

Designing Dirac solutions from geometry: A light-cone blueprint via covariant relativistic dynamical inversion

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We present a manifestly covariant route to *designing* exact Dirac solutions from geometric data. Given the spinor density ρ , four-velocity v^μ , spin s^μ , Yvon-Takabayashi (YT) angle β_s , and the local Lorentz rotor, covariant relativistic dynamical inversion (CRDI) returns a real four-potential A_μ for which the prescribed spinor is a solution to the Dirac equation. We investigate the relationship between the parameters of the spinor and the electromagnetic potentials that satisfy the corresponding Dirac equation. Specifically, we examine the impact of a nontrivial variation of the YT angle and the boost matrix embedded within the spinor on the distortion of electromagnetic fields, and seek analytically tractable null-sphere solutions on the hypersurfaces $u = ct - r$. We show how this approach facilitates a generalized parametrization of established solutions, providing a pathway to generate novel, physically meaningful configurations.

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I. INTRODUCTION

Exact Dirac solutions play a pivotal role in relativistic quantum theory and strong-field physics [1,2]. Two archetypes dominate: localized stationary Coulomb states and time-dependent Volkov/Furry solutions that are delocalized [3,4]. Between these extremes, one often wants localized, time-dependent, analytically tractable states in structured fields for external-field QED, benchmarking numerics, and wave engineering.

An appealing strategy is to treat Dirac as an *inverse problem*: given a desired spinorial motion/geometry, reconstruct a background that admits it. Several communities have developed such inversions. For instance, *algebraic inversions* solve the minimally coupled Dirac equation linearly for the unknown four-potential A_μ in terms of a prescribed spinor ψ and its bilinears. Early consistency concerns were noted by Eliezer [5]. Constructive formulations then showed how to obtain A_μ by enforcing algebraic constraints among bilinear covariants: Booth, Legg, and Jarvis recast the Dirac equation as a rank-deficient linear system and resolved the singularity with Fierz-type identities, providing explicit algebraic formulas [6]; Inglis and Jarvis later gave a fully gauge-invariant bilinear-covariant formulation and symmetry reductions that deliver exact families [7]. Related numerical work by Radford constructed localized self-consistent Maxwell-Dirac states, illustrating the usefulness of algebraic inversions as building blocks [8].

Inverse scattering and spectral programs generalize Gel'fand-Levitan/Marchenko ideas to Dirac operators. For spherically symmetric central potentials, reconstructions from phase shifts and bound states were established in the 1950s–1970s [9,10]. On the line, Gel'fand-Levitan-Marchenko-type equations and uniqueness theorems provide rigorous inversions for one-dimensional Dirac systems [11]. Classic monographs and later developments supply a rigorous foundation and extensions [12,13].

Relativistic dynamical inversion (RDI) and its manifestly covariant extension (CRDI) [14,15] target the same functional inversion—given ψ , find a real A_μ such that $[\gamma^\mu(i\hbar\partial_\mu - qA_\mu) - mc]\psi = 0$ —but do so with a *geometric* design language. Writing the spinor in Hestenes polar form $\Psi = \sqrt{\rho} e^{i\gamma_s\beta_s/2}\mathcal{R}$ exposes scalars (ρ, β_s), bilinear (kinematic) fields (v^μ, s^μ), and a local *Lorentz transformation* (rotor) $\mathcal{R}$ with an associated spin connection; CRDI expresses A_μ locally in terms of rotor rates and bilinears and, crucially, is formulated directly on tetrads, enabling non-Cartesian coordinates and accelerated frames without loss of covariance [16–18].

In turn, the geometric form of the Hestenes spinor allows for a transparent physical interpretation, where the rotor $\mathcal{R}$ corresponds to the Lorentz boost and rotation transformations, which transports the initial electron state to the state in the instantaneous rest frame. The latter, in the case of quasiclassical dynamics, can be deduced from the classical motion (solving the rotor equation), while Yvon-Takabayashi (YT) angle β_s is settled to fit the properties of the specific spin-orbit coupling in the system. Thus, employing given physical symmetries of a system, it is possible to construct geometrically the spinor (the method dubbed relativistic dynamical construction (RDC) in Ref. [19]), then check the validity of this heuristic procedure by means of CRDI. The ability of the RDC-CRDI method to reproduce well-established solutions to the Dirac equation has been demonstrated for the following cases homogeneous magnetic fields (Landau levels), inhomogeneous

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magnetic fields and the Redmond solution [20], as well as the ground state of the hydrogen atom [15].

Drawing from the insights gained so far, and to develop further our geometrical intuition within the RDC-CRDI framework, in this paper we explore in detail the connection between the parameters of Ψ and the electromagnetic vector potential A_μ that solves the corresponding Dirac equation. Doing so may help us find effective ways to generalize the parametrization leading to well-established solutions in order to generate new, physically meaningful ones. We will focus in particular on the YT angle β_s and on the boost matrix contained in the rotor $\mathcal{R}$, providing explicit families of stationary exact solutions. Our aim is to understand how the specific form of these parameters influences the electromagnetic fields within the Dirac equation. Moreover, aiming at constructing an analytical solution for the dipole-pulse theory (DPT) of Gonoskov *et al.* [21,22], we also design analytically tractable null-sphere solutions (retarded time $u = ct - r$). Furthermore, we provide a geometric “steering/throttle” interpretation of the fields built from the given Dirac spinor ψ , tying it back to the geometrical information displayed in Ψ .

This paper is organized as follows: Section II translates our notation to the standard Dirac equation, defines the kinematic fields as bilinear covariants, and summarizes the CRDI inversion. Section III discusses a family of static solutions with and without a YT angle and investigates the properties of the applied EM fields and corresponding sources. Section IV introduces the null-foliated ansatz and gives the compact potential. Section V discusses further general solutions. We conclude in Sec. VI and discuss further applications of the method. Appendix A motivates the ansatz geometrically, while Appendix B provides the detailed distributional/removal analysis for the null-foliated ansatz. Throughout, we treat A_μ as an externally generated background. CRDI provides A_μ such that the prescribed spinor is a solution to the Dirac equation in that background. Once A_μ is known, we compute $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and the Maxwell source required to support it, $J^\mu \equiv \nabla_\nu F^{\nu\mu}$.

II. CRDI CONVENTIONS AND RELATION TO THE STANDARD DIRAC EQUATION

A. Standard Dirac equation and the Dirac-Hestenes spinor

We start from the standard minimally coupled Dirac equation for the standard Dirac spinor ψ ,

$$[\gamma^\mu (i\hbar \nabla_\mu - qA_\mu) - mc]\psi = 0, \quad (1)$$

with metric signature $(+, -, -, -)$. In the Hestenes (spacetime-algebra) formulation one equivalently represents the same spinorial degrees of freedom by an even multivector (a Dirac-Hestenes spinor) Ψ . The map between ψ and Ψ is given by

$$\Psi = \begin{pmatrix} \psi_1 & -\psi_2^* & \psi_3 & \psi_4^* \\ \psi_2 & \psi_1^* & \psi_4 & -\psi_3^* \\ \psi_3 & \psi_4^* & \psi_1 & -\psi_2^* \\ \psi_4 & -\psi_3^* & \psi_2 & \psi_1^* \end{pmatrix},$$

and all bilinears computed from Ψ coincide with those computed from ψ (see Refs. [16–18]).

In this notation the matrix form of the Dirac equation is given by

$$(\hbar c \gamma^\mu \nabla_\mu \Psi \gamma_2 \gamma_1 - ce A_\mu \gamma^\mu \Psi) = mc^2 \Psi \gamma_0. \quad (2)$$

A more detailed explanation of the Hestenes equation in the context of this paper can be found in [15]. The right-multiplication by the bivector $\gamma_2 \gamma_1$ plays the role of the imaginary unit in the standard complex formulation, yielding an equation that is equivalent to Eq. (1).

B. Bilinear covariants and kinematic fields

The *kinematic fields* used in CRDI are the standard Dirac bilinear covariants. Define the Dirac current and axial (spin) current

$$\begin{aligned} j^\mu &= \bar{\psi} \gamma^\mu \psi = \frac{\text{Tr}[\gamma_\mu \Psi \Psi^\dagger \gamma^0]}{4}, \\ s^\mu &= \bar{\psi} \gamma^\mu \gamma^5 \psi = \frac{\text{Tr}[\gamma_\mu \Psi \alpha^3 \Psi^\dagger \gamma^0]}{4}. \end{aligned} \quad (3)$$

Since j^μ is timelike, the Lorentz scalar density and unit vectors are

$$\rho \equiv \sqrt{j_\mu j^\mu}, \quad v^\mu \equiv \frac{j^\mu}{\rho}, \quad s^\mu \equiv \frac{s^\mu}{\rho}. \quad (4)$$

These satisfy the standard Fierz-implied constraints

$$v_\mu v^\mu = 1, \quad s_\mu s^\mu = -1, \quad v_\mu s^\mu = 0. \quad (5)$$

Thus ρ is a Lorentz scalar and v^μ, s^μ are bilinear four-vectors determined by the spinor field.

The Hestenes polar form of the spinor is

$$\Psi = \sqrt{\rho} e^{i\gamma_5 \beta_s / 2} \mathcal{R}, \quad \mathcal{R} \in \text{Spin}(1, 3), \quad (6)$$

with Yvon-Takabayashi (YT) angle β_s and local Lorentz rotor $\mathcal{R}$. Let $R_{ij\mu}$ be the spin connection extracted from $\mathcal{R}$ via $\mathcal{R} \partial_\mu \mathcal{R}^{-1} = \frac{1}{2} R_{ij\mu} \sigma^{ij}$ with $\sigma^{ij} = \frac{1}{4} [\gamma^i, \gamma^j]$. Define $R_\mu \equiv R^a_{\mu a}$ and $B_\mu \equiv \frac{1}{2} \varepsilon_{\mu\alpha\beta\gamma} R^{\alpha\beta\gamma}$. Then the CRDI inversion reads [15]

$$\begin{aligned} eA^\eta &= -mc \cos \beta_s v^\eta + \frac{\hbar}{2} [(\partial_\mu \beta_s + B_\mu) v^\mu s^\eta \\ &\quad - (\partial_\mu \beta_s + B_\mu) s^\mu v^\eta - (\partial_\mu \ln \rho + R_\mu) s_\alpha v_\nu \varepsilon^{\mu\alpha\nu\eta}], \end{aligned} \quad (7)$$

and requiring $A_\mu \in \mathbb{R}$ demands

$$\nabla_\mu j^\mu = 0, \quad \nabla_\mu s^\mu = -\frac{2mc}{\hbar} \rho \sin \beta_s. \quad (8)$$

When $\beta_s = 0$ (the sector emphasized here for the time-dependent solutions), both j^μ and s^μ are conserved vector densities.

Let us now explain in more detail the geometrical meaning of Eq. (7), tying it directly to the matrix spinor. Through $\mathcal{R}$ (the local Lorentz frame comoving with the electron) Ψ builds the *oriented timelike plane* Σ built from s^μ and v^μ . The vector potential is then the sum of (i) a longitudinal piece along the flow of v_μ , with magnitude set by the *effective mass* $m \cos \beta_s$, (ii) a piece (the second and third terms) that tells how much the internal phase varies on the plane Σ , and (iii) the amplitude

and local Lorentz frame expansion on the plane orthogonal to Σ given by the last term. It is literally *vector potential* = *longitudinal drift plus how the local boosted and rotated orientation of the spin changes*.

CRDI can be used in “design mode” (RDC): one may prescribe geometric fields $(\rho, v^\mu, s^\mu, \beta_s, \mathcal{R})$ provided they satisfy the bilinear constraints and the compatibility conditions (8). The inversion (7) then yields a real A_μ compatible with the prescribed spinor geometry. Note that the geometric fields mentioned above can be constructed employing the symmetries of a physical problem via Lie derivatives [19], while the further CRDI confirms the initial construction conjecture or point out the deficiencies. As $\mathcal{R}$ incorporates the Lorentz-boost and rotation transformations, which connects the initial spinor to the rest frame one, in the quasiclassical case, it can be readily derived by solving the classical rotor equation [19].

III. STATIC FAMILY OF SOLUTIONS

Within the CRDI-RDC framework, we first explore the spinor blueprint that builds upon the success of the method describing a free hydrogen atom [15], and that in a magnetic field [20]. Aiming at generalization, we consider a static, localized $S_{1/2}$ type family in spherical coordinates with the inverse Lorentz factor as a controlling function and scrutinize the resulting potentials and the associated Maxwell fields to establish a connection to real physical setups.

Thus, we adopt spherical coordinates (t, r, θ, ϕ) with signature $(+, -, -, -)$, and introduce the localized $S_{1/2}$ spinor $[G(r) > 0]$, similar to that of a hydrogen atom in a ground state [15]:

$$\psi = e^{-\frac{G(r)}{2}} \begin{pmatrix} \kappa(\cos \frac{\theta}{2} - \sin \frac{\theta}{2})\sqrt{a(r)+1} \frac{e^{-i(\epsilon t)/\hbar - i\phi/2}}{r\sqrt{2}\sin\theta} \\ \kappa(\sin \frac{\theta}{2} + \cos \frac{\theta}{2})\sqrt{a(r)+1} \frac{e^{-i(\epsilon t)/\hbar + i\phi/2}}{r\sqrt{2}\sin\theta} \\ i\kappa(\sin \frac{\theta}{2} + \cos \frac{\theta}{2})\sqrt{1-a(r)} \frac{e^{-i(\epsilon t)/\hbar - i\phi/2}}{r\sqrt{2}\sin\theta} \\ i\kappa(\sin \frac{\theta}{2} - \cos \frac{\theta}{2})\sqrt{1-a(r)} \frac{e^{-i(\epsilon t)/\hbar + i\phi/2}}{r\sqrt{2}\sin\theta} \end{pmatrix}, \quad (9)$$

where κ is a real amplitude/normalization parameter setting the overall scale of ρ and j^μ (hence also the scale of the external sources J^μ inferred from Maxwell’s equations) and ϵ is a constant energy offset appearing in the global time-dependent phase; in the inverted potential it appears as

an additive constant in A_t and therefore fixes an energy/gauge reference. The only difference between the spinor (9) and the one studied in [15] [see Eq. (164) therein] is that now $0 < a(r) \leq 1$ is a function instead of a constant.

Let us now investigate a bit more closely the form of the Dirac spinor (9) written now in the Hestenes representation. With $\beta_s = 0$, the polar form of the matrix spinor is $\Psi = \sqrt{\rho} \mathcal{R}$, where

$$r^2 \rho = \kappa^2 \csc \theta a(r) e^{-G(r)}, \quad \mathcal{R} = \mathcal{B} U e^{-\gamma^2 \gamma^1 \epsilon t / \hbar}. \quad (10)$$

The boost $\mathcal{B}$ and rotation U matrices are, respectively

$$\mathcal{B} = \exp \left[\frac{\operatorname{arctanh} f}{2} (-\sin \phi \gamma^1 \gamma^0 + \cos \phi \gamma^2 \gamma^0) \right], \\ U = e^{-\gamma^2 \gamma^1 \phi / 2} e^{-\gamma^3 \gamma^1 \theta / 2} e^{-\gamma^3 \gamma^1 \pi / 4}, \quad (11)$$

where $f = -\sqrt{1 - a(r)^2}$. The bilinears are $s_\mu = (0, 0, -r, 0)$ and

$$v_\mu = \left\{ \frac{1}{a(r)}, 0, 0, \frac{r \sin \theta}{a(r)} \sqrt{1 - a(r)^2} \right\}. \quad (12)$$

These relations are remarkable, and worthy of more careful analysis. By defining $\beta(r) \equiv v(r)/c = \sqrt{1 - a(r)^2}$ then $a(r) = \sqrt{1 - \beta(r)^2} = \gamma(r)^{-1}$, we deduce that $a(r)$ is not a free parameter; it is in fact the *inverse Lorentz factor* of the circulating probability/charge current, which relate the spinor of Eq. (9) to the physical situation. When $a \rightarrow 1$, the flow is slow; when $a \rightarrow 0$, the flow becomes ultrarelativistic.

The flow described above generate electromagnetic fields, described by the inverted vector potential

$$eA_t = \frac{\hbar \gamma(r)}{2} \left[-\beta(r)' + \beta(r)G'(r) - \frac{2cm}{\hbar} \right] + \frac{\epsilon}{c}, \quad (13)$$

$$eA_\phi = \frac{\hbar \gamma(r)}{2} r \sin \theta \left[G'(r) - \frac{2cm}{\hbar} \beta(r) \right], \quad (14)$$

with $A_r = A_\theta = 0$ and the envelop function $G(r) > 0$.

The physical deficiency of these potentials is the angular singularity for $\theta = 0$ and $\theta = \pi$. In fact, rewriting the vector potential (13) in the compact form

$$eA_t = H(r) + \frac{\epsilon}{c}, \quad eA_\phi = \sin \theta F(r) \quad (15)$$

leads to the EM tensor

$$F^{\mu\nu} = \begin{pmatrix} 0 & H'(r) & 0 & 0 \\ -H'(r) & 0 & 0 & \frac{\csc(\theta)F'(r)}{r^2} \\ 0 & 0 & 0 & \frac{F(r)\cot(\theta)\csc(\theta)}{r^4} \\ 0 & -\frac{\csc(\theta)F'(r)}{r^2} & -\frac{F(r)\cot(\theta)\csc(\theta)}{r^4} & 0 \end{pmatrix} \quad (16)$$

and an azimuthal current density

$$J^\phi = \frac{F(r) \csc^3(\theta) - r^2 \csc(\theta) F''(r)}{r^4}, \quad (17)$$

both displaying a singularity for $\theta = 0, \theta = \pi$.

However, there is a remedy for this deficiency, the EM fields and sources are regularizable via proper choice of the YT angle. It was proposed in Ref. [15] that a nonzero YT leads to spin-orbit coupling since, in the $\beta_s = 0$ example, the spatial part of the spin and velocity components of the electron are

orthogonal. However, there is in fact a simple choice of $\beta_s(\theta)$ (note that in [15] the YT angle is also a function of θ) without spoiling such orthogonality, namely $\beta_s = \theta - \pi/2$. The new Dirac spinor is then $\psi_\beta = e^{-i\gamma^5(\theta-\pi/2)/2}\psi e^{-mcr\sin\theta/\hbar}$. The extra term $e^{-mcr\sin\theta/\hbar}$ ensures that the continuity equation for $\mathfrak{s}_\mu$ (8) is obeyed. The angular singularity is successfully removed by the envelop function

$$G(r) = - \int^r dx \frac{\sqrt{1-a(x)^2}}{x} = - \int^r dx \frac{\beta(x)}{x},$$

yielding the regularized vector potential

$$eA_t = \frac{\hbar a'(r)}{2\sqrt{1-a(r)^2}} - cmg(r)\sin(\theta) + \frac{\hbar a(r)}{2r} + \frac{\varepsilon}{c}, \quad (18)$$

$$eA_\phi = cmrg(r)\sin^2(\theta), \quad (19)$$

with $A_r = A_\theta = 0$, and

$$g(r) = \frac{1 - \sqrt{1-a(r)^2}}{a(r)} = \sqrt{\frac{1-\beta(r)}{1+\beta(r)}} \in (0, 1]. \quad (20)$$

It is noteworthy that $g(r)$ is just the relativistic Doppler factor.

IV. NULL-FOLIATED BLUEPRINT AND COMPACT POTENTIAL

Pursuing our goal of developing the Dirac solution for an electron in the DPT electromagnetic field [21,22], we propose a time-dependent version of the previously studied spinor, in order to investigate how such modification will affect the derived vector potential.

Let $u \equiv ct - r$ be the retarded time parametrizing our family of exact solutions. The localized *blueprint* $S_{1/2}$ spinor reads (with upper/lower two-spinors explicit)

$$\psi = e^{-\frac{mcr}{\hbar}} \begin{pmatrix} \kappa(\cos\frac{\theta}{2} - \sin\frac{\theta}{2})\sqrt{a(u)+1} \frac{e^{-i(\tilde{\varphi}(u)+et/\hbar)-i\phi/2}}{r\sqrt{2}\sin\theta} \\ \kappa(\sin\frac{\theta}{2} + \cos\frac{\theta}{2})\sqrt{a(u)+1} \frac{e^{-i(\tilde{\varphi}(u)+et/\hbar)+i\phi/2}}{r\sqrt{2}\sin\theta} \\ i\kappa(\sin\frac{\theta}{2} + \cos\frac{\theta}{2})\sqrt{1-a(u)} \frac{e^{-i(\tilde{\varphi}(u)+et/\hbar)-i\phi/2}}{r\sqrt{2}\sin\theta} \\ i\kappa(\sin\frac{\theta}{2} - \cos\frac{\theta}{2})\sqrt{1-a(u)} \frac{e^{-i(\tilde{\varphi}(u)+et/\hbar)+i\phi/2}}{r\sqrt{2}\sin\theta} \end{pmatrix}, \quad (21)$$

where κ and ε remain the same as for the stationary solutions. The function $a(u)$ is still the inverse Lorentz factor. The function $\tilde{\varphi}(u)$ is a retarded-time gauge function (chosen below for compactness of A_μ). In fact, this is just the previous spinor with $a(r) \rightarrow a(u)$ and $G = 2mcr/\hbar$ with an extra gauge function. Appendix A motivates the ansatz (21) from the imposed kinematics, while Appendix B analyzes the resulting distributional source structure.

Choosing the phase $\tilde{\varphi}(u) = \frac{1}{2}\sin^{-1}[a(u)]$ yields the *compact* potential

$$eA_t = mc \quad g(u) + \frac{\varepsilon}{c}, \quad eA_\phi = -mc r g(u) \sin\theta, \quad A_r = A_\theta = 0, \quad (22)$$

where $g(u)$ is just (20) with r replaced by u .

The electron's probability density and current for (21) are

$$\rho v^t = \psi^\dagger \psi = \frac{\kappa^2}{r^2} \csc\theta e^{-2cmr/\hbar}, \quad \rho v^r = 0, \quad \rho v^\theta = 0, \quad (23)$$

$$\rho v^\phi = -\frac{\kappa^2}{r^3} \csc^2\theta \sqrt{1-a(u)^2} e^{-2cmr/\hbar}, \quad (24)$$

so the electron's probability flow is purely azimuthal with drift $V^\phi/c = \sqrt{1-a(u)^2} \csc\theta/r$. Since $\beta_s = 0$, both $\nabla_\mu j^\mu = 0$ and $\nabla_\mu \mathfrak{s}^\mu = 0$ hold identically by (8).

With $g(u)$ as in Eq. (20), the field tensor $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ has the compact matrix form (indices ordered as ct, r, θ, ϕ)

$$F_{\mu\nu} = mc \begin{pmatrix} 0 & g'(u) & 0 & -r \sin\theta g'(u) \\ -g'(u) & 0 & 0 & \sin\theta [rg'(u) - g(u)] \\ 0 & 0 & 0 & -r \cos\theta g(u) \\ r \sin\theta g'(u) & \sin\theta [g(u) - rg'(u)] & r \cos\theta g(u) & 0 \end{pmatrix}, \quad (25)$$

with prime denoting d/du . In an orthonormal spherical frame $\{\hat{r}, \hat{\theta}, \hat{\phi}\}$ the nonzero EM field components are

$$eE_{\hat{r}} = cm g'(u), \quad eE_{\hat{\phi}} = -cm g'(u), \quad (26)$$

$$eB_{\hat{r}} = cm \frac{\cot\theta}{r} g(u), \quad eB_{\hat{\theta}} = -mc[g(u)/r - g'(u)]. \quad (27)$$

The corresponding external four-current (Maxwell source) is

$$J^\mu = \nabla_\nu F^{\nu\mu} = \left\{ cm \left(\frac{2g'(u)}{r} - g''(u) \right), -cmg''(u), 0, \frac{cm \csc(\theta)\mathcal{F}}{r^3} \right\}, \quad (28)$$

where $\mathcal{F} = 2rg'(u) + g(u) \csc^2(\theta)$. This electromagnetic field is an externally generated guiding field constructed so that the

prescribed Dirac spinor (21) solves the Dirac equation in that background. The inferred source J^μ is therefore the external charge/current distribution required to support this pulse-like field. It contains (i) a null-front sheet contribution, (ii) a core contribution at $r = 0$, and (iii) an axis-supported *azimuthal* current carried by J^ϕ . Details and regularization by excision are given in Appendix B.

Turning to the physical interpretation, each fixed value $u = u_0$ labels an outgoing hypersurface. The equation $u_0 = ct - r$ implies $r = ct - u_0$, i.e., a sphere whose radius increases linearly with t . Thus a profile written as $g(u)$ is a profile that translates outward at speed c ; derivatives $g'(u)$ quantify how the profile changes from one such hypersurface to the next. If $g'(u) \neq 0$ on some interval of u , then the resulting fields are time dependent and carry energy flux. For the null-foliated blueprint spinor (21), the orthonormal components of the external EM fields imply

$$\begin{aligned} S_{\hat{r}} &= \frac{1}{\mu_0} (\mathbf{E} \times \mathbf{B}) \cdot \hat{r} = -\frac{1}{\mu_0} E_{\hat{\phi}} B_{\hat{\theta}} \\ &= \frac{m^2 c^2}{\mu_0 e^2} \sin \theta g'(u) \left(g'(u) - \frac{g(u)}{r} \right), \end{aligned} \quad (29)$$

which approaches $S_{\hat{r}} \rightarrow (m^2 c^2 / \mu_0 e^2) \sin \theta [g'(u)]^2$ for $r \gg |g|$.

However, the obtained spinor still does not solve the DPT problem, as the inverted fields are not source free. Given A_μ , the Maxwell source is given by $J^\mu \equiv \nabla_\nu F^{\nu\mu}$ and is interpreted here as an external current. For the null-foliated blueprint spinor (21), the analytic expressions for $F_{\mu\nu}$ and J^μ are smooth on the punctured domain

$$r > 0, \quad \theta \in (0, \pi), \quad (30)$$

and the apparent singularities at $r = 0$ and $\theta = 0, \pi$ correspond to *distributional* sources (Dirac-delta supported contributions) when Maxwell's equations are extended to all space. The source structure (28) decomposes into (i) a causal null-front sheet on the outgoing hypersurfaces $u = \text{const}$ [seen in the $g''(u)$ part of J^t and J^r], (ii) a core contribution at the origin (seen in the $1/r$ part of J^t), and (iii) an *axis-supported azimuthal* current: the ϕ component J^ϕ diverges strongly near $\theta = 0, \pi$ and concentrates on the z axis in a distributional sense. Appendix B gives the detailed near-axis analysis and explicit regularization (finite-radius tube around the axis and a small ball around the origin).

Nevertheless, for pulselike fronts with $g'(\pm\infty) = 0$, the total charge in (28) reduces to a boundary term and vanishes (Appendix B). In that case, the external sources required to support the field have zero net charge and can be interpreted as multipole-type charge/current distributions.

V. OTHER EXAMPLES

We list explicit potentials, fields, and sources corresponding to the null-foliated construction above.

A. Example II: General radial envelope $G(r)$

Here we simply mention how to generalize the envelop function, without going into details of the EM fields and

charge and current densities. In the blueprint spinor (21), for the more general radial envelope $G(r) > 0$ one finds

$$eA_t = \frac{\hbar}{2a(u)} \left(\sqrt{1 - a(u)^2} G'(r) - \frac{2cm}{\hbar} \right) + \frac{\varepsilon}{c}, \quad (31)$$

$$eA_\phi = \frac{r \sin \theta}{2a(u)} [\hbar G'(r) - 2cm \sqrt{1 - a(u)^2}], \quad (32)$$

with $A_r = A_\theta = 0$ and the previous example is recovered for $G(r) = 2mcr/\hbar$.

B. Example III: Longitudinal rapidity profile $G(u)$ (includes constant-field limit)

Note that, in the previous example, the envelop function depends only on r . In order to generalize it to also depend on u , the electron's current must also have a j_r component so that the continuity equation is obeyed. In order to do so, one must apply a boost along $\hat{r}$ to the blueprint spinor (21). One first replaces the boost in the general radial envelope spinor by $\tilde{B} = e^{-G(u)\gamma^0\gamma^r/2}\mathcal{B}$ and then the density by $r^2\tilde{\rho} = \kappa^2 a(u) e^{-G(u)} \csc(\theta)$ while keeping the rest unchanged. The spin vector remains the same and the new velocity vector becomes

$$\tilde{v}_\mu = \left\{ \frac{\cosh(G(u))}{a(u)}, \frac{\sinh(G(u))}{a(u)}, 0, \frac{r\sqrt{1 - a(u)^2} \sin(\theta)}{a(u)} \right\}. \quad (33)$$

For the gauge phase obeying

$$\tilde{\varphi}' = \frac{a'(u)}{2\sqrt{1 - a(u)^2}} - \frac{cm \sinh(G(u))}{\hbar a(u)}, \quad (34)$$

we get the vector potential

$$e\tilde{A}_\mu = \left\{ \frac{\varepsilon}{c} - \frac{cme^{G(u)}}{a(u)}, 0, 0, -\frac{cmr\sqrt{1 - a(u)^2} \sin(\theta)}{a(u)} \right\}. \quad (35)$$

C. Constant-field limit

Consider $a(u) = 1$ and define $G(u) = \ln[\alpha(u - c_1)]$ for constants α and c_1 . In the matrix spinor, add a gauge phase $\vartheta(t)$ and choose

$$\tilde{\varphi}' = \frac{cm \sinh(G(u))}{\hbar}, \quad \vartheta(t) = \frac{\alpha c^2 m t (ct - 2c_1) - 2t\varepsilon}{2\hbar}. \quad (36)$$

Then the vector potential reduces to $eA_t = mc\alpha r$, which leads to a constant electric field $F_{tr} = mc\alpha$ supported by the charge density $J_t = 2mc\alpha/r$.

VI. CONCLUSIONS AND OUTLOOK

This paper develops a manifestly covariant “design mode” for the Dirac equation: instead of starting from a given electromagnetic four-potential and solving for a spinor, one gives the Dirac spinor corresponding to a specific local Lorentz transformation, which encapsulates the geometrical data (RDC) and then uses CRDI to reconstruct a real four-potential for which the prescribed spinor is an exact Dirac solution.

The RDC prescription uses the symmetries of the physical problem, together with quasiclassical intuition, making the RDC-CRDI framework a powerful heuristic tool for solving the Dirac equation, with CRDI providing the final confirmation step. The main aim of this paper was not to find a concrete solution for a given external field; rather, we analyzed how the properties of the RDC spinor are reflected in the electromagnetic fields reconstructed via CRDI. The resulting insight will sharpen the RDC intuition and facilitate the construction of new physically interesting Dirac solutions within the RDC-CRDI framework.

We underline the transparent geometrical interpretation of the RDC-CRDI framework. In this language, the inverted potential cleanly separates into contributions tied to longitudinal “drift” along the current, internal phase variation on the Σ plane, and changes of the local boosted and rotated frame, with reality enforced by the compatibility/continuity conditions for the bilinear currents.

Within that framework, we provide two complementary blueprints. First, a static localized $S_{1/2}$ type family in spherical coordinates makes transparent how the function a controls the circulating flow speed, acting as the inverse Lorentz factor of the spinor current. The associated inverted potentials are compact, but the corresponding Maxwell tensor and source reveal an axis singularity; introducing a simple nonzero Yvon-Takabayashi angle profile removes this angular singularity while preserving the orthogonality structure of spin and velocity.

Second, and most distinctively, we introduce a null-foliated (retarded-time) localized ansatz parametrized by outgoing light-cone hypersurfaces $u = ct - r$. With an explicit retarded-time phase choice, CRDI yields an especially compact guiding potential, controlled by a single profile $g(u)$ that equals a relativistic Doppler factor. The resulting fields have a clear radiative character with outward energy flux at large radius, and a source structure that decomposes into a causal null-front sheet, an origin contribution, and an axis-supported azimuthal current (distributional in the idealized analytic form). Pulselike choices with $g'(\pm\infty) = 0$ give zero net charge, so the required external support can be interpreted as multipole-type charge/current distributions.

Further generalizations include: (1) nonzero Yvon-Takabayashi angle ($\beta_s \neq 0$) for time dependent problems, where $\nabla_\mu(\rho s^\mu) = -(2mc/\hbar)\rho \sin\beta$ couples spin transport to mass/phase; (2) other foliations and symmetries (planar, cylindrical, helical) for beam, waveguide, and vortex states; (3) media/boundary-supported realizations where $F_{\mu\nu}$ is supported by materials rather than idealized filaments; and (4) curved spacetimes via tetrads.

DATA AVAILABILITY

There are no publicly available research data or software supporting this manuscript. Requests for further information or data should be sent to the authors.

APPENDIX A: SPHERE KINEMATICS AND MOTIVATION FOR THE NULL-FOLIATED ANSATZ

This appendix summarizes a geometric motivation for the null-foliated blueprint and explains why the compact gauge

choice $A_r = A_\theta = 0$ and $A_\phi \propto r \sin\theta$ is compatible with the imposed kinematics.

1. From spinor data to compact potentials

In the $\beta_s = 0$ sector, the polar form $\Psi = \sqrt{\rho} \mathcal{R}$ yields directly physical bilinear fields: a Lorentz scalar density ρ , a unit four-velocity v^μ (so that $j^\mu = \rho v^\mu$), and a unit spin s^μ . CRDI then provides a real four-potential $A_\mu[\rho, v, s; R_\mu, B_\mu]$ making the chosen ψ an exact Dirac solution; the electromagnetic field is $F = dA$.

For the general radial-envelope light-cone spinor, the bilinears read

$$r^2 \rho = \kappa^2 \csc\theta a(u) e^{-G(r)}, \quad s_\mu = \{0, 0, -r, 0\},$$

$$v_\mu = \left\{ \frac{1}{a(u)}, 0, 0, \frac{r \sin\theta}{a(u)} \sqrt{1 - a(u)^2} \right\}. \quad (\text{A1})$$

Thus, on each sphere $r = \text{const}$ the flow is purely azimuthal while the spin points along $-\hat{\theta}$.

2. Bundle/geodesic picture (classical kinematic constraints)

A classical (kinematic) picture that leads to the same angular dependence as our compact A_ϕ can be formulated using the Abelian Wong equation for a particle of charge-to-mass ratio $\kappa_{em} \equiv q/m$,

$$\nabla_u u^\mu = \kappa_{em} F^\mu{}_\nu u^\nu, \quad \kappa_{em} = \frac{q}{m}, \quad (\text{A2})$$

or equivalently its index-lowered form

$$u^\nu \nabla_\nu u_\mu = \kappa_{em} F_{\mu\nu} u^\nu. \quad (\text{A3})$$

This does *not* replace CRDI; it only motivates why the θ dependence $A_\phi \propto r \sin\theta$ is natural when one demands motion at fixed (r, θ) with only (t, ϕ) components.

Choose a gauge adapted to the kinematics,

$$A_r = 0, \quad A_\theta = 0, \quad A = A_t(u, r) dt + A_\phi(u, r, \theta) d\phi. \quad (\text{A4})$$

Then the only curvature components needed for the normal-force balance are

$$F_{tr} = -\partial_r A_t, \quad F_{r\phi} = \partial_r A_\phi, \quad F_{\theta\phi} = \partial_\theta A_\phi. \quad (\text{A5})$$

Projecting Eq. (A3) onto the normal directions $\{\partial_r, \partial_\theta\}$ for kinematics with $u^r = u^\theta = 0$ yields the polar balance equation

$$-r^2 \sin\theta \cos\theta (u^\phi)^2 = \kappa_{em} F_{\theta\phi} u^\phi = \kappa_{em} (\partial_\theta A_\phi) u^\phi. \quad (\text{A6})$$

For the bilinear kinematics of our ansatz this integrates to

$$A_\phi(u, r, \theta) = \frac{1}{\kappa_{em}} \frac{r \sqrt{1 - a(u)^2}}{a(u)} \sin\theta + C(u, r), \quad (\text{A7})$$

where $C(u, r)$ is a gauge function independent of θ . This reproduces the same angular dependence as the compact CRDI potential in Eq. (22).

3. Steering and throttle on a sphere (diagnostic picture)

Define the θ -independent quantity

$$\mathcal{Q}(u, r) \equiv \frac{e A_\phi(u, r, \theta)}{r \sin\theta}, \quad (\text{A8})$$

which is well-defined for the compact gauge family $A_\phi \propto r \sin \theta$. On a sphere $r = \text{const}$, a latitude (fixed θ) is not a geodesic away from the equator; maintaining it requires an in-surface sideways acceleration with geodesic curvature $k_g(\theta) = |\cot \theta|/r$. If the flow speed is $v_\phi = r \sin \theta \Omega$, the required sideways push is $a_\theta^{\text{need}} = v_\phi^2 k_g$. With $v = v_\phi \hat{\phi}$, the Lorentz force has a $\hat{\theta}$ component $q v_\phi B_{\hat{r}}$. Equating the required and provided normal accelerations gives a diagnostic turning-rate relation,

$$q v_\phi B_{\hat{r}} = \gamma m v_\phi^2 \frac{|\cot \theta|}{r} \implies \gamma m \Omega(r, u) = -\frac{2q}{e} Q(u, r), \quad (\text{A9})$$

while tangential acceleration along the ring is driven by the azimuthal electric component,

$$\frac{d}{dt}(\gamma m v_\phi) = q E_{\hat{\phi}} = -\frac{q r \sin \theta}{e} \partial_t Q. \quad (\text{A10})$$

APPENDIX B: SOURCES AND NEAR-AXIS REGULARIZATION

From Eq. (28) the charge density and radial current are

$$J^t = cm \left(\frac{2g'(u)}{r} - g''(u) \right), \quad J^r = -cm g''(u), \quad u = ct - r. \quad (\text{B1})$$

Integrating J^t over space gives

$$\begin{aligned} Q(t) &= 4\pi cm \int_0^\infty (2r g'(u) - r^2 g''(u)) dr \\ &= 4\pi cm \int_{-\infty}^\infty \frac{d}{du} [r^2 g'(u)] du = 4\pi cm [r^2 g'(u)]_{u=-\infty}^{u=+\infty}. \end{aligned} \quad (\text{B2})$$

For pulselike $g(u)$ with $g'(\pm\infty) = 0$, the net charge vanishes. Thus the $1/r$ term in J^t represents a finite, causal surface charge density on the outgoing hypersurfaces $u = \text{const}$, while the total charge reduces to a boundary term.

The only nonzero angular current component is J^ϕ :

$$J^\phi = \frac{cm}{r^3} \csc \theta \mathcal{F}, \quad \mathcal{F} = 2r g'(u) + g(u) \csc^2 \theta. \quad (\text{B3})$$

Near the axis ($\theta \rightarrow 0$) one has $J^\phi \sim (cm/r^3) g(u) \csc^3 \theta$, which concentrates on the z axis in a distributional sense and represents an *axis-supported azimuthal* current. Consistently, the magnetic component behaves as

$$e B_{\hat{r}} = cm \frac{\cot \theta}{r} g(u) \sim cm \frac{g(u)}{\rho}, \quad \rho = r \sin \theta, \quad (\text{B4})$$

i.e., a $1/\rho$ divergence in the cylindrical distance ρ to the axis. This is the characteristic signature of a line-supported source. In our case the line support is carried by J^ϕ (azimuthal direction). A physically meaningful regularization is to excise a tube $\rho < a$ around the axis and replace the singular filament by a finite-radius distribution that carries an azimuthal current

density $J^\phi(\rho, u)$ (or equivalently an azimuthal surface current on the tube). The exterior fields coincide with the analytic expressions on the punctured domain.

1. Angular cutoff and logarithm in integrated quantities

The EM energy density is

$$u_{\text{EM}} = \frac{1}{2\mu_0} \left(\frac{E^2}{c^2} + B^2 \right). \quad (\text{B5})$$

Near the axis, the dominant term scales as

$$u_{\text{EM}} \sim \frac{1}{2\mu_0} \left(\frac{cm}{e} \right)^2 \frac{g(u)^2}{\rho^2}. \quad (\text{B6})$$

At fixed r , $\rho = r \sin \theta$, and the angular integral over a cone of half-angle $\theta_{\min}$ yields

$$\int_{\theta_{\min}}^{\theta_0} \ln \left(\frac{1}{\theta_{\min}} \right). \quad (\text{B7})$$

Equivalently, excising a cylinder of radius a corresponds to $\theta_{\min} \simeq a/r$, giving the familiar $\ln(r/a)$ dependence. Thus integrated fluxes depend only logarithmically on the cutoff scale.

2. Removal of origin and axis (distributional sources)

The analytic solution is defined on $r > 0$ and $\theta \in (0, \pi)$:

(1) At the origin $r = 0$, the null-foliation degenerates and Gauss-type integrals can pick up a point support (core contribution).

(2) On the z axis ($\theta = 0, \pi$), the fields and J^ϕ concentrate on the axis, signaling an axis-supported distributional current. A small tubular neighborhood should be excised or regularized.

In this view, the removed regions carry the distributional sources, while the fields are smooth on the punctured domain.

3. Gauss surface for the null-front contribution (from F directly)

Maxwell's equations in differential-form notation are

$$dF = 0, \quad d^*F = \mu_0^* J. \quad (\text{B8})$$

Let Σ_t be the $t = \text{const}$ hypersurface. Gauss's law follows from integrating $d^*F = \mu_0^* J$ over a time slab and shrinking it:

$$\int_{\partial \Sigma} {}^*F = \mu_0 \int_{\Sigma} J = \mu_0 \int_{\Sigma} J^t \sqrt{\gamma} d^3x. \quad (\text{B9})$$

In spherical coordinates (Example I), choosing Σ to be the ball of radius R gives

$$Q(t) = \int_{\Sigma} J^t d^3x = 4\pi cm [r^2 g'(u)]_{u=-\infty}^{u=+\infty}, \quad (\text{B10})$$

consistent with the computation above: for pulselike fronts ($g'(\pm\infty) = 0$), the net charge is zero even though J^t contains a $1/r$ term (a causal surface density on $u = \text{const}$).

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