

Steady-state dynamics of quantum frequency combs in microring resonators

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Optical frequency combs are utilized in a wide range of optical applications, including atomic clocks, interferometers, and various sensing technologies. They are often generated via four-wave mixing in chip-integrated microring resonators, a method that requires low optical input power due to the high-quality factor of the resonator, making it highly efficient. While the classical properties of optical frequency combs are well established, this work investigates the quantum-mechanical characteristics of the individual comb modes. We derive closed-form analytical expressions describing the squeezing, second-order correlation, and joint spectral intensity between the generated signal and idler modes. This comprehensive theoretical framework enables an intuitive understanding and optimization of the quantum features across the comb, revealing conditions for substantial squeezing and entanglement relevant for quantum information processing. Our findings highlight the profound impact of design and dispersion on these quantum properties and offer a foundational tool for chip-integrated quantum applications, including quantum sensing, computing, and communication.

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I. INTRODUCTION

Optical frequency combs consist of multiple, evenly spaced frequencies within a single wave packet. These distinct characteristics are highly valuable for a wide range of applications, including atomic clocks [1–3], optical spectroscopy [4,5], lidar [6], and other sensing technologies [7], mostly by exploiting their spectral properties.

In modern applications, optical frequency combs are often generated in microring resonators via four-wave mixing (FWM), where two pump photons are absorbed to generate a signal and an idler mode [8–10]. In its simplest form, a microring resonator consists of a ring-shaped optical waveguide coupled to a straight waveguide, as depicted in Fig. 1. With a proper design, a small input power P_{in} in the straight waveguide can lead to a high optical power within the resonator [11]. An optical frequency comb is generated inside the ring

resonator when the pump power P_{in} exceeds a certain optical threshold power P_{th} and the energy and momentum conservation conditions for the FWM process are fulfilled [10]. A crucial benefit of using microring resonators for optical frequency-comb generation is their compact form factor combined with a low threshold power P_{in} , as detailed in [12]. Depending on the ring resonator's design, a frequency comb with specific spectral characteristics can be measured at the output of the straight waveguide [13]. The classical behavior of optical frequency combs is well understood and comprehensive overviews of their generation and applications can be found in [10,14].

Beyond their classical features, recent works also explored the quantum behavior of optical frequency combs generated via FWM. It has been shown that the modes within the frequency comb can exhibit squeezing and entanglement [15–25]. These quantum features are of particular interest for applications in quantum sensing, computing, and communication. For instance, squeezing can be exploited to achieve noise reduction in sensing applications [26–29], famously demonstrated in gravitational wave detection [30–32]. Similarly, entanglement is crucial for fault-tolerant photonic quantum computing [21,33–35] and for realizing quantum communication protocols [23]. Typically, only the quantum features between one signal and one idler mode are utilized.

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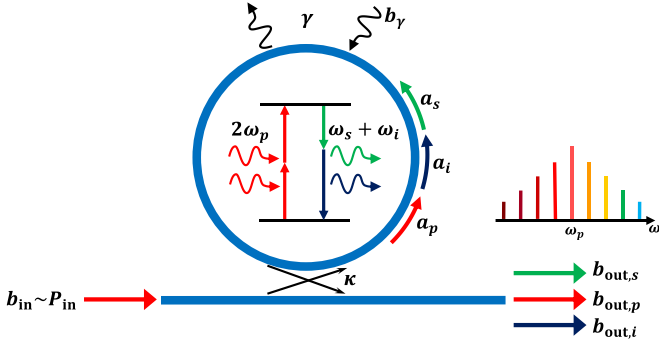


FIG. 1. Schematic setup of a microring resonator which is used to generate a frequency comb via FWM. The input mode b_{in} with the optical power P_{in} couples via the coupling rate κ into the resonator and forms the resonator mode a_p that receives losses through the loss rate γ and κ . Via FWM two modes a_p are absorbed and a signal and idler mode pair a_s and a_i are emitted that form the outcoupled frequency comb consisting of the modes $b_{\text{out},p}$, $b_{\text{out},s}$, and $b_{\text{out},i}$.

However, if an optical frequency comb is generated with quantum features across multiple modes, it is often termed a quantum frequency comb (QFC) [15–20,36].

Despite the established concept of QFCs, simple closed-form equations describing the quantum properties of each mode in a chip-integrated microring resonator, particularly their dependence on geometry and a description, are missing. Therefore, this work aims to derive theoretical equations that describe the dynamics of a quantum frequency comb generated in a microring resonator below threshold. We start first by modeling the classical frequency-comb generation in Sec. II A and then proceed to the quantum dynamics in Sec. II B, starting with linearized equations to describe the output modes. This enables us to derive equations for the squeezing, second-order correlation, and joint spectral intensity of the generated modes. Subsequently, our results are discussed in Sec. III, followed by a summary in Sec. IV.

We demonstrate that the frequency detuning of individual modes, and thus the dispersion of the ring resonator, is critical for both the classical behavior of the frequency combs and their quantum features. Most interestingly, while only a few modes with negligible frequency detuning exhibit a high photon number, a significant number of modes still show substantial quantum behavior. This indicates that significant quantum features of the optical modes can be achieved in both the normal and anomalous dispersion regimes of the ring resonator. Crucially, depending on the specific design, either the mode or particle entanglement of particular optical modes can be enhanced or reduced. These findings lay the foundation for developing miniaturized chip-integrated photonic solutions for quantum light sources, which are essential for quantum-enhanced systems and emerging quantum technology applications.

II. FREQUENCY-COMB GENERATION IN MICRORING RESONATORS

In this section we start by introducing the microring resonator system. We then proceed with a classical description of frequency-comb generation in Sec. II A, with the primary

goal to understand the detuning of a resonator mode as a function of the optical input power, the ring geometry, and the material parameters. Subsequently, the quantum dynamics is introduced in Sec. II B, where equations describing the squeezing spectrum, the second-order correlation function, and the joint spectral intensity (JSI) are derived.

Using the system structure of the microring resonator illustrated in Fig. 1, we present our model for frequency-comb generation. A laser with the pump mode b_{in} , amplitude $b_{\text{in}} = \sqrt{P_{\text{in}}/\hbar\omega_p}$, and angular frequency ω_p is launched into the straight waveguide. This pump light couples into the ring resonator via the coupling rate κ , forming the resonator pump mode a_p , which is affected by losses through the loss rate γ and by outcoupling via κ . A high-power resonator mode a_p is achieved when ω_p matches the ring resonator's resonance frequency $\omega_{R,T_0} = 2\pi cm/n_{\text{eff}}L_{\text{eff}}$ at a reference temperature T_0 , where c is the speed of light in vacuum, m is the mode number, n_{eff} is the effective refractive index, and L_{eff} is the effective ring length [11]. A high-power resonator mode facilitates the generation of nonlinear effects, such as frequency-comb formation. To describe these effects and the dynamics of the optical modes within the resonator, we utilize the well-known Hamiltonian of the FWM process, given by

$$\begin{aligned}
 H_{\text{FWM}} = & \hbar(\omega_{p,T_0}a_p^\dagger a_p + \omega_{s,T_0}a_s^\dagger a_s + \omega_{i,T_0}a_i^\dagger a_i) \\
 & - i\hbar g_{\text{opt}}(a_p a_p a_s^\dagger a_i^\dagger - a_p^\dagger a_p^\dagger a_s a_i) \\
 & - \frac{\hbar g_{\text{opt}}}{2}(a_p^\dagger a_p^\dagger a_p a_p + a_s^\dagger a_s^\dagger a_s a_s + a_i^\dagger a_i^\dagger a_i a_i) \\
 & - 2\hbar g_{\text{opt}}(a_s^\dagger a_s a_p^\dagger a_p + a_i^\dagger a_i a_p^\dagger a_p + a_s^\dagger a_s a_i^\dagger a_i). \quad (1)
 \end{aligned}$$

The first line represents the resonance frequencies of the relevant pump (a_p), signal (a_s), and idler (a_i) modes. The second line describes the FWM process, the third line the self-phase modulation (SPM) process, and the last line the cross-phase modulation (XPM) [37]. Self-phase modulation describes the nonlinear interaction within a single mode, while XPM describes the nonlinear intensity-intensity interaction between different modes. Both effects cause a frequency detuning of the cold cavity resonance frequencies ω_{p,T_0} , ω_{s,T_0} , and ω_{i,T_0} , as will be shown in more detail in Sec. II A. The intensity of the nonlinear behavior, such as a FWM process, depends on the optical nonlinearity of the waveguide, quantified by $g_{\text{opt}} = \hbar\omega_p^2 v_g^2 n_2 / cA_{\text{eff}}L_{\text{eff}}$, where v_g is the group velocity, A_{eff} is the effective mode area, and n_2 is the nonlinear refractive index [28,37].

While some other works include sum terms for multiple signal and idler modes within the Hamiltonian of Eq. (1), in this work we implicitly include all signal and idler modes within a_s and a_i and differentiate between individual modes later using their respective detunings. Thus, we consider a model where the pump mode couples to each signal-idler mode pair separately but the different signal-idler mode pairs do not interact with each other. Due to the low photon numbers below threshold and the rather weak optical nonlinearities g_{opt} , this rotating-wave-like approximation is well valid for all practical purposes. The Hamiltonian of Eq. (1) forms the foundation for describing the classical dynamics of the frequency comb in Sec. II A, followed by the quantum dynamics in Sec. II B.

A. Classical dynamics

For the investigation of frequency-comb generation, we begin with a classical description. While this procedure is well known and the complex mode interactions of classical frequency combs above threshold are typically modeled using the Lugiato-Lefever equation [38,39], our derivation aims to analyze the detuning of individual resonator modes in greater detail. As we will demonstrate in Sec. II B, this detailed analysis is crucial for accurately describing the quantum behavior of the generated signal and idler modes.

We start our analysis by splitting each mode a_j into a classical amplitude α_j and a fluctuating part δa_j such that $a_j = \alpha_j + \delta a_j$. In this section our focus is solely on the classical component α_j . Therefore, we utilize the Hamiltonian from Eq. (1) and state the well-known classical equations below threshold [28,40,41], including damping $\Gamma = \kappa + \gamma$ introduced by the coupling rate κ and the optical loss rate γ , as

$$\frac{d\alpha_s}{dt} = i\Delta_s\alpha_s - \frac{\Gamma}{2}\alpha_s - ig_{\text{opt}}\alpha_p^2\alpha_i^*, \quad (2)$$

$$\frac{d\alpha_i}{dt} = i\Delta_i\alpha_i - \frac{\Gamma}{2}\alpha_i - ig_{\text{opt}}\alpha_p^2\alpha_s^*, \quad (3)$$

$$\frac{d\alpha_p}{dt} = i\Delta_p\alpha_p - \frac{\Gamma}{2}\alpha_p + \sqrt{\kappa}b_{\text{in}}, \quad (4)$$

including the introduction of the frequency detunings with

$$\begin{aligned} \Delta_s &= \omega_s - \omega_{s,T_0} + g_{\text{opt}}|\alpha_s|^2 + 2g_{\text{opt}}(|\alpha_p|^2 + |\alpha_i|^2) \\ &\quad + g_{\text{th}}(|\alpha_p|^2 + |\alpha_s|^2 + |\alpha_i|^2), \end{aligned} \quad (5)$$

$$\begin{aligned} \Delta_i &= \omega_i - \omega_{i,T_0} + g_{\text{opt}}|\alpha_i|^2 + 2g_{\text{opt}}(|\alpha_s|^2 + |\alpha_p|^2) \\ &\quad + g_{\text{th}}(|\alpha_p|^2 + |\alpha_s|^2 + |\alpha_i|^2), \end{aligned} \quad (6)$$

$$\begin{aligned} \Delta_p &= \omega_p - \omega_{p,T_0} + g_{\text{opt}}|\alpha_p|^2 + 2g_{\text{opt}}(|\alpha_s|^2 + |\alpha_i|^2) \\ &\quad + g_{\text{th}}(|\alpha_p|^2 + |\alpha_s|^2 + |\alpha_i|^2). \end{aligned} \quad (7)$$

While the pump mode in the resonator α_p is increased by the input mode b_{in} , the signal and idler modes α_s and α_i increase in dependence of α_p and the optical nonlinearity g_{opt} . However, each mode is attenuated due to the losses $\Gamma = \gamma + \kappa$ and due to the detunings Δ_j that are described with Eqs. (5)–(7). Each detuning equation includes the SPM and XPM effects, which depend on the amplitudes of all interacting modes. In the following, we focus on an operation below threshold where $\alpha_p \gg \alpha_s, \alpha_i$ and thus we neglect the SPM and XPM influences caused by the signal and idler modes. It is important to note that for complex multimode dynamics the SPM and XPM influences needs to be included for each mode individually, which however requires a numerical investigation and is beyond the scope of this work.

Following [42], we have further included thermal influences within the frequency detuning terms using the thermal nonlinearity $g_{\text{th}} = \hbar\omega_p^2 n_{\text{eff}} \gamma_{\text{abs}} a_{\text{th}} / 2kL_{\text{eff}}$, where γ_{abs} is the absorption loss rate, a_{th} is the temperature coefficient, and k is the thermal conductivity. Consequently, the optical and thermal nonlinearities contribute to optical and thermal SPM and XPM effects, respectively.

To analyze the dynamics of the classical FWM process, we combine the equations of motion for the signal modes in the matrix form

$$\frac{d}{dt} \begin{pmatrix} \alpha_s \\ \alpha_i^* \end{pmatrix} = \begin{pmatrix} i\Delta_s - \frac{\Gamma}{2} & -ig_{\text{opt}}\alpha_p^2 \\ ig_{\text{opt}}\alpha_p^{*2} & -i\Delta_i - \frac{\Gamma}{2} \end{pmatrix} \begin{pmatrix} \alpha_s \\ \alpha_i^* \end{pmatrix} = \mathbf{L} \cdot \begin{pmatrix} \alpha_s \\ \alpha_i^* \end{pmatrix}. \quad (8)$$

We utilize the nontrivial solution arising from the singularity of the matrix $\mathbf{L}$, which marks the FWM threshold. By setting $\det(\mathbf{L}) = 0$ and solving for α_p , we determine the necessary pump mode threshold power to

$$\alpha_{\text{th}}^2 = \frac{\sqrt{\Delta_i\Delta_s + \Gamma^2/4}}{g_{\text{opt}}}. \quad (9)$$

This equation is not the final solution for the threshold power, since it still depends on each mode via the detuning terms Δ_i and Δ_s . However, in the case of perfect resonance these detunings in Eq. (9) vanish and the minimum required threshold for the FWM process can be determined from the steady-state solution of Eq. (4) and the definition of b_{in} , resulting in

$$P_{\text{th}} = \frac{\Gamma^3 \hbar \omega_p}{8g_{\text{opt}}\kappa}. \quad (10)$$

To receive a more proper solution including the detunings, it is required to use the energy conservation of the FWM process $2\omega_p = \omega_i + \omega_s$ and to express the angular frequency of the signal mode in dependence of the resonance frequencies with

$$\omega_s = \omega_p - \frac{\omega_{i,T_0}}{2} + \frac{\omega_{s,T_0}}{2}. \quad (11)$$

This result can be combined with the Taylor series of the dispersion in microring resonators $\omega_\mu = \omega_0 + D_1\mu + 1/2D_2\mu^2$ from [13] to derive the frequency detuning for a resonator mode only in dependence of the pump mode. Therefore, we assume a symmetric behavior between the signal and idler mode, which leads to $\Delta_i = \Delta_s = \Delta_\mu$, and introduce the detuning of mode μ with

$$\Delta_\mu = \Delta_{p,0} - \frac{1}{2}D_2\mu^2 + 2g_{\text{opt}}|\alpha_p|^2 + g_{\text{th}}|\alpha_p|^2 \quad (12)$$

and the bare frequency detuning of the pump mode $\Delta_{p,0} = \omega_p - \omega_{p,T_0}$. Combining Eqs. (9) and (12) leads to the solution of the required pump amplitude in the ring resonator in dependence of the mode number and frequency detuning with

$$|\alpha_{\text{th}}|^2 = \frac{(D_2\mu^2 - 2\Delta_{p,0})(2g_{\text{opt}} + g_{\text{th}})}{6g_{\text{opt}}^2 + 8g_{\text{opt}}g_{\text{th}} + 2g_{\text{th}}^2} \pm \frac{\sqrt{g_{\text{opt}}(2\Delta_{p,0} - D_2\mu^2)^2 - \Gamma^2(4g_{\text{opt}}g_{\text{th}} + 3g_{\text{opt}}^2 - g_{\text{th}}^2)}}{6g_{\text{opt}}^2 + 8g_{\text{opt}}g_{\text{th}} + 2g_{\text{th}}^2}. \quad (13)$$

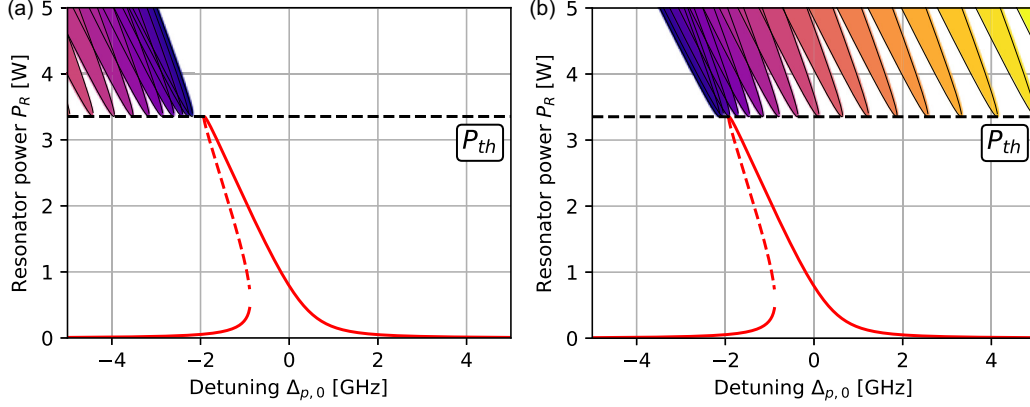


FIG. 2. Energy conservation of the FWM process in microring resonators in dependence of the detuning of the bare detuning of the pump mode $\Delta_{p,0} = \omega_p - \omega_{p,T_0}$ for various mode numbers μ and with $\kappa = 300$ MHz, $\gamma = 200$ MHz, $g_{th} = 10$ Hz, $g_{opt} = 1.5$ Hz, and $\omega_p = 1550$ nm. The red line indicates the optical power of the pump mode P_p , while the colored areas correspond to the required threshold power of a mode μ , starting with $\mu = 0$ for the blue area. The black dashed line indicates the minimum required threshold power P_{th} . (a) Normal dispersion ring resonator with $D_2 = -60$ MHz. (b) Anomalous dispersion ring resonator with $D_2 = 60$ MHz.

This result yields two solutions, which together form an area. To analyze this in detail, the result of Eq. (13) is presented in Fig. 2 as a function of the pump mode's bare frequency detuning $\Delta_{p,0}$ and for various mode numbers μ . The solution assuming a normal dispersion ring resonator is depicted in Fig. 2(a) and that for an anomalous dispersion in Fig. 2(b). The threshold amplitude $|\alpha_{th}|^2$ is converted to optical power, as detailed in Appendix A. Each differently colored area represents a solution for a distinct mode number μ , starting with $\mu = 0$ in dark blue. Additionally, the minimum threshold power P_{th} is indicated by the black dashed line and the optical power P_p of the pump mode α_p is shown in red, obtained by solving the steady state of Eq. (4). The SPM and XPM detunings cause the tilting of the modes and the observed bistability in the pump mode.

It is crucial to understand that a classical frequency-comb mode is generated when the red line representing the pump power P_p intersects the line of a possible signal or idler mode $|\alpha_{th}|^2$ for a given mode number μ . Obviously, this overlap is easier achievable for a ring resonator with anomalous dispersion. In a normal dispersion ring resonator, the dispersion D_2 causes each mode number μ to have an increasing frequency distance from the pump mode, preventing the lines from ever overlapping. Instead, the frequency detuning for each mode number Δ_μ increases with the total optical power in the resonator due to SPM and XPM effects. Only with anomalous dispersion, D_2 causes the detuning of a mode Δ_μ to shift towards the pump mode. This phenomenon is well known in the literature [10], which is why classical frequency combs are often generated in ring resonators utilizing anomalous dispersion [13].

To derive the mode number that first appears in a classical frequency comb, the detuning for each mode is analyzed in detail. Using Eq. (4) and assuming steady-state operation of the pump mode at injection locking with $|\alpha_p|^2 = \Gamma P_{in}/2g_{opt}P_{th}$, the detuning of the pump mode can be simplified to

$$\Delta_{p,0} = -\frac{g_{tot}\Gamma P_{in}}{2g_{opt}P_{th}}, \quad (14)$$

with the total nonlinearity $g_{tot} = g_{opt} + g_{th}$. The detuning for each mode number μ can be derived using the same conditions and Eq. (12), which leads to

$$\Delta_\mu = \frac{1}{2}D_2\mu - \frac{\Gamma(g_{th} + 2g_{opt})}{2g_{opt}}\frac{P_{in}}{P_{th}}. \quad (15)$$

A frequency comb is generated if the detunings match each other with $\Delta_\mu = \Delta_{p,0}$, which leads to the effective mode detuning for a signal and idler mode as the difference $\Delta_{\mu,eff} = \Delta_\mu - \Delta_{p,0}$ with

$$\Delta_{\mu,eff} = \frac{1}{2}D_2\mu^2 - \frac{\Gamma P_{in}}{2P_{th}}. \quad (16)$$

The mode number of the first appearing frequency-comb mode can be determined by setting $\Delta_{\mu,eff} = 0$. This leads to the well-known equation

$$\mu = \pm \sqrt{\frac{\Gamma P_{in}}{D_2 P_{th}} \frac{P_{in}=P_{th}}{P_{th}}} \pm \sqrt{\frac{\Gamma}{D_2}}, \quad (17)$$

which has already been derived in [10]. However, for the quantum description of the frequency comb, the effective detuning $\Delta_{\mu,eff}$ is crucial to describe the quantum behavior of the individual modes in Sec. II B. Interestingly, Eqs. (16) and (17) depend only on the optical nonlinearity g_{opt} via P_{th} , while the thermal nonlinearity g_{th} cancels out. This is expected, since the influence of the thermal detuning is the same across all modes.

B. Quantum dynamics

As discussed in the preceding section, a classical frequency comb appears when the pump power P_{in} is at least P_{th} . However, quantum fluctuations and optical modes exhibiting quantum effects are generated even at lower input power $P_{in} < P_{th}$. These quantum modes are analyzed in detail in the following sections, with a particular focus on the squeezing, the second-order correlation function, and the JSI of the quantum frequency comb.

Since we are primarily interested in the modes coupled out from the resonator into the straight waveguide to form the

waveguide mode b_{out} , we begin by applying the input-output theory. Therefore, the Hamiltonian of Eq. (1) is linearized by replacing a_p with the classical mode amplitude α_p and the equations of motion for the outcoupled signal and idler modes

are derived. This derivation has already been performed in our previous work [28], leading to the expression for the outcoupled modes of the ring resonator in the frequency domain

$$\mathbf{B}_{\text{out}}(\omega) = -\frac{1}{\sqrt{\kappa}} \left[\left(\boldsymbol{\Omega} - \mathbf{K} - \frac{\kappa}{2} \mathbf{I}_4 \right) \left(\boldsymbol{\Omega} - \mathbf{K} + \frac{\kappa}{2} \mathbf{I}_4 \right)^{-1} [\sqrt{\kappa} \mathbf{B}_{\text{in}}(\omega) + \sqrt{\gamma} \mathbf{B}_{\gamma}(\omega)] - \sqrt{\gamma} \mathbf{B}_{\gamma}(\omega) \right], \quad (18)$$

where $\mathbf{B}_{\text{in}}$ represents the input vector, $\mathbf{B}_{\gamma}$ the loss vector, and $\mathbf{B}_{\text{out}}$ the output vector and $\mathbf{K}$ includes the geometry, detuning, and pump parameters with

$$\mathbf{B}_{\text{in}} = \begin{pmatrix} b_{\text{in},s} \\ b_{\text{in},s}^{\dagger} \\ b_{\text{in},i} \\ b_{\text{in},i}^{\dagger} \end{pmatrix}, \quad \mathbf{B}_{\gamma} = \begin{pmatrix} b_{\gamma,s} \\ b_{\gamma,s}^{\dagger} \\ b_{\gamma,i} \\ b_{\gamma,i}^{\dagger} \end{pmatrix}, \quad \mathbf{B}_{\text{out}} = \begin{pmatrix} b_{\text{out},s} \\ b_{\text{out},s}^{\dagger} \\ b_{\text{out},i} \\ b_{\text{out},i}^{\dagger} \end{pmatrix}, \quad (19)$$

$$\mathbf{K} = \begin{pmatrix} -i\Delta_{s,\mu,\text{eff}} - \frac{\gamma}{2} & 0 & 0 & \frac{\sigma}{2} \\ 0 & i\Delta_{s,\mu,\text{eff}} - \frac{\gamma}{2} & \frac{\sigma^*}{2} & 0 \\ 0 & \frac{\sigma}{2} & -i\Delta_{i,\mu,\text{eff}} - \frac{\gamma}{2} & 0 \\ \frac{\sigma^*}{2} & 0 & 0 & i\Delta_{i,\mu,\text{eff}} - \frac{\gamma}{2} \end{pmatrix}, \quad (20)$$

and the pump parameter σ , which is defined as

$$\sigma = 2g_{\text{opt}}\alpha_p^2 = \Gamma \frac{P_{\text{in}}}{P_{\text{th}}}. \quad (21)$$

A key distinction from [28] is our introduction of the effective mode detunings for the signal ($\Delta_{s,\mu,\text{eff}}$) and idler ($\Delta_{i,\mu,\text{eff}}$) modes in Eq. (20), which enables the calculation of the different mode numbers.

Equation (18) describes the output modes, including losses within the ring resonator. However, after coupling out to the straight waveguide, these modes are subject to additional losses before being utilized for specific operations. We model these postcoupling losses using the standard procedure involving a beam splitter, which combines the output modes with vacuum modes according to $\sqrt{\eta} \mathbf{B}_{\text{out}}(\omega) + \sqrt{1-\eta} \mathbf{B}_{\mathbf{v}}$. Here the vacuum vector is defined as $\mathbf{B}_{\mathbf{v}} = [b_v, b_v^{\dagger}]^{\top}$ and η denotes the efficiency.

Since the modes in $\mathbf{B}_{\mathbf{v}}$, $\mathbf{B}_{\text{in}}$, and $\mathbf{B}_{\gamma}$ correspond to vacuum modes when only one mode at ω_p is used to pump the ring resonator mode a_p , the expectation values of the output modes can be obtained by utilizing Eq. (18) and the well-known expectation values for vacuum modes $\langle b_v(t) \rangle = \langle b_v^{\dagger}(t) \rangle = \langle b_v(t)b_v(t') \rangle = \langle b_v^{\dagger}(t)b_v(t') \rangle = 0$ and $\langle b_v(t)b_v^{\dagger}(t') \rangle = \delta(t-t')$ [43,44]. It is important to note that due to linearization, Eq. (18) is valid only below the FWM threshold, specifically for an input power P_{in} up to 99.895% of P_{th} [28].

Assuming a symmetric behavior for the signal and idler modes with $\Delta_{s,\mu,\text{eff}} = \Delta_{i,\mu,\text{eff}} = \Delta_{\mu,\text{eff}}$, Eq. (18) can be used to derive, for example, the photon number of the fluctuating signal mode of the mode number μ to

$$\langle b_{\text{out},s}^{\dagger}(\omega)b_{\text{out},s}(\omega') \rangle = \frac{4\Gamma^3\kappa\eta P_{\text{in}}^2 P_{\text{th}}^2}{[4\Delta_{\mu,\text{eff}}^2 P_{\text{th}}^2 + \Gamma^2(P_{\text{th}}^2 - P_{\text{in}}^2)]^2} \delta'. \quad (22)$$

In the same way, the expectation values of each mode μ in the quantum frequency comb can be determined.

Finally, we can use the results obtained so far to analyze the quantum behavior of the generated fluctuations. We begin by examining the squeezing between the signal and idler modes generated via FWM. For this purpose, the two-mode quadrature operator is introduced as

$$X_Q = \frac{1}{\sqrt{2}} [(b_{\text{out},s} + b_{\text{out},i})e^{i\phi_{\text{LO}}} + (b_{\text{out},s}^{\dagger} + b_{\text{out},i}^{\dagger})e^{-i\phi_{\text{LO}}}] \quad (23)$$

and with the local oscillator phase ϕ_{opt} , which corresponds to the squeezing angle [45]. To analyze the squeezing, the variance of X_Q is calculated with

$$\begin{aligned} \langle V(\omega, \omega') \rangle &= \Delta X_Q(\omega, \omega') \\ &= \langle X_Q(\omega)X_Q(\omega') \rangle - \langle X_Q(\omega) \rangle \langle X_Q(\omega') \rangle \\ &\xrightarrow{b_{\text{in},s,i}=b_v} \langle X_Q(\omega)X_Q(\omega') \rangle. \end{aligned} \quad (24)$$

For simplicity, we assume a symmetric behavior between the generated signal and idler mode with $\langle b_{\text{out},s}^{\dagger}b_{\text{out},s} \rangle \approx \langle b_{\text{out},i}^{\dagger}b_{\text{out},i} \rangle$ and $\langle b_{\text{out},i}b_{\text{out},s} \rangle \approx \langle b_{\text{out},s}b_{\text{out},i} \rangle$. This leads to the simplified result of the variance with

$$\begin{aligned} \langle V(\omega, \omega') \rangle &= \langle b_{\text{out},s}b_{\text{out},i} \rangle e^{2i\phi_{\text{LO}}} + \langle b_{\text{out},s}^{\dagger}b_{\text{out},i}^{\dagger} \rangle e^{-2i\phi_{\text{LO}}} \\ &\quad + 2\langle b_{\text{out},s}^{\dagger}b_{\text{out},s} \rangle + 1 \end{aligned} \quad (25)$$

using the expectation value of the photon number from Eq. (22) and

$$\langle b_{\text{out},s}b_{\text{out},i} \rangle = \frac{2\kappa\Gamma\eta P_{\text{in}}P_{\text{th}}[\Gamma^2 P_{\text{in}}^2 - P_{\text{th}}^2(2\Delta_{\mu,\text{eff}} + i\Gamma)^2]}{[4\Delta_{\mu,\text{eff}}^2 P_{\text{th}}^2 + \Gamma^2(P_{\text{th}}^2 - P_{\text{in}}^2)]^2}. \quad (26)$$

Note that the expectation value of $\langle b_{\text{out},s}^{\dagger}b_{\text{out},i}^{\dagger} \rangle$ corresponds to the complex conjugate of $\langle b_{\text{out},s}b_{\text{out},i} \rangle$.

To analyze this result, the squeezing spectrum is presented in Fig. 3. A dominant squeezing of approximately -5.3 dB

and an antisqueezing of approximately 21 dB are observed at $\Delta_{\mu,\text{eff}} = 0$. It is crucial to understand that the squeezing behavior for any other mode number μ can be determined simply by using its corresponding effective detuning $\Delta_{\mu,\text{eff}}$. Additionally, it is interesting that the optimal squeezing angle ϕ_{LO} depends on the effective detuning $\Delta_{\mu,\text{eff}}$ and consequently on μ . Ideally, the optimum squeezing occurs at an angle of $\phi_{\text{LO}} = n(\pi/2)$ with $n = 1, 3, \dots$ and antisqueezing at $\phi_{\text{LO}} = n\pi$ with $n = 0, 1, 2, \dots$, as it is well known in

the literature. However, when generating a complete quantum frequency comb, this ideal scenario can only be achieved for a few signal and idler mode pairs. This limitation arises because it is impossible to attain $\Delta_{\mu,\text{eff}} = 0$ for every mode μ simultaneously, due to the combined effects of XPM, SPM, and the ring dispersion.

Using Eq. (25), we can determine the optimal squeezing angle for each mode number μ to achieve the best squeezing as

$$\phi_{\text{LO,opt}} = \begin{cases} -\frac{1}{2} \tan^{-1} \left(\frac{4\Delta_{\mu,\text{eff}}\Gamma}{4\Delta_{\mu,\text{eff}}^2 - \Gamma^2 [1 - (\frac{P_{\text{in}}}{P_{\text{th}}})^2]} \right), & 4\Delta_{\mu}^2 - \Gamma^2 - \sigma^2 > 0 \\ -\frac{1}{2} \tan^{-1} \left(\frac{4\Delta_{\mu,\text{eff}}\Gamma}{4\Delta_{\mu,\text{eff}}^2 - \Gamma^2 [1 - (\frac{P_{\text{in}}}{P_{\text{th}}})^2]} \right) + \frac{\pi}{2}, & 4\Delta_{\mu}^2 - \Gamma^2 - \sigma^2 \leq 0. \end{cases} \quad (27)$$

The result of $\phi_{\text{LO,opt}}$ is sketched as the white dashed line in Fig. 3. Obviously, the squeezing angle of the antisqueezing is detuned by $\pi/2$. This result is used to simplify the equations for the squeezing and antisqueezing at $\phi_{\text{LO,opt}}$ to

$$\begin{aligned} \frac{\langle V_s \rangle}{\langle V_{\text{vac}} \rangle} &= 1 + 2\langle b_{\text{out},s}^\dagger b_{\text{out},s} \rangle - 2|\sqrt{\langle b_{\text{out},s} b_{\text{out},i} \rangle}|^2 \\ &= 1 + \frac{8\eta\kappa\Gamma^3}{4\Delta_{\mu,\text{eff}}^2 + \Gamma^2(1 - \frac{P_{\text{in}}}{P_{\text{th}}})} \\ &\quad - 4 \left| \sqrt{\frac{\eta\kappa\Gamma P_{\text{in}} P_{\text{th}} [\Gamma^2 P_{\text{in}}^2 - P_{\text{th}}^2 (2\Delta_{\mu,\text{eff}} + i\Gamma)^2]}{[4\Delta_{\mu,\text{eff}}^2 P_{\text{th}}^2 + \Gamma^2 (P_{\text{th}}^2 - P_{\text{in}}^2)]^2}} \right|^2, \end{aligned} \quad (28)$$

$$\begin{aligned} \frac{\langle V_{\text{as}} \rangle}{\langle V_{\text{vac}} \rangle} &= 1 + 2\langle b_{\text{out},s}^\dagger b_{\text{out},s} \rangle + 2|\sqrt{\langle b_{\text{out},s} b_{\text{out},i} \rangle}|^2 \\ &= 1 + \frac{8\eta\kappa\Gamma^3}{4\Delta_{\mu,\text{eff}}^2 + \Gamma^2(1 - \frac{P_{\text{in}}}{P_{\text{th}}})} \\ &\quad + 4 \left| \sqrt{\frac{\eta\kappa\Gamma P_{\text{in}} P_{\text{th}} [\Gamma^2 P_{\text{in}}^2 - P_{\text{th}}^2 (2\Delta_{\mu,\text{eff}} + i\Gamma)^2]}{[4\Delta_{\mu,\text{eff}}^2 P_{\text{th}}^2 + \Gamma^2 (P_{\text{th}}^2 - P_{\text{in}}^2)]^2}} \right|^2. \end{aligned} \quad (29)$$

The derived equations simplify to the results presented in [25] for the case of $\mu = 0$, validating our equations. Neglecting the detuning with $\Delta_{\mu,\text{eff}} = 0$, the equations for the optimum squeezing and antisqueezing can also be derived as

$$\frac{\langle V_{s,\text{opt}} \rangle}{\langle V_{\text{vac}} \rangle} = 1 - \frac{4\kappa\eta P_{\text{in}} P_{\text{th}}}{\Gamma(P_{\text{in}} + P_{\text{th}})} \xrightarrow{P_{\text{in}} \rightarrow P_{\text{th}}} 1 - \frac{\eta\kappa}{\Gamma}, \quad (30)$$

$$\frac{\langle V_{\text{as,opt}} \rangle}{\langle V_{\text{vac}} \rangle} = 1 + \frac{4\kappa\eta P_{\text{in}} P_{\text{th}}}{\Gamma(P_{\text{in}} - P_{\text{th}})} \xrightarrow{P_{\text{in}} \rightarrow P_{\text{th}}} \infty. \quad (31)$$

The derived results for the photon number of the fluctuations, as well as the squeezing and antisqueezing, are presented in Fig. 4 for anomalous dispersion at various input powers, indicated by the normalized pump power $P_n = P_{\text{in}}/P_{\text{th}}$. It is important to note that the mode at $\mu = 0$ corresponds to a single-mode squeezed state, as the generated photons share the same angular frequency as the pump, ω_p . All other modes with $\mu \neq 0$ represent a generated signal and idler pair and thus a two-mode squeezed state. At a low input power of

$P_n = 50\%$, many modes are stimulated with a rather low photon number. As P_n increases, the photon number also rises, becoming significant only for a few modes that exhibit the smallest effective detuning $\Delta_{\mu,\text{eff}}$. While both squeezing and antisqueezing demonstrably increase with P_n , the squeezing is already significant for multiple modes even at a relatively low input power P_n . Therefore, a wide range of modes exhibit significant squeezing, even though only a few of them consist of a large photon number.

The case of a normal dispersion ring resonator is also very interesting and presented in Fig. 5. From the classical results in Fig. 2, it is known that the FWM threshold power can never be achieved for any mode number μ due to normal dispersion, thus preventing the generation of a classical frequency comb. This aligns with the photon number results shown in Fig. 5, which remain very small across all P_n values. The $\mu = 0$ mode exhibits the largest photon number, as it corresponds to the smallest $\Delta_{\mu,\text{eff}}$. Interestingly, significant squeezing is still observed for a few modes, though the overall squeezing remains smaller compared to that achieved with an anomalous dispersion ring resonator. Nevertheless, this is an important

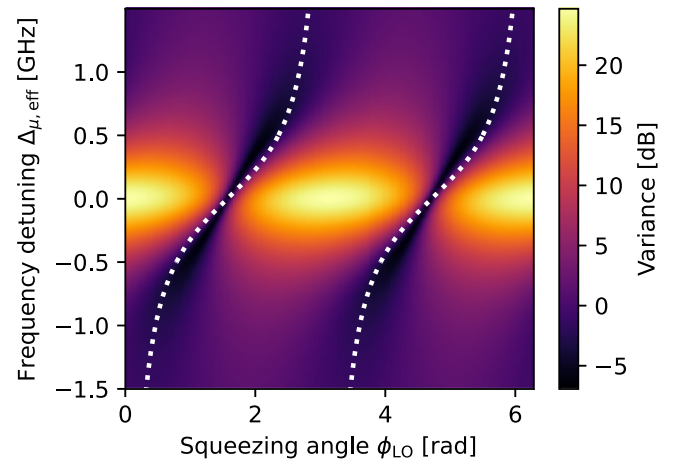


FIG. 3. Squeezing spectrum of the generated squeezed light in dependence of the effective frequency detuning $\Delta_{\mu,\text{eff}}$ and the squeezing angle ϕ_{LO} for $\kappa = 800$ MHz, $\gamma = 200$ MHz, $g_{\text{opt}} = 1.5$ MHz, $\eta = 1$, and $\lambda = 1550$ nm. The optimum squeezing angle $\phi_{\text{LO,opt}}$ is indicated by the white dotted line.

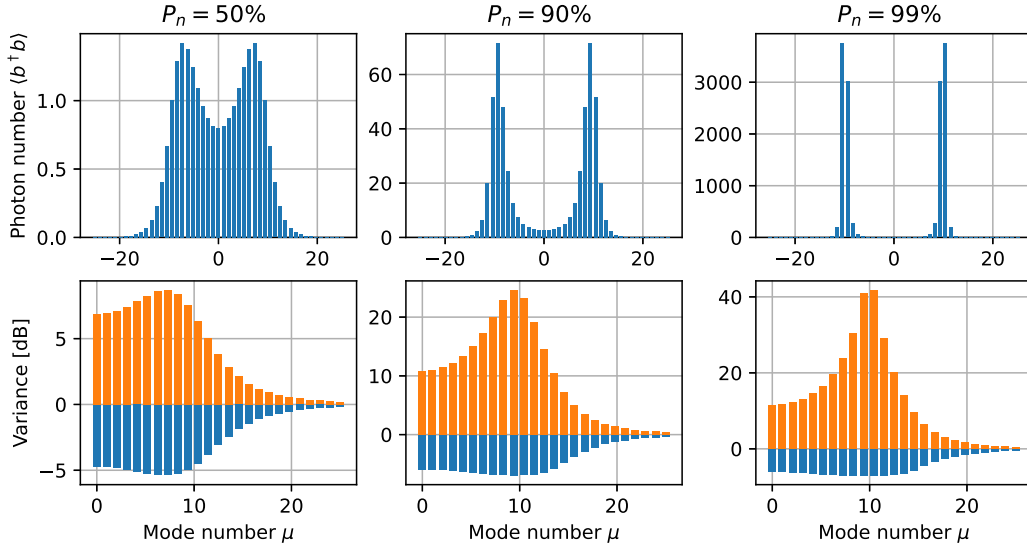


FIG. 4. Photon and squeezing spectrum of the quantum frequency comb for an anomalous dispersion ring resonator at the optimum squeezing angle $\phi_{LO,opt}$. The results are shown for various mode numbers μ , $\kappa = 800$ MHz, $\gamma = 200$ MHz, $g_{opt} = 1.5$ MHz, $D_2 = 10$ MHz, $\eta = 1$, $\lambda = 1550$ nm, and different normalized pump powers P_n . The orange values in the squeezing spectrum correspond to the antisqueezing, while the blue values correspond to the squeezing.

finding, as it demonstrates that a normal dispersion ring resonator can also be utilized to generate a quantum frequency comb. This could be particularly valuable if a single-mode squeezed state is desired, given that the squeezing results for the $\mu = 0$ mode are identical in the normal and anomalous regime.

Following the analysis of squeezing in the quantum frequency comb, we investigate the correlation between the modes. The second-order correlation function serves as an

indicator for the particle correlation, quantifying the two-photon correlation between a generated signal and idler photon. It can be determined using

$$g^{(2)}(t) = \frac{\langle b_{out,s}^\dagger(t) b_{out,i}^\dagger(t) b_{out,s}(t) b_{out,i}(t) \rangle}{\langle b_{out,s}^\dagger(t) b_{out,s}(t) \rangle \langle b_{out,i}^\dagger(t) b_{out,i}(t) \rangle}. \quad (32)$$

We evaluate $g_s^{(2)}(t)$ at $t = 0$, which is of interest in most applications. Note that the higher-order expectation values are

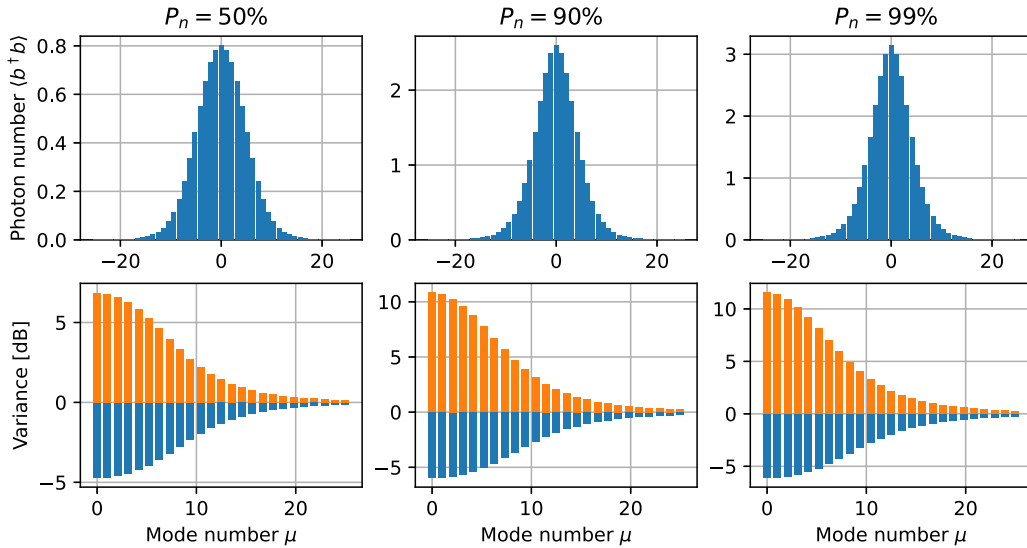


FIG. 5. Photon and squeezing spectrum of the quantum frequency comb for a normal dispersion ring resonator at the optimum squeezing angle $\phi_{LO,opt}$. The results are shown for various mode numbers μ , $\kappa = 800$ MHz, $\gamma = 200$ MHz, $g_{opt} = 1.5$ MHz, $D_2 = -10$ MHz, $\eta = 1$, $\lambda = 1550$ nm, and different normalized pump powers P_n . The orange values in the squeezing spectrum correspond to the antisqueezing, while the blue values correspond to the squeezing.

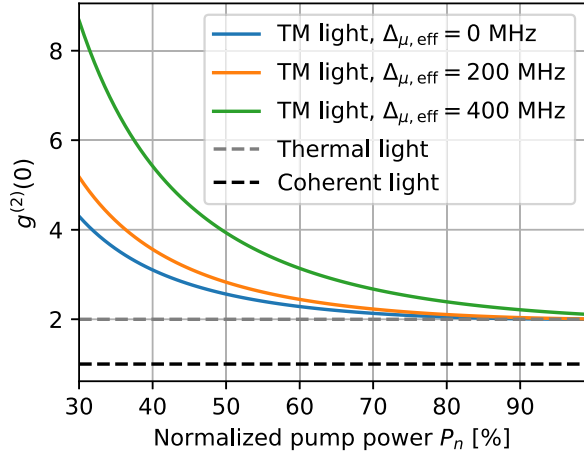


FIG. 6. Second-order correlation function $g^{(2)}(0)$ of coherent light (black dashed line), thermal light (gray dashed line) and two-mode light at different effective detuning values with $\Delta_{\mu,\text{eff}} = 0$ MHz (blue line), $\Delta_{\mu,\text{eff}} = 200$ MHz (orange line), and $\Delta_{\mu,\text{eff}} = 400$ MHz (green line) in dependence of the normalized pump power P_n . The signal and idler modes respectively correspond to thermal light, while both combined form the two-mode light. All values are shown for $\kappa = 800$ MHz, $\gamma = 200$ MHz, $g_{\text{opt}} = 1.5$ MHz, and $\lambda = 1550$ nm.

first reduced to second order using the cumulant expansion from [46] and then solved using the expectation values derived with Eq. (18). The result for the signal and idler modes, respectively, is $g_s^{(2)}(0) = 2$, indicating that both the signal and idler modes correspond to a thermal state. In comparison, the joint second-order correlation between the signal and idler

modes is given by

$$g_{si}^{(2)}(0) = \frac{1}{4} \left[\left(\frac{(4\Delta_{\mu,\text{eff}}^2 + \Gamma^2) P_{\text{th}}}{\Gamma^2 P_{\text{in}}} \right)^2 + \left(\frac{P_{\text{in}}}{P_{\text{th}}} \right)^2 - \frac{8\Delta_{\mu,\text{eff}}^2}{\Gamma^2} + 6 \right]. \quad (33)$$

The results are presented in Fig. 6 for various values of $\Delta_{\mu,\text{eff}}$ as a function of the normalized pump power P_n . Additionally, the values for thermal and coherent light, $g_{\text{thermal}}^{(2)}(0) = 2$ and $g_{\text{coherent}}^{(2)}(0) = 1$, respectively, are also shown. It can be observed that $g_{si}^{(2)}(0)$ increases with $\Delta_{\mu,\text{eff}}$ and decreases with increasing P_n with converging to a thermal state as $P_{\text{in}} \rightarrow P_{\text{th}}$. The reason for this behavior is that at low pump power P_n , only a few signal and idler photons are generated. However, if a signal photon is generated, there is a high probability that the corresponding idler photon is the particle-entangled partner. For higher input power, a larger number of photons are generated, leading to a reduction in the correlation between individual signal and idler photons.

Besides two-photon correlation, spectral mode correlation is another important feature analyzed in this work. It can be assessed using the JSI, which represents the joint signature probability distribution and indicates the probability of detecting a signal mode at a certain frequency if an idler mode is detected at another frequency. Thus, it is a value describing the correlation between the two modes as a function of frequency or time [47].

For the derivation of the JSI, the effective detunings of the signal and idler modes are considered individually and thus the JSI can be determined as

$$\begin{aligned} \Phi &= \langle b_{\text{out},s}^\dagger(\omega) b_{\text{out},i}^\dagger(\omega) b_{\text{out},s}(\omega) b_{\text{out},i}(\omega) \rangle \\ &= \frac{4\kappa^2 \Gamma^2 P_{\text{in}}^2 P_{\text{th}}^2 [\Gamma^4 P_{\text{in}}^4 + 2\Gamma^2 P_{\text{in}}^2 P_{\text{th}}^2 (4\Delta_i \Delta_s + 3\Gamma^2) + P_{\text{th}}^4 (4\Delta_i^2 + \Gamma^2)(4\Delta_s^2 + \Gamma^2)]}{[\Gamma^4 P_{\text{in}}^4 + 2\Gamma^2 P_{\text{in}}^2 P_{\text{th}}^2 (4\Delta_i \Delta_s - \Gamma^2) + P_{\text{th}}^4 (4\Delta_i^2 + \Gamma^2)(4\Delta_s^2 + \Gamma^2)]^2}. \end{aligned} \quad (34)$$

The result of Eq. (34) is shown in Fig. 7 for normalized pump powers of $P_n = 95\%$ and 99% . The plotted values are normalized by the maximum JSI value of 125 396 342.22. It

can be observed that the JSI increases with P_n and decreases significantly with an asymmetric detuning of the signal or idler mode, while exhibiting greater stability for symmetric

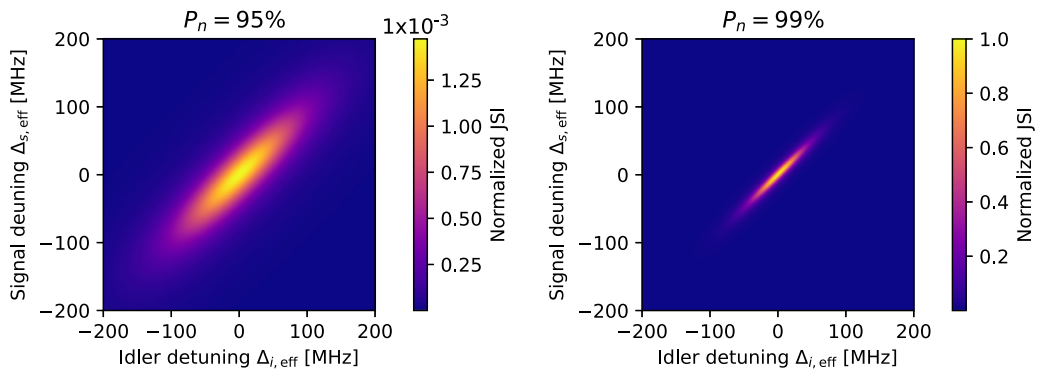


FIG. 7. Joint spectral intensity in dependence of the effective detuning of the signal and idler mode $\Delta_{s,\mu,\text{eff}}$ and $\Delta_{i,\mu,\text{eff}}$ at different normalized pump power values P_n . All values are shown for $\kappa = 800$ MHz, $\gamma = 200$ MHz, $g_{\text{opt}} = 1.5$ MHz, and $\lambda = 1550$ nm.

detuning. This behavior can be explained with the energy and momentum conservation of the FWM process $\omega_s + \omega_i = 2\omega_p$. Consequently, if the signal mode is detuned by a certain amount, the idler mode must undergo an inverse frequency shift to satisfy the conservation relation $\omega_s + \omega_i = 2\omega_p$.

C. Entanglement

Finally, the last investigated quantum feature of frequency combs is the most famous one with the entanglement, which is important for many quantum information and computing protocols. While $g^{(2)}$ and the JSI analyze the correlation between the photon pairs and modes, the entanglement between the signal and idler modes is investigated using the standard Simon criterion for continuous-variable quantum systems [48].

Therefore, the canonical operators of the signal and idler modes are defined first

$$Q_{s,i} = \frac{b_{\text{out},s} + b_{\text{out},s}^\dagger}{\sqrt{2}}, \quad P_{s,i} = \frac{b_{\text{out},s} - b_{\text{out},s}^\dagger}{i\sqrt{2}}, \quad (35)$$

which correspond the quadrature operators of the individual modes and are used to calculate the covariance matrix of the modes with

$$\mathbf{C} = \begin{pmatrix} C_A & C_C \\ C_C^T & C_B \end{pmatrix} = \begin{pmatrix} \langle \{Q_s, Q_s\} \rangle & \langle \{Q_s, P_s\} \rangle & \langle \{Q_s, Q_i\} \rangle & \langle \{Q_s, P_i\} \rangle \\ \langle \{P_s, Q_s\} \rangle & \langle \{P_s, P_s\} \rangle & \langle \{P_s, Q_i\} \rangle & \langle \{P_s, P_i\} \rangle \\ \langle \{Q_i, Q_s\} \rangle & \langle \{Q_i, P_s\} \rangle & \langle \{Q_i, Q_i\} \rangle & \langle \{Q_i, P_i\} \rangle \\ \langle \{P_i, Q_s\} \rangle & \langle \{P_i, P_s\} \rangle & \langle \{P_i, Q_i\} \rangle & \langle \{P_i, P_i\} \rangle \end{pmatrix}. \quad (36)$$

The operators $Q_{s,i}$ and $P_{s,i}$ can be combined in a vector $\mathbf{X} = (Q_s, P_s, Q_i, P_i)$ with the canonical commutation relation in the symplectic form $[X_i, X_j] = i\Omega_{i,j}$ with

$$\Omega = \begin{pmatrix} \omega & 0 \\ 0 & \omega \end{pmatrix}, \quad \omega = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \quad (37)$$

As in the previous sections, a symmetric behavior between the signal and the idler mode is assumed and the resulting nonvanishing expectation values of $\mathbf{C}$ are

$$\begin{aligned} \langle \{Q_s, Q_s\} \rangle &= \langle \{P_s, P_s\} \rangle = \frac{1}{2} + \langle b_{\text{out},s}^\dagger b_{\text{out},s} \rangle, \\ \langle \{Q_s, Q_i\} \rangle &= \frac{1}{2} (\langle b_{\text{out},i}^\dagger b_{\text{out},s} \rangle + \langle b_{\text{out},i} b_{\text{out},s}^\dagger \rangle), \\ \langle \{Q_s, P_i\} \rangle &= \langle \{P_s, Q_i\} \rangle = \frac{i}{2} (\langle b_{\text{out},i}^\dagger b_{\text{out},s} \rangle - \langle b_{\text{out},i} b_{\text{out},s}^\dagger \rangle), \\ \langle \{P_s, P_i\} \rangle &= -\frac{1}{2} (\langle b_{\text{out},i}^\dagger b_{\text{out},s} \rangle + \langle b_{\text{out},i} b_{\text{out},s}^\dagger \rangle), \\ \langle \{Q_i, Q_i\} \rangle &= \langle \{P_i, P_i\} \rangle = \frac{1}{2} + \langle b_{\text{out},i}^\dagger b_{\text{out},i} \rangle, \end{aligned} \quad (38)$$

with $\langle (a_s^\dagger)^2 \rangle = \langle a_s^2 \rangle = 0$ and the already derived results for the expectation values in Eqs. (22) and (26). Note that the matrix is symmetric with C_C due to the symmetry between the signal and the idler mode.

The entanglement between two states or modes is present if the two modes are not separable, and this indicates a

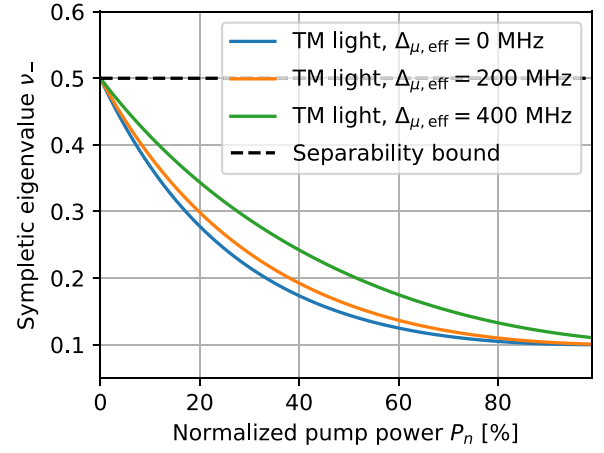


FIG. 8. Smallest symplectic eigenvalue v_- of two-mode light at different effective detuning values with $\Delta_{\mu,\text{eff}} = 0$ MHz (blue line), $\Delta_{\mu,\text{eff}} = 200$ MHz (orange line), and $\Delta_{\mu,\text{eff}} = 400$ MHz (green line) in dependence of the normalized pump power P_n . The separability bound is indicated by the black dashed line and all values $v_- < \frac{1}{2}$ violate it, showing entanglement. All values are shown for $\kappa = 800$ MHz, $\gamma = 200$ MHz, $g_{\text{opt}} = 1.5$ MHz, and $\lambda = 1550$ nm.

strong correlation between both modes. Simon's positive partial transpose (PPT) criterion states that if a state is separable, the partially transposed counterpart of $\mathbf{C}$, which is determined by $\tilde{\mathbf{C}} = \Lambda \mathbf{C} \Lambda$ and $\Lambda = \text{diag}(1, 1, 1, -1)$, is also separable [48–50]. It has been shown that this separability can be analyzed by the calculation of the smallest symplectic eigenvalue of $\mathbf{C}$, which is performed using the invariants of Serafini *et al.* [51] with

$$v_{\pm} = \sqrt{\frac{\Delta(\mathbf{C}) \pm \sqrt{\Delta(\mathbf{C})^2 - 4 \det(\mathbf{C})}}{2}}, \quad (39)$$

$$\begin{aligned} \Delta(\mathbf{C}) &= \det(C_A) + \det(C_B) - 2 \det(C_C) \\ &= \frac{1}{2} + 2 \langle b_{\text{out},i}^\dagger b_{\text{out},s}^\dagger \rangle \langle b_{\text{out},i} b_{\text{out},s} \rangle \\ &\quad + 2 \langle b_{\text{out},s}^\dagger b_{\text{out},s} \rangle^2 + 2 \langle b_{\text{out},s}^\dagger b_{\text{out},s} \rangle, \end{aligned} \quad (40)$$

and

$$\begin{aligned} \det(\mathbf{C}) &= (\langle b_{\text{out},i}^\dagger b_{\text{out},s}^\dagger \rangle \langle b_{\text{out},i} b_{\text{out},s} \rangle \\ &\quad - \langle b_{\text{out},s}^\dagger b_{\text{out},s} \rangle^2 - \langle b_{\text{out},s}^\dagger b_{\text{out},s} \rangle - \frac{1}{4})^2. \end{aligned} \quad (41)$$

In the investigated use case, the generated signal and idler modes are entangled if the Heisenberg uncertainty condition $\mathbf{C} + i\Omega/2 \geq \frac{1}{2}$ is violated and thus if the separability condition $v_- \geq \frac{1}{2}$ is violated [51,52]. Note that all our results use the convention $[Q, P] = i$, the vacuum variance equal to $\frac{1}{2}$, and $\hbar = 1$. To convert this to the shot-noise units with a vacuum variance of 1, it is required to multiply all elements of $\mathbf{C}$ and the symplectic eigenvalues by 2 and shift the separability threshold to $v_- > 1$ [48,51,52].

The results for v_- are shown in Fig. 8 and it can be seen that with a rising pump power v_- reduces; the separability is violated at $P_n > 0$. Thus, the entanglement increases as

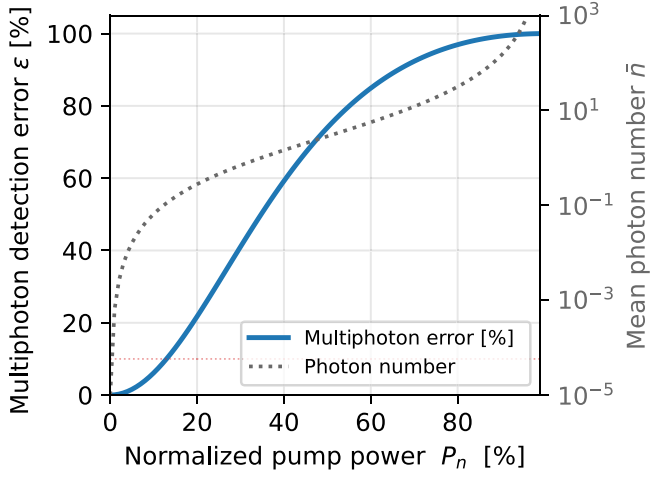


FIG. 9. Multiphoton detection error rate (blue solid line), which rises with the photon number of the generated signal and idler mode (gray dotted line).

expected with P_n and v_- converges to

$$v_{-, \min} \xrightarrow{P_{\text{in}} \rightarrow P_{\text{th}}} \frac{\gamma}{2\Gamma}. \quad (42)$$

This results in the requirement of $\kappa \gg \gamma$ and thus low losses are required to achieve a strong entanglement.

For an efficient utilization of the entanglement, it is crucial to choose the best-suited detection method. The common methods measure either the mode entanglement, the entanglement between the signal and idler modes, or the particle entanglement, the entanglement between the signal and the idler photon pair. While for the mode entanglement an intensity measurement at a certain frequency is performed, particle-entanglement measurements require a particle resolution. The intensity measurements can be performed efficiently using a homodyne or heterodyne measurement. However, particle-entanglement detection measurements are dominated by the multiphoton detection error, which can be given as

$$\varepsilon = \frac{\bar{n}}{\bar{n} + 1}, \quad (43)$$

with the photon number of the generated signal and idler modes $\bar{n} = \langle a_s^\dagger a_s \rangle + \langle a_i^\dagger a_i \rangle$ [53]. The error rate ε is shown in Fig. 9 together with the photon number $\bar{n}$. Generally, only a small ε is tolerated and thus particle-entanglement measurements are only used at very small values for P_n , while the best operation to exploit mode entanglement is at $P_{\text{in}} \approx P_{\text{th}}$, as it is well known in the literature [16–21,23,35]

III. DISCUSSION

In this work we derived closed-form analytical expressions describing the squeezing, the second-order correlation, the JSI, and the entanglement for each mode within a quantum frequency comb. These quantities are predominantly employed in typical quantum technology applications. Squeezing, for instance, is frequently utilized in sensing applications. Our results demonstrate that significant single- and two-mode squeezing can be generated in both normal and

anomalous dispersion regimes. While dispersion is irrelevant for single-mode squeezed light, optimal two-mode squeezing is exclusively achieved under anomalous dispersion, as only then $\Delta_{\mu, \text{eff}} = 0$ is possible. If the squeezing of multiple modes in the quantum frequency comb is utilized, it is important to note that each mode corresponds to a different squeezing angle as shown in Fig. 3. This difference can be critical for applications where modes are mixed, as matching the squeezing angles of various modes is essential to enhance or reduce the overall squeezing effects.

Furthermore, we analyzed the behavior of particle correlation using the second-order correlation function and mode correlation using the JSI. Interestingly, our results reveal an inverse behavior between the second-order correlation and the JSI and consequently between particle and mode correlation. Additionally, also the entanglement between the signal and idler modes was investigated using Simon’s criterion and it has been shown that the mode entanglement rises with the pump power. However, since the multiphoton detection error ε rises also with the pump power, the particle entanglement can only be utilized at a low pump power or a large $\Delta_{\mu, \text{eff}}$. This indicates that either the particle or mode entanglement of a given mode can be predominantly utilized, a characteristic determined by the design and operation of the ring resonator. Nevertheless, for the entire quantum frequency comb, it is possible to generate modes exhibiting high mode entanglement as well as modes consisting of a high particle entanglement through clever dispersion engineering and thus precise control over the $\Delta_{\mu, \text{eff}}$ of the individual modes.

IV. SUMMARY

In this work we derived closed-form analytical expressions describing the squeezing, the second-order correlation function, the JSI, and the entanglement for each mode in a quantum frequency comb generated using a microring resonator. These equations enable the design and operation of microring resonators to enhance specific quantum features while suppressing others. We demonstrated that microring resonators can be used to generate a broad variety of quantum features for individual optical modes. Thus, the derived equations provide a foundational framework for various chip-integrated quantum optics applications, facilitating the development of quantum sensing, computing, and communication.

ACKNOWLEDGMENTS

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DATA AVAILABILITY

The data that support the findings of this article are not publicly available. The data are available from the authors upon reasonable request.

APPENDIX A: CONVERSION OF RESONATOR MODES TO OPTICAL POWERS

To convert the classical amplitude α of a resonator mode a to optical powers, it is crucial to understand that α and a describe the mode over the whole resonator volume. Thus, they are unitless and do not directly correspond to an optical power as the waveguide mode b , which corresponds to a flux in units of $\sqrt{\text{Hz}}$. To solve this, the resonator modes are converted using the transmission rate t , which describes the amount of α that stays in the resonator using

$$\alpha_{\sqrt{\text{Hz}}} = \sqrt{t}\alpha, \quad (\text{A1})$$

which is the same as described in Appendix B of [28]. Then the amplitude can be converted to an optical power with the known relation

$$\alpha_{\sqrt{\text{Hz}}} = \sqrt{\frac{P}{\hbar\omega}}. \quad (\text{A2})$$

Note that the waveguide modes b can always be transferred to an optical power using Eq. (A2).

APPENDIX B: INPUT-OUTPUT THEORY

To describe the outputs modes of the ring resonator b_{out} with Eq. (18), the input-output theory is performed. First, the equations of motion describing the signal and idler modes in the ring resonator are derived using the Hamiltonian of Eq. (1) with

$$\begin{aligned} \frac{da_{s,i}(t)}{dt} = & -i\omega_{s,i}a_{s,i}(t) + \frac{\sigma}{2}a_{i,s}^\dagger(t) - \frac{\gamma}{2}a_{s,i}(t) - \frac{\kappa}{2}a_{s,i}(t) \\ & + \sqrt{\kappa}b_{\text{in},s,i}(t) + \sqrt{\gamma}b_{\gamma,s,i}(t). \end{aligned} \quad (\text{B1})$$

The boundary condition that has already been derived in [40,41] is used to describe the output modes b_{out} with

$$b_{\text{in},s,i}(t) = \sqrt{\kappa}a_{s,i}(t) - b_{\text{out},s,i}(t) \quad (\text{B2})$$

and leads with Eq. (B1) to the equation of motion including the output modes with

$$\begin{aligned} \frac{da_{s,i}(t)}{dt} = & -i\omega_{s,i}a_{s,i}(t) + \frac{\sigma}{2}a_{i,s}^\dagger(t) - \frac{\gamma}{2}a_{s,i}(t) + \frac{\kappa}{2}a_{s,i}(t) \\ & - \sqrt{\kappa}b_{\text{out},s,i}(t) + \sqrt{\gamma}b_{\gamma,s,i}(t). \end{aligned} \quad (\text{B3})$$

For a more compact form, Eqs. (B1) and (B3) are given in matrix notation with

$$\frac{d}{dt}\mathbf{A}(t) = \left(\mathbf{K} + \frac{\kappa}{2}\mathbb{1}\right)\mathbf{A}(t) + \sqrt{\kappa}\mathbf{B}_{\text{in}}(t) + \sqrt{\gamma}\mathbf{B}_{\gamma}(t) \quad (\text{B4})$$

and

$$\frac{d}{dt}\mathbf{A}(t) = \left(\mathbf{K} - \frac{\kappa}{2}\mathbb{1}\right)\mathbf{A}(t) - \sqrt{\kappa}\mathbf{B}_{\text{out}}(t) + \sqrt{\gamma}\mathbf{B}_{\gamma}(t). \quad (\text{B5})$$

Using the notation from Eqs. (19) and (20) and the vector including the resonator modes

$$\mathbf{A} = \begin{pmatrix} a_s \\ a_s^\dagger \\ a_i \\ a_i^\dagger \end{pmatrix}, \quad (\text{B6})$$

the Fourier transformation of Eqs. (B4) and (B5) is performed, which leads to

$$\left(\boldsymbol{\Omega} - \mathbf{K} - \frac{\kappa}{2}\mathbf{I}_4\right)\mathbf{A}(\omega) = -\sqrt{\kappa}\mathbf{B}_{\text{out}}(\omega) + \sqrt{\gamma}\mathbf{B}_{\gamma}(\omega), \quad (\text{B7})$$

$$\left(\boldsymbol{\Omega} - \mathbf{K} + \frac{\kappa}{2}\mathbf{I}_4\right)\mathbf{A}(\omega) = \sqrt{\kappa}\mathbf{B}_{\text{in}}(\omega) + \sqrt{\gamma}\mathbf{B}_{\gamma}(\omega). \quad (\text{B8})$$

Finally, the resonator modes $\mathbf{A}$ are eliminated by combining Eqs. (B8) and (B7), which leads to Eq. (18) of the main text.

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