

Anomalous multigap topological phases in periodically driven quantum rotors

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We demonstrate that periodically driven quantum rotors provide a promising and broadly applicable platform to implement multigap topological phases, where groups of bands can acquire topological invariants due to non-Abelian braiding of band degeneracies. By adiabatically varying the periodic kicks to the rotor we find nodal-line braiding, which causes sign flips of topological charges of band nodes and can prevent them from annihilating, indicated by nonzero values of the patch Euler class. In particular, we report on the emergence of an anomalous Dirac string phase arising in the strongly driven regime, a truly out-of-equilibrium phase of the quantum rotor. This phase emanates from braiding processes involving all (quasienergy) gaps and manifests itself with edge states at zero angular momentum. Our results reveal direct applications in state-of-the-art experiments of quantum rotors, such as linear molecules driven by periodic far-off-resonant laser pulses or artificial quantum rotors in optical lattices, whose extensive versatility offers precise modification and observation of novel non-Abelian topological properties.

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I. INTRODUCTION

Over the past decade, research in topological physics has transitioned from primarily investigating topological insulators and semimetals [1–10] to a broader exploration of platforms [11–14]. Recently, multigap phases [15–22] have emerged as a new class of topological matter that generically goes beyond single-band characterizations. In such systems, groups of bands (band subspaces) can acquire multigap invariants [15–22]. The most notable example is the Euler class arising in systems with minimum $N \geq 3$ bands featuring a real Hamiltonian due to the presence of the $\mathcal{PT}$ (combination of parity and time reversal) or $C_2\mathcal{T}$ (twofold rotations and time reversal) symmetry, which bestows non-Abelian frame charges to band singularities [15,16,23]. These charges can be changed by braiding nodes residing between adjacent pairs of bands in momentum space. The braiding can thus result in nodes with the same charges in a two-band subspace ("gap") that cannot annihilate each other, which is a physical quantity captured by the Euler invariant. While still a nascent field, multigap topologies are increasingly finding their way to a multitude of settings, including phononic

systems [24,25], electronic systems [16,26,27], magnetic systems [28,29], metamaterials [30–34], and out-of-equilibrium contexts [35–40] including an anomalous variant necessitating periodic driving [37].

On another front, it has recently been shown that periodically kicked quantum rotors can exhibit Floquet topological phases [41–44]. This platform, in particular, is interesting given that these systems, which are driven far from equilibrium, are markedly different from solid-state systems, yet they offer clean settings with a number of knobs to control for realizing novel condensed-matter phenomena. A remarkable example as a clear advantage in this regard is the number of (Floquet) bands N , which is governed by the number of kicks within a single revival time of the rotor [45–47] and can be selected arbitrarily. This flexibility lays the groundwork for the exploration of hitherto uncharted multiband topological phenomena and particularly those out of equilibrium. The time evolution of the quantum rotor, induced through the periodic driving, can harbour topological charges (singularities) that manifest as significant changes in observables, such as the orientation of a rotating molecule [48]. These changes are protected by an inversion symmetry, $\mathcal{P}$. We here show that preserving time-reversal symmetry ($\mathcal{T}$) in kicked quantum rotors allows for these topological singularities to have non-Abelian properties for $N \geq 2$ [15,16,23].

We investigate non-Abelian braiding of topological singularities by periodically kicking quantum rotors. Upon tuning the strength of the laser pulses (kicks), we demonstrate that non-Abelian charges of band nodes can be changed by moving them around each other while the rotor is evolving. In particular, this allows us to realize an anomalous Dirac string phase with no equilibrium counterpart [37], in which *all* gaps

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in the Floquet spectrum of the rotor, including the anomalous π -gap, harbour topological singularities and contribute to the braiding process. Importantly, these multigap phases manifest themselves with zero-angular-momentum edge states as smoking-gun signatures. Unlike previous studies that focused on real-space lattices [37], our findings open a new avenue for current experimental studies of quantum rotors, spanning a broad range including linear molecules driven by far-off-resonant laser pulses, quantum nanorotors, and synthetic quantum rotors [42,49–52]. The adaptability of these experiments, where even the number of bands can be controlled, offers unique opportunities beyond previously explored settings and allows for extensive tuning of system parameters, e.g., by adjusting the laser strength, providing a new platform for studying nontrivial multiband topology.

II. METHOD

We consider three-dimensional quantum rotors, such as linear molecules, periodically driven by ultrashort off-resonant pulses [53]. The Hamiltonian of the free rotor can be expressed in terms of the squared angular momentum operator, $H_0 = \pi \hat{\mathbf{L}}^2 / \tau_B$, with the rotational period τ_B . We set $\hbar = 1$ throughout the text. For pulses significantly shorter than the rotor's rotational period [54], the effective potential for one alignment pulse can be represented as [55–57] $\hat{V}(P_1, P_2) = P_1 \cos(\hat{\theta}) + P_2 \cos^2(\hat{\theta})$, with pulse strengths P_1 and P_2 and the angle $\hat{\theta}$ between the axis of the rotor and the linear polarization of the laser. As this potential does not alter the magnetic quantum number m , we focus on the case $m = 0$ [58]. In this picture, angular momentum states $l \geq 0$ serve as lattice sites in a semi-infinite lattice, tunneling among which is controlled by the periodic laser pulses [48].

We employ Floquet theory to analyze the one-period time-translation operator [57,59],

$$U_{\text{KR}}(P_1, P_2) = e^{-i\pi \hat{\mathbf{L}}^2 \tau / (2\tau_B)} e^{i\hat{V}(P_1, P_2)} e^{-i\pi \hat{\mathbf{L}}^2 \tau / (2\tau_B)}, \quad (1)$$

where the period between the pulses is τ . We here consider the explicit time-reversal invariant Floquet gauge with the kick applied in the center [60]. Furthermore, we focus on the quantum resonance case with $\tau = 2\tau_B/N$ for $N \geq 3$, where the system exhibits N quasienergy bands [61].

The laser pulses introduce hopping in the synthetic lattice formed by the angular momentum states, which results in an effective tight-binding model [48] and allows us to define the quasimomentum $k \in [0, 2\pi]$ [62]. Depending on the kicking strength, there is tunneling between not only two nearest-neighbor sites but also next-to-nearest and further neighbors, which is indeed crucial for the existence of nontrivial nodal lines in the system [35].

For the kicked rotor, the stroboscopic Hamiltonian, $H_{\text{KR}} = i \log(U_{\text{KR}}) / \tau$, with Floquet states $\psi_n(k)$ and quasienergies $\epsilon_n(k)\tau \in [-\pi, \pi]$ reads

$$H_{\text{KR}}(k) \psi_n(k) = \epsilon_n(k) \psi_n(k), \quad (2)$$

where $n \in \{1 \dots N\}$ labels the band index. We, hereafter, absorb the Floquet period in the definition of H_{KR} and quasienergy for simplicity. Resolved as a phase, the 2π periodicity of the Floquet spectrum allows for novel topological phenomena, including band-touching points in the so-called

anomalous π -gap between bands $n = 1$ and $n = N$. Here, the π -gap refers to the quasienergy interval centered at $\epsilon = \pi$ that appears due to the 2π -periodicity of the Floquet spectrum, which has no static analog. This gives rise to anomalous Floquet topologies with no equilibrium counterpart which have been attracting great attention as they offer unique properties [11,12,37,63–65].

We emphasise that H_{KR} obeys an inversion symmetry $\mathcal{P}$ and a time-reversal symmetry $\mathcal{T}$ with

$$\mathcal{P} H_{\text{KR}}(k) = H_{\text{KR}}(-k) \mathcal{P}, \quad \mathcal{T} H_{\text{KR}}(k) = H_{\text{KR}}^*(-k) \mathcal{T}, \quad (3)$$

which allows us to adopt a real, Euler, definition in the following. As was shown in Ref. [48], the single-kicked rotor exhibits only linear nodal lines. Since the double-kicked rotor in general breaks time-reversal symmetry, we consider a triple kick of the form

$$U_{\text{TKR}}(\mathbf{P}) = U_{\text{KR}}(P_1, P_2) U_{\text{KR}}(P_3, P_4) U_{\text{KR}}(P_1, P_2) \quad (4)$$

as the simplest model to demonstrate the braiding of nodal lines. Here, the Floquet period becomes $T = 3\tau = 3\tau_B/(2N)$ as the relevant stroboscopic timescale in the remainder of the discussion. The triple-kicked rotor (4) has four parameters $\mathbf{P} = (P_1, \dots, P_4)$ for the strength of the kicks that we control to induce band-touching points between adjacent bands [see Fig. 1(a)].

III. QUANTUM ROTOR AND ANOMALOUS DIRAC STRING PHASE

The triple-kicked rotor Hamiltonian (H_{TKR}) can be cast into a real form due to the $\mathcal{PT}$ symmetry established in Eqs. (3)–(4). Focusing on the $N = 3$ -band resonance case, the space of the flattened Hamiltonians is given by $\text{O}(3)/\text{O}(1)^3 = \text{O}(3)/\mathbb{Z}_2^3 = \text{SO}(3)/D_2$, where the eigenstates form a real and orthonormal frame, a "dreibein," upon fixing its orientation [16]. While the two eigenstates involved in a band-touching point are not uniquely defined at the node, they pick up a π -Berry phase upon circling around it, corresponding to also the frame rotating by π around an axis fixed by the two-band subspace [40]. These nodes can be characterized by a global frame charge taking value in the first homotopy group $\pi_1(\text{SO}(3)/D_2) = \mathbb{Q}$, which is the non-Abelian quaternion group $\mathbb{Q} = \{+1, \pm i, \pm j, \pm k, -1\}$ [23]. Most importantly, the sign of the node's frame charge can be reversed upon braiding with a node hosted in either adjacent gaps (i.e., the gap directly above or below), hence directly reflecting the anticommuting algebra of the quaternion elements [16,17,37].

The non-Abelian braiding of band singularities naturally necessitates a two-dimensional parameter space hosting the braided loops. In kicked quantum rotors, while quasimomentum k corresponds to the first dimension, we establish a second dimension by modulating the kicking strengths over time in a periodic way as a function of $\alpha = \alpha(t) \in [0, 2\pi]$ as illustrated in Fig. 1(a) for a $N = 3$ -band model, $H_{\text{TKR}}(k, \alpha)$.

Under this protocol, the initial state $|\phi_0\rangle$ evolves as

$$|\phi_{N_\gamma}\rangle = U_{\text{TKR}}[\mathbf{P}(\alpha_{N_\gamma})] \cdots U_{\text{TKR}}[\mathbf{P}(\alpha_1)] |\phi_0\rangle, \quad (5)$$

where N_γ denotes the number of stroboscopic iterations completing a cycle for $\alpha_{N_\gamma} = 2\pi$. By tuning the kicking strengths, we observe both extended bulk and localized edge states as

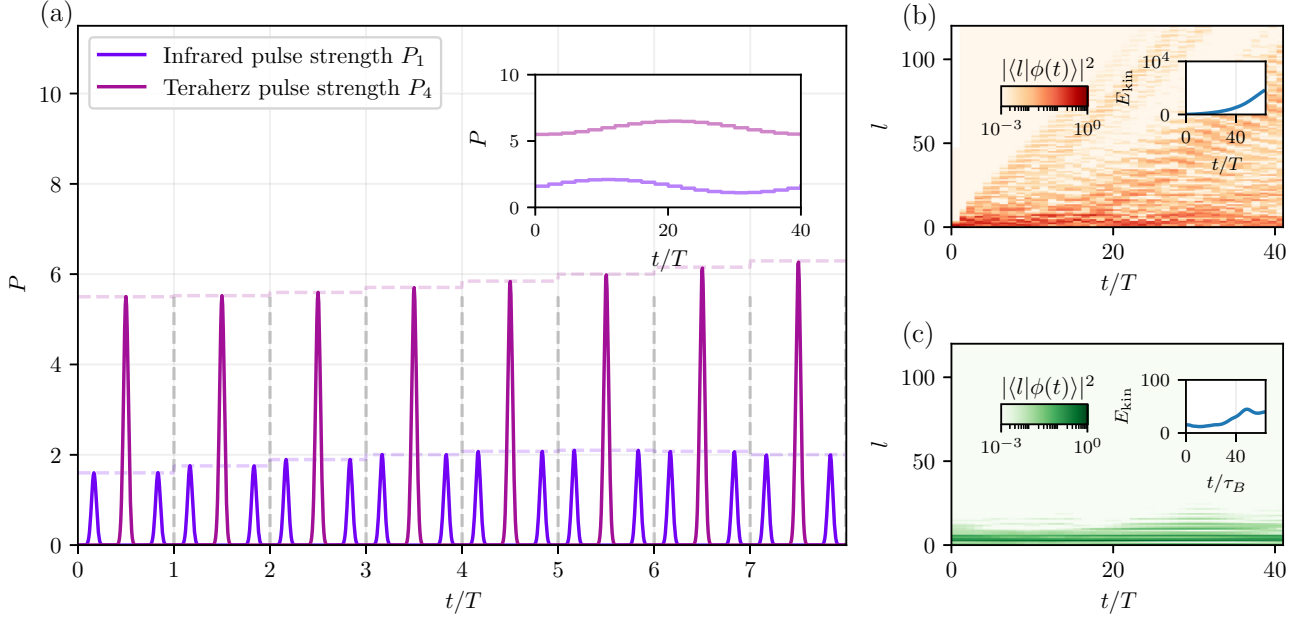


FIG. 1. Triple-kicked rotor (4) with three bands ($N = 3$) and $P_2 = P_3 = 0$, which reduces the protocol to two infrared pulses and one terahertz pulse within each Floquet period. We vary the pulse strengths stroboscopically on a circular path, $P_1(\alpha) = 1.6 + \sin(\alpha)/2$ and $P_4(\alpha) = 6.0 - \cos(\alpha)/2$, where $\alpha \in [0, 2\pi]$ serves as our synthetic second (momentum) dimension. (a) Pulse sequence as a function of Floquet period. The inset shows the variation of pulse strengths over the entire protocol, spanning $N_\gamma = 40$ iterations with three pulses per sequence within each of which α is adjusted as $\alpha_n = 2\pi n/N_\gamma$. In panels (b) and (c) we show the full time evolution of a driven rotor under this protocol [Eq. (5)]. (b) Starting with the typical thermal state (with $\pi/(\tau_B k_B T) = 0.17$), we observe resonant energy absorption ($E_{\text{kin}} = \langle \hat{\mathbf{L}}^2 \rangle \approx 5 \times 10^3$), as expected for a generic initial state in this system. Conversely, panel (c) showcases an edge state (with the same initial energy) that persists throughout the protocol and absorbs little energy ($\langle \hat{\mathbf{L}}^2 \rangle \approx 50$). Significantly, this is an anomalous π -gap topological edge state, witnessing the Dirac string phase depicted in Fig. 2(d), and exists even though all bulk Zak phases vanish (see text).

shown in Figs. 1(b) and 1(c), respectively. As expected for a quantum rotor kicked at quantum resonance, the generic thermal state $\phi_0(l) \propto \exp[-\pi l(l+1)/(\tau_B k_B T)]$ at temperature T absorbs substantial energy from the driving over time and spreads in momentum space. In contrast, an edge state with identical initial energy remains localized under the same driving protocol. By adiabatically connecting to the static regime [14,37,67], we identify this edge state as anomalous, residing in the π -gap [see Fig. 2(d) for the corresponding spectrum]. Crucially, it exhibits topological protection as it can only be eliminated through gap closure and braiding. We now elucidate this relationship and topological characterization in greater detail.

An equivalent way of tracking the non-Abelian braiding in the multigap Euler topology involves Dirac strings (DSs) [24,30,37]. In general, the gauge of the eigenstates cannot be chosen smoothly across the whole momentum space $\mathbf{k} = (k, \alpha)$ in the presence of spectral degeneracies whose π -Berry phase imposes a line (DS) along which the eigenstates change their sign, as illustrated in Fig. 2(b). While DSs are gauge objects, i.e., their exact position in $\mathbf{k}$ cannot be fixed, crossing a DS induces a π Zak phase, $\gamma_n = \oint \langle \psi_n(\mathbf{k}) | \nabla_{\mathbf{k}} \psi_n(\mathbf{k}) \rangle \cdot d\mathbf{k}$, which is calculated along noncontractible paths and indeed is a gauge-invariant observable quantity [68]. Similarly, also the sign of a frame charge changes upon crossing a DS lying in an adjacent gap [see Figs. 3(d) and 3(e)], successfully encoding the braiding of band nodes [37,40]. We note that time-reversal symmetry and inversion in our system enforces that nodal points either come in pairs at $(-k, k)$ connected by DSs or

occur at high-symmetry points $k = 0$ and $k = \pi$ [48]. Namely, the nodal points carry topological charges and can only be created or annihilated in pairs.

By tuning pulse strengths in the triple-kicked rotor we create band nodes and move them around in the $\mathbf{k}$ -space before annihilating them across the Brillouin zone (BZ), leaving their DS behind as shown in Figs. 2(b) and 2(c). This manifests as a π jump in the Zak phase along the orthogonal direction, γ_n^k , in bands $n=1$ and $n=2$, and we observe the corresponding edge states in gap 1 [69]. Figure 2(a) demonstrates the phase diagram with the Zak phases of each band marked. Zak phases of bands change upon crossing each nodal line, of which we keep track by defining a gap function (modulo 2π [70]),

$$\delta_n(\mathbf{k}, \mathbf{P}) = |\epsilon_n(\mathbf{k}, \mathbf{P}) - \epsilon_{n+1}(\mathbf{k}, \mathbf{P})|_{2\pi}, \quad (6)$$

vanishing at nodal points. We identify four distinct Zak-phase configurations.

A crucial point underpinned in Fig. 2(a) is that Zak phases on their own are insufficient to capture the out-of-equilibrium topological phenomena [37,39], as they are topological invariants associated with bands only. This stems from the fact that the periodically kicked quantum rotor can harbor a nodal line also in the anomalous π -gap (δ_3). Starting from the trivial phase (gray area at the bottom) and following a path traversing various phases (arrow), we arrive at a phase again with vanishing Zak phases upon crossing nodal lines in each gap. However, these two phases (gray-shaded areas) are indeed topologically distinct as one cannot go from one to the other without gap closures [39]. This phase is enabled

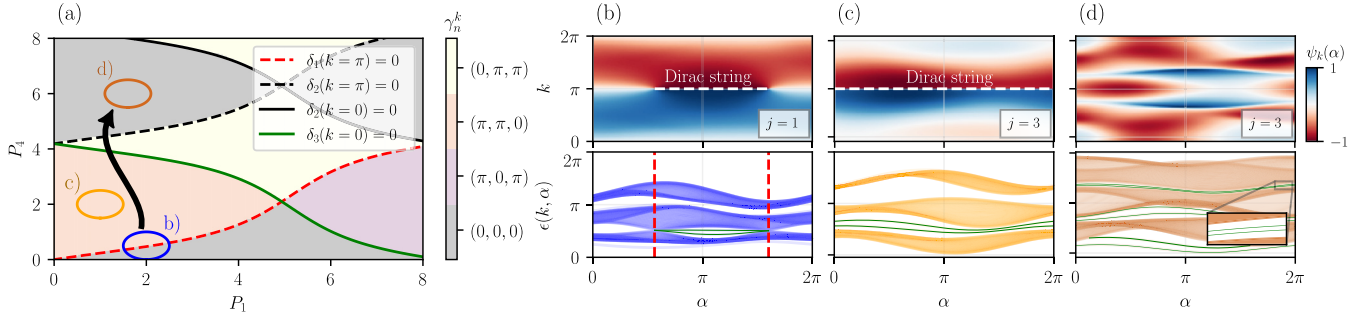


FIG. 2. (a) Phase diagram for the triple-kicked rotor with $P_2 = P_3 = 0$, with Zak phases of each band γ_n^k along the k direction. We identify four different configurations, separated by nodal lines at $k \in \{0, \pi\}$ where the gap δ_n in Eq. (6) vanishes. Additional nodal lines at arbitrary $(k, -k)$ are not shown for clarity (see Ref. [66] for all phase crossings). Colored circles represent the cyclic modulation in $\mathbf{P}(\alpha)$ (cf. Fig. 1), which results in the formation of a DS if it crosses over a nodal line as in panel (b). (b)–(d) The j th component of the $n = 2$ eigenstate (upper panels) and the corresponding quasienergy spectrum (lower panels) along the path marked by an arrow in panel (a) traversing through several topological phase transitions. (c) Two nodal points annihilate across the BZ, leaving their DS behind, which changes the Zak phase along the perpendicular direction to $\gamma_1^k = \gamma_2^k = \pi$ and induces edge states in this gap with open boundary conditions. (d) After crossing two more nodal lines in the two remaining gaps, the Zak phase of each band is set back to zero thanks to the nodal line in the anomalous gap (δ_3) enabled by periodic kicking. This is an anomalous Dirac string phase with edge modes in each gap, despite the trivial Zak phases which are the band invariants at equilibrium and cannot capture this topological regime on their own. This highlights the inability of single-band Zak phases to characterize Floquet topological phases, where one needs to keep track of nodal lines and Dirac strings in a gap-specific way.

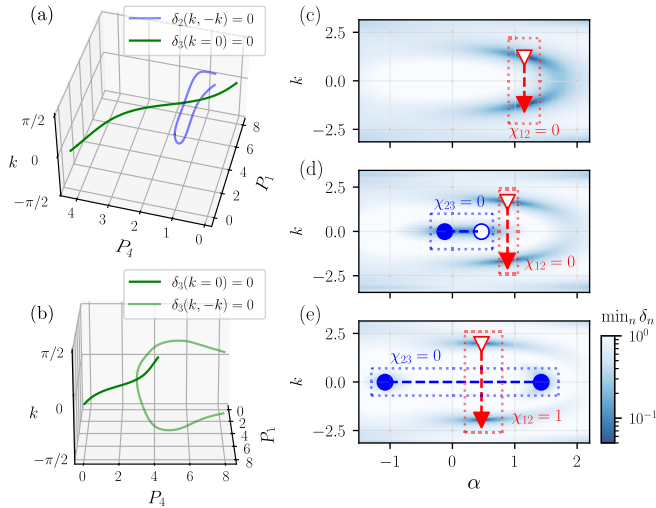


FIG. 3. (a) Non-Abelian braiding in (k, α) space with the same parameters as in Fig. 2. Examples of nodal lines, showing how the nodal line in gap 2 (blue line) winds around the nodal line of gap 3 (green) during braiding. (b) Formation of a $(k, -k)$ nodal line around a $k=0$ nodal line. Other nodal lines are not shown for better visibility [66]. (c)–(e) The triple-kicked quantum rotor can realize braiding processes and the nontrivial patch Euler class for various parametrizations, here shown over a different path, $P(\alpha, \beta) = f[\cos(\alpha), \sin(\alpha), \beta]$, with some smooth function $f(x, y, \beta)$ interpolating between different positions of band nodes (see text). The minimal gap size in the background reveals (c) a pair of nodes with opposite charges (depicted with empty and solid triangles, $\chi_{12} = 0$) in gap 1 for $\beta = 0.15$ and (d) one more pair with opposite charges in the adjacent gap (circles) for $\beta = 0.21$. (e) Moving a band node across the DS (dashed line) in the adjacent gap flips its sign, here $\beta = 0.3$. The nodes in gap 1 (triangles) now cannot annihilate within the red dotted area as one has to go through the DS in gap 2, changing their sign, which is captured by the nontrivial Euler class $\chi_{12} = 1$ within this patch.

by the out-of-equilibrium nature of the system and called an anomalous Dirac string (ADS) phase with nodal lines in all gaps in the quasienergy spectrum [37], which can be identified by keeping track of the DSs in a gap-resolved way [14]. Indeed, the ADS phase reveals itself in the kicked quantum rotor with edge states in each gap, including the anomalous π -gap, in Fig. 2(d).

IV. NONTRIVIAL PATCH EULER CLASS

The periodically kicked rotor offers a rich phase space with abundant possibilities for different nodal-line structures. While we focus on the $k=0$ and $k=\pi$ nodal lines that change Zak phases along the k direction in Fig. 2 to highlight the main phases and in particular the anomalous nature of the system, $\mathcal{PT}$ symmetry allows also for nodal points at $(k, -k)$ pairs resulting in Zak-phase changes along the α direction as shown in Fig. 3 (see Ref. [66] for the full nodal-line structure). The interplay of nodal lines from different band gaps enables non-Abelian braiding of band singularities, flipping their frame charges when they cross a nodal line in an adjacent gap.

The obstruction against annihilation of a pair of band nodes between bands n and $n+1$ within a patch $D \in \mathbb{R}^k$ can be captured by the patch Euler class

$$\chi_{n,n+1}^D = \frac{1}{2\pi} \left(\oint_D \text{Eu}(\mathbf{k}) d\mathbf{k} \wedge d\alpha - \oint_{\partial D} \mathcal{A} \cdot d\mathbf{k} \right) \in \mathbb{Z}. \quad (7)$$

Here, the Euler two-form is defined as $\text{Eu}(\mathbf{k}) = \langle \partial_k \psi_n | \partial_\alpha \psi_{n+1} \rangle - \langle \partial_\alpha \psi_n | \partial_k \psi_{n+1} \rangle$ with corresponding one-form $\mathcal{A}(k, \alpha) = \langle \psi_n | \nabla \psi_{n+1} \rangle$. Nodes after being created in pairs from vacuum [Figs. 3(c) and 3(d)] carry opposite charges, hence a vanishing patch Euler class. However, frame charges associated with nodal lines of adjacent gaps anticommute and can be changed upon crossing their DSs, here demonstrated by tuning the kicking parameters [71]. In the resulting configuration [Fig. 3(e)], the nodes in gap 1 (red triangles)

cannot annihilate within the marked patch as the nodal lines have twisted around each other, revealed by a nontrivial patch Euler class.

V. CONCLUSIONS

We have demonstrated that periodically kicked quantum rotors provide a versatile platform for realizing multigap topological phases. By controlling kicking strengths, one can induce nodal lines in each gap, including the π -gap, culminating in an anomalous Dirac string phase [37] in the strongly driven regime. This out-of-equilibrium phase is revealed by zero-angular-momentum edge states in all gaps. In addition, we have shown experimentally feasible protocols to braid such nodal lines residing between different gaps, which enables non-Abelian braiding of band singularities and, accordingly, a finite patch Euler class.

Our work extends beyond static topological matter, offering insights into novel driven phenomena in the clean, highly controllable setting of quantum rotors. The flexibility in selecting the number of Floquet bands and precise control over system parameters make quantum rotors an ideal test-bed for exploring multiband topological phenomena. These findings open avenues for future research, including the exploration of systems with $N > 3$ bands [17,72], the impact of disorder and interactions on topological properties, and the potential for realizing exotic quantum states through engineered non-Abelian braiding processes. Moreover, the direct applicability to state-of-the-art experiments with driven linear molecules and artificial quantum rotors in optical lattices promises immediate opportunities for experimental validation and further exploration of these intriguing topological phases.

Furthermore, the topological structure of the driven rotor leaves clear fingerprints in several *experimentally accessible* observables. In the anomalous Dirac-string phase the

edge modes are present in *all* three quasienergy gaps and the subspace of these edge modes can be identified through standard energy-absorption protocols. This phase is further distinguished by edge modes inside the π -gap and by a trivial Zak phase for all bands, the latter of which is measurable via full wave-function tomography. Note that, in contrast to the previously identified anomalous Dirac-string phase in a kagome lattice [37], the one in our system gives rise to a different edge state spectrum due to the geometry of the effective lattice realized by the kicked rotor that is distinct from kagome [73]. Moreover, each topological phase transition produces pronounced changes in the time-resolved orientation and alignment signals of the molecular wave packet, as demonstrated in Ref. [48]. Finally, the non-Abelian braiding of Dirac cones can be observed interferometrically, as recently shown in Ref. [40]. Taken together, the wide tunability of quantum simulation platforms renders the predicted non-Abelian topological features directly accessible to current and future experiments.

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- $$|x - y|_{2\pi} = \min_{n \in \mathbb{Z}} |x - y + 2\pi n|,$$
- which gives the minimal distance on the circle between radians x and y .
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