u , in $X(u)$, whenever X is part of the time-ordered exponential argument despite the fact that we take X to be time independent, to signify that X *cannot* be pulled outside the integral in the time-ordered exponential. Such a bound can be conveniently established using standard Trotterization techniques [64] by introducing the operator

$$F(z, s) := e^{\frac{iXz}{2}} (\mathcal{T}e^{-i \int_s^{s+z} Y(u)du})^{-1} e^{\frac{iXz}{2}} \mathcal{T}e^{-i \int_s^{s+z} X(u)+Y(u)du} - 1, \quad (\text{B114})$$

which compares the original time-evolution increment followed by an inverse Trotterized increment to the identity. To relate $F(\Delta t, z)$ to the operator difference we wish to bound, we use $F(0, s) = 0$ to write $F(\Delta t, s) = \int_0^{\Delta t} dz \frac{d}{dz} F(z, s)$. Using the integral triangle inequality, we find

$$\|F(\Delta t, s)\| \leq \int_0^{\Delta t} dz \left\| \frac{d}{dz} F(z, s) \right\|. \quad (\text{B115})$$

Exploiting the invariance of the operator norm under multiplication by unitary operators to multiply both sides by $e^{-\frac{iX\Delta t}{2}} (\mathcal{T}e^{-i \int_s^{s+\Delta t} Y(u)du}) e^{-\frac{iX\Delta t}{2}}$, we thus find

$$\begin{aligned} & \left\| e^{-\frac{iX\Delta t}{2}} (\mathcal{T}e^{-i \int_s^{s+\Delta t} Y(u)du}) e^{-\frac{iX\Delta t}{2}} - \mathcal{T}e^{-i \int_s^{s+\Delta t} X(u)+Y(u)du} \right\| \\ & \leq \int_0^{\Delta t} dz \left\| \frac{d}{dz} F(z, s) \right\|. \end{aligned} \quad (\text{B116})$$

We note that the a sum over ℓ on the left-hand side with $s = t_{\ell-1}$ gives an upper bound on the Trotter error by Eqs. (B109), (B110), and (B113).

As the next step, we bound $\left\| \frac{d}{dz} F(z, s) \right\|$. Through direct evaluation and use the triangle inequality, one can verify

$$\begin{aligned} \left\| \frac{d}{dz} F(z, s) \right\| & \leq \int_0^z dw \int_0^w du \left[\frac{1}{2} \|[[X, Y(s+z)], Y(s+w)]\| \right. \\ & \quad + \frac{1}{4} \|[[X, [X, Y(z+s)]]\| \\ & \quad + \frac{1}{2} \int_w^z dv \|[[X, Y'(v+s)], Y(s+u)]\| \\ & \quad \left. + \frac{1}{2} \|[[X, Y'(w+s)]]\| \right]. \end{aligned} \quad (\text{B117})$$

Notice that integrating this equation leads to the well-known Suzuki-Trotter error bound when $Y(t)$ is constant [94]. We now analyze the terms of integrand in (B117) individually. For brevity, we focus on the first term here. The remaining three terms can be bounded by similar arguments.

To bound the first term in Eq. (B117), we first use our definitions of X and Y to obtain

$$\begin{aligned} & \|[[X, Y(t)], Y(s)]\| \\ & = \gamma \sum_{\alpha, \beta} \|[[H_S, S_{\alpha}] \otimes A_{T;\alpha}(t), S_{\beta} \otimes A_{T;\beta}(s)]\|. \end{aligned} \quad (\text{B118})$$

Hence,

$$\begin{aligned} & \|[[X, Y(s+z)], Y(s+w)]\| \\ & \leq 2\gamma \sum_{\alpha, \beta} \|[[H_S, S_{\alpha}]] \| A_{T;\alpha}(s+z) \| \|S_{\beta}\| \|A_{T;\beta}(s+w)\|. \end{aligned} \quad (\text{B119})$$

Recall from the definition of $A_{T;\beta}(t)$ [Eq. (36)] that $\|A_{T;\beta}(t)\| \leq \frac{1}{\sqrt{\Delta\xi}} \sum_{n=-\infty}^{\infty} \sum_{\lambda} |g_{f;\beta\lambda}(t - \xi_n)| \Delta\xi$. Thus, using the inequality $f(y) \leq \frac{1}{\delta} \int_{x_0}^{x_0+\delta} dx f(x) + \int_{x_0}^{x_0+\delta} dx |f'(x)|$ for all $x_0 \leq y \leq x_0 + \delta$ and f absolutely continuous, we find

$$\begin{aligned} \|A_{T;\beta}(t)\| & \leq \frac{1}{\sqrt{\Delta\xi}} \int_{-\infty}^{\infty} ds' \left[\sum_{\lambda} |g_{f;\beta\lambda}(t - s')| \right. \\ & \quad \left. + \Delta\xi |g'_{f;\beta\lambda}(t - s')| \right]. \end{aligned} \quad (\text{B120})$$

Defining $\mathbf{s} = (s_1, s_2, \dots, s_{N_C})$, $\dot{\mathbf{s}} = (\dot{s}_1, \dot{s}_2, \dots, \dot{s}_{N_C})$, with

$$\begin{aligned} s_{\alpha} & = \|S_{\alpha}\| = 1, \\ \dot{s}_{\alpha} & = \|[H_S, S_{\alpha}]\|, \end{aligned} \quad (\text{B121})$$

and using the Cauchy-Schwarz inequality, we thus find

$$\begin{aligned} \sum_{\beta} \|S_{\beta}\| \|A_{T;\beta}(t)\| & \leq \frac{1}{\sqrt{\Delta\xi}} \|\mathbf{s}\|_2 \int_{-\infty}^{\infty} ds' [\|\mathbf{g}_f(t - s')\|_{2,1} \\ & \quad + \Delta\xi \|\mathbf{g}'_f(t - s')\|_{2,1}], \end{aligned} \quad (\text{B122})$$

with $\|\cdot\|_2$ denoting the standard Euclidean norm. Defining $C_f := \frac{\sqrt{\gamma} \int_{-\infty}^{\infty} ds \|\mathbf{g}'_f(s)\|_{2,1}}{\Gamma^{3/2}}$, it follows from the definition of Γ_f that

$$\sqrt{\gamma} \sum_{\beta} \|S_{\beta}\| \|A_{T;\beta}(t)\| \leq \|\mathbf{s}\|_2 \frac{1}{\sqrt{\Delta\xi}} \left(\frac{\sqrt{\Gamma_f}}{2} + \Delta\xi \Gamma^{3/2} C_f \right). \quad (\text{B123})$$

By a similar line of arguments we find

$$\begin{aligned} & \sqrt{\gamma} \sum_{\alpha} \|[H_S, S_{\alpha}]\| \|A_{T;\alpha}(t)\| \\ & \leq \sqrt{\gamma \Delta\xi} \|\dot{\mathbf{s}}\| \sum_{n=-\infty}^{\infty} \|\mathbf{g}_f(t - \xi_n)\|_{2,1}. \end{aligned} \quad (\text{B124})$$

Thus it now follows that

$$\|[[X, Y(z+s)], Y(s+w)]\| \leq 2\|\dot{s}\|_2\|s\|_2 \left(\frac{\sqrt{\Gamma_f}}{2} + \Delta\xi\Gamma^{3/2}C_f \right) \sqrt{\gamma} \sum_{n=-\infty}^{\infty} \|g_f(z+s-\xi_n)\|_{2,1}. \quad (\text{B125})$$

Using this, (B117), (B113), and (B116), we find, for the first error term,

$$\begin{aligned} & \sum_{\ell=1}^M \int_0^{\Delta t} dz \int_0^z dw \int_0^w du \frac{1}{2} \|[[X, Y(t_\ell+z)], Y(t_\ell+w)]\| \\ & \leq \frac{(\Delta t)^2}{2} \|\dot{s}\|_2\|s\|_2 \left(\frac{\sqrt{\Gamma_f}}{2} + \Delta\xi\Gamma^{3/2}C_f \right) \sqrt{\gamma} \sum_{n=-\infty}^{\infty} \sum_{\ell=1}^M \int_0^{\Delta t} dz \|g_f(t_{\ell-1}+z-\xi_n)\|_{2,1}. \end{aligned} \quad (\text{B126})$$

We next use that $\sum_{\ell=1}^M \int_0^{\Delta t} dz \|g_f(t_{\ell-1}+z-\xi_n)\|_{2,1} = \int_{t_0}^{M\Delta t} dz \|g_f(t_{\ell-1}+z-\xi_n)\|_{2,1}$ and $\sum_{n=-\infty}^{\infty} \int_{t_0-\xi_n}^{t-\xi_n} dt f(t) \leq (\frac{T}{\Delta\xi} + 1) \int_{-\infty}^{\infty} dt f(t)$, for any measurable, non-negative function f ; see Appendix E3, Eq. (E15), where a proof is provided. These facts together imply

$$\sum_{\ell=1}^M \int_0^{\Delta t} dz \int_0^z dw \int_0^w du \frac{1}{2} \|[[X, Y(t_\ell+z)], Y(t_\ell+w)]\| \leq \frac{(\Delta t)^2}{2\Delta\xi} (T + \Delta\xi) \|\dot{s}\|_2\|s\|_2 \left(\frac{\sqrt{\Gamma_f}}{2} + \Delta\xi\Gamma^{3/2}C_f \right) \frac{\sqrt{\Gamma_f}}{2}. \quad (\text{B127})$$

Analyzing the remaining terms of (B117) in a similar way gives us

$$\sum_{\ell=1}^M \int_0^{\Delta t} dz \int_0^z dw \int_0^w du \frac{1}{4} \|X, [X, Y(z+t_\ell)]\| \leq \frac{(\Delta t)^2}{8\Delta\xi} (T + \Delta\xi) \|\dot{s}\|_2 \sqrt{\Delta\xi} \frac{\sqrt{\Gamma_f}}{2}, \quad (\text{B128})$$

$$\sum_{\ell=1}^M \int_0^{\Delta t} dz \int_0^z dw \int_0^w du \frac{1}{2} \|X, Y'(w+t_\ell)\| \leq \frac{(\Delta t)^2}{4\Delta\xi} (T + \Delta\xi) \|\dot{s}\|_2 \sqrt{\Delta\xi} \Gamma^{3/2}C_f, \quad (\text{B129})$$

and

$$\sum_{\ell=1}^M \int_0^{\Delta t} dz \int_0^z dw \int_0^w du \frac{1}{2} \int_w^z dv \|[[X, Y'(v+t_\ell)], Y(u+t_\ell)]\| \leq \frac{(\Delta t)^2}{2\Delta\xi} (T + \Delta\xi) \|\dot{s}\|_2\|s\|_2 \left(\frac{\sqrt{\Gamma_f}}{2} + \Delta\xi\Gamma^{3/2}C_f \right) \frac{\Gamma^{3/2}C_f\Delta t}{3}, \quad (\text{B130})$$

where $\ddot{s} = (\ddot{s}_1, \ddot{s}_2, \dots, \ddot{s}_{N_C})$ with $\ddot{s}_\alpha = \|[H_S, [H_S, S_\alpha]]\|$. Hence, by combining Eqs. (B127)–(B130) with (B113), (B116), and (B117), we obtain the following bound for the total accumulated Trotter error bound for the ATA:

$$\varepsilon_4 \leq 2 \frac{(\Delta t)^2}{\Delta\xi} (T + \Delta\xi) \left[\sqrt{\Delta\xi} \left(\frac{\sqrt{\Gamma_f}\|\ddot{s}\|_2}{8} + \frac{\|\dot{s}\|_2\Gamma^{3/2}C_f}{4} \right) + \frac{\|\dot{s}\|_2\|s\|_2}{2} \left(\frac{\sqrt{\Gamma_f}}{2} + \frac{1}{3}\Gamma^{3/2}C_f\Delta t \right) \left(\frac{\sqrt{\Gamma_f}}{2} + \Gamma^{3/2}C_f\Delta\xi \right) \right]. \quad (\text{B131})$$

Next we note that $\|s\|_2^2 = N_C$. Using this result, along with $\sqrt{\Gamma_f} \leq \frac{2^{3/4}}{\sqrt{\pi}} \sqrt{\Gamma}$, and noticing that $\int_{-\infty}^{\infty} ds |g'_f(s)|_{2,1} \leq \frac{2^{3/4}}{\sqrt{\pi}} \int_{-\infty}^{\infty} ds |g'_f(s)|_{2,1}$ [95], we find

$$\varepsilon_4 \leq \frac{2^{3/4}}{\sqrt{\pi}} \frac{(\Delta t)^2}{\Delta\xi} (T + \Delta\xi) \left[\sqrt{\Delta\xi\Gamma} \left(\frac{\|\ddot{s}\|_2}{4} + \frac{\|\dot{s}\|_2\Gamma C}{2} \right) + \|\dot{s}\|_2 \sqrt{N_C}\Gamma \left(\frac{1}{2} + \frac{1}{3}C\Gamma\Delta t \right) \left(\frac{1}{2} + C\Gamma\Delta\xi \right) \right], \quad (\text{B132})$$

with $C = \sqrt{\gamma}\Gamma^{-3/2}(\int_{-\infty}^{\infty} ds |g'_f(s)|_{2,1})$. We see from the second-order Trotter error in (B132) how small Δt must be chosen to guarantee the desired accuracy $\varepsilon_4 = O(\varepsilon\Gamma T)$. Assuming $\Delta\xi \leq \frac{\varepsilon}{\Gamma}$, we find the following requirement on Δt :

$$\Delta t \leq \min \left[\left(\frac{\varepsilon\Delta\xi}{C\Gamma\|\dot{s}\|_2\sqrt{N_C}} \right)^{\frac{1}{3}}, \left(\frac{\varepsilon}{C^2\Gamma^2\|\dot{s}\|_2\sqrt{N_C}} \right)^{\frac{1}{3}}, \sqrt{\frac{\varepsilon\Delta\xi}{\|\dot{s}\|_2\sqrt{N_C}}}, \sqrt{\frac{\varepsilon}{C\Gamma\|\dot{s}\|_2\sqrt{N_C}}}, \sqrt{\frac{\varepsilon\sqrt{\Gamma\Delta\xi}}{\|\dot{s}\|_2}}, \sqrt{\frac{\varepsilon\sqrt{\Gamma\Delta\xi}}{C\Gamma\|\dot{s}\|_2}} \right]. \quad (\text{B133})$$

In the limit where $\varepsilon \rightarrow 0$, the dominant inequality is

$$\Delta t \leq \sqrt{\frac{\varepsilon\Delta\xi}{\|\dot{s}\|_2\sqrt{N_C}}}. \quad (\text{B134})$$

This establishes the Trotter error bound used in Eq. (31) of the main text.

APPENDIX C: BOUNDS ON THE FILTERED BATH CHARACTERISTIC SCALES

In this Appendix, we prove that the relevant interaction rate and timescale for the filtered jump correlator, $\mathbf{g}_f(t)$, can be bounded by those of its unfiltered counterpart, $\mathbf{g}(t)$. This is important as the errors encountered in the Appendixes above often depend on these timescales. The results of this section justify our reexpression of all bounds in terms of the corresponding rates and timescales associated with $\mathbf{g}$; Γ and τ [see, e.g., (17) of the main text]. The characteristic interaction rate and correlation time of $\mathbf{g}_f$ are defined as

$$\begin{aligned}\Gamma_f &:= 4\gamma \left[\int_{-\infty}^{\infty} dt \|\mathbf{g}_f(t)\|_{2,1} \right]^2, \\ \tau_f &:= \frac{\int_{-\infty}^{\infty} dt \|\mathbf{g}_f(t)\|_{2,1}}{\int_{-\infty}^{\infty} dt \|\mathbf{g}_f(t)\|_{2,1}}.\end{aligned}\quad (\text{C1})$$

In this Appendix, we prove that

$$\begin{aligned}\Gamma_f &\leq 14.41\Gamma, \\ \Gamma_f \tau_f &\leq 14.41\Gamma\tau + 40.75\Gamma/\Omega.\end{aligned}\quad (\text{C2})$$

We do not claim optimality of the numerical constants, and they may be further reduced by a different choice of $\tilde{\varphi}(\omega)$. Choosing $\Omega \geq 1/\tau$, we therefore have

$$\Gamma_f \tau_f = O(\Gamma\tau). \quad (\text{C3})$$

To establish the bound above, we will, for convenience, use the following notation:

$$\begin{aligned}\int_{f(t)>c} h &:= \int_{-\infty}^{\infty} dt \theta(f(t) - c)h(t), \\ \int_{f(t)<c} h &:= \int_{-\infty}^{\infty} dt \theta(c - f(t))h(t),\end{aligned}\quad (\text{C4})$$

for any functions $f(t)$ and $h(t)$, with $\theta(\cdot)$ denoting the Heaviside step function. Similarly, we use the notation

$$\int_{\substack{f_1(t)>c_1 \\ f_2(t)>c_2}} h := \int_{-\infty}^{\infty} dt \theta(f_1(t) - c_1) \theta(f_2(t) - c_2) h(t), \quad (\text{C5})$$

with obvious generalizations to other directions of the inequalities in the integration domain.

To bound Γ_f , we invoke Minkowski's integral inequality, which states

$$\begin{aligned}\sqrt{\sum_{\alpha} \left| \int_{-\infty}^{\infty} ds g_{\alpha\beta}(t-s) \varphi(s) \right|^2} \\ \leq \int_{-\infty}^{\infty} ds |\varphi(s)| \sqrt{\sum_{\alpha} |g_{\alpha\beta}(t-s)|^2}.\end{aligned}\quad (\text{C6})$$

Squaring both side of this inequality and integrating over t , we therefore have

$$\Gamma_f \leq \|\varphi\|_1^2 \Gamma. \quad (\text{C7})$$

We now show that, by our choice of φ in Eq. (B8), we have

$$\|\varphi\|_1 \leq \frac{4 \cdot 2^{3/4}}{\sqrt{\pi}} \approx 3.8 \approx \sqrt{14.41}, \quad (\text{C8})$$

from which the first bound in Eq. (C2) follows. (Note that the above bound is valid with any matrix-valued function, $\mathbf{h}$, with integrable $\|\mathbf{h}\|_{2,1}$, in place of $\mathbf{g}$.) To prove (C8), we first note

$$\|\varphi\|_1 \leq \int_{|t|<s_0} |\varphi| + \int_{\substack{|t^2\varphi|>s_1 \\ |t|>s_0}} |\varphi| + \int_{\substack{|t^2\varphi|<s_1 \\ |t|>s_0}} |\varphi|, \quad (\text{C9})$$

where s_0 and s_1 are free parameters to be chosen below. Using the Cauchy-Schwarz inequality in the first term, we find $\int_{|t|<s_0} |\varphi| \leq \sqrt{2s_0} \|\varphi\|_2 = \sqrt{s_0/\pi} \|\tilde{\varphi}\|_2 \leq \sqrt{2s_0\Omega/\pi}$. For second term, notice that

$$\int_{\substack{|t^2\varphi|>s_1 \\ |t|>s_0}} |\varphi| \leq \frac{1}{s_1} \int_{-\infty}^{\infty} dt |t^2 \varphi(t)|^2. \quad (\text{C10})$$

Using $(s_1)^{-1} \int_{-\infty}^{\infty} dt |t^2 \varphi(t)|^2 = \frac{1}{2s_1\pi} \int_{-\infty}^{\infty} d\omega |t \frac{d}{d\omega} \tilde{\varphi}(\omega)|^2$ and $\frac{1}{2s_1\pi} \int_{-\infty}^{\infty} d\omega |t \frac{d}{d\omega} \tilde{\varphi}(\omega)|^2 \leq \frac{\Omega}{2s_1\pi} (\frac{4}{\Omega})^2$, we find

$$\int_{\substack{|t^2\varphi|>s_1 \\ |t|>s_0}} |\varphi| \leq \frac{8}{\pi s_1 \Omega}. \quad (\text{C11})$$

For the third term in (C9) we observe that

$$\int_{\substack{|t^2\varphi|<s_1 \\ |t|>s_1}} |\varphi| \leq 2s_1 \int_{s_0}^{\infty} dt \frac{1}{t^2} = 2 \frac{s_1}{s_0}. \quad (\text{C12})$$

Thus, $\|\varphi\|_1 \leq \sqrt{2s_0\Omega/\pi} + \frac{8}{\pi s_1 \Omega} + 2 \frac{s_1}{s_0}$. Minimizing the right-hand side with respect to s_0 and s_1 , we find Eq. (C8) at $s_0 = 4\sqrt{2}/\Omega$ and $s_1 = 4 \cdot 2^{1/4}/(\sqrt{\pi}\Omega)$.

We prove the second bound in Eq. (C2) by a similar approach to the first bound: First we note that

$$\Gamma_f \tau_f = \gamma \int_{-\infty}^{\infty} dt \|\mathbf{g}_f(t)\|_{2,1} \int_{-\infty}^{\infty} dt \|\mathbf{g}_f(t)\|_{2,1}. \quad (\text{C13})$$

We identify the last factor as $\sqrt{\Gamma_f/4\gamma}$, which we have bounded above. To bound the first factor, we use that $\int_{-\infty}^{\infty} ds \mathbf{g}(t-s) \varphi(s) = \int_{-\infty}^{\infty} ds (t-s) \mathbf{g}(t-s) \varphi(s) + \int_{-\infty}^{\infty} ds \mathbf{g}(t-s) s \varphi(s)$. A straightforward computation, using Minkowski's integral inequality, as in (C6), then shows

$$\begin{aligned}\int_{-\infty}^{\infty} dt \|\mathbf{g}_f(t)\|_{2,1} &\leq \|\varphi\|_1 \int_{-\infty}^{\infty} dt \|\mathbf{g}(t)\|_{2,1} \\ &\quad + \|t\varphi\|_1 \int_{-\infty}^{\infty} dt \|\mathbf{g}(t)\|_{2,1}.\end{aligned}\quad (\text{C14})$$

With our choice of φ in Eq. (B8), we show below that

$$\|t\varphi\|_1 \leq \frac{2^{1/4}}{\sqrt{\pi}} \frac{16}{\Omega} \approx 10.74 \frac{1}{\Omega}. \quad (\text{C15})$$

Equation (C2) follows by combining this result with (C13), (C15), and (C8).

To prove (C15), observe first that

$$\|t\varphi\|_1 \leq \int_{|t|<r_0} |t\varphi| + \int_{\substack{|t^3\varphi|>r_1^2 \\ |t|>r_0}} |t\varphi| + \int_{\substack{|t^3\varphi|<r_1^2 \\ |t|>r_0}} |t\varphi|, \quad (\text{C16})$$

where r_0 and r_1 are free parameters to be chosen below. For the first term in (C16) we use the Cauchy-Schwarz inequality again to bound $\int_{|t|<r_0} |t\varphi| \leq \sqrt{2r_0} \|t\varphi\|_2$. Using Placherel's Theorem, which states that the Fourier transform preserves

the L^2 -norm up to a normalization constant, it hence follows that

$$\int_{|t| < r_0} |t\varphi| \leq \sqrt{\frac{r_0}{\pi}} \left\| \frac{d}{d\omega} \tilde{\varphi} \right\|_2 \leq \sqrt{\frac{8r_0}{\pi\Omega}}, \quad (\text{C17})$$

where the latter inequality follows from the explicit form of $\tilde{\varphi}$ given in Eq. (B8). For the second term in (C16) we use

$$\int_{\substack{|t^3\varphi| > r_1^2 \\ |t| > r_0}} |t\varphi| \leq \frac{1}{r_1^2} \int_{-\infty}^{\infty} dt |t^2\varphi(t)|^2. \quad (\text{C18})$$

Using again Plancherel's Theorem, $\int_{-\infty}^{\infty} dt |t^2\varphi(t)|^2 = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega \left| \frac{d^2}{d\omega^2} \tilde{\varphi}(\omega) \right|^2$, and the explicit $\tilde{\varphi}$ from Eq. (B8), it follows that

$$\int_{-\infty}^{\infty} dt |t^2\varphi(t)|^2 \leq \frac{4^4}{2\pi r_1^2 \Omega^3}. \quad (\text{C19})$$

For the last term in (C16) we have

$$\int_{\substack{|t^3\varphi| < r_1^2 \\ |t| > r_0}} |t\varphi| \leq 2(r_1)^2 \int_{r_0}^{\infty} dt \frac{1}{t^2} = 2 \frac{r_1^2}{r_0}. \quad (\text{C20})$$

Combining the results above, we find

$$\|t\varphi\|_1 \leq \sqrt{\frac{8r_0}{\pi\Omega}} + \frac{4^4}{2\pi r_1^2 \Omega^3} + 2 \frac{r_1^2}{r_0}. \quad (\text{C21})$$

Minimizing the right-hand side with respect to r_0 and r_1 [achieved with $r_0 = 8\sqrt{2}/\Omega$ and $r_1 = 4 \cdot 2^{3/8}/(\pi^{1/4}\Omega)$] we obtain Eq. (C15).

APPENDIX D: TROTTERIZING ANCILLA-SYSTEM COUPLINGS WHEN OPERATORS COUPLED TO BATHS COMMUTE

In this Appendix, we describe how the system-ancilla coupling can be Trotterized for multiple quantum noise channels with commuting $\{S_\alpha\}$.

To recap, for multiple channels, we use the following operator to advance the coupling in the m th Trotter step:

$$V_m := \mathcal{T} \exp \left[-i \int_{t_{m-1}}^{t_m} ds \sqrt{\gamma} \mathbf{S} \cdot \mathbf{A}_T(s) \right]. \quad (\text{D1})$$

[See Eq. (37) of the main text]. If $\{S_\alpha\}$ mutually commute, then we may factorize V_m into unitaries acting on individual ancillas,

$$V_m = \prod_{n,\lambda} V_{m;\lambda n}, \quad (\text{D2})$$

where

$$V_{m;\lambda n} := \mathcal{T} \exp \left[-i \sqrt{\gamma} \int_{I_{mn}} ds \mathbf{S} \cdot \mathbf{A}_{T;\lambda n}(s) \right], \quad (\text{D3})$$

and where the integration domain I_{mn} is the intersection between the m th Trotter step interval and the time cutoff interval for ancilla n , $I_{mn} := [t_{m-1}, t_m] \cap [\xi_n - \tau_c, \xi_n + \tau_c]$. Finally, the α th entry of $\mathbf{A}_{T;\lambda n}$ is given by

$$A_{T;\alpha\lambda n}(s) := \sqrt{\Delta\xi} g_{f;\alpha\lambda}^*(s - \xi_n) \sigma_{\lambda n}^+ + \text{H.c.} \quad (\text{D4})$$

The ATA for multiple commuting quantum noise channels can then be executed with the pseudocode in Fig. 4 by compiling V_m as a product of gates acting only on single ancillas.

APPENDIX E: DERIVATIONS OF COMMONLY USED IDENTITIES

In this Appendix, we derive the bounds in Eqs. (B50), (B51), (B61), (B75), (B103), and (B112) that are used throughout Appendixes B and C.

1. Dyson substitutions: Proof of Eqs. (B50) and (B51)

In this section, we derive general identities for Dyson substitutions of unitary superoperators given in Eqs. (B50) and (B51). Specifically, we show that given a Hamiltonian $H(t) = H_1(t) + H_2(t)$, the unitary superoperator $\mathcal{U}(t, t_0)$ generated by $H(t)$ —i.e., the time-evolution superoperator—satisfies

$$\mathcal{U}(t, t_0) = \mathcal{U}_1(t, t_0) - i v_d \int_{t_0}^t ds \mathcal{U}_1(t, s) H_2^d(s) \mathcal{U}(s, t_0), \quad (\text{E1})$$

and

$$\mathcal{U}(t, t_0) = \mathcal{U}_1(t, t_0) - i v_d \int_{t_0}^t ds \mathcal{U}(t, s) H_2^d(s) \mathcal{U}_1(s, t_0), \quad (\text{E2})$$

where $\mathcal{U}_1(s, t_0)$ denotes the unitary superoperator generated by $H_1(t)$. In Eqs. (E1) and (E2), we refer to the right-hand sides as H_2 -Dyson substitutions of $\mathcal{U}$.

To prove (E1), recall that, given a Hamiltonian $H(t)$, the corresponding time-evolution superoperator is

$$\mathcal{U}(t, t_0) = \mathcal{T} \exp \left(-i \sum_{d \in \{L, R\}} v_d \int_{t_0}^t ds H^d(s) \right), \quad (\text{E3})$$

where L denotes left multiplication and R denotes right multiplication and $v_L = -v_R = 1$. From now on, we contract indices, so it is implicit that

$$v_d H^d := \sum_{d \in \{L, R\}} v_d H^d. \quad (\text{E4})$$

The derivative of $\mathcal{U}$ evaluated at time s is (by definition)

$$\left. \frac{\partial \mathcal{U}(t, t_0)}{\partial t} \right|_s = -i v_d H^d(s) \mathcal{U}(s, t_0). \quad (\text{E5})$$

Clearly $\mathcal{U}(t_0, t_0) = \mathcal{I}$, where $\mathcal{I}$ is the identity superoperator. Hence, integrating the above equation, we find

$$\mathcal{U}(t, t_0) = \mathcal{I} - i v_d \int_{t_0}^t ds H^d(s) \mathcal{U}(s, t_0). \quad (\text{E6})$$

Likewise, the derivative with respect to t_0 yields

$$\left. \frac{\partial \mathcal{U}(t, t_0)}{\partial t_0} \right|_s = i v_d \mathcal{U}(t, s) H^d(s). \quad (\text{E7})$$

Integrating this derivative from t to t_0 and exchanging the limits of integration in exchange for a sign change, we find

$$\mathcal{U}(t, t_0) = \mathcal{I} - i v_d \int_{t_0}^t ds \mathcal{U}(t, s) H^d(s). \quad (\text{E8})$$

With the above preliminaries established, we now prove Eqs. (E1) and (E2): If the Hamiltonian is written as $H(t) = H_1(t) + H_2(t)$, then the superoperator associated with the unitary time-evolution operator is then given by Eq (E3). On the other hand, consider the superoperator

$$\tilde{\mathcal{U}}(t, t_0) = \mathcal{U}_1(t, t_0) - i v_d \int_{t_0}^t ds \mathcal{U}_1(t, s) H_2^d(s) \mathcal{U}(s, t_0). \quad (\text{E9})$$

By straightforward differentiation we find

$$\frac{\partial}{\partial t}(\tilde{\mathcal{U}}(t, t_0) - \mathcal{U}(t, t_0)) = -i v_d H_1^d(t)(\tilde{\mathcal{U}}(t, t_0) - \mathcal{U}(t, t_0)). \quad (\text{E10})$$

It follows that

$$\tilde{\mathcal{U}}(t, t_0) - \mathcal{U}(t, t_0) = \mathcal{U}_1(t, t_0)(\tilde{\mathcal{U}}(t_0, t_0) - \mathcal{U}(t_0, t_0)) = 0$$

and hence the superoperator $\mathcal{U}(t, t_0)$ is equal to $\tilde{\mathcal{U}}(t, t_0)$, and (E1) follows. Equation (E2) follows by a similar argument.

2. Bound on the trace norm of a partial trace: Proof of Eq. (B61)

In this section we prove Eq. (B61), i.e., we provide a bound for the trace norm of a partial trace of a product of bounded operators, one of which is a trace-class operator. This bound is used to bound the operator norm of the error maps in Appendixes B 2, B 3, and B 4. Specifically Eq. (B61) states that for any trace class operator ρ on a composite Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_B$ and bounded operators X and Y on that same Hilbert space, we have

$$\|\text{Tr}_B[X \rho Y]\|_{\text{tr}} \leq \|X\| \|Y\| \|\rho\|_{\text{tr}}, \quad (\text{E11})$$

where Tr_B denotes the partial trace over $\mathcal{H}_B$. We will often think of ρ as a quantum state; however, the inequality is generally valid for any trace class operator.

To prove Eq. (E11), we use the fact that there exist two orthonormal bases for $\mathcal{H}_A$, $\{|\psi_j\rangle\}_j$, and $\{|\phi_j\rangle\}_j$, such that

$$\|\text{Tr}_B[X \rho Y]\|_{\text{tr}} = \sum_j \langle \psi_j | \text{Tr}_B[X \rho Y] | \phi_j \rangle. \quad (\text{E12})$$

This follows from the singular value decomposition of the operator $\text{Tr}_B[X \rho Y]$ on $\mathcal{H}_A$. We next express $\langle \psi_j | \text{Tr}_B[X \rho Y] | \phi_j \rangle = \sum_a \langle \psi_j | \otimes \langle a | O | \phi_j \rangle \otimes | a \rangle$, with $\{|a\rangle\}$ any orthonormal basis on $\mathcal{H}_B$, and note that there exist a unitary operator, U , such that $U|\psi_i\rangle = |\phi_i\rangle$. Thus,

$$\begin{aligned} \|\text{Tr}_B[X \rho Y]\|_{\text{tr}} &= \sum_{j,a} \langle \phi_j | \otimes \langle a | (U \otimes I) X \rho Y | \phi_j \rangle \otimes | a \rangle \\ &= \text{Tr}[Y(U \otimes I) X \rho], \end{aligned} \quad (\text{E13})$$

where we have used the cyclic property of the trace in the last step. Taking the absolute value on both sides, and using Hölder's inequality for Schatten- p norms, $|\text{Tr}[AB]| \leq \|A\| \|B\|_{\text{tr}}$, we obtain

$$|\text{Tr}[Y(U \otimes I) X \rho]| \leq \|Y(U \otimes I) X\| \|\rho\|_{\text{tr}}. \quad (\text{E14})$$

Using submultiplicativity of the operator norm along with $\|U \otimes I\| = 1$ has unit operator norm leads to Eq. (E11).

3. Proof of Eq. (B75)

In this section, we prove the bound in Eq. (B75) for a non-negative function, $f(t)$:

$$\sum_{n=-\infty}^{\infty} \int_{t_0-\xi_n}^{t-\xi_n} dt f(t) \leq \left(\frac{T}{\Delta\xi} + 1 \right) \int_{-\infty}^{\infty} dt f(t), \quad (\text{E15})$$

where $T = t - t_0 \geq 0$ and $\xi_n = n\Delta\xi$. To show this inequality, we introduce the characteristic function on an interval

$$\chi_{[t_a, t_b]}(t) := \begin{cases} 1, & t \in [t_a, t_b] \\ 0, & \text{Else} \end{cases}. \quad (\text{E16})$$

This allows us to write the above integrals as

$$\sum_{n=-\infty}^{\infty} \int_{t_0-\xi_n}^{t-\xi_n} dt f(t) = \int_{-\infty}^{\infty} dt f(t) \sum_{n=-\infty}^{\infty} \chi_{[t_0-\xi_n, t-\xi_n]}(t). \quad (\text{E17})$$

The sum over the indicator functions is uniformly bounded

$$\sum_{n=-\infty}^{\infty} \chi_{[t_0-\xi_n, t-\xi_n]}(t) \leq \left(\frac{T}{\Delta\xi} + 1 \right). \quad (\text{E18})$$

Due to the non-negativity of $f(t)$, we thus arrive at the desired inequality.

4. Proof of Eq. (B103)

Here we prove the bound in Eq. (B103), which states

$$\sum_{\lambda, \alpha, \alpha'} |g_{\alpha\lambda}(s)| |g_{\alpha'\lambda}(u)| \leq \|g(-s)\|_{2,1} \|g(-u)\|_{2,1}, \quad (\text{E19})$$

To establish this result, we first use the Cauchy-Schwarz inequality to obtain

$$\begin{aligned} \sum_{\lambda, \alpha, \alpha'} |g_{\alpha\lambda}(s)| |g_{\alpha'\lambda}(u)| \\ \leq \sum_{\alpha, \alpha'} \left(\sum_{\lambda} |g_{\alpha\lambda}(s)|^2 \right)^{\frac{1}{2}} \left(\sum_{\lambda} |g_{\alpha'\lambda}(u)|^2 \right)^{\frac{1}{2}}. \end{aligned} \quad (\text{E20})$$

Using that $g_{\alpha'\lambda}(u) = g_{\lambda\alpha'}^*(-u)$ we hence find

$$\begin{aligned} \sum_{\lambda, \alpha, \alpha'} |g_{\alpha\lambda}(s)| |g_{\alpha'\lambda}(u)| \\ \leq \sum_{\alpha, \alpha'} \left(\sum_{\lambda} |g_{\alpha\lambda}(-s)|^2 \right)^{\frac{1}{2}} \left(\sum_{\lambda} |g_{\lambda\alpha'}(-u)|^2 \right)^{\frac{1}{2}}. \end{aligned} \quad (\text{E21})$$

We recognize the above right-hand side as the product of two 2,1-norms, $\|M\|_{2,1} := \sum_{\lambda} (\sum_{\alpha} |M_{\alpha\lambda}|^2)^{1/2}$, of g , i.e.,

$$\sum_{\lambda, \alpha, \alpha'} |g_{\alpha\lambda}(s)| |g_{\alpha'\lambda}(u)| \leq \|g(-s)\|_{2,1} \|g(-u)\|_{2,1}, \quad (\text{E22})$$

which is the desired result.

5. Bound on the accumulated Trotter error: Proof of Eq. (B112)

Here prove the bound in Eq. (B112):

$$\|\rho_T(t_M) - \rho_{\text{Trotter}}(t_M)\|_{\text{Tr}} \leq 2 \sum_{\ell=1}^M \left\| \mathcal{T} e^{-i \int_{t_{\ell-1}}^{t_\ell} dt' H_T(t')} - e^{-\frac{iH_S \Delta t}{2}} V_\ell e^{-\frac{iH_S \Delta t}{2}} \right\|, \quad (\text{E23})$$

with $t_M = t_0 + M \Delta t$.

To conveniently prove the bound above we introduce the following notation: We let

$$U_T(t, t_0) = \mathcal{T} e^{-i \int_{t_0}^t dt' H_T(t')}, \quad (\text{E24})$$

denote full system-ancillas ATR time-evolution operator, with H_T given in Eq. (B83), and let

$$G_m = e^{-\frac{iH_S \Delta t}{2}} V_m e^{-\frac{iH_S \Delta t}{2}} \quad (\text{E25})$$

denote the full system-ancilla ATR Trotter step unitary. Furthermore, we let

$$\tilde{\rho}_T(t) = U_T(t, t_0)(\rho(t_0) \otimes |\mathbf{0}\rangle\langle\mathbf{0}|)U_T^\dagger(t, t_0), \quad (\text{E26})$$

denote the evolution of the density matrix of the composite system-ancilla system generated by $H_T(t)$ and likewise let

$$\tilde{\rho}_{\text{Trotter}}(t_\ell) = \left[\prod_{m=1}^{\ell} G_m \right] \rho(t_0) \otimes |\mathbf{0}\rangle\langle\mathbf{0}| \left[\prod_{m'=1}^{\ell} G_{m'} \right]^\dagger \quad (\text{E27})$$

denote the corresponding time-evolved state of the composite system with the Trotterized evolution. We finally introduce the shorthand $U_{T;\ell} := U_T(t_\ell, t_{\ell-1})$.

With the notation above, it clearly follows that $\tilde{\rho}_T(t_\ell) = U_{T;\ell} \tilde{\rho}_T(t_{\ell-1}) U_{T;\ell}^\dagger$, and $\tilde{\rho}_{\text{Trotter}}(t_\ell) = G_\ell \tilde{\rho}_{\text{Trotter}}(t_{\ell-1}) G_\ell^\dagger$. Therefore,

$$\tilde{\rho}_T(t_\ell) - \tilde{\rho}_{\text{Trotter}}(t_\ell) = U_{T;\ell} \tilde{\rho}_T(t_{\ell-1}) U_{T;\ell}^\dagger - G_\ell \tilde{\rho}_{\text{Trotter}}(t_{\ell-1}) G_\ell^\dagger. \quad (\text{E28})$$

Adding and subtracting $U_{T;\ell} \tilde{\rho}_{\text{Trotter}}(t_{\ell-1}) U_{T;\ell}^\dagger$, taking the trace norm, and using the triangle inequality we find

$$\begin{aligned} \|\tilde{\rho}_T(t_\ell) - \tilde{\rho}_{\text{Trotter}}(t_\ell)\|_{\text{Tr}} &\leq \|U_{T;\ell}[\tilde{\rho}_T(t_{\ell-1}) - \tilde{\rho}_{\text{Trotter}}(t_{\ell-1})]U_{T;\ell}^\dagger\|_{\text{Tr}} \\ &\quad + \|U_{T;\ell} \tilde{\rho}_{\text{Trotter}}(t_{\ell-1}) U_{T;\ell}^\dagger - G_\ell \tilde{\rho}_{\text{Trotter}}(t_{\ell-1}) G_\ell^\dagger\|_{\text{Tr}}. \end{aligned} \quad (\text{E29})$$

As the trace-norm is unitarily invariant, the first term of (E29) gives

$$\begin{aligned} \|U_{T;\ell}[\tilde{\rho}_T(t_{\ell-1}) - \tilde{\rho}_{\text{Trotter}}(t_{\ell-1})]U_{T;\ell}^\dagger\|_{\text{Tr}} &= \|\tilde{\rho}_T(t_{\ell-1}) - \tilde{\rho}_{\text{Trotter}}(t_{\ell-1})\|_{\text{Tr}}. \end{aligned} \quad (\text{E30})$$

For the second term of (E29) we add and subtract $G_\ell \tilde{\rho}_{\text{Trotter}}(t_{\ell-1}) U_{T;\ell}^\dagger$ and use the triangle inequality to find

$$\begin{aligned} \|U_{T;\ell} \tilde{\rho}_{\text{Trotter}}(t_{\ell-1}) U_{T;\ell}^\dagger - G_\ell \tilde{\rho}_{\text{Trotter}}(t_{\ell-1}) G_\ell^\dagger\|_{\text{Tr}} &\leq \|U_{T;\ell} - G_\ell\|_{\text{Tr}} \|\tilde{\rho}_{\text{Trotter}}(t_{\ell-1})\|_{\text{Tr}} \\ &\quad + \|G_\ell \tilde{\rho}_{\text{Trotter}}(t_{\ell-1}) (U_{T;\ell} - G_\ell)^\dagger\|_{\text{Tr}}. \end{aligned} \quad (\text{E31})$$

Using that $\|X \tilde{\rho}_{\text{Trotter}} Y\|_{\text{Tr}} \leq \|X\| \|Y\|$, for X and Y bounded operators [Eq. (E11) with $\mathcal{H}_B = \mathbb{C}$ and $\|\tilde{\rho}_{\text{Trotter}}\|_{\text{Tr}} = 1$], we thus find

$$\|U_{T;\ell} \tilde{\rho}_{\text{Trotter}}(t_{\ell-1}) U_{T;\ell}^\dagger - G_\ell \tilde{\rho}_{\text{Trotter}}(t_{\ell-1}) G_\ell^\dagger\|_{\text{Tr}} \leq 2 \|U_{T;\ell} - G_\ell\|. \quad (\text{E32})$$

Thus,

$$\begin{aligned} \|\tilde{\rho}_T(t_\ell) - \tilde{\rho}_{\text{Trotter}}(t_\ell)\|_{\text{Tr}} &\leq \|\tilde{\rho}_T(t_{\ell-1}) - \tilde{\rho}_{\text{Trotter}}(t_{\ell-1})\|_{\text{Tr}} + 2 \|U_{T;\ell} - G_\ell\|. \end{aligned} \quad (\text{E33})$$

By induction and by using $\|\tilde{\rho}_T(t_0) - \tilde{\rho}_{\text{Trotter}}(t_0)\|_{\text{Tr}} = 0$, it follows that

$$\begin{aligned} \|\tilde{\rho}_T(t_M) - \tilde{\rho}_{\text{Trotter}}(t_M)\|_{\text{Tr}} &\leq 2 \sum_{\ell=1}^M \left\| \mathcal{T} e^{-i \int_{t_{\ell-1}}^{t_\ell} dt' H_T(t')} - e^{-\frac{iH_S \Delta t}{2}} V_\ell e^{-\frac{iH_S \Delta t}{2}} \right\|. \end{aligned} \quad (\text{E34})$$

Equation (B112) [or (E23)] now follows by noting that

$$\|\rho_T(t_M) - \rho_{\text{Trotter}}(t_M)\|_{\text{Tr}} \leq \|\tilde{\rho}_T(t_M) - \tilde{\rho}_{\text{Trotter}}(t_M)\|_{\text{Tr}}, \quad (\text{E35})$$

which follows from the bound $\|\text{Tr}_B(\tilde{\rho})\|_{\text{Tr}} \leq \|\tilde{\rho}\|_{\text{Tr}}$, valid for any trace-class operator $\tilde{\rho}$ on a composite Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_B$ [see Eq. (E11), with $\mathcal{H}_B$ being the ancillas Hilbert space, and with $X = Y = I$].

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- [53] For two functions f_1 and f_2 their convolution is $[f_1 * f_2](x) := \int_{-\infty}^{\infty} dx' f_1(x - x') f_2(x')$ when the right-hand side is well defined.
- [54] The filtering is a convenient step to account for the time-discretization of the sum; for sufficiently small $\Delta\xi$, we have $g(t) \approx g_f(t)$ (with exact equality if the bath has a hard UV cutoff below $\pi/\Delta\xi$). Note that it might be possible to obtain similar proofs for unfiltered functions, for $\Delta\xi \rightarrow 0$. We leave such an investigation for future work.
- [55] This follows from the triangle inequality, $\|A\| \leq \sum_n |g_f(n\Delta\xi)| \sqrt{\Delta\xi}$, along with $\sqrt{\gamma} \sum_n |g_f(n\Delta\xi)| \sqrt{\Delta\xi} \sim \sqrt{\Gamma/\Delta\xi}$.
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- [93] The two operators in (B40) and (B47) are equivalent in the sense that (B40) is isospectral to (B47) when the latter is restricted to the invariant subspace of the oscillator space, where all oscillators are in the span of the vacuum state and first excited state. As the initial state of the oscillators is taken to belong to this subspace, so are the entire dynamics. Therefore, (B40) and (B47) generate the same dynamics, with our choice of initial state for the ancillas/oscillators.
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- [95] This follows by $\mathbf{g}'_f(s) = [\mathbf{g}' * \varphi](s)$ and a similar argument to that given in Appendix C in (C6)–(C8).