

Role of nonclassicality in mediated spatial quantum correlations

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The study of nonclassicality is essential to understand the quantum-to-classical transition in physical systems. Recently, a witness of nonclassicality has been proposed, linking the ability of a system (the mediator) to create quantum correlations between two quantum probes with its nonclassicality, intended as the existence of at least two noncommuting variables. Here we propose an inequality that quantitatively links the increase in quantum correlations between the probes to a function of the noncommutativity of the mediator's observables. We test the inequality for various degrees of nonclassicality of the mediator, from fully quantum to fully classical. This quantum-to-classical transition is simulated via a phase-flip channel applied to the mediator, inducing an effective reduction of the noncommutativity of its variables. Our results provide a general framework for witnessing nonclassicality, assessing the nonclassicality of a system via its intrinsic properties, independently of the specific chosen interaction dynamics.

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I. INTRODUCTION

Understanding what makes quantum systems different from classical systems has been a central issue in quantum information and quantum foundations. One way to characterize the quantumness of a physical system is via its nonclassicality, here defined as the fact that it has at least two variables that do not commute, i.e., that cannot be measured simultaneously to arbitrarily high accuracy by the same device [1]. A system would thus be defined as classical if all its degrees of freedom can be represented by an Abelian algebra, where all the operators commute with each other. This definition is in line with several models embedding classicality in a quantum framework, e.g., Koopmanian models [2], or imposing supeselection rules that effectively forbid noncommuting observables, as in [3].

How could one determine whether a system is nonclassical? The most obvious solution would be to directly measure the investigated system and check if two or more observables obey the uncertainty principle, one of the pillars of quantum theory [4]. However, it is not always possible to directly measure such observables, especially when facing complex systems with an untractable dynamics, like biological ones or systems for which quantum theory may break down, as in the case of gravity. As a result, the question of if, and where,

the cut between classical and quantum world should be traced is crucial for the universality of quantum theory. Assuming such a cut exists, the system one should face is known as a hybrid system, where a quantum sector $\mathcal{Q}$ made of fully fledged quantum systems (qubits, for example) interacts with a system M whose quantum nature is unknown, potentially classical. In this setup, the quantum sector could be exploited to design indirect tests of the unknown sector's nonclassicality, capable of inferring the necessary quantum description of M by ruling out the hypothesis that could be described by a fully classical model. Furthermore, these tests should work as sufficient conditions for M 's nonclassicality: By observing the effects of M on $\mathcal{Q}$, one could experimentally test if M must be described by the quantum formalism. Recently, new experiments have been proposed to indirectly witness the nonclassicality of a mediator of interactions between two quantum probes [5]. These experiments suggest that the creation of entanglement between two quantum probes, which interact locally [6] only through a mediator M , can serve as a witness of its nonclassical nature (see Fig. 1). An immediate application of this argument is a witness to the nonclassicality of the gravitational field, as described in [7,8]; however, the witness is more generally applicable [1].

Interestingly, a general method for quantitatively assessing the witness, independent of the chosen model for the mediator and its interactions with the probes, is currently lacking. In fact, the witness of nonclassicality, as originally proposed, does not explicitly link the amount of entanglement generated by the probes to the nonclassicality of the mediator independently of the choice of the interaction Hamiltonians. This prompts the need for a broader quantitative framework capable of encompassing a wide range of possible models for the systems and the interactions involved. Addressing this need is one of the primary motivations for this work.

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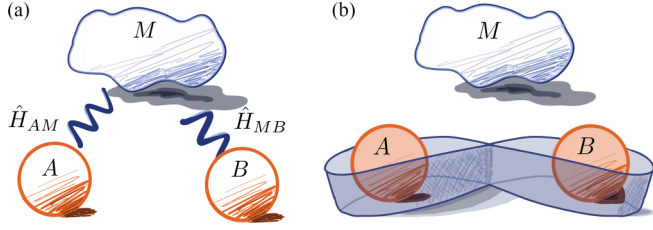


FIG. 1. (a) Two probes A and B in a separable initial state interacting via the mediator M through the Hamiltonians $\hat{H}_{AM}$ and $\hat{H}_{MB}$. (b) Two probes A and B in a final entangled state after the mediated interaction occurs. Detecting the entanglement between A and B is then sufficient to infer the nonclassicality of M [7,8].

A significant step towards this direction has been taken. In Refs. [9,10] the authors established a causal relation between the increase of mediated quantum correlation shared by the probes and the nonclassicality of interactions, expressed as the commutator of the interaction Hamiltonians $\hat{H}_{AM}$ and $\hat{H}_{MB}$. Here we first establish a quantitative connection between the gain in quantum correlations of the two probes, $\Delta Q_{\text{corr } A:B}$, and a function of the nonclassicality of interactions, specifically in the scenario where the mediator is unknown and experimentally inaccessible. Subsequently, we reformulate the nonclassicality of interactions in terms of the mediator's nonclassicality and thus express it as a function of the commutators of the mediator's observables. When compared to the nonclassicality of interactions, the nonclassicality of the mediator is an intrinsic property of the system rather than specific to the probes' relation to the mediator. Additionally, the inequality is general because it is valid for any mediated interaction involving the two probes and the mediator.

We also investigate the impact of classicalization on the mediator's observables, modeled as a phase-flip channel applied to the mediator. We show that as the phase-flip parameter p approaches $\frac{1}{2}$, the commutator value approaches zero; this simulates a transition from a fully quantum system to a fully classical system, which, as we prove, is unable to create quantum correlations between the probes.

Our result offers a deeper understanding of the mediator's role as a quantum channel capable of distributing entanglement between two communicating parties, as it focuses on an intrinsic characterization of the mediator rather than how it dynamically interacts with the probes.

II. CHARACTERIZATION OF NONCLASSICAL INTERACTIONS

Suppose our closed system is made of two quantum probes A and B and a mediator M , all with finite-dimensional Hilbert spaces. The interactions are assumed to be local, in the sense that there is no direct A - B coupling [6], and the total Hamiltonian takes the form

$$\hat{H} = \hat{H}_{AM} + \hat{H}_{MB}, \quad (1)$$

where the interaction terms $\hat{H}_{AM}$ and $\hat{H}_{MB}$ are otherwise left unspecified. Since the dynamics of AMB is assumed to be closed, the time evolution is unitary and is generated by (1).

In particular, the global evolution operator is

$$U_{AMB}(t) = e^{-it(\hat{H}_{AM} + \hat{H}_{MB})}, \quad (2)$$

with the associated unitary channel defined as

$$\Lambda_{AMB}(\rho) = U_{AMB}(t)\rho U_{AMB}^\dagger(t). \quad (3)$$

In the nonclassicality witness scenario, the mediator M is experimentally inaccessible; hence, to infer its possible nonclassical nature, we need to test its effects through measurements on the two probes. The quantity of interest is the possible increase in quantum correlations between the probes induced by their interaction with the mediator. To retain the highest possible level of generality, a broad class of quantum correlation measures Q is considered. Specifically, Q is assumed to be a gd -continuous function, a property shared by commonly used quantum correlation measures such as entropy of entanglement and logarithmic negativity [11,12]. Being gd continuous implies that Q is monotonic under local processing and satisfies the continuity condition

$$|Q(x) - Q(y)| \leq g(d(x, y)), \quad (4)$$

where g is an invertible monotonically increasing function with the property $\lim_{s \rightarrow 0} g(s) = 0$ and d is a distance between states x and y .

We then define the increase in quantum correlation between the two quantum probes A and B by

$$\Delta Q_{\text{corr } A:B} = Q_{A:B}(\rho_t) - \mathcal{B}(Q(\rho_0)). \quad (5)$$

The quantity $Q_{A:B}(\rho_t)$ is the amount of accessible, i.e., measurable, quantum correlation of the evolved state of the two probes' state ρ_t after the interaction. The evolved state is defined as $\rho_t = \text{tr}_M[\Lambda_{AMB}(\rho_{AMB}(0))]$, while $\mathcal{B}(Q(\rho_0))$ is the maximum possible initial quantum correlation between A and B given their initial state $\rho_0 = \text{tr}_M(\rho_{AMB})$. An exact definition of $\mathcal{B}$ is given in Appendix A [see (A27) and (A28)].

In [9,10] it has been shown that if the local interactions are classical, meaning that $\hat{H}_{AM}$ and $\hat{H}_{MB}$ commute, then the amount of quantum correlation Q between A and B after the interaction cannot exceed the maximum initial correlation, so that

$$[\hat{H}_{AM}, \hat{H}_{MB}] = 0 \implies \Delta Q_{\text{corr } A:B} \leq 0. \quad (6)$$

Before introducing the next results of [9,10], we need the notion of a decomposable map. A map that acts on the tripartite system, $\lambda_{AMB}(\rho)$, is decomposable if there exist two completely positive and trace-preserving (CPTP) maps λ_{AM} and λ_{MB} such that

$$\lambda_{AMB}(\rho) = \lambda_{MB} \circ \lambda_{AM}(\rho). \quad (7)$$

In the nonclassicality witness scenario, we cannot fully determine the map induced by the local interactions (1), as the mediator is unknown and inaccessible. We can only access the marginal map $\Lambda_{AB} = \text{tr}_M(\Lambda_{AMB})$ given how the mediator acts on the quantum probes. The global map Λ_{AMB} can be found among the Stinespring dilations of Λ_{AB} , with the appropriate dimension m . It was found in [9,10] that if the interactions are classical, then there exists a Stinespring dilation of Λ_{AB} of dimension $m \geq d_M$ which is decomposable and where d_M is the dimension of the mediator's Hilbert space. We call

the space of all such maps' dilations $\overline{DEC}(m)$, and we can summarize this implication as

$$[\hat{H}_{AM}, \hat{H}_{MB}] = 0 \Rightarrow \Lambda_{AB} \in \overline{DEC}(m). \quad (8)$$

Crucially, the nondecomposability of the induced map Λ_{AB} admits a direct dynamical interpretation, as it implies the nonclassicality of the mediated interaction

$$\Lambda_{AB} \notin \overline{DEC}(m) \iff [\hat{H}_{AM}, \hat{H}_{MB}] \neq 0. \quad (9)$$

With these definitions in place, one of the main results of [9,10] states that if the total quantum correlation shared between A and B increases, then the marginal map Λ_{AB} is nondecomposable, that is,

$$\Delta Q_{\text{corr } A:B} > 0 \implies \Lambda_{AB} \notin \overline{DEC}(m). \quad (10)$$

We stress that $\Delta Q_{\text{corr } A:B}$ depends only on quantities related to the two probes A and B , which can be measured in the laboratory. Moreover, no further assumption is made on $\hat{H}$, $\hat{H}_{AM}$, and $\hat{H}_{MB}$ beyond locality and finite dimensionality, and the connection between the increase of quantum correlations and the nonclassicality of interactions is therefore valid for any of the allowed local Hamiltonians.

Let us now proceed with our discussion by introducing one of our results, which sets the increase in quantum correlation between the probes as a lower bound to a function of the nonclassicality of interactions. This result is derived under the same assumptions of the previous works [9,10]: A , B , and M are defined within finite-dimensional Hilbert spaces, and the Hamiltonians satisfy locality conditions, meaning that no direct interaction between A and B is allowed. Furthermore, the allowed Hamiltonians remain unspecified throughout the derivation, ensuring the generality of the result.

This result is obtained by building on the distance-based inequality derived in [10], which bounds the experimentally accessible quantity $g^{-1}(\Delta Q_{\text{corr } A:B})$ in terms of the distance between a dilation of the induced marginal map Λ_{AB} and a suitable dilation of a decomposable map. Here g^{-1} denotes the inverse of the function g appearing in the definition of gd continuity of the correlation measure Q . As shown in Appendix B, this distance can be further bounded by the spectral norm of a difference of unitaries associated with the local interaction Hamiltonians, namely,

$$\|U_{AMB}(t) - U_{AM}(t)U_{MB}(t)\|_{\infty}, \quad (11)$$

where $U_{AMB}(t)$ is defined in (2), while $U_{AM}(t)$ and $U_{MB}(t)$ are defined as

$$U_{AM}(t) = e^{-it\hat{H}_{AM}}, \quad U_{MB}(t) = e^{-it\hat{H}_{MB}}. \quad (12)$$

This quantity measures, in the operator norm, the deviation of the true global evolution of AMB from the decomposed evolution obtained by concatenating the evolutions generated by $\hat{H}_{AM}$ and $\hat{H}_{MB}$.

To make the dependence on noncommutativity explicit, we consider the Zassenhaus factorization of the global unitary

$$U_{AMB}(t) = U_{AM}(t)U_{MB}(t) \prod_{n=2}^{\infty} \exp[(-it)^n Z_n] \quad (13)$$

and define the associated Zassenhaus correction unitary

$$U_C(t) = \prod_{n=2}^{\infty} \exp[(-it)^n Z_n] = e^{-it\hat{H}_C}, \quad (14)$$

where $\hat{H}_C$ is the corresponding Hermitian generator of $U_C(t)$. Appendix B shows that

$$\|U_{AMB}(t) - U_{AM}(t)U_{MB}(t)\|_{\infty} = \|U_C(t) - \mathbb{I}\|_{\infty} \quad (15)$$

and that the latter admits a closed functional form in terms of $\hat{H}_C$.

We now present the outcome of this derivation, a quantitative relation that links $g^{-1}(\Delta Q_{\text{corr } A:B})$ directly to a function of the nonclassicality of interactions.

Result 1. Consider a closed finite-dimensional tripartite system $A \oplus M \oplus B$ with mediated Hamiltonian $\hat{H} = \hat{H}_{AM} + \hat{H}_{MB}$. For any gd -continuous quantum correlation measure Q , for the increase in quantum correlation $\Delta Q_{\text{corr } A:B}$ defined in (5) and the Zassenhaus correction Hamiltonian $\hat{H}_C$ defined in (14), we have

$$g^{-1}(\Delta Q_{\text{corr } A:B}) \leq 2 \left\| \sin\left(\frac{t}{2}\hat{H}_C\right) \right\|_{\infty} \quad \forall t. \quad (16)$$

As shown in Appendix A, $\hat{H}_C$ is completely determined by the interaction Hamiltonians $\hat{H}_{AM}$ and $\hat{H}_{MB}$ and it vanishes whenever they commute, i.e., $[\hat{H}_{AM}, \hat{H}_{MB}] = 0 \Rightarrow \hat{H}_C = 0$. In the short-time regime, a second-order expansion in t , detailed in Appendix B, yields an especially transparent form of the bound, since it makes the dependence on the commutator $[\hat{H}_{AM}, \hat{H}_{MB}]$ fully explicit:

$$g^{-1}(\Delta Q_{\text{corr } A:B}) \leq 2 \left\| \sin\left(\frac{it^2}{4}[\hat{H}_{AM}, \hat{H}_{MB}]\right) \right\|_{\infty}. \quad (17)$$

This inequality relates the gain in quantum correlations between A and B to a function of the commutator $[\hat{H}_{AM}, \hat{H}_{MB}]$. Equivalently, by the same expansion, the leading nontrivial order in t implies that the generator entering (16) is

$$\hat{H}_C \approx \frac{it}{2}[\hat{H}_{AM}, \hat{H}_{MB}]. \quad (18)$$

For later convenience, we introduce a compact notation for the right-hand side of the short-time bound. We define the map

$$f(H) = g\left(2 \left\| \sin\left(\frac{it^2}{4}H\right) \right\|_{\infty}\right) \quad (19)$$

so that (17) can be equivalently written as

$$\Delta Q_{\text{corr } A:B} \leq f([\hat{H}_{AM}, \hat{H}_{MB}]). \quad (20)$$

This definition is purely notational; once the correlation measure Q (hence g) and the interaction time t are fixed, the bound depends on the dynamics only through the commutator $[\hat{H}_{AM}, \hat{H}_{MB}]$.

Taken together, the inequalities (16) and (17) establish a quantitative link between the gain in quantum correlations of the two probes and the nonclassicality of the underlying interactions, in the setting where the mediator is unknown and experimentally inaccessible. This constitutes the first main result of this work.

Before proceeding further, it is worth making a few remarks on the main inequality (16). If the interactions are classical, namely, if $[\hat{H}_{AM}, \hat{H}_{MB}] = 0$, then Appendix A shows that the Zassenhaus correction is trivial, $U_C(t) = \mathbb{I}$, and hence $\hat{H}_C = 0$ [see (A13)]. Consequently, since $\sin(\frac{t}{2}\hat{H}_C) = 0$, the right-hand side of (16) vanishes and we obtain $g^{-1}(\Delta Q_{\text{corr } A:B}) \leq 0$. Since g is monotonically increasing, invertible, and satisfies $\lim_{s \rightarrow 0} g(s) = 0$, it follows that

$$\Delta Q_{\text{corr } A:B} \leq 0. \quad (21)$$

In particular, the condition (10) cannot be satisfied. In agreement with [10], we summarize this consequence as follows.

Corollary 1. Classical interactions cannot increase quantum correlations. If $[\hat{H}_{AM}, \hat{H}_{MB}] = 0$, then $\Delta Q_{\text{corr } A:B} \leq 0$.

Conversely, if an experiment reveals an increase in quantum correlations, namely, $\Delta Q_{\text{corr } A:B} > 0$, then, by the properties of g recalled above, $g^{-1}(\Delta Q_{\text{corr } A:B}) > 0$. By (16), this can only occur if the right-hand side is strictly positive; hence $\hat{H}_C \neq 0$. Using (A14) in Appendix A, we conclude that the interaction Hamiltonians cannot commute, i.e. $[\hat{H}_{AM}, \hat{H}_{MB}] \neq 0$. This is consistent with the explicit second-order form (17), where the leading nontrivial contribution is governed by the commutator $[\hat{H}_{AM}, \hat{H}_{MB}]$.

Corollary 2. An increase in quantum correlation rules out classical interactions. If $\Delta Q_{\text{corr } A:B} > 0$, then necessarily $[\hat{H}_{AM}, \hat{H}_{MB}] \neq 0$ or equivalently $\hat{H}_C \neq 0$.

A further remark concerns the size and achievability of the right-hand side of (16). Since $\|\sin(X)\|_\infty \leq 1$ for any Hermitian operator X , the bound is limited as

$$2 \left\| \sin\left(\frac{t}{2}\hat{H}_C\right) \right\|_\infty \leq 2, \quad (22)$$

so that its maximum is universal, i.e., independent of the specific choice of $\hat{H}_{AM}$ and $\hat{H}_{MB}$.

Moreover, for any fixed $\hat{H}_C \neq 0$, this maximal value can be attained by a suitable choice of the interaction time t . As shown in Appendix B [see Eq. (B20)], the earliest time at which this occurs is controlled by the largest eigenvalue of $\hat{H}_C$ and scales as $t_* = \pi / \max_j |\lambda_j|$, where $\{\lambda_j\}$ are the eigenvalues of $\hat{H}_C$. In this sense, while the maximal value of the bound is universal, the shortest interaction time needed to approach it depends on the spectral structure of $\hat{H}_C$.

Finally, in the short-time regime where $\hat{H}_C \approx \frac{it}{2}[\hat{H}_{AM}, \hat{H}_{MB}]$ [see (B24) in Appendix B], it is natural to expect that larger noncommutativity, as quantified, for instance, by $\|[\hat{H}_{AM}, \hat{H}_{MB}]\|_\infty$, tends at leading order to increase the initial growth rate of the right-hand side of the bound (17) as a function of t and may therefore allow the maximum to be approached more rapidly. A systematic characterization of these timescales is left for future work.

III. FROM NONCLASSICAL INTERACTIONS TO THE NONCLASSICAL MEDIATOR

The goal of the following section will be to generally express the nonclassicality of interactions, i.e., noncommutativity of interaction Hamiltonians, the quantity appearing in (17), in terms of the nonclassicality of the mediator, namely, the noncommutativity of the mediator's observables,

shifting from a model-dependent point of view to a model-independent one.

The starting point is to define more precisely the most general $\hat{H}_{AM}$ and $\hat{H}_{MB}$ and then explicitly calculate their commutator, to show that it ultimately depends on the commutator of the mediator's observables. This shows that the ability of a mediator to create quantum correlations between two spatially separated quantum systems is due to its nonclassicality and not to the way it interacts with these probes.

We start by defining the quantum probes A and B and the mediator M . Following the assumptions of finite dimensions, we have two possible very general choices for the Hilbert spaces $\mathcal{H}_A$, $\mathcal{H}_M$, and $\mathcal{H}_B$. The first is to consider a generic d -dimensional Hilbert space, while the second possibility is to choose a tensor product of qubit Hilbert spaces $\mathcal{H}_2 \otimes \cdots \otimes \mathcal{H}_2$ that, given the universality of quantum computation, would allow us to use a certain number of qubits to simulate any quantum system of finite dimension [13]. To simplify the notation and without loss of generality, we set the mediator M to be a chain of T qubits with Hilbert space dimension $d_M = 2^T$, while the probes A and B are d_A - and d_B -dimensional systems described by Hilbert spaces with $\text{SU}(N)$ operator algebra. Thus, the most general $\hat{H}_{AM}$ and $\hat{H}_{MB}$ Hamiltonians will be

$$\begin{aligned} \hat{H}_{AM} &= \sum_{i,j_1,j_2,\dots,j_T} \alpha_{i,j_1,j_2,\dots,j_T} (a_i \otimes m_{j_1} \otimes m_{j_2} \otimes \cdots \otimes m_{j_T} \otimes \mathbb{I}_B), \\ \hat{H}_{MB} &= \sum_{l_1,l_2,\dots,l_T,k} \beta_{l_1,l_2,\dots,l_T,k} (\mathbb{I}_A \otimes m_{l_1} \otimes m_{l_2} \otimes \cdots \otimes m_{l_T} \otimes b_k), \end{aligned} \quad (23)$$

where α and β are real-valued coefficients, a_i and b_k are $\text{SU}(d_A)$ and $\text{SU}(d_B)$ operators including the identity, and $\{m_{j_l}\}_{j_l=0,\dots,3} = \{\mathbb{I}, X, Y, Z\}$ is the set of the Pauli operators plus the identity for the first qubit of the mediator in $\hat{H}_{AM}$, where the number used as a subscript indicates which qubit of the mediator the operator refers to. To distinguish the mediator's operators appearing in $\hat{H}_{MB}$ from those appearing in $\hat{H}_{AM}$, the subscript l replaces j . Most importantly, the identity operators appearing in the tensor products come from imposing the locality condition on the interactions. This avoids any direct interaction term of the pair AB , thus enforcing locality.

Considering a mediator M composed of T qubits, the commutator of the interaction Hamiltonians admits the explicit expansion

$$\begin{aligned} [\hat{H}_{AM}, \hat{H}_{MB}] &= \sum_{i,j_1,j_2,\dots,j_T,l_1,l_2,\dots,l_T,k} \alpha_{i,j_1,j_2,\dots,j_T} \beta_{l_1,l_2,\dots,l_T,k} a_i \\ &\quad \otimes ([m_{j_1}, m_{l_1}] \otimes m_{j_2} m_{l_2} \otimes \cdots \otimes m_{j_T} m_{l_T} \\ &\quad + m_{l_1} m_{j_1} \otimes [m_{j_2}, m_{l_2}] \otimes \cdots \otimes m_{j_T} m_{l_T} \\ &\quad + m_{l_1} m_{j_1} \otimes m_{l_2} m_{j_2} \otimes \cdots \otimes [m_{j_T}, m_{l_T}]) \otimes b_k. \end{aligned} \quad (24)$$

Equation (24) constitutes one of the main results of this work. Its full derivation is reported in Appendix C. Most importantly, Eq. (24) shows that $[\hat{H}_{AM}, \hat{H}_{MB}]$ can be nonvanishing only if at least one commutator of mediator observables appearing in the two couplings is nonzero, i.e., only if there exists at least one pair such that $[m_{j_l}, m_{l_l}] \neq 0$ for

some mediator-qubit label r . In addition, Appendix C provides a dedicated analysis of the two-qubit case, identifying which combinations of mediator observables can lead to a nonvanishing Hamiltonian commutator. We summarize this implication as follows.

Result 2. Nonclassical interactions imply a nonclassical mediator. Consider a mediated Hamiltonian $\hat{H} = \hat{H}_{AM} + \hat{H}_{MB}$ with $\hat{H}_{AM}$ and $\hat{H}_{MB}$ of the form (23); then their commutator is given by (24). If $[\hat{H}_{AM}, \hat{H}_{MB}] \neq 0$, then there exists at least one pair of mediator observables appearing in the two Hamiltonians whose commutator is nonvanishing. In particular, in the expansion (24), at least one mediator commutator satisfies $[m_{j_r}, m_{l_r}] \neq 0$ for some mediator-qubit label r .

Equivalently, if all mediator observables that appear in the interaction terms commute with each other, then the interaction Hamiltonians commute. This gives the following corollary.

Corollary 3. A classical mediator implies classical interactions. If for all mediator operators appearing in (23) one has $[m_{j_r}, m_{l_r}] = 0$ for every mediator-qubit label r , then $[\hat{H}_{AM}, \hat{H}_{MB}] = 0$.

Importantly, the converse does not generally hold: A nonclassical mediator does not force $[\hat{H}_{AM}, \hat{H}_{MB}] \neq 0$, because the Hamiltonians may include only a commuting subset of mediator observables (or admit cancellations). To explain the logic of the implication, let us provide a simple example. Let M be a single qubit, so that it supports noncommuting observables, e.g., $[X, Z] \neq 0$. Choose local couplings involving only the same commuting mediator observable Z ,

$$\hat{H}_{AM} = a_i \otimes Z \otimes \mathbb{I}_B, \quad \hat{H}_{MB} = \mathbb{I}_A \otimes Z \otimes b_k, \quad (25)$$

with generic Hermitian operators a_i and b_k acting on the two probes. Then

$$[\hat{H}_{AM}, \hat{H}_{MB}] = 0, \quad (26)$$

even though the mediator is nonclassical.

Remark 1. Mediator nonclassicality is necessary but not sufficient to guarantee nonclassical interactions.

To identify when the commutator (24) can be nonvanishing, it is useful to analyze the mediator contribution explicitly in the two-qubit case. The detailed derivation is given in Appendix C; here we summarize the outcome. Depending on whether the mediator indices satisfy $l_1 \neq j_1$ or $l_1 = j_1$, the nonvanishing terms in the mediator factor in (24) are controlled by the commutator of the first-qubit operators $[m_{j_1}, m_{l_1}]$ or by the commutator of the second-qubit operators $[m_{j_2}, m_{l_2}]$, respectively. In particular, in this two-qubit case, $[\hat{H}_{AM}, \hat{H}_{MB}] \neq 0$ can occur only if at least a pair of the noncommuting mediators' operators appears in the two Hamiltonians $\hat{H}_{AM}$ and $\hat{H}_{MB}$.

In general, for the specific chosen Hamiltonians, the presence of a single commutator for one of the mediator's qubits is sufficient to ensure that the commutator of the Hamiltonians is nonvanishing. This motivates the following remark.

Remark 2. A single quantum system within a composite classical-quantum mediator M can be enough to mediate quantum correlations between the probes A and B .

We now make the functional dependence of the short-time quantitative bound (17) explicit by rewriting its right-hand

side using the expansion (24) of $[\hat{H}_{AM}, \hat{H}_{MB}]$ for a mediator M composed of T qubits.

Recalling here for convenience the bound (20) in terms of the map f introduced in (19), we have $\Delta Q_{\text{corr } A:B} \leq f([\hat{H}_{AM}, \hat{H}_{MB}])$. The functional form of f is fixed once the correlation measure Q (hence g) and the interaction time t are fixed, and Eq. (20) holds for any choice of local Hamiltonians $\hat{H}_{AM}$ and $\hat{H}_{MB}$.

Indeed, the coefficients α and β defining the Hamiltonians in (23) select which tensor-product factors of probe and mediator observables are activated in the interaction and with what weights. Crucially, the explicit form of the Hamiltonians' commutator in Eq. (24) shows that each potentially nonvanishing contribution contains at least one commutator of mediator observables. Therefore, for any fixed choice of $\hat{H}_{AM}$ and $\hat{H}_{MB}$, the right-hand side of (20) depends on the mediator only through the collection of commutators of those mediator observables that appear in the two interaction terms. For notational convenience, we define this dependence as

$$f([\hat{H}_{AM}, \hat{H}_{MB}]) \equiv f_H([m_{AM}, m_{MB}]), \quad (27)$$

where $[m_{AM}, m_{MB}]$ denotes the collection $\{[m_{j_r}, m_{l_r}]\}_{r=1}^T$ of mediator commutators appearing in the expansion of $[\hat{H}_{AM}, \hat{H}_{MB}]$ and the subscript H serves as a reminder that the explicit expression of $f_H([m_{AM}, m_{MB}])$ is obtained only after specifying the Hamiltonians, i.e., the coefficients α and β , while its functional form is universally fixed by (19) and by the general form of the Hamiltonians' commutator (24). Accordingly, we can rewrite (20) in the new form

$$\Delta Q_{\text{corr } A:B} \leq f_H([m_{AM}, m_{MB}]). \quad (28)$$

In this sense, once Q and t are fixed, an experimentally measured gain $\Delta Q_{\text{corr } A:B}$ is upper bounded by a function controlled by the noncommutativity of the mediator observables, which are dynamically involved in the interaction.

In what follows, we explore what happens when we send each of the commutators $[m_{j_r}, m_{l_r}]$ to 0. This is equivalent to considering a progressively more classical mediator and formally restricting the upper bound to the possible value of $\Delta Q_{\text{corr } A:B}$.

IV. A CLASSICAL MEDIATOR CANNOT CREATE QUANTUM CORRELATIONS

In real experimental scenarios, physical systems under investigation cannot be perfectly protected from interactions with the environment, leading to some errors and noise that inevitably modify our system by making it classical and thus with only commuting observables. It is thus crucial to understand the effects of this process on the capability of the unknown system to mediate quantum coherence. The results presented in this work are perfect for the scope, as (28) shows that the upper bound to the probes' quantum correlation gain mediated by M is indeed a function of its nonclassicality. Thus, we consider here a model that simulates a system with a lower nonclassicality compared to the full quantum case. This is reflected in the commutativity of M 's variables, thus ultimately on M 's nonclassicality according to the definition adopted in this work.

A common noisy channel that might be considered for this purpose is the phase-flip channel, which can act on our qubits, making the commutator's value of their observables vanish. This happens when the value of p , a free parameter of the phase flip, is $\frac{1}{2}$. This possibility is interesting because, as we will see, it sets the upper bound of the inequality (28) to 0, effectively preventing the mediator from increasing the quantum correlation across the two probes. This result suggests that a completely decohered quantum channel cannot increase the quantum correlations between two probes, as one can physically expect.

Given these considerations, we will assume a noise model whose overall effect is to suppress the noncommutativity of the mediator's observables progressively, essentially making them more classical under our definition. In particular, we allow the qubits of the mediator in our system to be affected separately by independent phase-flip channels. This operation can be viewed as inducing a permanent modification of the mediator's effective Hilbert space structure and of the interaction Hamiltonians H_{AM} and H_{MB} ; however, providing a physical mechanism that gives rise to such a modification lies beyond the scope of the present work and is left for future investigation. Note that the phase-flip channel considered here is applied before the local interactions between A and M and between M and B occur, acting as a preparation of the mediator.

To formalize this description, we resort to the Heisenberg picture of quantum mechanics, using the descriptor formalism [14,15]. Since we have defined a base of observables, it will be enough to see how they are affected by the phase flip.

Let q_x denote an operator representing a tensor product of the X component of a qubit in our system with the identity operators on all other subsystems, for example, $q_{x_{j_1}} = \mathbb{I}_A \otimes X_{M_1} \otimes \mathbb{I}_{M_2} \otimes \mathbb{I}_{M_3} \otimes \cdots \otimes \mathbb{I}_B$, where the subscript j_1 in q_x indicates the first qubit of the mediator coupled with the probe A in the Hamiltonian $\hat{H}_{AM}$. Then we have also q_z and q_y , with $q_x^2 = q_y^2 = q_z^2 = I$ and the products $q_z q_x = i q_y$ and $q_x q_z = -i q_y$ which define the algebraic properties. If a gate $U(t_n)$ operates between time t_n and t_{n+1} , we define $\mathcal{O}_x(t_{n+1}) = U(t_n)^\dagger \mathcal{O}_x(t_n) U(t_n)$ as the operator representing the observable $\mathcal{O}_x$ after its action. The initial conditions are fixed by choosing particular values for $q_x(t_0)$, $q_y(t_0)$, $q_z(t_0)$, and the Heisenberg state ρ_H . Using the algebraic relations for the observables defined above, the state of a qubit at time t_n is then completely specified by giving at least two components, e.g., $\{q_x(t), q_z(t)\}$ [16].

Let us recall how a phase flip acts on a qubit and its observables. For a general observable $\mathcal{O}_x(t)$, the phase-flip channel acts as

$$\hat{E}(\mathcal{O}_x(t)) = \sum_a M_a(t)^\dagger \mathcal{O}_x(t) M_a(t), \quad (29)$$

where the M_a are the Kraus operators of the phase-flip channel: $M_0(t) = \sqrt{p}I$ and $M_1(t) = \sqrt{1-p}q_z(t)$. After this operation, the generators of the algebra of the qubit undergoing decoherence are affected in the following way [16]:

$$\begin{aligned} \hat{E}(q_x(t)) &= (2p-1)q_x(t), \\ \hat{E}(q_y(t)) &= (2p-1)q_y(t), \\ \hat{E}(q_z(t)) &= q_z(t). \end{aligned} \quad (30)$$

We can now move to the Hamiltonians and apply a phase flip $\hat{E}$ to both $\hat{H}_{AM}$ and $\hat{H}_{AB}$, one per mediator's qubit in M . After the calculations provided in Appendix D, the Hamiltonians read

$$\begin{aligned} \hat{H}_{AM} &= \hat{E}_T(\cdots \hat{E}_2(\hat{E}_1(\hat{H}_{AM}))) \\ &= \sum_{a_i, j_1, \dots, j_T} \alpha'_{a_i, j_1, \dots, j_T} (q_{a_i} q_{j_1} \cdots q_{j_T} q_{\mathbb{I}_B}). \end{aligned} \quad (31)$$

Here each descriptor is an observable in the AMB Hilbert space, so the tensor product is no longer required. As before, the qubit of the B probe is set to be an identity operator, meaning that the descriptor $q_{\mathbb{I}_B}$ is simply a tensor product of identity operators of all the qubits in A , M , and B . Similarly for $\hat{H}_{MB}$,

$$\begin{aligned} \hat{H}_{MB} &= \hat{E}_T(\cdots \hat{E}_2(\hat{E}_1(\hat{H}_{MB}))) \\ &= \sum_{l_1, \dots, l_T, b_k} \beta'_{l_1, \dots, l_T, b_k} (q_{\mathbb{I}_A} q_{l_1} \cdots q_{l_T} q_{b_k}). \end{aligned} \quad (32)$$

These resulting Hamiltonians account for the multiple phase flips that have been applied, by storing an extra $(2p-1)$ factor in the primed coefficients α' and β' each time an X or Y observable is considered for a qubit mediator [see the discussion around (D6) in Appendix D]. We now calculate the commutator for the new degraded Hamiltonians by simply following the same steps as the previous commutator in (24) as explained in Appendix D, obtaining

$$\begin{aligned} [\hat{H}_{AM}, \hat{H}_{MB}] &= \sum_{a_i, j_1, j_2, \dots, j_T, l_1, l_2, \dots, l_T, b_k} \alpha'_{a_i, j_1, j_2, \dots, j_T} \beta'_{l_1, l_2, \dots, l_T, b_k} q_{a_i} \\ &\times ([q_{j_1}, q_{l_1}] q_{j_2} q_{l_2} \cdots q_{j_T} q_{l_T} + q_{l_1} q_{j_1} [q_{j_2}, q_{l_2}] \cdots q_{j_T} q_{l_T} \\ &+ q_{l_1} q_{j_1} q_{l_2} q_{j_2} \cdots [q_{j_T}, q_{l_T}]) q_{b_k}. \end{aligned} \quad (33)$$

This expression has the same algebraic structure as the commutator in (24), with the only difference that the coefficients α and β are replaced by the phase-flip-degraded coefficients α' and β' , which incorporate factors of $(2p-1)$ whenever an X or Y mediator observable appears. In particular, the dependence of the correlation-gain bound on decoherence enters through the degraded Hamiltonian commutator in (33). In direct analogy with (28), we therefore write

$$\Delta Q_{\text{corr } A:B} \leq f_{H'}([m_{AM}, m_{MB}], p), \quad (34)$$

where $f_{H'}([m_{AM}, m_{MB}], p)$ denotes the right-hand side of the bound obtained after specifying the interaction Hamiltonians, i.e., the coefficients α and β , and applying the phase-flip channel with parameter p to the mediator observables so that the mediator commutators in the expansion are rescaled by the corresponding $(2p-1)$ factors encoded in α' and β' .

As discussed in the preceding section, only certain combinations of mediator observables appearing in the Hamiltonians can contribute to a nonvanishing commutator. Crucially, any potentially nonvanishing term in (33) contains at least one commutator of mediator observables, which always involves at least an X or Y component; thus, it is accompanied by one or two factors of $(2p-1)$. As a consequence, the degraded commutator vanishes at $p = \frac{1}{2}$, so that

$$f_H([m_{AM}, m_{MB}], \frac{1}{2}) = 0 \text{ and}$$

$$\Delta Q_{\text{corr } A:B} \leq 0. \quad (35)$$

Therefore, if an experiment observes $\Delta Q_{\text{corr } A:B} > 0$ under the assumptions of the witness, the mediator cannot be fully classicalized in this phase-flip sense and must retain noncommuting observables dynamically involved in the interaction.

More generally, different noise models acting on M would modify the effective Hamiltonian commutator depending on the way they affect the mediator and its interaction with the probes. A detailed study of these decoherence channels is left for future work.

V. DISCUSSION

In this work we have proposed an inequality quantitatively connecting the nonclassicality of a mediator of interactions with its ability to create quantum correlations between two probes. In particular, we considered a closed finite-dimensional tripartite system AMB evolving under an interaction Hamiltonian $\hat{H} = \hat{H}_{AM} + \hat{H}_{MB}$, where M mediates the interaction between the probes A and B (see Fig. 1). We have taken as our starting point the results in [9,10], which relate an increase of quantum correlations $\Delta Q_{\text{corr } A:B}$ to the nonclassicality of the interactions, expressed as the noncommutativity of $\hat{H}_{AM}$ and $\hat{H}_{MB}$.

Our first main contribution is Result 1, which provides a quantitative upper bound on the experimentally accessible increase in quantum correlations $\Delta Q_{\text{corr } A:B}$, defined in (5), in terms of a closed function of the mediated dynamics [see (16)]. In the short-time regime, this bound admits the explicit commutator form (17), making the dependence on the noncommutativity of the interaction Hamiltonians transparent. Equivalently, the short-time bound can be written in the more compact form $\Delta Q_{\text{corr } A:B} \leq f([\hat{H}_{AM}, \hat{H}_{MB}])$, as already given in (20), where f is defined in (19) and is fixed by the chosen correlation measure Q and by the interaction time t , while the dynamics enters only through the Hamiltonian commutator. Altogether, under the sole assumptions of locality and finite dimensionality, Result 1 yields a quantitative link between the gain in quantum correlations of the probes and the nonclassicality of the interactions, as clearly remarked by Corollary 2.

Our second main contribution consists of Eq. (24) and Result 2, which reformulate the nonclassicality of interactions in terms of the mediator's nonclassicality. Combining (24) with the short-time bound in the compact form (20) yields (28), namely, $\Delta Q_{\text{corr } A:B} \leq f_H([m_{AM}, m_{MB}])$. Here the increase of quantum correlations between the probes is upper bounded by a function of the commutators of the mediator observables dynamically involved in the specific interaction.

Taken together, these contributions establish a general quantitative witness of mediator nonclassicality: By Corollary 2, observing $\Delta Q_{\text{corr } A:B} > 0$ rules out commuting local interactions; Result 2 then ensures that at least one pair of mediator observables dynamically involved in the coupling is noncommuting, while the observed increase in quantum correlations is constrained by the bound (16) for all t .

These results provide a general quantitative framework for the witness of nonclassicality in [7,8]. In those proposals,

the creation of entanglement between two probes interacting locally only through a mediator is a sufficient condition to infer the mediator's nonclassicality. Here, within the same mediated-interaction setting, we make this connection quantitative and independent of the particular choice of interaction Hamiltonians by relating the observed gain in quantum correlations to a function controlled by the noncommutativity of the mediator's observables.

We further explored the impact of decoherence on the mediator's observables, particularly through a specific type of phase-flip channel. We showed that as the phase-flip parameter p approaches $\frac{1}{2}$, the commutativity of the mediator's observables reaches zero, effectively yielding the mediator to be classical and preventing its ability to increase quantum correlations between two probes.

Future work can expand on our findings in several ways. One promising direction is to interpret the mediator as a quantum channel. In particular, if one chooses as a correlation measure Q the relative entropy of entanglement $E_{A:B}$, it is natural to define the corresponding correlation gain $\Delta E_{A:B}$. With this choice, our general bound (28) reads

$$\Delta E_{A:B} \leq f_H([m_{AM}, m_{MB}]), \quad (36)$$

where f_H is fixed once t and the Hamiltonians are specified, and it is controlled by the commutators of the mediator observables dynamically involved in the coupling. A natural question is whether a gain in relative entropy of entanglement $\Delta E_{A:B}$ can be related to a quantum capacity or to the entanglement generation capacity of the mediator M , as suggested in [17]. This involves investigating whether one can translate the two sides of (36) into bounds on channel capacities of M . More precisely, one would like to establish quantitative relations of the form

$$Q_M^{(\min)}(\Delta E_{A:B}) \leq Q_M^{(\max)}(f_H([m_{AM}, m_{MB}])), \quad (37)$$

where $Q_M^{(\min)}$ denotes a chosen capacity of the mediator and is an experimentally inferred lower bound constructed from the observed entanglement gain, while $Q_M^{(\max)}$ is a theoretical upper bound controlled by the mediator commutators activated by the interaction. Interestingly, this transition could be obtained by relating (36) in terms of coherent information, whose supremum is indeed the quantum channel capacity [18,19].

Within this context, two distinct situations arise. One is testing the nonclassicality of an unknown system (e.g., a biological system) which is suspected to have quantum properties, by attempting quantum communication with it and measuring quantum correlation gain in the probes. Alternatively, utilizing the mediator for entanglement distribution protocols across a quantum network, one can theoretically calculate a bound for its maximum ability to deliver entanglement between two probes without prior testing. In both cases, our inequality can provide bounds on the quantum capacity of the mediator based on its observable and Hilbert space structure.

Additionally, in the context of superselected models for gravity [3], where the classicality of gravity emerges from imposing certain superselection rules to commutators of its quantum observables, our result can be used as a falsification

test: If quantum correlations are detected between probes, our inequality (28) implies that quantum gravitational observables m_j and m_l are dynamically involved, potentially leading to the exclusion of certain semiclassical models for gravity.

Finally, the symmetry between two-point space and time quantum correlations [20] suggests the possibility of extending the results of this work to mediated temporal quantum correlation between a single quantum probe at times t_0 and $t > t_0$. The potential differences in the functional form of the upper bound (28) can inform on the symmetrical role of space and time in quantum mechanics and offer a common framework to treat the witness of nonclassicality in [1,7,8] and its temporal single-probe version, as discussed in [21–23].

In summary, our proposed inequality gives a general quantitative framework for the recently proposed witnesses of nonclassicality [7,8]. We have also demonstrated the quantum-to-classical transition of the mediator by studying a decohering map applied to the mediator. Our work thus provides a fundamental link between the mediator's nonclassical features and the observed quantum correlations, independent of the specific dynamics of the interactions and the choice of the type of systems used. Its generality makes it suitable for a range of possible applications, from quantum metrology to quantum biology and even quantum gravity.

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APPENDIX A: DEFINITIONS AND NOTATION

This Appendix briefly revisits essential definitions and notation used throughout the paper and introduces additional notation required for the derivations presented later in this Appendix.

System configuration. The system under investigation, considered as closed, consists of two quantum probes A and B and a mediator M only. Each subsystem is described by a finite-dimensional Hilbert space, denoted by $\mathcal{H}_A$, $\mathcal{H}_M$, and $\mathcal{H}_B$, with dimensions $d_A = \dim(\mathcal{H}_A)$, $d_M = \dim(\mathcal{H}_M)$, and $d_B = \dim(\mathcal{H}_B)$, respectively. To enforce locality, i.e., to prohibit direct interactions between A and B , we assume that the system Hamiltonian has the mediated form

$$\hat{H} = \hat{H}_{AM} + \hat{H}_{MB}. \quad (\text{A1})$$

In particular, no direct A - B coupling term is present. Notice that the Hamiltonians H_{AM} and H_{MB} also include the free evolution of A and B .

Unitaries, Zassenhaus factorization, and U_C . The unitary evolution associated with (A1) is

$$U_{AMB}(t) = e^{-i\hat{H}t} = e^{-i(\hat{H}_{AM} + \hat{H}_{MB})t}. \quad (\text{A2})$$

We also define the local evolutions

$$U_{AM}(t) = e^{-i\hat{H}_{AM}t}, \quad U_{MB}(t) = e^{-i\hat{H}_{MB}t}. \quad (\text{A3})$$

Using the Zassenhaus expansion, one can factorize $U_{AMB}(t)$ as

$$U_{AMB}(t) = U_{AM}(t)U_{MB}(t) \prod_{n=2}^{\infty} \exp[(-it)^n Z_n], \quad (\text{A4})$$

where the Zassenhaus exponents Z_n are uniquely determined by the pair $\{\hat{H}_{AM}, \hat{H}_{MB}\}$. More precisely, for each $n \geq 2$, Z_n is a homogeneous Lie polynomial of degree n in the two generators $\hat{H}_{AM}$ and $\hat{H}_{MB}$ [24], i.e., a finite linear combination (with rational coefficients) of nested commutators built from $\{\hat{H}_{AM}, \hat{H}_{MB}\}$:

$$[V_1, [V_2, \dots, [V_{n-1}, V_n], \dots]] : V_i \in \{\hat{H}_{AM}, \hat{H}_{MB}\}. \quad (\text{A5})$$

In particular, the leading Zassenhaus exponent is

$$Z_2 = -\frac{1}{2}[\hat{H}_{AM}, \hat{H}_{MB}]. \quad (\text{A6})$$

The unitary $U_C(t)$, introduced in the main text in (14), is given by the Zassenhaus correction appearing in (A4). We define

$$U_C(t) = \prod_{n=2}^{\infty} \exp[(-it)^n Z_n] = U_{MB}^\dagger(t) U_{AM}^\dagger(t) U_{AMB}(t) \quad (\text{A7})$$

and also the corresponding Hamiltonian generator $\hat{H}_C$ implicitly by

$$U_C(t) = e^{-it\hat{H}_C}. \quad (\text{A8})$$

Moreover, if the interaction Hamiltonians commute, then every nested commutator built from $\{\hat{H}_{AM}, \hat{H}_{MB}\}$, as introduced in (A5), vanishes. Since each Z_n is a homogeneous Lie polynomial of degree $n \geq 2$ in these two generators, this implies

$$[\hat{H}_{AM}, \hat{H}_{MB}] = 0 \implies Z_n = 0 \forall n \geq 2, \quad (\text{A9})$$

and therefore each factor in the Zassenhaus product is the identity

$$\exp[(-it)^n Z_n] = \mathbb{I} \forall n \geq 2. \quad (\text{A10})$$

Hence, the entire Zassenhaus correction is trivial,

$$U_C(t) = \mathbb{I}, \quad \text{and consequently} \quad \hat{H}_C = 0. \quad (\text{A11})$$

Independently, in the commuting case, the Baker-Campbell-Hausdorff formula yields the exact factorization of the global unitary

$$\begin{aligned} U_{AMB}(t) &= e^{-it(\hat{H}_{AM} + \hat{H}_{MB})} = e^{-it\hat{H}_{AM}} e^{-it\hat{H}_{MB}} \\ &= U_{AM}(t)U_{MB}(t) \quad \text{for } [\hat{H}_{AM}, \hat{H}_{MB}] = 0. \end{aligned} \quad (\text{A12})$$

By the definition (A7), this exact factorization gives $U_C(t) = \mathbb{I}$ and hence $\hat{H}_C = 0$, in agreement with the Zassenhaus-based argument above.

We therefore summarize these implications as

$$[\hat{H}_{AM}, \hat{H}_{MB}] = 0 \implies U_C(t) = \mathbb{I} \implies \hat{H}_C = 0. \quad (\text{A13})$$

Equivalently, a nontrivial Zassenhaus correction implies non-commuting interactions, namely,

$$U_C(t) \neq \mathbb{I} \implies \hat{H}_C \neq 0 \implies [\hat{H}_{AM}, \hat{H}_{MB}] \neq 0. \quad (\text{A14})$$

States and induced map. Let $\rho_{AMB}(0)$ denote the initial global state. The reduced initial state of the probes is

$$\rho_0 = \rho_{AB}(0) = \text{tr}_M[\rho_{AMB}(0)]. \quad (\text{A15})$$

The evolved global state is

$$\rho_{AMB}(t) = U_{AMB}(t)\rho_{AMB}(0)U_{AMB}^\dagger(t); \quad (\text{A16})$$

hence the reduced final state of the probes is

$$\rho_t = \rho_{AB}(t) = \text{tr}_M[\rho_{AMB}(t)]. \quad (\text{A17})$$

The corresponding global unitary channel is

$$\Lambda_{AMB}(\rho) = U_{AMB}(t)\rho U_{AMB}^\dagger(t). \quad (\text{A18})$$

Accordingly, the induced (marginal) map on the probes, Λ_{AB} , is defined by

$$\Lambda_{AB}(\rho) = \text{tr}_M[\Lambda_{AMB}(\rho)]. \quad (\text{A19})$$

Decomposability and its variants. Decomposability formalizes the idea that the mediated dynamics can be realized in a single sequential “round”: first an operation involving AM , followed by an operation involving MB . At the level of channels, a CPTP map λ_{AMB} on the tripartite system is said to be decomposable if there exist CPTP maps λ_{AM} and λ_{MB} such that

$$\lambda_{AMB}(\rho) = \lambda_{MB} \circ \lambda_{AM}(\rho). \quad (\text{A20})$$

The ordering in (A20) is part of the statement: Decomposability asserts the existence of at least one factorization of this form and it does not in general imply that the reverse order $\lambda_{AMB} = \lambda_{AM} \circ \lambda_{MB}$ is also available.

For closed dynamics, the corresponding notion is defined directly at the level of unitaries. A unitary U_{AMB} is decomposable if it can be written as

$$U_{AMB} = U_{MB}U_{AM} \quad (\text{A21})$$

for some unitaries U_{AM} and U_{MB} . When U_{AMB} is generated by a time-independent Hamiltonian $U_{AMB}(t) = e^{-it\hat{H}}$, one may also distinguish the stronger property of a continuous commuting decomposition, namely, the existence of continuous families $U_{AM}(t)$ and $U_{MB}(t)$ such that

$$U_{AMB}(t) = U_{AM}(t)U_{MB}(t) = U_{MB}(t)U_{AM}(t),$$

$$[U_{AM}(t), U_{MB}(t)] = 0 \forall t. \quad (\text{A22})$$

This strengthening is relevant because, as shown in [10], commuting decompositions capture precisely the sense in which a unitary evolution can be regarded as generated by commuting local Hamiltonians, whereas plain decomposability alone is strictly weaker.

From tripartite decomposability to inaccessible mediators. In our setting, only the marginal evolution of the probes is operationally accessible, namely, the CPTP map Λ_{AB} , through measurements performed on A and B directly. Accordingly,

to assess whether Λ_{AB} belongs to the decomposable class (A20), we consider suitable tripartite dilations, introducing an explicit mediator of bounded dimension. Following [10], we say that Λ_{AB} admits a decomposable m dilation if there exist a mediator system M with $d_M \leq m$, an initial state σ_M , and a decomposable channel λ_{ABM} such that

$$\Lambda_{AB}(\rho) = \text{tr}_M[\lambda_{ABM}(\rho \otimes \sigma_M)] \forall \rho. \quad (\text{A23})$$

We denote by $\overline{DEC}(m)$ the set of all maps Λ_{AB} admitting such a decomposable m dilation.

Classicality of interactions and decomposability. For closed mediated dynamics generated by a local Hamiltonian of the type (A1), a natural notion of mediated classical interaction is the commutativity of the coupling terms,

$$[\hat{H}_{AM}, \hat{H}_{MB}] = 0. \quad (\text{A24})$$

In this case, the Baker-Campbell-Hausdorff formula implies an exact factorization of the global unitary for all t , as already shown in (A12). So the evolution is continuously decomposable with order-independent (commuting) factors in the sense of (A22).

Quantum correlation measure Q , correlation gain $\Delta Q_{\text{corr } A:B}$, and initial correlation $\mathcal{B}(Q(\rho_0))$. Throughout this work, Q denotes a generic bipartite quantifier of quantum correlations. We assume that Q belongs to the class of gd -continuous functions, which follow a continuity property with respect to a contractive distance d , namely, there exists an invertible, monotonically increasing function g , with $\lim_{s \rightarrow 0} g(s) = 0$, such that for any pair of states x and y one has

$$|Q(x) - Q(y)| \leq g(d(x, y)). \quad (\text{A25})$$

In addition, Q is taken to be monotonic under local processing.

With ρ_0 and ρ_t defined in (A15) and (A17), we quantify the correlation gain between A and B by

$$\Delta Q_{\text{corr } A:B} = Q_{A:B}(\rho_t) - \mathcal{B}(Q(\rho_0)). \quad (\text{A26})$$

The term $Q_{A:B}(\rho_t)$ is directly accessible from measurements on the probes, whereas $\mathcal{B}(Q(\rho_0))$ accounts for the maximum potential correlation within the system before any interaction.

The quantity $\mathcal{B}(Q(\rho_0))$ is defined as

$$\mathcal{B}(Q(\rho_0)) = \sup_{\sigma_{AM}} Q_{A:M}(\sigma_{AM}) + \sup_{\sigma_{MB}} Q_{M:B}(\sigma_{MB}) + I_{A:B}(\rho_0), \quad (\text{A27})$$

where the supremum runs over all AM and MB states, while $I_{A:B}(\rho_0)$ is calculated as

$$I_{A:B}(\rho_0) = \inf_{\sigma_A \otimes \sigma_B} g(d(\rho_0, \sigma_A \otimes \sigma_B)), \quad (\text{A28})$$

and the infimum runs over all product states $\sigma_A \otimes \sigma_B$. Here σ_A and σ_B denote generic initial states of A and B , respectively.

APPENDIX B: CONNECTING THE MEDIATED QUANTUM CORRELATIONS TO THE NONCLASSICALITY OF INTERACTIONS

In this Appendix we derive Result 1 and Eq. (16), starting from the distance-based inequality used in [10]. The argument proceeds in two steps. First, we show that the quantity $g^{-1}(\Delta Q_{\text{corr } A:B})$ can be upper bounded by the spectral norm

of a suitable difference of unitaries $\|U - V\|_\infty$ associated with the global and local interaction terms of (A2) and (A3). Second, we rewrite this unitary difference in terms of the Zassenhaus correction $U_C(t)$ and its generator $\hat{H}_C$ and we show that it reduces to the spectral norm of $U_C(t) - \mathbb{I}$, which can be evaluated with a closed functional form.

We begin the derivation by recalling the distance-based inequality from [10],

$$g^{-1}(\Delta Q_{\text{corr } A:B}) \leq d(\tilde{\Lambda}(\rho_{AB}(0) \otimes \sigma_M), \tilde{\lambda}(\rho_{AB}(0) \otimes \sigma_M)), \quad (\text{B1})$$

which holds for arbitrary choices of the dilations $\tilde{\Lambda}$ and $\tilde{\lambda}$ of the marginal map Λ_{AB} and for all decomposable maps $\lambda_{AB} \in \overline{DEC}(m)$. In particular, since the right-hand side provides an upper bound for any admissible dilation pair, we may take a convenient choice for $\tilde{\Lambda}$ and for a decomposable $\tilde{\lambda}$. We pick the unitary dilation induced by the actual global evolution and a decomposable unitary dilation, namely,

$$\begin{aligned} \tilde{\Lambda}(\rho) &= \Lambda_{AMB}(\rho) = U_{AMB} \rho U_{AMB}^\dagger, \\ \tilde{\lambda}(\rho) &= \lambda_{AMB}(\rho) = (U_{AM} U_{MB}) \rho (U_{MB}^\dagger U_{AM}^\dagger), \end{aligned} \quad (\text{B2})$$

where we keep the time argument t implicit to lighten the notation. We also take d to be the trace distance,

$$d_{\text{tr}}(X, Y) = \frac{1}{2} \|X - Y\|_1. \quad (\text{B3})$$

For compactness, we define

$$U = U_{AMB}, \quad V = U_{AM} U_{MB}, \quad \rho = \rho_{AB}(0) \otimes \sigma_M. \quad (\text{B4})$$

With these choices, Eq. (B1) becomes

$$\begin{aligned} g^{-1}(\Delta Q_{\text{corr } A:B}) &\leq d_{\text{tr}}(\Lambda_{AMB}(\rho), \lambda_{AMB}(\rho)) \\ &= \frac{1}{2} \|U \rho U^\dagger - V \rho V^\dagger\|_1, \end{aligned} \quad (\text{B5})$$

which we can rewrite as

$$\begin{aligned} &\frac{1}{2} \|U \rho U^\dagger - V \rho V^\dagger\|_1 \\ &= \frac{1}{2} \|(U - V) \rho U^\dagger + V \rho (U^\dagger - V^\dagger)\|_1 \\ &\leq \frac{1}{2} [\|(U - V) \rho U^\dagger\|_1 + \|V \rho (U^\dagger - V^\dagger)\|_1], \end{aligned} \quad (\text{B6})$$

where we added and subtracted $V \rho U^\dagger$ and used the triangle inequality for $\|\cdot\|_1$. Next, using the ideal property of Schatten norms $\|AXB\|_1 \leq \|A\|_\infty \|X\|_1 \|B\|_\infty$ [25,26], we obtain

$$\begin{aligned} &\frac{1}{2} [\|(U - V) \rho U^\dagger\|_1 + \|V \rho (U^\dagger - V^\dagger)\|_1] \\ &\leq \frac{1}{2} (\|U - V\|_\infty \|\rho\|_1 \|U^\dagger\|_\infty + \|V\|_\infty \|\rho\|_1 \|U^\dagger - V^\dagger\|_\infty). \end{aligned} \quad (\text{B7})$$

Finally, since $\|\rho\|_1 = 1$ and U and V are unitary so that $\|U\|_\infty = \|V\|_\infty = \|U^\dagger\|_\infty = 1$ and $\|U^\dagger - V^\dagger\|_\infty = \|U - V\|_\infty$, we conclude that

$$g^{-1}(\Delta Q_{\text{corr } A:B}) \leq \|U - V\|_\infty. \quad (\text{B8})$$

Let us now express the right-hand side of (B8) in terms of the Zassenhaus correction $U_C(t)$ introduced in Appendix A. Using (A4) and (A7), we have

$$U_{AMB}(t) = U_{AM}(t) U_{MB}(t) U_C(t). \quad (\text{B9})$$

Therefore, given the definitions of U and V in (B4), we have

$$\begin{aligned} &g^{-1}(\Delta Q_{\text{corr } A:B}) \\ &\leq \|U - V\|_\infty = \|U_{AMB} - U_{AM} U_{MB}\|_\infty \\ &= \|U_{AM} U_{MB} (U_C - \mathbb{I})\|_\infty \leq \|U_{AM} U_{MB}\|_\infty \|U_C - \mathbb{I}\|_\infty \\ &= \|U_C - \mathbb{I}\|_\infty, \end{aligned} \quad (\text{B10})$$

where we used submultiplicativity of $\|\cdot\|_\infty$ and the fact that products of unitaries have a spectral norm equal to 1. Thus, we obtain

$$g^{-1}(\Delta Q_{\text{corr } A:B}) \leq \|U_C - \mathbb{I}\|_\infty. \quad (\text{B11})$$

To manipulate the right-hand side, we recall the spectral norm definition

$$\|X\|_\infty = \sqrt{\lambda_{\max}(X^\dagger X)}. \quad (\text{B12})$$

Evaluating the square of (B12) with $X = U_C(t) - \mathbb{I}$ yields

$$\begin{aligned} \|U_C - \mathbb{I}\|_\infty^2 &= \lambda_{\max}[(U_C - \mathbb{I})^\dagger (U_C - \mathbb{I})] \\ &= \lambda_{\max}[(e^{it\hat{H}_C} - \mathbb{I})(e^{-it\hat{H}_C} - \mathbb{I})] \\ &= \lambda_{\max}(2\mathbb{I} - e^{it\hat{H}_C} - e^{-it\hat{H}_C}) \\ &= \lambda_{\max}[2(\mathbb{I} - \cos(t\hat{H}_C))], \end{aligned} \quad (\text{B13})$$

where we used the definition (A8) for U_C and the operatorial identity $e^{iY} + e^{-iY} = 2\cos(Y)$ for a Hermitian Y . Using the trigonometric identity $1 - \cos(Y) = 2\sin^2(Y/2)$ and taking square roots gives

$$\|U_C - \mathbb{I}\|_\infty = 2 \left\| \sin\left(\frac{t}{2} \hat{H}_C\right) \right\|_\infty, \quad (\text{B14})$$

and therefore we find Result 1 given in (16),

$$g^{-1}(\Delta Q_{\text{corr } A:B}) \leq 2 \left\| \sin\left(\frac{t}{2} \hat{H}_C\right) \right\|_\infty. \quad (\text{B15})$$

The closed form (B15) also allows us to characterize the maximal value of the bound and the earliest interaction time at which it can be reached. Since $\|\sin(X)\|_\infty \leq 1$ for any Hermitian X , the right-hand side of (B15) is bounded by 2. Let $\{\lambda_j\}$ denote the eigenvalues of $\hat{H}_C$. Then we have

$$\left\| \sin\left(\frac{t}{2} \hat{H}_C\right) \right\|_\infty = \max_j \left| \sin\left(\frac{t}{2} \lambda_j\right) \right|. \quad (\text{B16})$$

Fix j with $\lambda_j \neq 0$. The modulus $|\sin(\frac{t}{2} \lambda_j)|$ attains its maximum value 1 if and only if

$$\frac{t}{2} \lambda_j = \frac{\pi}{2} + k\pi, \quad k \in \mathbb{Z}, \quad (\text{B17})$$

since $\sin(\frac{\pi}{2} + k\pi) = (-1)^k$. Accordingly, the set of interaction times for which the j th eigenvalue saturates the modulus is

$$t_{j,k} = \frac{\pi + 2k\pi}{\lambda_j}, \quad k \in \mathbb{Z}, \quad (\text{B18})$$

and the corresponding smallest positive time is obtained by taking $k = 0$ and using the modulus,

$$t_j^* = \frac{\pi}{|\lambda_j|}. \quad (\text{B19})$$

Therefore, the earliest time at which the operator norm can reach its maximum is

$$t_* = \min_j t_j^* = \min_j \frac{\pi}{|\lambda_j|} = \frac{\pi}{\max_j |\lambda_j|}. \quad (\text{B20})$$

Let $\lambda_{\max}$ be an eigenvalue such that $|\lambda_{\max}| = \max_j |\lambda_j|$. Then at $t = t_*$ we have

$$\left| \sin\left(\frac{t_*}{2} \lambda_{\max}\right) \right| = 1 \quad \text{and hence} \quad \left| \sin\left(\frac{t_*}{2} \hat{H}_C\right) \right|_{\infty} = 1. \quad (\text{B21})$$

In particular, for $\hat{H}_C \neq 0$, the right-hand side of (B15) attains its maximal value 2 at $t = t_*$.

Finally, although (B15) holds exactly for all interaction times t , it is useful to extract a more explicit commutator-based expression in the short-time regime. In particular, truncating U_C , as defined in (A7), to the leading nontrivial Zassenhaus term ($n = 2$) yields

$$U_C(t) = \exp[(-it)^2 Z_2]. \quad (\text{B22})$$

Using the explicit expression for Z_2 given in (A6), we then obtain

$$U_C(t) = \exp[-it\hat{H}_C(t)] = \exp\left(\frac{t^2}{2}[\hat{H}_{AM}, \hat{H}_{MB}]\right), \quad (\text{B23})$$

which implies

$$\hat{H}_C = \frac{it}{2}[\hat{H}_{AM}, \hat{H}_{MB}]. \quad (\text{B24})$$

Substituting this approximation into (B15) gives the explicit commutator form

$$g^{-1}(\Delta Q_{\text{corr } A:B}) \leq 2 \left\| \sin\left(\frac{it^2}{4}[\hat{H}_{AM}, \hat{H}_{MB}]\right) \right\|_{\infty}, \quad (\text{B25})$$

which matches the small- t version (17).

APPENDIX C: NONCLASSICALITY OF INTERACTIONS DEPENDS ON THE NONCLASSICALITY OF THE MEDIATOR

In this Appendix we show how the commutator between the local interaction Hamiltonians $\hat{H}_{AM}$ and $\hat{H}_{MB}$ in a tripartite system AMB depends on the nonclassicality of the investigated system M . We recall that we define a system as nonclassical when its description requires a minimum of two noncommuting observables, i.e., m_{j_1} and m_{j_2} such that $[m_{j_1}, m_{j_2}] \neq 0$ and $j_1 \neq j_2$.

We begin by writing the most general form of the local interaction Hamiltonians $\hat{H}_{AM}$ and $\hat{H}_{MB}$ used in the main text (23).

$$\begin{aligned} \hat{H}_{AM} &= \sum_{i,j_1,j_2,\dots,j_T} \alpha_{i,j_1,j_2,\dots,j_T} (a_i \otimes m_{j_1} \otimes m_{j_2} \otimes \dots \otimes m_{j_T} \otimes \mathbb{I}_B) \\ \hat{H}_{MB} &= \sum_{l_1,l_2,\dots,l_T,k} \beta_{l_1,l_2,\dots,l_T,k} (\mathbb{I}_A \otimes m_{l_1} \otimes m_{l_2} \otimes \dots \otimes m_{l_T} \otimes b_k). \end{aligned}$$

We can now write the commutator

$$[\hat{H}_{AM}, \hat{H}_{MB}] = \left[\sum_{i,j_1,j_2,\dots,j_T} \alpha_{i,j_1,j_2,\dots,j_T} (a_i \otimes m_{j_1} \otimes m_{j_2} \otimes \dots \otimes m_{j_T} \otimes \mathbb{I}_B), \sum_{l_1,l_2,\dots,l_T,k} \beta_{l_1,l_2,\dots,l_T,k} (\mathbb{I}_A \otimes m_{l_1} \otimes m_{l_2} \otimes \dots \otimes m_{l_T} \otimes b_k) \right]. \quad (\text{C1})$$

To simplify the notation, we factorize the sums and the coefficients out of the commutator by using its linearity,

$$[\hat{H}_{AM}, \hat{H}_{MB}] = \sum_{i,j_1,j_2,\dots,j_T,l_1,l_2,\dots,l_T,k} \alpha_{i,j_1,j_2,\dots,j_T} \beta_{l_1,l_2,\dots,l_T,k} [a_i \otimes m_{j_1} \otimes m_{j_2} \otimes \dots \otimes m_{j_T} \otimes \mathbb{I}_B, \mathbb{I}_A \otimes m_{l_1} \otimes m_{l_2} \otimes \dots \otimes m_{l_T} \otimes b_k]. \quad (\text{C2})$$

Using the linearity of the commutator and distributivity of the tensor product again, we can factorize probe operators out as well,

$$[\hat{H}_{AM}, \hat{H}_{MB}] = \sum_{i,j_1,j_2,\dots,j_T,l_1,l_2,\dots,l_T,k} \alpha_{i,j_1,j_2,\dots,j_T} \beta_{l_1,l_2,\dots,l_T,k} a_i \otimes [m_{j_1} \otimes m_{j_2} \otimes \dots \otimes m_{j_T}, m_{l_1} \otimes m_{l_2} \otimes \dots \otimes m_{l_T}] \otimes b_k. \quad (\text{C3})$$

Now we can only focus on the commutator of mediator observables. We are also going to use the following property of the commutator:

$$[A \otimes B, C \otimes D] = CA \otimes [B, D] + [A, C] \otimes BD. \quad (\text{C4})$$

Here $A, C \in \mathcal{H}$ and $B, D \in \mathcal{H}'$, with $\mathcal{H}$ and $\mathcal{H}'$ denoting two different Hilbert spaces. Then we set $A = m_{j_1}$, $C = m_{l_1}$, $B = m_{j_2} \otimes \dots \otimes m_{j_T}$, and $D = m_{l_2} \otimes \dots \otimes m_{l_T}$, obtaining

$$\begin{aligned} &[m_{j_1} \otimes m_{j_2} \otimes \dots \otimes m_{j_T}, m_{l_1} \otimes m_{l_2} \otimes \dots \otimes m_{l_T}] \\ &= m_{l_1} m_{j_1} \otimes [m_{j_2} \otimes \dots \otimes m_{j_T}, m_{l_2} \otimes \dots \otimes m_{l_T}] \\ &+ [m_{j_1}, m_{l_1}] \otimes m_{j_2} m_{l_2} \otimes \dots \otimes m_{j_T} m_{l_T}. \end{aligned} \quad (\text{C5})$$

Now we can recursively apply the same formula (C4) to the commutator in the first term of the sum in Eq. (C2), coming to the final result for the commutator of the Hamiltonians

$$\begin{aligned} &[\hat{H}_{AM}, \hat{H}_{MB}] \\ &= \sum_{i,j_1,j_2,\dots,j_T,l_1,l_2,\dots,l_T,k} \alpha_{i,j_1,j_2,\dots,j_T} \beta_{l_1,l_2,\dots,l_T,k} a_i \\ &\otimes ([m_{j_1}, m_{l_1}] \otimes m_{j_2} m_{l_2} \otimes \dots \otimes m_{j_T} m_{l_T} \\ &+ m_{l_1} m_{j_1} \otimes [m_{j_2}, m_{l_2}] \otimes \dots \otimes m_{j_T} m_{l_T} \\ &+ m_{l_1} m_{j_1} \otimes m_{l_2} m_{j_2} \otimes \dots \otimes [m_{j_T}, m_{l_T}]) \otimes b_k. \end{aligned} \quad (\text{C6})$$

When considering just two qubits for the mediator, the commutator of the two Hamiltonians reduces to

$$[\hat{H}_{AM}, \hat{H}_{MB}] = \sum_{i,j_1,j_2,l_1,l_2,k} \alpha_{i,j_1,j_2} \beta_{l_1,l_2,k} a_i \otimes ([m_{j_1}, m_{l_1}] \otimes m_{j_2} m_{l_2} + m_{l_1} m_{j_1} \otimes [m_{j_2}, m_{l_2}]) \otimes b_k. \quad (C7)$$

To determine when the two-qubit interaction Hamiltonians commute or not, it is sufficient to analyze the mediator-dependent factor appearing in (C7):

$$[m_{j_1}, m_{l_1}] \otimes m_{j_2} m_{l_2} + m_{l_1} m_{j_1} \otimes [m_{j_2}, m_{l_2}]. \quad (C8)$$

Indeed, the probe operators a_i and b_k factorize, so $[\hat{H}_{AM}, \hat{H}_{MB}] \neq 0$ only if the factor (C8) is nonvanishing. We therefore simplify the combination of commutators and products of mediators' observables into a more compact form

$$\begin{aligned} & [m_{j_1}, m_{l_1}] \otimes m_{j_2} m_{l_2} + m_{l_1} m_{j_1} \otimes [m_{j_2}, m_{l_2}] \\ &= (m_{j_1} m_{l_1} - m_{l_1} m_{j_1}) \otimes m_{j_2} m_{l_2} \\ &+ m_{l_1} m_{j_1} \otimes (m_{j_2} m_{l_2} - m_{l_2} m_{j_2}) \\ &= m_{j_1} m_{l_1} \otimes m_{j_2} m_{l_2} - m_{l_1} m_{j_1} \otimes m_{j_2} m_{l_2} \\ &+ m_{l_1} m_{j_1} \otimes m_{j_2} m_{l_2} - m_{l_1} m_{j_1} \otimes m_{l_2} m_{j_2} \\ &= m_{j_1} m_{l_1} \otimes m_{j_2} m_{l_2} - m_{l_1} m_{j_1} \otimes m_{l_2} m_{j_2}. \end{aligned} \quad (C9)$$

In the first two lines, we expand the commutators and distribute the tensor product over sums. After multiplying out, the second and third terms cancel. Next we use the Pauli-algebra anticommutation relations

$$m_{l_1} m_{j_1} = 2\delta_{l_1 j_1} \mathbb{I} - m_{j_1} m_{l_1}. \quad (C10)$$

Substituting (C10) into (C9) and collecting terms gives

$$\begin{aligned} & [m_{j_1}, m_{l_1}] \otimes m_{j_2} m_{l_2} + m_{l_1} m_{j_1} \otimes [m_{j_2}, m_{l_2}] \\ &= m_{j_1} m_{l_1} \otimes m_{j_2} m_{l_2} - (2\delta_{l_1 j_1} \mathbb{I} - m_{j_1} m_{l_1}) \otimes m_{l_2} m_{j_2} \\ &= m_{j_1} m_{l_1} \otimes (m_{j_2} m_{l_2} + m_{l_2} m_{j_2}) - 2\delta_{l_1 j_1} \mathbb{I} \otimes m_{l_2} m_{j_2}. \end{aligned} \quad (C11)$$

Combining the previous steps yields

$$[m_{j_1}, m_{l_1}] \otimes m_{j_2} m_{l_2} + m_{l_1} m_{j_1} \otimes [m_{j_2}, m_{l_2}] = \dots$$

$$= m_{j_1} m_{l_1} \otimes \{m_{j_2}, m_{l_2}\} - 2\delta_{l_1 j_1} \mathbb{I} \otimes m_{l_2} m_{j_2}. \quad (C12)$$

Having obtained (C12), we can now characterize when this mediator factor (C8) vanishes and hence when $[\hat{H}_{AM}, \hat{H}_{MB}] = 0$. This is conveniently done by distinguishing whether the first-qubit indices satisfy $l_1 \neq j_1$ or $l_1 = j_1$ (an equivalent split could be performed using the second mediator qubit).

Case analysis for the two-qubit mediator factor. Starting from (C12), we distinguish two cases.

(i) The condition $l_1 \neq j_1$ implies $\delta_{l_1 j_1} = 0$, leaving $m_{j_1} m_{l_1} \otimes \{m_{j_2}, m_{l_2}\}$ to be the only term that may be nonvanishing in (C12). This is the case when

$$\begin{aligned} m_{l_2} = m_{j_2} = \mathbb{I}, \\ m_{l_2} = m_{j_2} = X, \quad \text{or} \quad m_{l_2} = \mathbb{I}; \quad m_{j_2} = X, Y, Z; \\ m_{l_2} = m_{j_2} = Y, \quad \text{or} \quad m_{l_2} = X, Y, Z; \quad m_{j_2} = \mathbb{I}. \\ m_{l_2} = m_{j_2} = Z \end{aligned}$$

As a result, the second commutator appearing in (C8) vanishes, and this yields

$$\begin{aligned} & [m_{j_1}, m_{l_1}] \otimes m_{j_2} m_{l_2} + m_{l_1} m_{j_1} \otimes [m_{j_2}, m_{l_2}] \\ &= [m_{j_1}, m_{l_1}] \otimes m_{j_2} m_{l_2}. \end{aligned} \quad (C13)$$

Thus, in the case $l_1 \neq j_1$, the nonvanishing contribution to the mediator factor can be traced back to noncommutativity of the first-qubit operators, i.e., to $[m_{j_1}, m_{l_1}] \neq 0$.

(ii) For the condition $l_1 = j_1$, after some simple manipulations (C8) reads

$$\begin{aligned} & [m_{j_1}, m_{l_1}] \otimes m_{j_2} m_{l_2} + m_{l_1} m_{j_1} \otimes [m_{j_2}, m_{l_2}] \\ &= \mathbb{I} \otimes [m_{j_2}, m_{l_2}]. \end{aligned} \quad (C14)$$

Therefore, in this second case, when $l_1 = j_1$, the mediator factor (C9) is nonvanishing only if the second-qubit contributes with noncommuting observables, i.e., only if $[m_{j_2}, m_{l_2}] \neq 0$.

In summary, in the two-qubit expansion (C8), the commutator $[\hat{H}_{AM}, \hat{H}_{MB}]$ can be nonzero only if at least one pair of mediator Pauli operators, appearing in the two Hamiltonians, is noncommuting (either on the first or the second mediator qubit).

APPENDIX D: DEPHASING THE MEDIATOR'S QUBITS IN THE HEISENBERG PICTURE

In this Appendix we rewrite the Hamiltonians defined in (23) using the descriptor formalism [14,15] and recalculate their commutator after applying the dephasing channel to each of the mediator's qubits.

Explicitly, the two interaction Hamiltonians can be written in the form

$$\hat{H}_{AM} = \sum_{a_i, j_1, \dots, j_T} \alpha_{a_i, j_1, \dots, j_T} (q_{a_i} q_{j_1} \cdots q_{j_T} q_{\mathbb{I}_B}), \quad (D1)$$

$$\hat{H}_{MB} = \sum_{l_1, \dots, l_T, b_k} \beta_{l_1, \dots, l_T, b_k} (q_{\mathbb{I}_A} q_{l_1} \cdots q_{l_T} q_{b_k}). \quad (D2)$$

As explained in Sec. III, the tensor product is replaced by a standard product since all the descriptors belong to AMB Hilbert space. To preserve the locality condition, the identity operators of the original Hamiltonians (23) are replaced by the descriptors $q_{\mathbb{I}_B}$ and $q_{\mathbb{I}_A}$, which are a tensor product of identity operators of all the qubits in the system.

Let us now apply a dephasing channel, as defined in (29), to the first qubit of the mediator on the Hamiltonian $\hat{H}_{AM}$ expressed in terms of descriptors (D1),

$$\begin{aligned}\hat{E}_1(\hat{H}_{AM}) &\equiv \sum_a M_a(t)^\dagger \hat{H}_{AM} M_a(t) = \sqrt{p} I_{M_1} \hat{H}_{AM} I_{M_1} \sqrt{p} + \sqrt{1-p} q_{z_1}^\dagger(t) \hat{H}_{AM} q_{z_1}(t) \sqrt{1-p} \\ &= p \sum_{a_i, j_1, \dots, j_T} \alpha_{a_i, j_1, \dots, j_T} (q_{a_i} I_{M_1} q_{j_1} I_{M_1} \cdots q_{j_T} q_{\mathbb{I}_B}) + (1-p) \sum_{a_i, j_1, \dots, j_T} \alpha_{a_i, j_1, \dots, j_T} (q_{a_i} q_{z_1} q_{j_1} q_{z_1} \cdots q_{j_T} q_{\mathbb{I}_B}).\end{aligned}\quad (\text{D3})$$

We now expand the second sum on the index j_1 and use the algebraic rule of the descriptors to compute $q_{z_1} q_{j_1} q_{z_1}$,

$$\begin{aligned}\hat{E}_1(\hat{H}_{AM}) &= p \sum_{a_i, j_1, \dots, j_T} \alpha_{a_i, j_1, \dots, j_T} (q_{a_i} q_{j_1} \cdots q_{j_T} q_{\mathbb{I}_B}) + (1-p) \left(\sum_{a_i, j_2, \dots, j_T} \alpha_{a_i, j_1=\mathbb{I}_1, j_2, \dots, j_T} (q_{a_i} q_{\mathbb{I}_1} \cdots q_{j_T} q_{\mathbb{I}_B}) \right. \\ &+ \sum_{a_i, j_2, \dots, j_T} \alpha_{a_i, j_1=z_1, j_2, \dots, j_T} (q_{a_i} q_{z_1} \cdots q_{j_T} q_{\mathbb{I}_B}) - \sum_{a_i, j_2, \dots, j_T} \alpha_{a_i, j_1=x_1, j_2, \dots, j_T} (q_{a_i} q_{x_1} \cdots q_{j_T} q_{\mathbb{I}_B}) \\ &\left. - \sum_{a_i, j_2, \dots, j_T} \alpha_{a_i, j_1=y_1, j_2, \dots, j_T} (q_{a_i} q_{y_1} \cdots q_{j_T} q_{\mathbb{I}_B}) \right).\end{aligned}\quad (\text{D4})$$

Now we expand the first sum on the index j_1 and sum the equivalent terms

$$\begin{aligned}\hat{E}_1(\hat{H}_{AM}) &= \sum_{a_i, j_2, \dots, j_T} \alpha_{a_i, j_1=\mathbb{I}_1, j_2, \dots, j_T} (q_{a_i} q_{\mathbb{I}_1} \cdots q_{j_T} q_{\mathbb{I}_B}) + \sum_{a_i, j_2, \dots, j_T} \alpha_{a_i, j_1=z_1, j_2, \dots, j_T} (q_{a_i} q_{z_1} \cdots q_{j_T} q_{\mathbb{I}_B}) \\ &+ (2p-1) \sum_{a_i, j_2, \dots, j_T} \alpha_{a_i, j_1=x_1, j_2, \dots, j_T} (q_{a_i} q_{x_1} \cdots q_{j_T} q_{\mathbb{I}_B}) + (2p-1) \sum_{a_i, j_2, \dots, j_T} \alpha_{a_i, j_1=y_1, j_2, \dots, j_T} (q_{a_i} q_{y_1} \cdots q_{j_T} q_{\mathbb{I}_B}).\end{aligned}\quad (\text{D5})$$

Eventually, we absorb the four summation symbols and redefine a new coefficient α' , which takes into account a factor $(2p-1)$ when multiplied by q_x or q_y . This new primed coefficient now considers the effect of the phase flip on the first qubit of the mediator:

$$\hat{E}_1(\hat{H}_{AM}) = \sum_{a_i, j_1, \dots, j_T} \alpha'_{a_i, j_1, \dots, j_T} (q_{a_i} q_{j_1} \cdots q_{j_T} q_{\mathbb{I}_B}).\quad (\text{D6})$$

Further phase-flip operators $\hat{E}_2, \hat{E}_3, \dots, \hat{E}_T$, behave equivalently to $\hat{E}_1$, modifying coefficients when the respective observables q_x and q_y appear, with an extra $(2p-1)$. Having derived the dephased Hamiltonians, we can proceed to derive their commutator, which appears in (33). This is a straightforward task, as we can substitute back the tensor product of the observables $a_i \otimes m_{j_1} \otimes m_{j_2} \otimes \cdots \otimes m_{j_T} \otimes \mathbb{I}_B$ and perform the same derivation of Appendix C which led to (C6), obtaining the final result

$$\begin{aligned}[\hat{H}_{AM}, \hat{H}_{MB}] &= \sum_{a_i, j_1, j_2, \dots, j_T, l_1, l_2, \dots, l_T, b_k} \alpha'_{a_i, j_1, j_2, \dots, j_T} \beta'_{l_1, l_2, \dots, l_T, b_k} a_i \otimes ([m_{j_1}, m_{l_1}] \otimes m_{j_2} m_{l_2} \otimes \cdots \otimes m_{j_T} m_{l_T} \\ &\times m_{l_1} m_{j_1} \otimes [m_{j_2}, m_{l_2}] \otimes \cdots \otimes m_{j_T} m_{l_T} m_{l_1} m_{j_1} \otimes m_{l_2} m_{j_2} \otimes \cdots \otimes [m_{j_T}, m_{l_T}]) \otimes b_k,\end{aligned}\quad (\text{D7})$$

which can be reexpressed using the descriptor formalism as

$$\begin{aligned}[\hat{H}_{AM}, \hat{H}_{MB}] &= \sum_{a_i, j_1, j_2, \dots, j_T, l_1, l_2, \dots, l_T, b_k} \alpha'_{a_i, j_1, j_2, \dots, j_T} \beta'_{l_1, l_2, \dots, l_T, b_k} q_{a_i} ([q_{j_1}, q_{l_1}] q_{j_2} q_{l_2} \cdots q_{j_T} q_{l_T} + q_{l_1} q_{j_1} [q_{j_2}, q_{l_2}] \cdots q_{j_T} q_{l_T} \\ &+ q_{l_1} q_{j_1} q_{l_2} q_{j_2} \cdots [q_{j_T}, q_{l_T}]) q_{b_k}.\end{aligned}\quad (\text{D8})$$

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