

Propagation of partially spatially coherent laser beams in instantaneous Kerr media

Bahman Hafizi ^{1,*}, Joshua J. Isaacs,¹ Daniel Younis ¹, Luke A. Johnson,² Dmitri Kaganovich ¹ and John P. Palastro ³

¹*Naval Research Laboratory, Washington, DC 20375, USA*

²*The MITRE Corporation, McLean, Virginia 22102, USA*

³*Laboratory for Laser Energetics, University of Rochester, Rochester, New York 14623, USA*



(Received 19 August 2025; accepted 27 January 2026; published 17 February 2026)

The propagation of intense, partially spatially coherent laser beams in a medium with instantaneous third-order susceptibility is studied analytically and numerically. For sufficiently high power relative to that required for nonlinear self-focusing, the propagation initially proceeds in two stages. In the first stage, spatial coherence builds up, and in the second stage, the number of speckles reduces. Once the degree of coherence is sufficiently high, whole-beam self-focusing occurs. The beam power is mostly confined within the initial spot radius. Two analytical approaches for describing the evolution of the beam are presented. The method of moments leads to an analytical solution for the rms spot radius that is in excellent agreement with simulations. This method does not require any knowledge of the field statistics beyond the initial conditions and provides no information about the evolution of the individual speckles. The other approach employs a self-similar solution for the second-order coherence function of the field and assumes that the fourth-order coherence function is factorizable and obeys complex circular Gaussian random statistics. The latter method also leads to an analytical expression for the spot radius, but its predictions for the qualitative evolution of the speckles disagree with wave-optics simulations.

DOI: [10.1103/bnyf-rrmf](https://doi.org/10.1103/bnyf-rrmf)

I. INTRODUCTION

A wide range of physical processes create partially spatially coherent light, including refraction off turbulent eddies in atmospheric, oceanic, or aero-optic environments; spontaneous generation in multitransverse mode and random lasers; or scatter off biological and photonic structures. Partially spatially coherent light can also be prepared using spatial light modulators, deformable mirrors, or a random surface such as ground glass. As discussed in Sec. IV, it is possible that high-power, partially coherent beams can find applications such as in mitigating the effects of turbulence. For a serious evaluation of these, however, it is important to first understand the more fundamental phenomenon of nonlinear propagation of partially coherent beams in a homogeneous medium.

The objective here is to adapt and employ the methodology of statistical optics [1–10] to analyze the propagation of partially spatially coherent beams in a nonlinear medium. This will emphasize the importance that the initial field statistics play in nonlinear propagation and their subsequent evolution. Two analytical approaches are considered. The approaches make predictions that are useful for applications such as the beam spot radius and power-in-the-bucket, among others. The first approach makes use of the method of moments and leads

to an expression for the ensemble-averaged rms laser beam radius based solely on the knowledge of the field statistics at the transmitter (initial plane). This approach provides no information about the evolution of the individual speckles (note that the term “speckle” is used to refer generally to discrete islets of beam intensity). The second approach makes use of a self-similar form for the second-order coherence function of the field and assumes the statistics remain invariant. This approach leads to an expression for the beam radius that is very similar in form to that in the first approach. Two characteristic length scales emerge from these theories. One is the usual Rayleigh range associated with the whole-beam spot size. The other is typically shorter and is the geometric mean of (i) the diffraction range associated with the radius of the whole beam and (ii) the radius of the speckles. An important, but incorrect, prediction of the second approach is that the number of speckles in the cross-section of the beam is conserved. These results are tested by wave-optics simulations, which show the gradual disappearance of speckles along the propagation path—roughly on the scale given by the mean geometric diffraction length—which contradicts the prediction of the self-similar approach. The simulations reveal that the breakdown of this approach is, in part, because the statistics evolve along the propagation path.

The remainder of this article is organized as follows. Section II presents the formulation of the two approaches employed to obtain analytical expressions for the radius of the beam envelope. This section addresses in detail the approximations that are inherent to the two approaches. The analysis is benchmarked against wave-optics simulations in Sec. III. Section IV discusses the results along with possible generalizations and potential applications.

*Contact author: bahman.u.hafizi.civ@us.navy.mil

II. MAIN EQUATIONS AND PHYSICAL INTERPRETATION

Consider propagation of an intense laser beam in the steady-state regime in a transparent medium with Kerr nonlinearity. Up until the point where the beam collapses and before a self-focusing arrest mechanism (e.g., photoionization) is triggered, the paraxial description applies and the electric field $\mathbf{E}(\mathbf{r}, t)$ can be represented in the form

$$\mathbf{E}(\mathbf{r}_\perp, z, t) = \frac{1}{2}E(\mathbf{r}_\perp, z, t)e^{i\psi(z,t)}\hat{\mathbf{e}}_x + \text{c.c.}, \quad (1)$$

where $\psi(z, t) = k_0 z - \omega_0 t$ is the rapidly varying phase, $k_0 = n_0 \omega_0 / c$; n_0 is the linear refractive index; and the complex-valued envelope $E(\mathbf{r}_\perp, z, t)$ is a slowly-varying function of $\mathbf{r}_\perp = (x, y)$ and t , which varies even more slowly along the propagation direction z .

Here, the response of the medium is assumed to be instantaneous (Appendix A) and third order in the field strength, as is the case with the Kerr effect in air as well as a condensed medium such as water. Then, in terms of the independent variables z , $\tau = t - z/v_g$, the propagation equation for $E(\mathbf{r}, t)$ takes the form [11–17]

$$\left(2ik_0 \frac{\partial}{\partial z} + \nabla_\perp^2\right)E(\mathbf{r}_\perp, z) = -2\bar{n}_2|E(\mathbf{r}_\perp, z)|^2E(\mathbf{r}_\perp, z), \quad (2)$$

where the normalized nonlinear refractive index $\bar{n}_2 = n_0 c / 4P_{\text{NL}}$ is related to the critical power for self-focusing, $P_{\text{NL}} = \lambda_0^2 / (2\pi n_0 n_2)$, and $\lambda_0 = 2\pi c / \omega_0$ is the laser wavelength in free space. The transverse Laplacian is understood to represent differentiation along $\mathbf{r}_\perp$. Equation (2) is a simplified model of propagation in the paraxial approximation. The operator on the left-hand side (LHS) neglects a mixed, spatiotemporal derivative, as well as chromatic effects (and is therefore appropriate in the long-pulse limit) [18–20]. Generalizations of Eq. (2), their applications in physics, and their mathematical structure have been studied in Refs. [21,22].

Equation (2) has the form of a nonlinear Schrödinger equation. Making use of the action functional [23], the Hamiltonian

$$H(E^*, E, z) = \frac{1}{2} \int d^2 r_\perp (\nabla_\perp E^* \cdot \nabla_\perp E - \bar{n}_2 E^* E^* E E) \quad (3)$$

leads to Eq. (2) by the standard canonical formalism. The second term in the integrand of Eq. (3) describes parametric processes such as self-focusing and transverse beam breakup. The latter is sometimes referred to as the filamentation instability [13].

Solution of Eq. (2) depends on the initial conditions at the transmitter, assumed to be in the plane $z = 0$. Partial spatial coherence in the transmitter plane can arise in several ways. In high-power applications, it is a natural outcome of combining light from a large array of diodes. Partial spatial coherence can also be imposed on purpose. For example, the actuators of a deformable mirror can be independently controlled to modify the phase of an incident beam according to specified statistics [24]. For analysis, it is useful to employ a statistical optics description of the propagation, wherein an ensemble of many realizations of the initial incoherence is considered. An important distinction pertains to the statistics of the incident radiation, for example, whether they are laserlike or pseudothermal, etc. [7,25]. Here, the first-order statistics of

the incident beam at the transmitter are assumed to be laserlike (with a Poisson distribution for the photon count) and the second-order statistics are chosen to be that of a Gaussian random-phase screen. In this paper, statistics which are fourth-order and higher in the field are not independently specified and are the subject of future studies. Physically, field coherence at the fourth order determines beam centroid wander and intensity scintillation—the control of which is critical for many applications [26–28]. The fourth-order coherence function can describe entangled photon pairs [29,30], relevant in turbulence-free ghost imaging [9].

A. Beam radius based on averaged description of beams

In the method of moments [31], the mean-square long-term radius is defined by

$$\langle R_L^2 \rangle = \frac{1}{P} \int d^2 r_\perp r_\perp^2 \langle I(\mathbf{r}_\perp, z) \rangle. \quad (4)$$

Here $\langle \cdot \rangle$ denotes an average over the ensemble associated with the partial spatial coherence of the electric field. The parameter P , defined by

$$P = \int d^2 r_\perp I(\mathbf{r}_\perp, z), \quad (5)$$

is the conserved power in the beam. Making use of the definition in Eq. (4), one finds that

$$\frac{d\langle H \rangle}{dz} = 0, \quad (6)$$

$$\frac{d^2 \langle R_L^2 \rangle}{dz^2} = \frac{cn_0}{2\pi k_0^2 P} \langle H \rangle, \quad (7)$$

where the ensemble average of the Hamiltonian in Eq. (3) is

$$\langle H \rangle = \frac{1}{2} \int d^2 r_\perp [\langle \nabla_\perp E^* \cdot \nabla_\perp E \rangle - \bar{n}_2 \langle |E|^4 \rangle]. \quad (8)$$

Since $\langle H \rangle$ is conserved, it is entirely determined by the field and its statistics in the initial plane of propagation. In this plane the fluctuating complex-valued electric field amplitude may be represented as

$$E(\mathbf{r}_\perp, z = 0) = E_0 e^{-r_\perp^2/R_0^2} e^{i\phi(\mathbf{r}_\perp)}, \quad (9)$$

where the random phase ϕ is defined in Appendix B. The probability density function for the irradiance

$$I(\mathbf{r}_\perp) \equiv n_0 c E^* E / 8\pi$$

(averaged over an optical period) is $\delta[I(\mathbf{0}_\perp) - n_0 c |E_0|^2 / 8\pi]$, i.e., the initial intensity is constant between realizations. This fully specifies the first-order statistics (i.e., the statistical properties at a single point in space) of an ideal laser and allows the second term on the right-hand side (RHS) of Eq. (8) to be determined. Using Eq. (9) in Eq. (5) leads to an expression for the power

$$P = \frac{cn_0}{16} |E_0|^2 R_0^2. \quad (10)$$

Note that for the form of the initial field given in Eq. (9), the power is, like the initial intensity, constant between realizations of the random phase, $\langle P \rangle = P$.

The appearance the grad operator in the first term on the RHS of Eq. (8) indicates that it is necessary to know the second-order statistics (i.e., the statistical properties at two spatial points) of the laser beam. The second-order statistics are chosen to be that of a Gaussian random-phase screen, given by the complex degree of spatial coherence in the initial plane [6,7]. With this, the Hamiltonian can be evaluated at the transmitter (Appendix B)

$$\langle H \rangle = \frac{8\pi}{cn_0} \frac{P}{R_0^2} \left(1 + 2\phi_0^2 \frac{R_0^2}{\rho_0^2} - \frac{P}{P_{NL}} \right), \quad (11)$$

where ρ_0 is the coherence length and $\langle \phi^2 \rangle = \phi_0^2$ is the phase variance [Eq. (B7) in Appendix B]. Making use of Eq. (11), Eq. (7) integrates to

$$\langle R_L^2 \rangle = \frac{1}{2} R_0^2 + \frac{2z^2}{k_0^2 R_0^2} \left(1 + 2\phi_0^2 \frac{R_0^2}{\rho_0^2} - \frac{P}{P_{NL}} \right). \quad (12)$$

Notice that the first term in this expression has the form $R_0^2/2$ as a result of the way R_0 is introduced in Eq. (9) (cf. Ref. [32]). The terms in parentheses on the right represent the effects of coherent diffraction, partial spatial coherence, and self-focusing, respectively. The case of a spatially coherent beam is recovered by taking $\phi_0 = 0$ or $\rho_0 \rightarrow \infty$.

A high-power collimated beam will, on average, collapse after a distance

$$Z_{sf} = \frac{Z_{R0}}{\sqrt{P/P_{NL} - 1 - \phi_0^2 N_s}},$$

where

$$Z_{R0} \equiv \frac{1}{2} k_0 R_0^2 \quad (13)$$

is the Rayleigh range and

$$N_s \equiv 2 \frac{R_0^2}{\rho_0^2} \quad (14)$$

is a measure of the number of phase inhomogeneities (extrema) in the near field [33]. In terms of N_s , and for later reference, it is informative to rewrite Eq. (12) in the form

$$\langle R_L^2 \rangle = \frac{R_0^2}{2} \left[1 + \frac{z^2}{Z_{R0}^2} + \frac{z^2}{Z_{vCZ}^2} \left(1 - \frac{1}{\phi_0^2 N_s} \frac{P}{P_{NL}} \right) \right], \quad (15)$$

where

$$Z_{vCZ} \equiv \frac{1}{2} k_0 R_0 \frac{\rho_0}{\sqrt{2}\phi_0} \quad (16)$$

is the characteristic length scale associated with a partially coherent beam [33]. Equation (15) reveals that, in the low-power limit, a speckled beam has an effective diffraction length that is the geometric mean of the diffraction ranges associated with (i) the radius of the whole beam and (ii) that of the speckles therein. The vCZ suffix employed here alludes to the van Cittert–Zernike theorem, according to which, in the linear regime, the coherence of light improves with propagation over a similar range [7].

B. Beam radius based on self-similar solution for the second-order coherence function

The moment approach provides an analytical expression for the spot radius that is expected to be “exact”—as will be confirmed in Sec. III. It has the virtue of being entirely agnostic to the evolution of the statistics. Its major drawback is that it provides no information about the evolution of speckles.

To overcome this, it is necessary to analyze the coherence functions of the field [33]. A brief review is provided to introduce the notation. The second-order coherence function is defined by

$$\Gamma_E(\mathbf{r}_{\perp 1}, \mathbf{r}_{\perp 2}, z) \equiv \langle E(\mathbf{r}_{\perp 1}, z) E^*(\mathbf{r}_{\perp 2}, z) \rangle. \quad (17)$$

This is often referred to as the mutual coherence function (MCF) [7,26,34]. To derive an equation for the variation of the MCF, it is convenient to introduce the shorthand notation $E_i \equiv E(\mathbf{r}_{\perp i}, z)$, $i = 1, 2$. Multiplying Eq. (2) for E_1 by E_2^* and subtracting from it the equation for E_2^* multiplied by E_1 yields

$$\left[2ik_0 \frac{\partial}{\partial z} + (\nabla_{\perp 1}^2 - \nabla_{\perp 2}^2) \right] \Gamma_E = -2\bar{n}_2 (|E_1|^2 - |E_2|^2) E_1 E_2^*. \quad (18)$$

Solving this equation requires an expression for the ensemble average on the right-hand side. This average is expressible in terms of the fourth-order coherence function

$$\begin{aligned} \Gamma_4(\mathbf{r}_{\perp i}, \mathbf{r}_{\perp j}; \mathbf{r}_{\perp k}, \mathbf{r}_{\perp \ell}; z) \\ \equiv \langle E(\mathbf{r}_{\perp i}, z) E(\mathbf{r}_{\perp j}, z) E^*(\mathbf{r}_{\perp k}, z) E^*(\mathbf{r}_{\perp \ell}, z) \rangle \end{aligned} \quad (19)$$

(by appropriate selection of the indices i, j, k, ℓ), which is *a priori* unknown. The equation for the fourth-order coherence function will, in turn, couple to that for the sixth-order coherence function, etc. [35].

The general solution of a nonlinear partial differential equation, such as Eq. (18), is not known. A common approximation is to employ a self-similar solution involving a number of parameters that are functions of z and that are determined by an equal number of ordinary differential equations [33,36,37].

For the problem of interest here, it is useful to first make a change of variables to $\boldsymbol{\rho} = \mathbf{r}_{\perp 1} - \mathbf{r}_{\perp 2}$ and $\mathbf{R} = (\mathbf{r}_{\perp 1} + \mathbf{r}_{\perp 2})/2$, defining the relative and centroid coordinates, respectively. The former captures the fast transverse variations associated with realizations of the speckle patterns of a bounded beam, while the latter captures the slow spatial variations of the average beam irradiance. A physically motivated solution is then an exponential function of $\boldsymbol{\rho}$ and of $\mathbf{R}$. Here, it is assumed that it is sufficient to retain terms up to the second degree in the exponent

$$\begin{aligned} \Gamma_E(\boldsymbol{\rho}, \mathbf{R}, z) = |E_0|^2 e^{-2R^2/W^2(z) - \rho^2/2W^2(z)} \\ \times \exp \left[-\phi_0^2 (1 - e^{-\rho^2/\rho_c^2(z)}) \right] e^{i\boldsymbol{\rho} \cdot \mathbf{R} \psi(z)} e^{S(z)}. \end{aligned} \quad (20)$$

In this expression, $W(z)$ is a measure of the beam width and $\rho_c(z)$ is the transverse coherence length—the z dependence of both attesting to the self-similar nature of the proposed solution. Equation (20) is adapted from that in Ref. [33], written for a Gaussian phase screen, and makes use of the expression for the complex degree of coherence, Eq. (B7) in

Appendix B. The function ψ is a cross-term phase factor and e^S controls the overall envelope diffraction.

The relevance of the new variables is manifested by taking limits. The broad-beam limit is given by $\nabla_{\mathbf{R}}\Gamma_E = 0$. This requires $W \rightarrow \infty$ and $\psi \rightarrow 0$, and then Γ_E limits to the complex degree of coherence for a Gaussian phase screen with homogeneous statistics. For $\rho_c \rightarrow \infty$ or $\phi_0 \rightarrow 0$, the (coherent) product of two (bounded) Gaussian beams is recovered.

The final step to solve Eq. (18) is to decide how to determine the fourth-order coherence function $\Gamma_4(\mathbf{r}_{\perp i}, \mathbf{r}_{\perp j}; \mathbf{r}_{\perp k}, \mathbf{r}_{\perp \ell}; z)$. Besides the ansatz expressed by Eq. (20), the other key step in closing the system of equations is to employ the factorization

$$\langle E_i E_j E_k^* E_\ell^* \rangle \rightarrow \langle E_i E_k^* \rangle \langle E_j E_\ell^* \rangle + \langle E_i E_\ell^* \rangle \langle E_j E_k^* \rangle \quad (21)$$

to express the right-hand side of Eq. (18) in terms of $\Gamma_E(\boldsymbol{\rho}, \mathbf{R}, z)$. This is expected to be valid provided the field statistics of E are those of a complex circular zero-mean Gaussian random field [$E \sim \mathcal{CN}(0, \sigma^2) \cdot \mathbb{E}(E) = 0$], as is the case of pseudothermal light. In Ref. [33] it is argued that if this is valid at the transmitter and, as the beam propagates, the nonlinear phase shift arising from the induced fluctuations of the refractive index is small, then the factorization in Eq. (21) can hold throughout the propagation. Reference [33] is concerned with media with noninstantaneous Kerr nonlinearity, and the smallness of the phase shift is aided by the response time of the medium being long compared to the coherence time of the field fluctuations (see Appendix C and [38]).

In this paper the initial field statistics is chosen to be laser-like and there is no guarantee that it will evolve toward that in [33]. Nevertheless, with the availability of a state-of-the-art wave-optics code, it is important to compare the consistency of the self-similar solution with simulation results. In this vein, inserting Eqs. (20) and (21) into Eq. (18) and, equating like powers of $\rho^n R^m$ ($n, m = 0, 1, 2$), leads to [39,40]

$$\begin{aligned} \frac{d^2 W}{dz^2} &= \frac{4}{k_0^2 W^3} + \frac{8}{k_0^2 W} \left[\frac{\phi_0^2}{\rho_c^2} - \frac{2P/P_{\text{NL}}}{W^2} \right], \\ \frac{d\rho_c}{dz} &= \frac{\rho_c}{W} \frac{dW}{dz}, \\ \frac{dS}{dz} &= -\frac{2}{k_0} \psi, \\ \frac{d\psi}{dz} &= -\frac{\psi^2}{k_0} + \frac{4}{k_0 W^4} + \frac{8\phi_0^2}{k_0 \rho_c^2 W^2} - \frac{8}{k_0 W^4} \frac{2P}{P_{\text{NL}}}. \end{aligned} \quad (22)$$

To integrate this set of equations, it should first be noted that the second equation has the simple solution

$$\frac{W^2(z)}{\rho_c^2(z)} = \text{const.}, \quad (23)$$

which predicts that the number of speckles is fixed. Making use of Eq. (23), it follows that

$$W^2 = W_0^2 \left\{ 1 + \frac{z^2}{\tilde{Z}_{R0}^2} + \frac{z^2}{\tilde{Z}_{\text{vCZ}}^2} \left[1 - \frac{4P/P_{\text{NL}}}{\phi_0^2 N_s} \right] \right\}, \quad (24)$$

where $\tilde{Z}_{R0}$ and $\tilde{Z}_{\text{vCZ}}$ differ from the expressions in Eqs. (13) and (16) in that they are evaluated with the initial spot radius $W_0 \equiv W(z=0)$ and speckle radius $\rho_{c0} \equiv \rho_c(z=0)$. Apart

TABLE I. Simulation parameters.

Symbol	Description	Value(s)
λ	Central wavelength	0.8 μm
R_0	$1/e^2$ intensity radius	2 cm
Z_{R0}	Rayleigh range	1.57 km
P/P_{NL}	Normalized power	0.5–4
ρ_0	Coherence length	0.05–0.2 cm
ϕ_0	Square root of phase variance	0–0.3
N	Number of realizations	200
$N_{x,y}$	Number of grid points	8192
$L_{x,y}$	Computational domain size	$64R_0$

from the numerical coefficients, Eq. (24) is formally identical to Eq. (15). (The numerical discrepancy is discussed in Refs. [41,42].)

To summarize, the averaged description in Sec. II A leads to an analytical expression for the rms radius, Eq. (15), based solely on the knowledge of the field statistics at the transmitter, as captured in the expression for the Hamiltonian in Eq. (11). This approach provides no information about the evolution of the individual speckles or the coherence functions. The approach in Sec. II B assumes that (i) the MCF has a self-similar form and (ii) complex circular Gaussian random statistics are obeyed. An analytical expression for the beam radius, Eq. (24), that is very similar in form to that in Eq. (15) is obtained. It predicts that the number of speckles in the beam cross-section is conserved. To test these results, it is necessary to perform wave-optics simulations.

III. BENCHMARKING THEORY AGAINST SIMULATIONS

A number of simplifications have been made in the analysis presented above. While the analysis in Sec. II A is essentially exact, the self-similar solution in Sec. II B has a number of requirements. One is that the field statistics do not change much with propagation. Another requirement is that the degree of incoherence of the field is sufficiently high. The latter is valid if there are a large number of speckles in the beam cross-section, $N_s \gg 1$, in which case the nonlinear phase shift can be small (see Appendix C).

Simulations of beam propagation have been carried out using the PyCAP library, which solves Eq. (2) for the parameters listed in Table I (see Appendix D and [15,17]). To provide some context, the parameters here are similar to those in recent work considering coherent laser beam propagation in a turbulent atmosphere [17]. Here, the beams start collimated and there is no turbulence. It is ensured that when nonlinear collapse is expected, it occurs for all random realizations of the initial phase screen (a deliberate choice to maintain consistency). The total power ranges over $0.5P_{\text{NL}}-4P_{\text{NL}}$, where the critical power for self-focusing is defined as that which satisfies the self-guiding condition for a Gaussian beam. Each combination of phase screen parameters (ρ_0 and ϕ_0) is simulated for a random ensemble of $N = 200$ realizations, and the figures are classified according to ρ_0 and ϕ_0 . The coherence time is long and, for some of the parameters, the nonlinear phase shift (Appendix C) can be sufficiently large to alter the beam statistics with propagation. For the range of phase

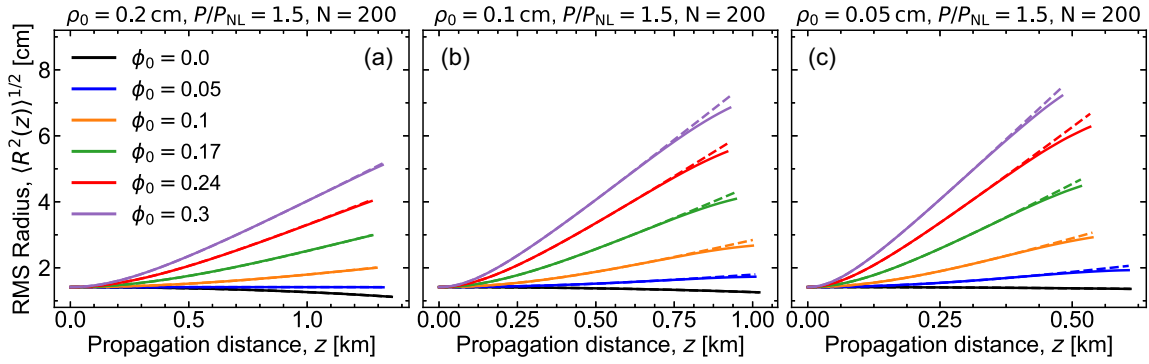


FIG. 1. Comparison of the rms beam radius $(R^2(z))^{1/2}$ between theory [Eq. (15), dashed] and simulations (PyCAP, solid); the two are in excellent agreement. The coherence length values are (a) $\rho_0 = 0.2$ cm, (b) $\rho_0 = 0.1$ cm, and (c) $\rho_0 = 0.05$ cm, with different (square root of) phase variances ϕ_0 as shown. Based on convergence tests (see Fig. 6 in Appendix D), the small discrepancy between theory and simulation is an artifact of nonconservation of beam energy in the finite computational domain. The simulation curves terminate where energy is lost due to the beam impinging on the transverse absorbing boundary.

variances considered, $\phi_0 \ll 1$, and hence the propagation is strictly in the nongeometrical optics limit [7,43,44].

Figure 1 shows the ensemble-averaged rms radius as a function of propagation distance. Shown along with the simulation results (solid curves) is the analytical formula in Eq. (15), evaluated for the corresponding parameters (dashed curves). For some parameters the beam diverges, while for others it self-focuses.

Note that the shapes of the solid curves in Fig. 1 are affected by the presence of finite, absorbing boundaries. Because energy at large radii tend to be disproportionately represented in any calculation of the second moment [as in Eq. (4)], energy near to and absorbed in the boundaries, even at very low intensity, results in an underestimation of the rms radius in simulation as compared with the analytical result. This is the reason for the discrepancy between the analytical and simulation curves. The analysis and simulations are in close agreement provided the simulation box is sufficiently large.

Figure 1 provides a global picture of beam evolution, displaying information about the beam radius. To gain an understanding of the intrabeam dynamics, Fig. 2 shows snapshots of the beam intensity in the transverse plane at a few representative points along the propagation path for a single random realization of the Gaussian phase statistics. Initially, the intensity shows the expected speckles from the diffraction of phase fluctuations. In order of magnitude, the number of speckles in the first image, top row, is consistent with $N_s = 2R_0^2/\rho_0^2 = 200$ [Eq. (14)].

A consequence of the analysis in Sec. II B is that the number of speckles is invariant under propagation [cf. Eq. (23)]. It is evident from Fig. 2 and the associated visualizations that this does not hold for the simulation parameters.

As the beam propagates, the individual speckles undergo diffractive spreading within the beam cross-section. Some energy propagates outward, forming a low-power halo, while the majority remains at the center, merging and smoothing in a single bell-shaped irradiance profile. Indeed, one observes significant “mode cleaning” in Fig. 2. In connection with this, Fig. 3(a) shows that very little power moves out of the original area of the beam. Similarly, Fig. 3(b)—a

“power-in-the-bucket” plot—shows that due to self-focusing, a relatively large fraction of the power is concentrated near the axis.

To gain understanding of the beam statistics and its development, consider the normalized intensity variance

$$\sigma_I^2(\mathbf{r}_\perp, z) \equiv \frac{\langle I(\mathbf{r}_\perp, z)I(\mathbf{r}_\perp, z) \rangle}{\langle I(\mathbf{r}_\perp, z) \rangle \langle I(\mathbf{r}_\perp, z) \rangle} - 1. \quad (25)$$

This function is useful since it can be displayed in $\mathbf{r}_\perp$ space for visualization at various axial locations z along the propagation path. At a given axial location, σ_I^2 is a direct measure of the level of scintillation across the beam profile. Starting out with laserlike beam statistics, as considered here, $\sigma_I^2 = 0$. If, upon propagation, the statistics tend toward that of pseudothermal light satisfying Eq. (21), then $\sigma_I^2 \rightarrow 1$. More generally, the correlation function of intensity fluctuations is defined as [26]

$$B_I(\mathbf{r}_{\perp 1}, \mathbf{r}_{\perp 2}, z) \equiv \Gamma_4(\mathbf{r}_{\perp 1}, \mathbf{r}_{\perp 2}; \mathbf{r}_{\perp 1}, \mathbf{r}_{\perp 2}, z) - \Gamma_E(\mathbf{r}_{\perp 1}, \mathbf{r}_{\perp 1}, z)\Gamma_E(\mathbf{r}_{\perp 2}, \mathbf{r}_{\perp 2}, z) \quad (26)$$

Because of its high dimensionality, this function is difficult to visualize in its entirety. The intensity variance, derived by contracting over the arguments of $B_I(\mathbf{r}_{\perp 1}, \mathbf{r}_{\perp 2}, z)$ and dividing through by Γ_E^2 , is a function of a single coordinate and is more readily displayed.

For the example in Fig. 1(a), transverse line-outs of the normalized intensity variance $\sigma_I^2(x, 0, z)$ are plotted in Figs. 4(a) and 4(b) at various locations along the propagation range. The parameters of Fig. 4(a) correspond to a beam that is undergoing expansion, and it is observed that the intensity variance decreases with propagation distance in the central region. On the other hand, Fig. 4(b) is that of a self-focusing beam, and the intensity variance in the central region grows significantly. Evidently, self-focusing has a major impact on the field statistics. With 200 realizations in the simulations, a relative standard error on the order of $1/\sqrt{200}$ in the plots of σ_I^2 in Fig. 4 is expected, which may also have been amplified by the logarithmic scale of the plots.

In Ref. [33] it is noted that a necessary condition for the validity of the factorization in Eq. (21) is the smallness of a nonlinear phase shift, reproduced here in Eq. (C4) of

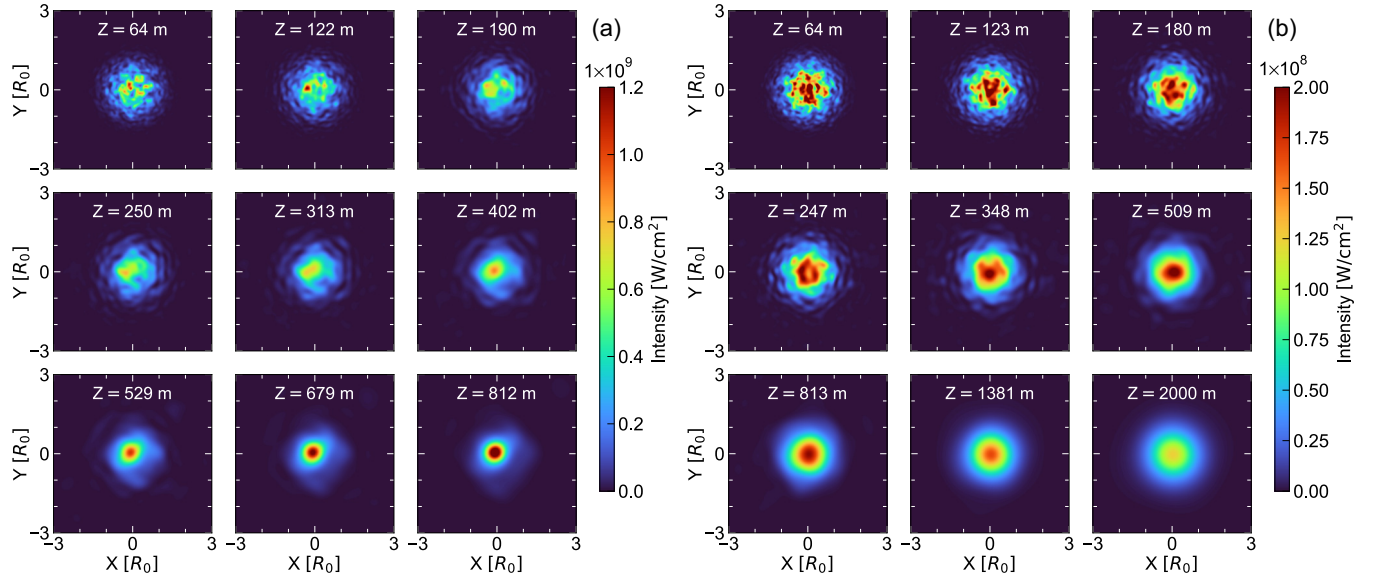


FIG. 2. The initial transverse incoherence in the laser beam fades during propagation, resulting in a single, coherent, high-intensity peak. The simulation parameters are $\rho_0 = 0.2$ cm, $\phi_0 = 0.3$, and (a) $P/P_{NL} = 1.5$, and (b) $P/P_{NL} = 0.5$. Animations of the evolving beam intensity $I(\mathbf{r}_\perp, z)$ are provided in visualizations 1–3 in the Supplemental Material [106].

Appendix C. As the speckles dwindle in number and the beam self-cleans, this phase shift becomes large and the field statistics change [35,38]. Figure 4(b) shows that there is considerable buildup of field coherence near the origin, where self-focusing concentrates the power.

IV. DISCUSSION

The objective of this work is to gain an understanding, at a fundamental level, of the phenomenon of nonlinear propagation of partially coherent beams in homogeneous media. While the model is simplified to shed light on the physics at a basic level, there are potential applications wherein this understanding may be useful. In the context of atmospheric and oceanic optics, for example, the scattering of laser light by turbulence can hinder applications, such as directed energy, power beaming, and control of lightning. Turbulence results in beam spreading, fragmentation (speckle), centroid

wander, and intensity scintillation (twinkling) [26,32,45–49]. As a result, mitigation of optical turbulence is an active area of research, studying, for example, sources of partially spatially coherent light [50–70] and structured light [71–79]. Conventional statistical optics approaches, however, such as speckle suppression for imaging may be insufficient for applications that require intense light at range because spatial incoherence drives beam divergence. The addition of nonlinear processes, mediated by an effective third-order susceptibility, can lead to dramatic optical phenomena, including self-focusing, coherent and incoherent filamentation, modulational instability, and solitons [11–15,33,35,38,39,80–95].

In this paper two approaches have been employed to analyze the propagation of partially spatially coherent laser beams in Kerr nonlinear media. The first approach uses the method of moments and has the advantage of not requiring any information about the beam statistics beyond those at the

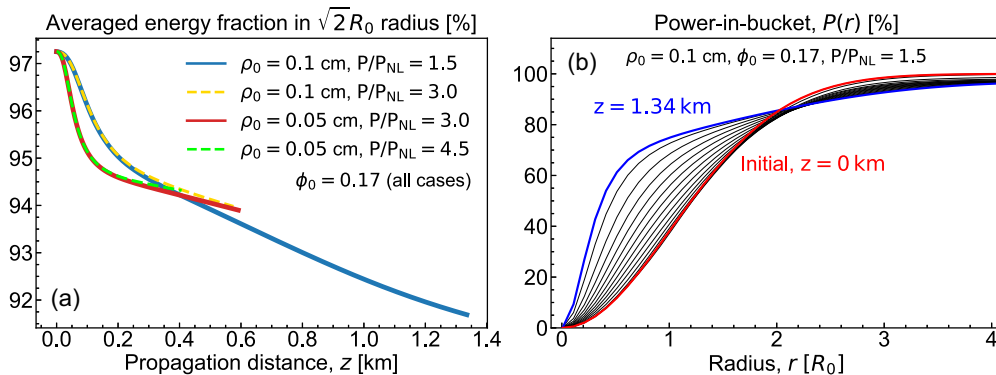


FIG. 3. As the incoherence fades, the beam energy remains localized within the initial spot, indicating that the reduction in the number of speckles is not due to diffractive losses. (a) Energy residing within the initial spot radius as a function of propagation distance for various powers and statistical parameters. (b) Fraction of power contained within a circle of increasing radius (power-in-bucket) for different propagation distances up to $z = 1.34$ km.

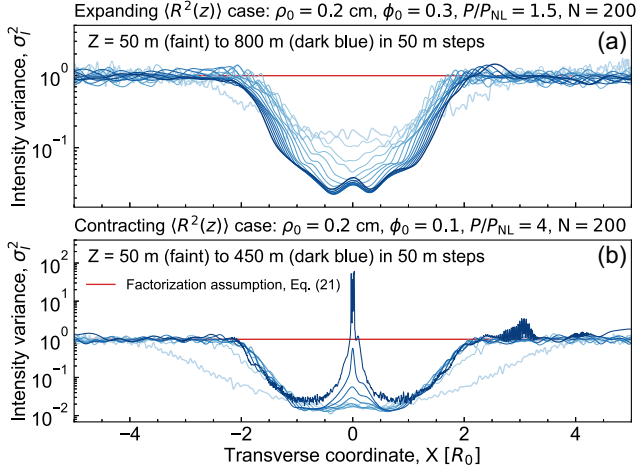


FIG. 4. The intensity variance drops with propagation distance until self-focusing leads to a localized intensity peak near $|\mathbf{r}_\perp| = 0$. This change in statistics invalidates the self-similar assumption employed for the analytical approach described in Sec. II B, illustrated by the red line. The coherence length and phase variance parameters are such that $\langle R^2(z) \rangle$ is (a) expanding or (b) contracting. Animations of the evolving beam intensity $I(\mathbf{r}_\perp, z)$ are provided in visualizations 1 and 2 of the Supplemental Material [106], the parameters of which respectively correspond to panels (a) and (b).

transmitter. The rms beam radius can be solved for analytically, but no other information about the beam, such as the distribution of speckles within it, can be ascertained after the initial plane. The second approach assumes that the statistics of the field remain invariant, that the second-order coherence function can be described by a self-similar form, and that the fourth-order coherence function is factorizable in terms of the second-order coherence function. With these, an analytical expression for the beam radius is obtained. An important, but incorrect, prediction of this approach is that the number of speckles is invariant.

The evolution of the beam radius predicted by wave-optics simulations is in excellent agreement with both approaches. However, it is observed that the speckles form and then gradually disappear as if mode cleaning were taking place. The change in the beam statistics is quantified by plotting the normalized intensity variance along the propagation range. This change is found to be very significant for a self-focusing beam as the collapse point is approached.

Mode cleaning has been reported on in the past in air and in GRIN (graded-index) multimode fibers [12,96–101]. It has been referred to as “self-cleaning,” “mode cleaning,” and as “Kerr beam self-cleaning,” the latter emphasizing the fundamental role played by the Kerr effect. In one study, the nonlinear interaction of the optical modes is modeled in analogy with two-dimensional hydrodynamic turbulence. Unlike three-dimensional (Kolmogorov) turbulence, two-dimensional turbulence is characterized by two cascades, the usual one to small scales as well as an inverse cascade to long scales [100]. The latter results in the formation of a condensate, which is devoid of speckles. A more direct connection between the change in statistics observed here and self-cleaning will be pursued in future work.

Self-focusing and transverse beam breakup instability lie at the heart of the nonlinear interaction considered here. Both are affected by refractive index fluctuations, whether they are linear or nonlinear. The former can be represented as an additive term $\delta n(\mathbf{r}_\perp, z)$ on the LHS of Eq. (2). The effects of turbulence are outside the scope of this presentation. However, propagation of an initially coherent beam through a turbulent medium naturally leads to the development of partial spatial incoherence. So, the interplay of an initially prescribed spatial incoherence with that imparted in a turbulent medium is of interest [45,102,103]. Analysis of the effects of turbulence requires knowledge of the coherence function of refractive index fluctuations

$$B_{\delta n}(\mathbf{r}_{\perp 1}, z_1; \mathbf{r}_{\perp 2}, z_2) \equiv \langle \delta n(\mathbf{r}_{\perp 1}, z_1) \delta n(\mathbf{r}_{\perp 2}, z_2) \rangle. \quad (27)$$

It is important to note that the averaging indicated by $\langle \cdot \rangle$ in this expression is associated with the turbulence characteristics of the medium in which the beam propagates. Clearly, it is physically distinct from that in Eq. (4). In the limit of vanishingly small turbulence and for a model random fluctuation of the linear refractive index, it is shown in Ref. [104] that the growth rate of the beam breakup instability is reduced.

For realistic levels of turbulence, it is possible to employ the Kolmogorov–von Kármán spectrum of eddies and the Markov approximation. *If the statistics of partial spatial coherence and of turbulence are independent and remain so during the propagation*, Eq. (18) generalizes to

$$\begin{aligned} & \left[2ik_0 \frac{\partial}{\partial z} + 2 \nabla_{\mathbf{R}} \cdot \nabla_{\boldsymbol{\rho}} + \frac{i\pi}{2k_0} H_n(\boldsymbol{\rho}) \right] \Gamma_E \\ & = -2\bar{n}_2 \langle (|E_1|^2 - |E_2|^2) E_1 E_2^* \rangle. \end{aligned} \quad (28)$$

This equation is written in terms of the independent variables introduced prior to Eq. (20). The function $H_n(\boldsymbol{\rho}) \geq 0$ accounts for the effects of turbulence, is proportional to the refractive index structure constant C_n^2 , and is an increasing function of separation $|\boldsymbol{\rho}|$ between the two points. Although the solution of Eq. (28) is not known, based on its form it is apparent that the phase mixing that comes about due to scattering off turbulent eddies leads to a decrease in the field coherence between two points $\Gamma_E \sim e^{-\pi H_n(\boldsymbol{\rho}) z / 4k_0^2}$, the more so the further apart the two points are.

It is also possible to extend the analysis in Secs. II A and II B to include the effects of eddy scattering. Equation (12) acquires an extra (positive definite) term that depends on C_n^2 and contributes to beam spreading. Similarly, Eq. (22) is modified by an extra, negative definite term, proportional to C_n^2 on the RHS of the second member of the set. Then, in place of Eq. (23), one finds

$$\frac{d}{dz} \frac{W^2(z)}{\rho_c^2(z)} > 0.$$

That is, the size of the speckles decreases relative to the beam size. This analysis assumes that the statistics of beam spatial incoherence and of turbulence are independent. Proof of this is a nontrivial task, specially in the presence of strong nonlinearity. By providing insight into the nonlinear propagation of partially spatially coherent beams in homogeneous media, this

work can be a foundation for understanding their propagation in more complicated situations.

Experimental verification of the analytical/numerical beam propagation results in this paper can be accomplished using a photorefractive medium such as SBN:75 crystal [88] and whose optical properties can be precisely controlled by the external electric field and temperature [105]. Although this crystal has a delayed Kerr response, using a rotating ground glass plate, the relative magnitude of the response time of the crystal and field coherence time can be varied, providing a large parameter space for exploration. The experimental setup and phase variance control can be conducted similarly to that in Ref. [24], with the photorefractive crystal introduced into the optical path.

ACKNOWLEDGMENTS

This work was supported by the U.S. Naval Research Laboratory and the Office of the Undersecretary of Defense (LUCI Program and A1910-097-017-004224). J.P.P. was funded by the University of Rochester, the Department of Energy National Nuclear Security Administration (DE-NA0004144), and the New York State Energy Research and Development Authority.

This report was prepared as an account of work sponsored by an agency of the U.S. Government. Neither the U.S. Government nor any agency thereof, nor any of their employees, makes any warranty, express or implied, or assumes any legal liability or responsibility for the accuracy, completeness, or usefulness of any information, apparatus, product, or process disclosed, or represents that its use would not infringe privately owned rights. Reference herein to any specific commercial product, process, or service by trade name, trademark, manufacturer, or otherwise does not necessarily constitute or imply its endorsement, recommendation, or favoring by the U.S. Government or any agency thereof. The views and opinions of authors expressed herein do not necessarily state or reflect those of the U.S. Government or any agency thereof.

DATA AVAILABILITY

The data that support the findings of this article are not publicly available upon publication because it is not technically feasible and/or the cost of preparing, depositing, and hosting the data would be prohibitive within the terms of this research project. The data are available from the authors upon reasonable request.

APPENDIX A: SELF-ACTION IN NONLINEAR DIELECTRICS

Intense lasers excite oscillation modes in dielectrics covering physical processes and bandwidths ranging from electronic Kerr, to molecular reorientation, to thermal response. The effects of these responses back on the radiation can be captured by representing the refractive index in the form $n = n_0 + n_{\text{NL}}$, where n_0 and n_{NL} are the linear and the nonlinear refractive indices, respectively. Following Debye, in media with third-order susceptibility of the Kerr type, the nonlinear refractive index n_{NL} is assumed to be given

by

$$\left(\tau_{\text{res}} \frac{\partial}{\partial \tau} + 1\right) n_{\text{NL}} = \frac{1}{8\pi} n_2 c n_0 |E|^2$$

with explicit solution

$$n_{\text{NL}} = \frac{n_2}{\tau_{\text{res}}} \frac{c n_0}{8\pi} \int_{-\infty}^{\tau} ds |E(s)|^2 e^{-(\tau-s)/\tau_{\text{res}}}, \quad (\text{A1})$$

where τ_{res} is the response time of the medium [13]. The relative magnitudes of τ_{res} , the pulse duration t_0 , and the (statistical) coherence time τ_{coh} of the field affect the propagation.

If τ_{coh} is short compared to τ_{res} , Eq. (A1) can be approximated as

$$n_{\text{NL}} \approx \frac{N_2}{\tau_{\text{res}}} \frac{c n_0}{8\pi} \int_{-\infty}^{\tau} ds \langle |E|^2 \rangle(s) e^{-(\tau-s)/\tau_{\text{res}}},$$

where $\langle |E|^2 \rangle$ is the time-smoothed irradiance, which, assuming ergodicity, is the same as the ensemble-averaged quantity [93]. Thus, the lag in the response leads to noninstantaneous effects. Most observations of incoherent solitons are in this regime [39,86–89,91,92], although there is also evidence for solitons in *nonlocal*, effectively instantaneous media [94].

In media such as air and water, the response can be nearly *instantaneous*, with $\tau_{\text{res}} < \tau_{\text{coh}}$, and Eq. (A1) simplifies to

$$n_{\text{NL}} \approx n_2 c n_0 |E|^2 / 8\pi.$$

APPENDIX B: PHYSICAL MODELS FOR PARTIAL COHERENCE

The fluctuating, complex-valued electric field amplitude on the entrance pupil may be represented in the form

$$E(\mathbf{r}_{\perp}, z=0) = E_0 e^{-r_{\perp}^2/R_0^2} e^{i\phi(\mathbf{r}_{\perp})}. \quad (\text{B1})$$

The factor $e^{i\phi(\mathbf{r}_{\perp})}$ in Eq. (B1) randomizes the phase but not the amplitude [7]. Making use of this, the MCF—defined in Eq. (17)—is given by

$$\Gamma_E(\mathbf{r}_{\perp 1}, \mathbf{r}_{\perp 2}, z=0) = |E_0|^2 e^{-(r_{\perp 1}^2 + r_{\perp 2}^2)/R_0^2} e^{-D_{\phi}(\mathbf{r}_{\perp 1}, \mathbf{r}_{\perp 2})/2},$$

where $D_{\phi}(\mathbf{r}_{\perp 1}, \mathbf{r}_{\perp 2}) = \langle [\phi(\mathbf{r}_{\perp 1}) - \phi(\mathbf{r}_{\perp 2})]^2 \rangle$ is the phase structure function [7,26]. In this paper the field statistics are assumed to be homogeneous in the wide sense, meaning that $D_{\phi}(\mathbf{r}_{\perp 1}, \mathbf{r}_{\perp 2}) = D_{\phi}(\mathbf{r}_{\perp 1} - \mathbf{r}_{\perp 2})$ [7].

Examples of the MCF at the transmitter (in the plane $z=0$) include [7]

(1) Fully spatially coherent beam

$$\Gamma_E(\mathbf{r}_{\perp 1}, \mathbf{r}_{\perp 2}) = E(\mathbf{r}_{\perp 1}) E^*(\mathbf{r}_{\perp 2}) \quad (\text{B2})$$

(2) Fully spatially incoherent source

$$\Gamma_E(\mathbf{r}_{\perp 1}, \mathbf{r}_{\perp 2}) = \frac{4\pi}{k_0^2} E(\mathbf{r}_{\perp 1}) E^*(\mathbf{r}_{\perp 2}) \delta^2(\mathbf{r}_{\perp 1} - \mathbf{r}_{\perp 2}) \quad (\text{B3})$$

(3) Schell model beam

$$\Gamma_E(\mathbf{r}_{\perp 1}, \mathbf{r}_{\perp 2}) = |E(\mathbf{r}_{\perp 1})| |E(\mathbf{r}_{\perp 2})| \mu_0(\mathbf{r}_{\perp 1} - \mathbf{r}_{\perp 2}) \quad (\text{B4})$$

where $\mu_0(\mathbf{r}_{\perp 1} - \mathbf{r}_{\perp 2})$, the complex degree of coherence of the field, is a function of the relative coordinates $\mathbf{r}_{\perp 1} - \mathbf{r}_{\perp 2}$ only.

For the Gaussian–Schell model,

$$D_\phi(\mathbf{r}_{\perp 1}, \mathbf{r}_{\perp 2}) = 2|\mathbf{r}_{\perp 1} - \mathbf{r}_{\perp 2}|^2/\rho_0^2, \quad (\text{B5})$$

where ρ_0 is the coherence length, and the complex degree of (spatial) coherence in Eq. (B4) is

$$\mu_0(\mathbf{r}_{\perp 1} - \mathbf{r}_{\perp 2}) = \exp(-|\mathbf{r}_{\perp 1} - \mathbf{r}_{\perp 2}|^2/\rho_0^2). \quad (\text{B6})$$

Ground glass is commonly used as a phase screen [39]. The phase $\phi(\mathbf{r}_\perp)$ is then determined by the optical depth, which means it is wavelength dependent, and the autocorrelation function can be assumed to be of the form

$$\langle \phi(\mathbf{r}_{\perp 1})\phi(\mathbf{r}_{\perp 2}) \rangle = \phi_0^2 e^{-|\mathbf{r}_{\perp 1} - \mathbf{r}_{\perp 2}|^2/\rho_0^2},$$

with vanishing mean $\langle \phi \rangle = 0$ and variance $\langle \phi^2 \rangle = \phi_0^2$. This is obtained for circularly symmetric Gaussian statistics [7], and the complex degree of coherence is

$$\mu_0(\mathbf{r}_{\perp 1}, \mathbf{r}_{\perp 2}) = \exp[-\phi_0^2(1 - e^{-|\mathbf{r}_{\perp 1} - \mathbf{r}_{\perp 2}|^2/\rho_0^2})]. \quad (\text{B7})$$

It should be noticed that ϕ_0 is not limited to $\leq \pi$. Indeed, $\phi_0 \gg 1$ corresponds to the geometrical optics limit where

$$\mu_0(\mathbf{r}_{\perp 1}, \mathbf{r}_{\perp 2}) \approx \exp[-\phi_0^2|\mathbf{r}_{\perp 1} - \mathbf{r}_{\perp 2}|^2/\rho_0^2]$$

and the effective coherence length is ρ_0/ϕ_0 [7,43,44]. In the limit $\phi_0 \ll 1$, on the other hand,

$$\mu_0(\mathbf{r}_{\perp 1}, \mathbf{r}_{\perp 2}) \approx 1 - \phi_0^2 + \phi_0^2 e^{-|\mathbf{r}_{\perp 1} - \mathbf{r}_{\perp 2}|^2/\rho_0^2} + \mathcal{O}(\phi_0^4),$$

and the coherence length is ρ_0 .

Consider now derivation of the expression for the ensemble-averaged Hamiltonian Eq. (11) for the Gaussian phase screen, starting from Eq. (3). The second term in the latter is readily evaluated by substituting in Eq. (B1), performing the resulting integration over a Gaussian function followed by ensemble averaging. A more subtle approach is required for the term with gradients. First, one notes that

$$\nabla_\perp E^* \cdot \nabla_\perp E = \lim_{\mathbf{r}_{\perp 1}, \mathbf{r}_{\perp 2} \rightarrow \mathbf{r}_\perp} \nabla_\perp E^*(\mathbf{r}_{\perp 1}) \cdot \nabla_\perp E(\mathbf{r}_{\perp 2}).$$

Next, ensemble averaging, noting that the operators of averaging and differentiation commute, one finds

$$\langle \nabla_\perp E^* \cdot \nabla_\perp E \rangle = \lim_{\mathbf{r}_{\perp 1}, \mathbf{r}_{\perp 2} \rightarrow \mathbf{r}_\perp} \nabla_\perp \cdot \nabla_\perp \langle E^*(\mathbf{r}_{\perp 1})E(\mathbf{r}_{\perp 2}) \rangle,$$

which, using the definition of coherence function in Eq. (17), takes the form

$$\langle \nabla_\perp E^* \cdot \nabla_\perp E \rangle = \lim_{\mathbf{r}_{\perp 1}, \mathbf{r}_{\perp 2} \rightarrow \mathbf{r}_\perp} \nabla_\perp \cdot \nabla_\perp \Gamma_E(\mathbf{r}_{\perp 1}, \mathbf{r}_{\perp 2}).$$

For circularly symmetric Gaussian statistics, the complex degree of coherence is given in Eq. (B7), whence

$$\Gamma_E(\mathbf{r}_{\perp 1}, \mathbf{r}_{\perp 2}) = |E_0|^2 \exp[-\phi_0^2(1 - e^{-|\mathbf{r}_{\perp 1} - \mathbf{r}_{\perp 2}|^2/\rho_0^2})].$$

Inserting this expression into the previous one, carrying out the differentiation followed by taking the limits, and finally performing the integration leads to the first two terms in Eq. (11). Note that the parameters ρ_0 and ϕ_0 define the phase screen discussed in Ref. [7]. They are the only parameters related to the statistics that are required for the simulations.

APPENDIX C: NONLINEAR PHASE SHIFT

The derivation of Eq. (22) relies on the validity of the factorization expressed in Eq. (21). For this, it is necessary that the nonlinear phase shift arising from the interaction between the field and the refractive index fluctuations, $\delta n_{\text{NL}} \equiv n_{\text{NL}} - \langle n_{\text{NL}} \rangle$, be small when integrated over the diffraction range $k_0 \rho_0^2$ of a speckle in the case of small phase variance $\phi_0 \leq 1$ [33]. More specifically, the phase shift

$$\delta \phi_{\text{NL}} \equiv k_0 \int_z^{z+k_0 \rho_0^2} dz' \delta n_{\text{NL}}(z') \quad (\text{C1})$$

must satisfy

$$\sqrt{\langle (\delta \phi_{\text{NL}})^2 \rangle} \ll \pi. \quad (\text{C2})$$

The refractive index fluctuation in Eq. (C2) is obtained following the discussion in Sec. II. Making use of Eq. (A1), one finds

$$\begin{aligned} \langle \delta n_{\text{NL}}(1) \delta n_{\text{NL}}(2) \rangle &= \left(\frac{n_2}{\tau_{\text{res}}} \frac{cn_0}{8\pi} \right)^2 \exp[-(\tau_1 + \tau_2)/\tau_{\text{res}}] \int_{-\infty}^{\tau_1} ds_1 \int_{-\infty}^{\tau_2} ds_2 \{ [\langle |E_1(s_1)|^2 \rangle - |E_1(s_1)|^2] [\langle |E_2(s_2)|^2 \rangle \\ &\quad - |E_2(s_2)|^2] \} \exp[(s_1 + s_2)/\tau_{\text{res}}] \end{aligned}$$

where, for brevity, $\delta n_{\text{NL}}(i) = \delta n_{\text{NL}}(\mathbf{r}_{\perp i}, z, \tau_i)$. Employing the notation $B_{ij}(\tau_1, \tau_2) \equiv \Gamma_E(\mathbf{r}_{\perp i}, \tau_1; \mathbf{r}_{\perp j}, \tau_2; z)$ in Ref. [33], Eq. (21) yields

$$\langle \delta n_{\text{NL}}(1) \delta n_{\text{NL}}(2) \rangle = \left(\frac{n_2}{\tau_{\text{res}}} \frac{cn_0}{8\pi} \right)^2 e^{-(\tau_1 + \tau_2)/\tau_{\text{res}}} \int_{-\infty}^{\tau_1} ds_1 \int_{-\infty}^{\tau_2} ds_2 |B_{12}(s_1, s_2)|^2 e^{(s_1 + s_2)/\tau_{\text{res}}}. \quad (\text{C3})$$

Assuming that the characteristic times for amplitude and phase fluctuations are comparable and equal to τ_{coh} , and employing the form

$$B_{12}(\tau_1, \tau_2) = \frac{16}{cn_0 R_0^2} \hat{B}_{12} e^{-|\tau_1 - \tau_2|/\tau_{\text{coh}}} \{ H[(\tau_1 + \tau_2)/2] - H[(\tau_1 + \tau_2)/2 - t_0] \},$$

results in

$$\langle \delta \phi_{\text{NL}}(1) \delta \phi_{\text{NL}}(2) \rangle = \left(\frac{4n_0}{\tau_{\text{res}}} \frac{P}{P_{\text{NL}}} \frac{\rho_0^2}{R_0^2} \right)^2 |\hat{B}_{12}|^2 e^{-(\tau_1 + \tau_2)/\tau_{\text{res}}} e^{-\frac{\tau_1 + \tau_2}{\tau_{\text{res}}}} \int_{-\infty}^{\tau_1} ds_1 \int_{-\infty}^{\tau_2} ds_2 e^{\frac{s_1 + s_2}{\tau_{\text{res}}}} e^{-\frac{2|s_1 - s_2|}{\tau_{\text{coh}}}}$$

and effecting the integrations

$$\sqrt{\langle(\delta\phi_{\text{NL}})^2\rangle} = 4n_0 \frac{P/P_{\text{NL}}}{R_0^2/\rho_0^2} |\hat{B}_{12}| \nu(t_0, \tau_{\text{coh}}, \tau_{\text{res}}), \quad (\text{C4})$$

where

$$\nu(t_0, \tau_{\text{coh}}, \tau_{\text{res}}) \equiv e^{-t_0/\tau_{\text{res}}} \sqrt{\frac{\tau_{\text{coh}} + 2\tau_{\text{res}} + (\tau_{\text{coh}} - 2\tau_{\text{res}})e^{2t_0/\tau_{\text{res}}} - 2\tau_{\text{coh}}e^{(-\frac{2}{\tau_{\text{coh}}} + \frac{1}{\tau_{\text{res}}})t_0}}{\tau_{\text{coh}}^2 - 4\tau_{\text{res}}^2}} \tau_{\text{coh}}. \quad (\text{C5})$$

The function $\nu(t_0, \tau_{\text{coh}}, \tau_{\text{res}})$ has the limiting forms

$$\nu(t_0, \tau_{\text{coh}}, \tau_{\text{res}}) = \begin{cases} \sqrt{\frac{\tau_{\text{coh}}}{2\tau_{\text{res}}}} (1 - e^{-\frac{2t_0}{\tau_{\text{res}}}}), & \tau_{\text{coh}} \ll \tau_{\text{res}} < t_0 \\ 1, & \tau_{\text{res}} \ll \tau_{\text{coh}} < t_0 \end{cases} \quad (\text{C6})$$

and is plotted in Fig. 5 for two cases.

Notice that the rms phase shift in Eq. (C4) is reduced in inverse proportion to the number of speckles. Moreover, as the laser beam self-cleans, the denominator in Eq. (C4) is reduced and hence the phase shift increases and becomes large. Finally, in a noninstantaneous medium such as that considered in [33], the phase shift is reduced in proportion to $\sqrt{\tau_{\text{coh}}/\tau_{\text{res}}}$ as indicated by the top line in Eq. (C6). For the instantaneous media under consideration in the present paper, however, $\nu(t_0, \tau_{\text{coh}}, \tau_{\text{res}}) = 1$.

APPENDIX D: PYTHON CODE FOR ATMOSPHERIC PROPAGATION (PyCAP)

To assess the analysis and its limitations, simulations of partially coherent beams propagating through a bulk Kerr medium were performed using the Naval Research Laboratory's PyCAP (Python Code for Atmospheric Propagation). For the simulations herein, PyCAP solves Eq. (2) for the laser field by a symmetrized Fourier split-step with an adaptive error-limiting propagation step.

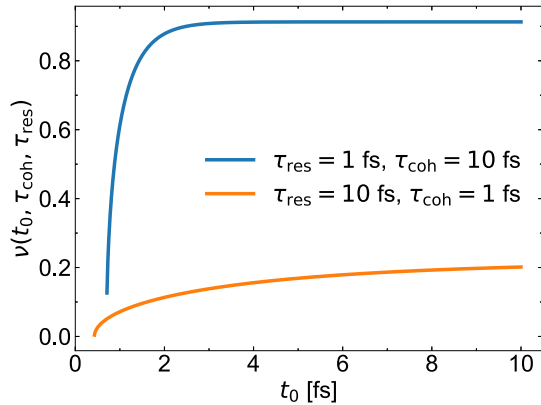


FIG. 5. The function $\nu(t_0, \tau_{\text{coh}}, \tau_{\text{res}})$, defined in Eq. (C5), vs the pulse duration t_0 . The blue curve is for a short response time, approximating an instantaneous medium, whereas the orange curve is for a long response time, representing an inertial medium.

Phase screens.—The initial laser field has a Gaussian transverse profile, with a random phase applied at $z = 0$:

$$E(\mathbf{r}_\perp, z = 0) = E_0 e^{-r_\perp^2/R_0^2} e^{i\phi_n(x,y)}.$$

The phase screens $\phi_n(x, y)$ (the subscript n is to indicate that this is the n th random realization) are generated using a fast Fourier transform (FFT) approach based on the Wiener–Khinchin theorem, discretized onto an $N \times N$ grid of spacing $\Delta x = \Delta y$ with N even. The theorem equates the power spectral density of a stationary process with the spectral decomposition of its autocorrelation. Here, the two-dimensional analog is employed, a method routinely used to create phase screens for modeling laser propagation in turbulence.

First a square matrix $C_{i,j}$ is created from the continuous, circularly-symmetric desired coherence function $B(r)$ of the phase

$$C_{i,j} = \phi_0^2 N^2 e^{-(x_i^2 + y_j^2)/\rho_0^2}.$$

Here, x_i and y_j are the discretized transverse position from $-(N/2)\Delta x$ to $(N/2 - 1)\Delta x$, ϕ_0^2 is the phase-screen variance, and ρ_0 is the phase spatial coherence length. (The function $B(r)$, $r \equiv \sqrt{x^2 + y^2}$, is the same as the function $2\sigma_\phi^2 \gamma_\phi(r)$ in Sec. 8.2.2 of Ref. [7].) The factor of N^2 is due to the discretization and is necessary for normalization. The power spectral density of the screen is then calculated using a two-dimensional FFT (x, y plane FFT, denoted by FFT2) of the coherence function, $S_{\ell,m} = \text{FFT2}(C_{i,j})$. For a particular random screen, the phase is calculated via an inverse two-dimensional FFT, $\phi_{i,j,n} = \text{Re}\{\text{IFFT2}[\sqrt{S_{\ell,m}}(a_{\ell,m;n} + ib_{\ell,m;n})]\}$, where a and b are Gaussian random variables with zero mean and unit variance. Taking the imaginary part rather than the real part of the result of the inverse FFT2 yields another, statistically independent phase screen. So, each forward-inverse FFT pair produces two independent screens with the desired statistics.

Figure 6 provides convergence test results for the data in Fig. 1. In the numerical integration of the nonlinear Schrödinger equation, as the number of transverse grid points $N_{x,y}$ and spatial extent $L_{x,y}$ are increased in proportion (preserving the resolution), the agreement between the numerical and analytical rms radius $\langle R^2(z) \rangle^{1/2}$ is preserved for a longer propagation distance, as shown in Fig. 6(a). The energy versus propagation distance of each parameter case is provided in Fig. 6(b), from which it is apparent that the numerical rms radius is highly sensitive to the absorption of field components at large radii.

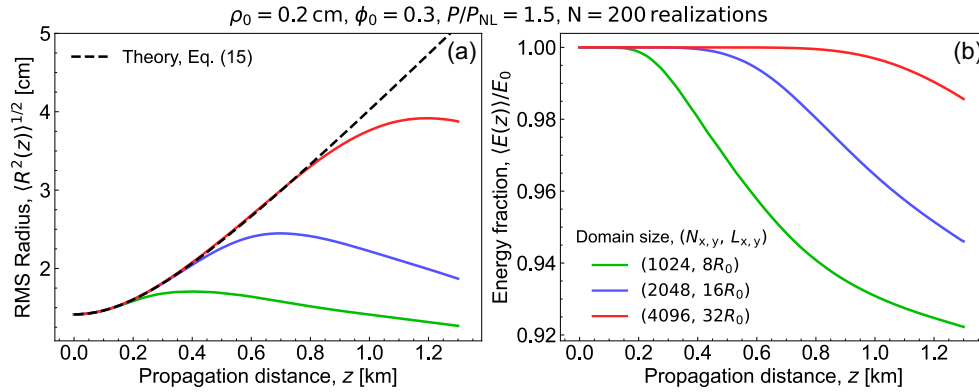


FIG. 6. Convergence test. Demonstrating that as the number of transverse points and spatial extent $(N_{x,y}, L_{x,y})$ increases, the numerical nonlinear Schrödinger equation result (solid) converges to the analytical result (dashed) of Eq. (15). The rms radius [panel (a)] is sensitive in the kilometer range to even a sub-2% energy loss [panel (b), red curve] due to absorbing boundary conditions.

- [1] R. G. Glauber, The quantum theory of optical coherence, *Phys. Rev.* **130**, 2529 (1963).
- [2] R. G. Glauber, Coherent and incoherent states of the radiation field, *Phys. Rev.* **131**, 2766 (1963).
- [3] V. P. Aksenov, V. A. Banakh, and V. L. Mironov, Fluctuations of laser radiation intensity reflected from a turbulent atmosphere, *Sov. J. Quantum Electron.* **6**, 1233 (1976).
- [4] F. Gori, G. Guattari, and C. Padovani, Modal expansion for j_0 -correlated Schell-model sources, *Opt. Commun.* **64**, 311 (1987).
- [5] F. Gori, M. Santarsiero, and A. Sona, The change of width for a partially coherent beam on paraxial propagation, *Opt. Commun.* **82**, 197 (1991).
- [6] L. Mandel and E. Wolf, *Optical Coherence and Quantum Optics* (Cambridge University Press, Cambridge, UK, 1995).
- [7] J. W. Goodman, *Statistical Optics*, 2nd ed. (Wiley, New York, 2015).
- [8] F. Gori and M. Santarsiero, Devising genuine spatial correlation functions, *Opt. Lett.* **32**, 3531 (2007).
- [9] R. E. Meyers, K. S. Deacon, and Y. Shih, Turbulence-free ghost imaging, *Appl. Phys. Lett.* **98**, 111115 (2011).
- [10] J. W. Goodman, *Speckle Phenomena in Optics: Theory and Applications*, 2nd ed. (SPIE Press, Bellingham, WA, 2020).
- [11] Y. R. Shen, *The Principles of Nonlinear Optics* (Wiley, Hoboken, NJ, 2002).
- [12] G. P. Agrawal, *Nonlinear Fiber Optics*, 6th ed. (Academic Press, Cambridge, MA, 2019).
- [13] R. W. Boyd, *Nonlinear Optics*, 4th ed. (Academic Press, Cambridge, MA, 2020).
- [14] H. Elder, A. Khizhnyak, V. Markov, and P. Sprangle, Nonlinear dynamics of the laser spot size and pulse duration in turbulence, in *Laser Communication and Propagation through the Atmosphere and Oceans X*, Vol. 11834, International Society for Optics and Photonics (SPIE, San Diego, CA, 2021), p. 118340Q.
- [15] J. Isaacs, B. Hafizi, L. A. Johnson, E. W. Rosenthal, L. Mrini, and J. Peñano, Modeling the propagation of a high-average-power train of ultrashort laser pulses, *Opt. Express* **30**, 22306 (2022).
- [16] B. Hafizi, J. P. Palastro, J. R. Peñano, D. F. Gordon, T. G. Jones, M. H. Helle, and D. Kaganovich, Stimulated Raman scattering and nonlinear focusing of high-power laser beams propagating in water, *Opt. Lett.* **40**, 1556 (2015).
- [17] B. Hafizi, J. Isaacs, J. P. Palastro, and J. R. Peñano, Nonlinear propagation of chirped laser pulses through a dispersive and turbulent atmosphere, *J. Opt. Soc. Am. B* **41**, 1457 (2024).
- [18] T. Brabec and F. Krausz, Nonlinear optical pulse propagation in the single-cycle regime, *Phys. Rev. Lett.* **78**, 3282 (1997).
- [19] M. Kolesik, J. V. Moloney, and M. Mlejnek, Unidirectional optical pulse propagation equation, *Phys. Rev. Lett.* **89**, 283902 (2002).
- [20] M. Kolesik and J. V. Moloney, Nonlinear optical pulse propagation simulation: From Maxwell's to unidirectional equations, *Phys. Rev. E* **70**, 036604 (2004).
- [21] L. Bergé, Wave collapse in physics: Principles and applications to light and plasma waves, *Phys. Rep.* **303**, 259 (1998).
- [22] L. Bergé and A. Couairon, Nonlinear propagation of self-guided ultra-short pulses in ionized gases, *Phys. Plasmas* **7**, 210 (2000).
- [23] V. V. Vorob'ev, Self-focusing of light beams without axial symmetry, *Radiophys. Quantum Electron.* **13**, 1472 (1970).
- [24] B. Schreyer, D. Younis, D. Kaganovich, L. A. Johnson, T. M. Antonsen, and B. Hafizi, Realization of nontrivial partial spatial coherence by a deformable mirror, *Appl. Opt.* **64**, 5014 (2025).
- [25] P. W. Milonni and J. H. Eberly, *Laser Physics* (Wiley, Hoboken, NJ, 2010).
- [26] V. I. Tatarskiĭ, The effects of the turbulent atmosphere on wave propagation, Tech. Rep. TT-68-50464, U.S. Dept. of Commerce, Springfield, Virginia (1971).
- [27] G. P. Berman, A. A. Chumak, and V. N. Gorshkov, Beam wandering in the atmosphere: The effect of partial coherence, *Phys. Rev. E* **76**, 056606 (2007).

- [28] G. P. Berman, V. N. Gorshkov, and S. V. Torous, Scintillation reduction for laser beams propagating through turbulent atmosphere, *J. Phys. B* **44**, 055402 (2011).
- [29] T. B. Pittman, Y. H. Shih, D. V. Strekalov, and A. V. Sergienko, Optical imaging by means of two-photon quantum entanglement, *Phys. Rev. A* **52**, R3429(R) (1995).
- [30] B. I. Erkmen and J. H. Shapiro, Unified theory of ghost imaging with Gaussian-state light, *Phys. Rev. A* **77**, 043809 (2008).
- [31] S. N. Vlasov, V. A. Petrishchev, and V. I. Talanov, Averaged description of wave beams in linear and nonlinear media (the method of moments), *Radiophys. Quantum Electron.* **14**, 1062 (1971).
- [32] R. L. Fante, Electromagnetic beam propagation in turbulent media, *Proc. IEEE* **63**, 1669 (1975).
- [33] G. A. Pasmanik, Self-interaction of incoherent light beams, *Sov. Phys. JETP* **39**, 234 (1974).
- [34] In passing, it should be noted that the analysis herein is purely classical where E and E^* commute.
- [35] V. A. Aleshkevich, G. D. Kozhoridze, and A. N. Matveev, Self-action of partly coherent laser radiation, *Sov. Phys. Usp.* **161**, 81 (1991).
- [36] I. Sreenivasiah, A. Ishimaru, and S. T. Hong, Two-frequency mutual coherence function and pulse propagation in a random medium: An analytic solution to the plane wave case, *Radio Sci.* **11**, 775 (1976).
- [37] C. S. Liu and V. K. Tripathi, Laser guiding in an axially nonuniform plasma channel, *Phys. Plasmas* **1**, 3100 (1994).
- [38] J. Garnier, L. Videau, C. Gouédard, and A. Migus, Propagation and amplification of incoherent pulses in dispersive and nonlinear media, *J. Opt. Soc. Am. B* **15**, 2773 (1998).
- [39] C. Sun, D. V. Dylov, and J. W. Fleischer, Nonlinear focusing and defocusing of partially coherent spatial beams, *Opt. Lett.* **34**, 3003 (2009).
- [40] V. P. Nayyar, Propagation of partially coherent Gaussian Schell-model sources in nonlinear media, *J. Opt. Soc. Am. B* **14**, 2248 (1997).
- [41] E. Esarey, P. Sprangle, J. Krall, and A. Ting, Self-focusing and guiding of short laser pulses in ionizing gases and plasmas, *IEEE J. Quantum Electron.* **33**, 1879 (1997).
- [42] P. Sprangle, J. R. Peñano, and B. Hafizi, Propagation of intense short laser pulses in the atmosphere, *Phys. Rev. E* **66**, 046418 (2002).
- [43] E. E. Salpeter, Interplanetary scintillations, I. Theory, *Astrophys. J.* **147**, 433 (1967).
- [44] R. Buckley, Diffraction by a random phase-changing screen: A numerical experiment, *J. Atmos. Terr. Phys.* **37**, 1431 (1975).
- [45] H. T. Yura, Physical model for strong optical-amplitude fluctuations in a turbulent medium, *J. Opt. Soc. Am.* **64**, 59 (1974).
- [46] A. Ishimaru, *Wave Propagation and Scattering in Random Media, Vol. 1: Single Scattering and Transport Theory, Vol. 2: Multiple Scattering, Turbulence, Rough Surfaces and Remote Sensing* (Academic Press, New York, 1978).
- [47] J. W. Hardy, *Adaptive Optics for Astronomical Telescopes* (Oxford University Press, New York, 1998).
- [48] V. I. Klyatskin, Electromagnetic wave propagation in a randomly inhomogeneous medium as a problem in mathematical statistical physics, *Phys.-Usp.* **47**, 169 (2004).
- [49] L. C. Andrews and R. L. Phillips, *Laser Beam Propagation Through Random Media* (SPIE Press, Bellingham, WA, 2005).
- [50] S. C. H. Wang, M. A. Plonus, and C. F. Ouyang, Irradiance scintillations of a partially coherent source in extremely strong turbulence, *Appl. Opt.* **18**, 1133 (1979).
- [51] E. Wolf, New theory of partial coherence in the space-frequency domain, Part I: Spectra and cross spectra of steady-state sources, *J. Opt. Soc. Am.* **72**, 343 (1982).
- [52] G. Gbur and E. Wolf, Spreading of partially coherent beams in random media, *J. Opt. Soc. Am. A* **19**, 1592 (2002).
- [53] J. C. Ricklin and F. M. Davidson, Atmospheric turbulence effects on a partially coherent Gaussian beam: Implications for free-space laser communication, *J. Opt. Soc. Am. A* **19**, 1794 (2002).
- [54] E. Wolf, Unified theory of coherence and polarization of random electromagnetic beams, *Phys. Lett. A* **312**, 263 (2003).
- [55] T. Shirai, A. Dogariu, and E. Wolf, Mode analysis of spreading of partially coherent beams propagating through atmospheric turbulence, *J. Opt. Soc. Am. A* **20**, 1094 (2003).
- [56] A. Dogariu and S. Amarande, Propagation of partially coherent beams: Turbulence-induced degradation, *Opt. Lett.* **28**, 10 (2003).
- [57] O. Korotkova, L. C. Andrews, and R. L. Phillips, Model for a partially coherent Gaussian beam in atmospheric turbulence with application in Lasercom, *Opt. Eng.* **43**, 330 (2004).
- [58] Y. Cai and S.-Y. Zhu, Ghost interference with partially coherent radiation, *Opt. Lett.* **29**, 2716 (2004).
- [59] T. Shirai, O. Korotkova, and E. Wolf, A method of generating electromagnetic Gaussian Schell-model beams, *J. Opt. A: Pure Appl. Opt.* **7**, 232 (2005).
- [60] G. Gbur and O. Korotkova, Angular spectrum representation for the propagation of arbitrary coherent and partially coherent beams through atmospheric turbulence, *J. Opt. Soc. Am. A* **24**, 745 (2007).
- [61] O. Korotkova and G. Gbur, Angular spectrum representation for propagation of random electromagnetic beams in a turbulent atmosphere, *J. Opt. Soc. Am. A* **24**, 2728 (2007).
- [62] Y. Yuan, Y. Cai, J. Qu, H. T. Eyyuboğlu, Y. Baykal, and O. Korotkova, M^2 -factor of coherent and partially coherent dark hollow beams propagating in turbulent atmosphere, *Opt. Express* **17**, 17344 (2009).
- [63] Y. Gu, O. Korotkova, and G. Gbur, Scintillation of nonuniformly polarized beams in atmospheric turbulence, *Opt. Lett.* **34**, 2261 (2009).
- [64] Y. Gu and G. Gbur, Reduction of turbulence-induced scintillation by nonuniformly polarized beam arrays, *Opt. Lett.* **37**, 1553 (2012).
- [65] G. Gbur, Partially coherent beam propagation in atmospheric turbulence [Invited], *J. Opt. Soc. Am. A* **31**, 2038 (2014).
- [66] Y. Cai, Y. Chen, and F. Wang, Generation and propagation of partially coherent beams with nonconventional correlation functions: A review [Invited], *J. Opt. Soc. Am. A* **31**, 2083 (2014).
- [67] M. W. Hyde, S. Basu, D. G. Voelz, and X. Xiao, Experimentally generating any desired partially coherent Schell-model source using phase-only control, *J. Appl. Phys.* **118**, 093102 (2015).

- [68] M. W. Hyde, S. Bose-Pillai, D. G. Voelz, and X. Xiao, Generation of vector partially coherent optical sources using phase-only spatial light modulators, *Phys. Rev. Appl.* **6**, 064030 (2016).
- [69] M. W. Hyde, Stochastic complex transmittance screens for synthesizing general partially coherent sources, *J. Opt. Soc. Am. A* **37**, 257 (2020).
- [70] M. W. Hyde, The behavior of partially coherent twisted space-time beams in atmospheric turbulence, *Front. Phys.* **10**, 1055401 (2023).
- [71] W. Nelson, J. P. Palastro, C. C. Davis, and P. Sprangle, Propagation of Bessel and Airy beams through atmospheric turbulence, *J. Opt. Soc. Am. A* **31**, 603 (2014).
- [72] M. A. Cox, C. Rosales-Guzmán, M. P. J. Lavery, D. J. Versfeld, and A. Forbes, On the resilience of scalar and vector vortex modes in turbulence, *Opt. Express* **24**, 18105 (2016).
- [73] M. Charnotskii, Propagation of polarized waves in inhomogeneous media, *J. Opt. Soc. Am. A* **33**, 1385 (2016).
- [74] A. Forbes, M. de Oliveira, and M. R. Dennis, Structured light, *Nat. Photonics* **15**, 253 (2021).
- [75] V. P. Aksenov, V. V. Dudurov, V. V. Kolosov, and G. A. Filimonov, Fluctuations of the orbital angular momentum of a laser beam registered by a finite-size receiver aperture after propagation through a turbulent atmosphere, *OSA Continuum* **4**, 1945 (2021).
- [76] I. Nape, K. Singh, A. Klug, W. Buono, C. Rosales-Guzmán, A. McWilliam, S. Franke-Arnold, A. Kritzinger, P. Forbes, A. Dudley, and A. Forbes, Revealing the invariance of vectorial structured light in complex media, *Nat. Photonics* **16**, 538 (2022).
- [77] Y. Shen and C. Rosales-Guzmán, Nonseparable states of light: From quantum to classical, *Laser Photonics Rev.* **16**, 2100533 (2022).
- [78] K. Singh, I. Nape, W. T. Buono, A. Dudley, and A. Forbes, A robust basis for multi-bit optical communication with vectorial light, *Laser Photonics Rev.* **17**, 2370027 (2023).
- [79] A. Klug, C. Peters, and A. Forbes, Robust structured light in atmospheric turbulence, *Adv. Photonics* **5**, 016006 (2023).
- [80] S. A. Akhmanov, A. P. Sukhorukov, and R. V. Khokhlov, Self-focusing and self-trapping of intense light beams in a nonlinear medium, *Sov. Phys. JETP* **23**, 1025 (1966).
- [81] S. A. Akhmanov, A. P. Sukhorukov, and R. V. Khokhlov, Self-focusing and diffraction of light in a nonlinear medium, *Sov. Phys. Usp.* **10**, 609 (1968).
- [82] V. A. Aleshkevich, S. S. Lebedev, and A. N. Matveev, Self-interaction of a noncoherent light beam, *Sov. J. Quantum Electron.* **11**, 647 (1981).
- [83] V. P. Kandidov and S. A. Shlenov, Statistics of semicoherent radiation in medium with cubic nonlinearity, *Radiophys. Quantum Electron.* **27**, 817 (1984).
- [84] V. P. Kandidov, Monte Carlo method in nonlinear statistical optics, *Phys.-Usp.* **39**, 1243 (1996).
- [85] V. P. Kandidov and S. A. Shlenov, Thermal self-action of laser beams and filamentation of pulses in turbulent atmosphere, *Atmos. Ocean. Opt.* **25**, 192 (2012).
- [86] G. C. Valley, M. Segev, B. Crosignani, A. Yariv, M. M. Fejer, and M. C. Bashaw, Dark and bright photovoltaic spatial solitons, *Phys. Rev. A* **50**, R4457(R) (1994).
- [87] M. Mitchell, Z. Chen, M.-f. Shih, and M. Segev, Self-trapping of partially spatially incoherent light, *Phys. Rev. Lett.* **77**, 490 (1996).
- [88] D. N. Christodoulides, T. H. Coskun, M. Mitchell, and M. Segev, Theory of incoherent self-focusing in biased photorefractive media, *Phys. Rev. Lett.* **78**, 646 (1997).
- [89] V. V. Shkunov and D. Z. Anderson, Radiation transfer model of self-trapping spatially incoherent radiation by nonlinear media, *Phys. Rev. Lett.* **81**, 2683 (1998).
- [90] O. Bang, D. Edmundson, and W. Królikowski, Collapse of incoherent light beams in inertial bulk Kerr media, *Phys. Rev. Lett.* **83**, 5479 (1999).
- [91] M. Soljačić, M. Segev, T. Coskun, D. N. Christodoulides, and A. Vishwanath, Modulation instability of incoherent beams in noninstantaneous nonlinear media, *Phys. Rev. Lett.* **84**, 467 (2000).
- [92] D. Kip, M. Soljačić, M. Segev, S. M. Sears, and D. N. Christodoulides, (1+1)-dimensional modulation instability of spatially incoherent light, *J. Opt. Soc. Am. B* **19**, 502 (2002).
- [93] J. Garnier, J.-P. Ayanides, and O. Morice, Propagation of partially coherent light with the Maxwell–Debye equation, *J. Opt. Soc. Am. B* **20**, 1409 (2003).
- [94] C. Rotschild, T. Schwartz, O. Cohen, and M. Segev, Incoherent spatial solitons in effectively instantaneous nonlinear media, *Nat. Photonics* **2**, 371 (2008).
- [95] H. Lajunen, V. Torres-Company, J. Lancis, E. Silvestre, and P. Andrés, Pulse-by-pulse method to characterize partially coherent pulse propagation in instantaneous nonlinear media, *Opt. Express* **18**, 14979 (2010).
- [96] B. Prade, M. Franco, A. Mysyrowicz, A. Couairon, H. Buersing, B. Eberle, M. Krenz, D. Seiffer, and O. Vasseur, Spatial mode cleaning by femtosecond filamentation in air, *Opt. Lett.* **31**, 2601 (2006).
- [97] P. Aschieri, J. Garnier, C. Michel, V. Doya, and A. Picozzi, Condensation and thermalization of classical optical waves in a waveguide, *Phys. Rev. A* **83**, 033838 (2011).
- [98] C. Milián, V. Jukna, A. Couairon, A. Houard, B. Forestier, J. Carbonnel, Y. Liu, B. Prade, and A. Mysyrowicz, Laser beam self-symmetrization in air in the multifilamentation regime, *J. Phys. B* **48**, 094013 (2015).
- [99] K. Krupa, A. Tonello, B. M. Shalaby, M. Fabert, A. Barthélémy, G. Millot, S. Wabnitz, and V. Couderc, Spatial beam self-cleaning in multimode fibres, *Nat. Photonics* **11**, 237 (2017).
- [100] E. V. Podivilov, D. S. Kharenko, V. A. Gonta, K. Krupa, O. S. Sidelnikov, S. Turitsyn, M. P. Fedoruk, S. A. Babin, and S. Wabnitz, Hydrodynamic 2D turbulence and spatial beam condensation in multimode optical fibers, *Phys. Rev. Lett.* **122**, 103902 (2019).
- [101] Z. Mohammadzahery, M. Jandaghi, E. Aghayari, and H. Nabavi, Observation of spatial nonlinear self-cleaning in a few-mode step-index fiber for special distributions of initial excited modes, *Sci. Rep.* **11**, 24350 (2021).
- [102] S. F. Clifford, G. R. Ochs, and R. S. Lawrence, Saturation of optical scintillation by strong turbulence, *J. Opt. Soc. Am.* **64**, 148 (1974).
- [103] W. Jian, Propagation of a Gaussian-Schell beam through turbulent media, *J. Mod. Opt.* **37**, 671 (1990).

- [104] A. Bourdier, J.-C. Diels, H. Ghalila, and O. Delage, Effect of atmospheric turbulence on modulational instability in laser-pulse propagation, *Physica D* **457**, 133974 (2024).
- [105] R. R. Neurgaonkar, W. K. Cory, J. R. Oliver, M. D. Ewbank, and W. F. Hall, Development and modification of photorefractive properties in the tungsten Bronze family crystals, *Opt. Eng.* **26**, 265392 (1987).
- [106] See Supplemental Material at <http://link.aps.org/supplemental/10.1103/bnyf-rrmf> for animations of the evolving beam above and below the critical power for self-focusing.