

Macroscopic QED and noise currents in time-varying media

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Macroscopic QED (MQED) is the field theory for computing quantum electromagnetic effects in dispersive media. Here, we extend MQED to treat *time-varying*, dispersive media. For a time-dependent Drude model, we find that the expected replacement $\varepsilon(\omega) \rightarrow \varepsilon(t, \omega)$ within standard MQED does not lead to a permittivity that is similar to that of the classical time-dependent Drude model and can lead to large polarization currents, becoming singular in the limit of a step change in carrier density. We show this unwanted behavior can be removed through instead modifying the reservoir dynamics, quantizing the resulting theory and finding the nonequilibrium, time-varying noise currents, which exhibit extra correlations due to temporal reflections within the material dynamics.

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The arrow of time is evident in most electromagnetic (EM) problems. Sources radiate *outward*-going waves, and matter responds only to EM fields in the *past*. By contrast, there is no such “arrow of space.” This inequivalence of space and time implies that waves with a *time*-dependent speed $c(t)$ cannot be understood through a simple $x \leftrightarrow t$ relabeling of waves with a *space*-dependent speed $c(x)$. For instance, an abrupt change of wave speed within a region of space causes reflection; the familiar conversion between waves propagating toward $x = +\infty$ and $-\infty$. A similar abrupt change in time does *not*, by analogy, generate waves that propagate into the past [1,2]. That would violate causality. This inequivalence is especially conspicuous in QED, where, unlike reflection from a spatial boundary, temporal reflection implies the creation of photons [3,4].

Here, we examine the theory of QED in a dispersive, time-varying material. We consider materials described by a time-dependent Drude model, although our conclusions can be applied more generally. Our first task is to find a classical Lagrangian (and by extension, a Hamiltonian) that accurately describes the field and material dynamics. This is a difficult task as the presence of dispersion implies dissipation, which—as is well known in the theory of macroscopic QED (MQED)—forces us to introduce a reservoir of simple harmonic oscillators to mimic the material dynamics. We examine two possible modifications to the field-reservoir Lagrangian to describe a time-modulated material response, finding that modulating the field-reservoir coupling leads to a permittivity that does not reproduce the dynamics of a classical time-dependent Drude model, as well as leading to anomalously large noise currents.

These fluctuating (noise) currents are a crucial prediction of MQED. In stationary systems these noise currents obey the fluctuation-dissipation theorem (FDT) [5,6], which derives

from the equilibrium between absorption and emission [as in the schematic in Fig. 1(a)]. As time-varying materials are, by definition, *not* in thermal equilibrium, the FDT does not apply. We might expect an abrupt modulation of the material to radiate, due to the abrupt change in the noise currents [Figs. 1(b) and 1(c)]. The second part of our work finds the evolution of these fluctuating currents during a modulation of the material parameters. Our findings complements the recent paper of Vázquez-Lozano *et al.*, who investigated modifications to these currents using perturbation theory and the assumption of a nondispersive modulation of the material [7].

Besides fundamental interest, the reason for paying such close attention to the quantum theory of time-varying materials comes from recent experiments, where large and abrupt changes in material parameters have been implemented at radio frequencies [2], in the microwave regime [8], at optical frequencies [9–12], and for acoustic [13,14], elastic [15], and water waves [1]. Note that dispersion is non-negligible in all of these experiments (which can in many cases can be analyzed using, e.g., transfer matrix methods [16]). In parallel, analytical and numerical calculations have shown a number of new effects in these materials, such as synthetic motion and drag [17,18], modification of spontaneous emission [19], temporal aiming [20], gain and wave compression [21,22], spectrum reshaping [23], nonreciprocity [24], and have found special pulses with enhanced or suppressed transmission [25]. For a comprehensive review see Ref. [26].

More recently, a number of authors have considered more subtle, quantum effects in time and space-time varying materials. These papers are mostly concerned with calculating the rate of photon creation [27–30]. Although there is a great deal of work on classical wave propagation through dispersive, time-varying media [31–35], the theory of quantum effects in these media—with a few exceptions (three notable papers being Refs. [36,37] and [29])—does not usually take dispersive effects into account.

I. DESCRIBING DISPERSION

In classical electromagnetism the susceptibility of a material is something we simply impose, making as lunatic a

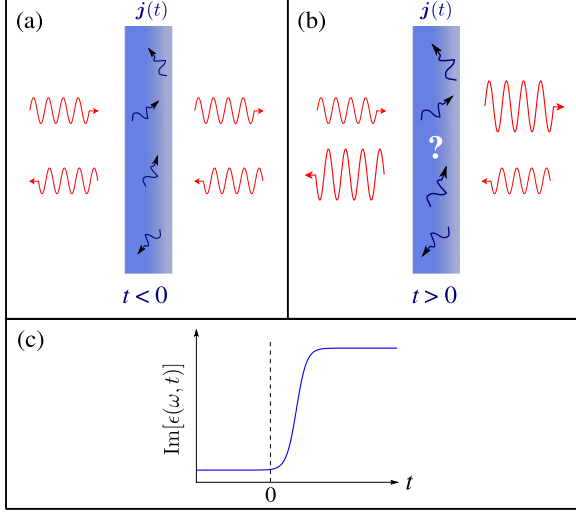


FIG. 1. Noise currents in time-varying media: (a) In equilibrium, absorption of electromagnetic energy is balanced by reemission, via fluctuating noise currents $\mathbf{j}$ within the material. These can be calculated using MQED. (b) If the coupling between field and medium is changed abruptly as in (c), absorption and reemission are no longer in balance and it is unclear how the noise currents should behave. Here, we examine extensions to MQED to treat this problem.

choice as we like. In quantum electromagnetism we are not so free. As soon as material dispersion—and by implication, dissipation [38]—is accounted for, the usual electromagnetic Hamiltonian becomes non-Hermitian, breaking the conservation of norm, and equivalently the operator commutation relations. The moral is that, to quantize a system properly, the Hamiltonian must include the material degrees of freedom.

MQED [39–41] represents the solution to this problem: a general approach to the quantum mechanics of the electromagnetic field in a dispersive, dissipative material. The theory can be seeded from different assumptions, and here we work from a Lagrangian, as in Ref. [41]. The Lagrangian of MQED is given as the integral of the Lagrangian density, which is broken into three pieces,

$$\mathcal{L} = \mathcal{L}_{\text{EM}} + \mathcal{L}_I + \mathcal{L}_R. \quad (1)$$

For simplicity, here we work in one spatial dimension plus time, where the Lagrangian density for the electromagnetic field equals the textbook expression [42],

$$\begin{aligned} \mathcal{L}_{\text{EM}} &= \frac{\varepsilon_0}{2} [\varepsilon_\infty E^2 - c^2 B^2] \\ &= \frac{\varepsilon_0}{2} \left[\varepsilon_\infty \left(\frac{\partial A}{\partial t} \right)^2 - c^2 \left(\frac{\partial A}{\partial x} \right)^2 \right], \end{aligned} \quad (2)$$

where ε_∞ is the high-frequency limit of the permittivity, and A is the vector potential with the electric field given by $\mathbf{E} = -\partial_t \mathbf{A} \mathbf{e}_y$ and the magnetic field by $\mathbf{B} = \partial_x \mathbf{A} \mathbf{e}_z$. Assuming the material responds linearly to the EM field, we add the Lagrangian density of a continuum of harmonic oscillators

(“the reservoir”), with each oscillator having amplitude X_ω ,

$$\mathcal{L}_R = \frac{1}{2} \int_0^\infty d\omega \left[\left(\frac{\partial X_\omega}{\partial t} \right)^2 - \omega^2 X_\omega^2 \right], \quad (3)$$

which, given the stupendous number of degrees of freedom, we hope might mimic the linear response of any material we choose. Indeed, for stationary media, this is the case. The EM field and reservoir are coupled via the following interaction:

$$\begin{aligned} \mathcal{L}_I &= -\frac{\partial A}{\partial t} \int_0^\infty d\omega \alpha(\omega) X_\omega \\ &= -\frac{\partial A}{\partial t} \int_0^\infty d\omega \sqrt{\frac{2\omega \varepsilon_0 \text{Im}[\varepsilon(\omega)]}{\pi}} X_\omega, \end{aligned} \quad (4)$$

where $\varepsilon(\omega)$ is the complex permittivity, which must be purely dissipative (i.e., no gain, $\text{Im}[\varepsilon] > 0$) for our Lagrangian to be real valued. Through applying Lagrange’s equations of motion to the above sum of Eqs. (2)–(4) and eliminating the reservoir as described in Ref. [41], the EM field obeys Maxwell’s equations, with the effective permittivity given by

$$\varepsilon(\omega) = \varepsilon_\infty + \text{P} \frac{1}{\pi} \int_{-\infty}^\infty \frac{\text{Im}[\varepsilon(\omega')]}{\omega' - \omega} d\omega' + i \text{Im}[\varepsilon(\omega)]. \quad (5)$$

The real part of this effective permittivity (5) is the Hilbert transform of the imaginary part, a relationship known as a Kramers-Kronig relation [38], which holds for any causal material response. Provided we assume that our material is purely dissipative and causal, the equations of motion derived from the Lagrangian density (1) reproduces Maxwell’s equations for any permittivity we like. Extensions of this approach can be applied to mimic the response of magnetic [41], bianisotropic [43], and moving media [44]. Versions of this theory have been used to calculate dispersion forces [45,46], spontaneous emission [47], and quantum friction [48,49]. For a review see Ref. [40].

Now suppose the material is time varying. For example, the large and rapid modulation of indium-tin-oxide (ITO) reported in Refs. [9–12], a material which is both lossy and highly dispersive. There are several ways to implement the time variation in the Lagrangian (1). We can include a time dependence in the field-reservoir coupling (4), or make the Lagrangian of the reservoir (3) explicitly time dependent, or some combination of the two. The remainder of the paper explores the consequences of these different choices.

Special case: Modulated metals

For illustration we take a Drude model with a time-varying plasma frequency. This model is commonly used to model, e.g., the EM response of time-modulated ITO, where the permittivity has been previously written as (see, e.g., Refs. [10,50])

$$\varepsilon_D(t, \omega) \sim \varepsilon_\infty - \frac{\omega_p^2(t)}{\omega(\omega + i\gamma)}, \quad (6)$$

with γ the collision rate and $\omega_p^2 = e^2 N / \varepsilon_0 m^*$ the plasma frequency, written in terms of the electron charge e , effective

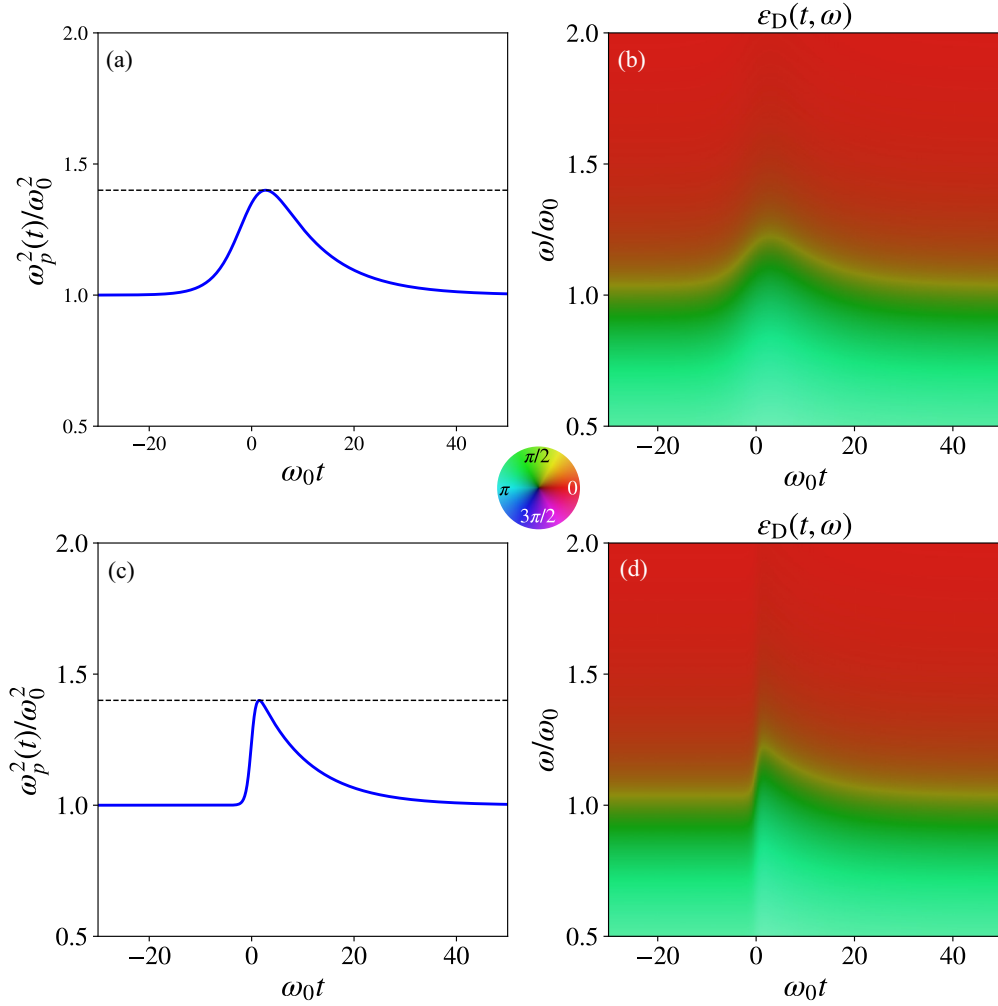


FIG. 2. Approximate Drude model: (c) and (d) show the assumed plasma frequency in Eq. (6), which varies as a smoothed step function, $\omega_p^2(t) = \omega_0^2 + \delta\omega_p^2 \frac{1}{2} [1 + \tanh(t/t_r)] e^{-t/t_d}$. In both cases $\delta\omega_p^2$ is chosen to give the same maximum plasma frequency $\omega_{\text{max}}^2 = 1.4\omega_0^2$, shown as the black dashed line. (a) and (b) show a plasma frequency changing slowly, $\omega_0 t_r = 5$ and $\omega_0 t_d = 10$, whereas (c) and (d) consider a rapid change, $\omega_0 t_r = 1$ and $\omega_0 t_d = 10$. (b) and (d) show the complex permittivity for these two cases (color shows phase and saturation magnitude, as indicated in the central panel).

mass m^* , and carrier density N . Although not widely stated, Eq. (6) is an approximation to the true permittivity, valid for a slowly modulated material response [Fig. 2 shows the behavior of the permittivity (6) for adiabatic and nonadiabatic modulations]. Here, we assume the modulation of the plasma frequency is due to a changing carrier density. An almost identical argument can be made for a fixed density of carriers with a time-varying effective mass, which is most relevant for ITO [10]. But through assuming a time-varying carrier density we can directly compare our results to Stepanov's earlier work on time-varying plasmas [51]. For us, the important finding in Ref. [51] is that, taking the current density for a collection of charges uniformly moving at velocity v , $j = N(t)ev$, and assuming each charge obeys the equation of motion, $m\dot{v} = eE - \gamma v$, the polarization $\partial_t P = j$ obeys the second-order differential equation,

$$N \frac{\partial}{\partial t} \left(\frac{1}{N} \frac{\partial P}{\partial t} \right) + \gamma \frac{\partial P}{\partial t} = \frac{Ne^2}{m} E, \quad (7)$$

which has the exact solution (neglecting homogeneous terms)

$$P(t) = \epsilon_0 \int_{-\infty}^t dt' \omega_p^2(t') \int_{-\infty}^{t'} dt'' e^{-\gamma(t'-t'')} E(t''). \quad (8)$$

When written in terms of the Fourier expansion of the electric field, $E(t) = (2\pi)^{-1} \int d\omega \tilde{E}(\omega) \exp(-i\omega t)$, Eq. (8) yields the following expression for the permittivity:

$$\begin{aligned} \epsilon_D(t, \omega) &= \epsilon_\infty + \frac{i}{\omega + i\gamma} \int_{-\infty}^t dt' \omega_p^2(t') e^{i\omega(t-t')} \\ &= \epsilon_\infty - \frac{1}{\omega(\omega + i\gamma)} [\omega_p^2(t) - \sigma_0(t, \omega)]. \end{aligned} \quad (9)$$

This expression equals Eq. (6), plus an additional “memory” term $\sigma_0 = \int_{-\infty}^t d\tau \exp[i\omega(t-\tau)] d_\tau \omega_p^2(\tau)$. Note that an important difference between Eq. (9) and the approximate expression (6) is that the polarization calculated from (9) is always continuous, ensuring that the time modulation induces a finite current, even in the limit of a step change (see Appendix). If we repeatedly integrate the

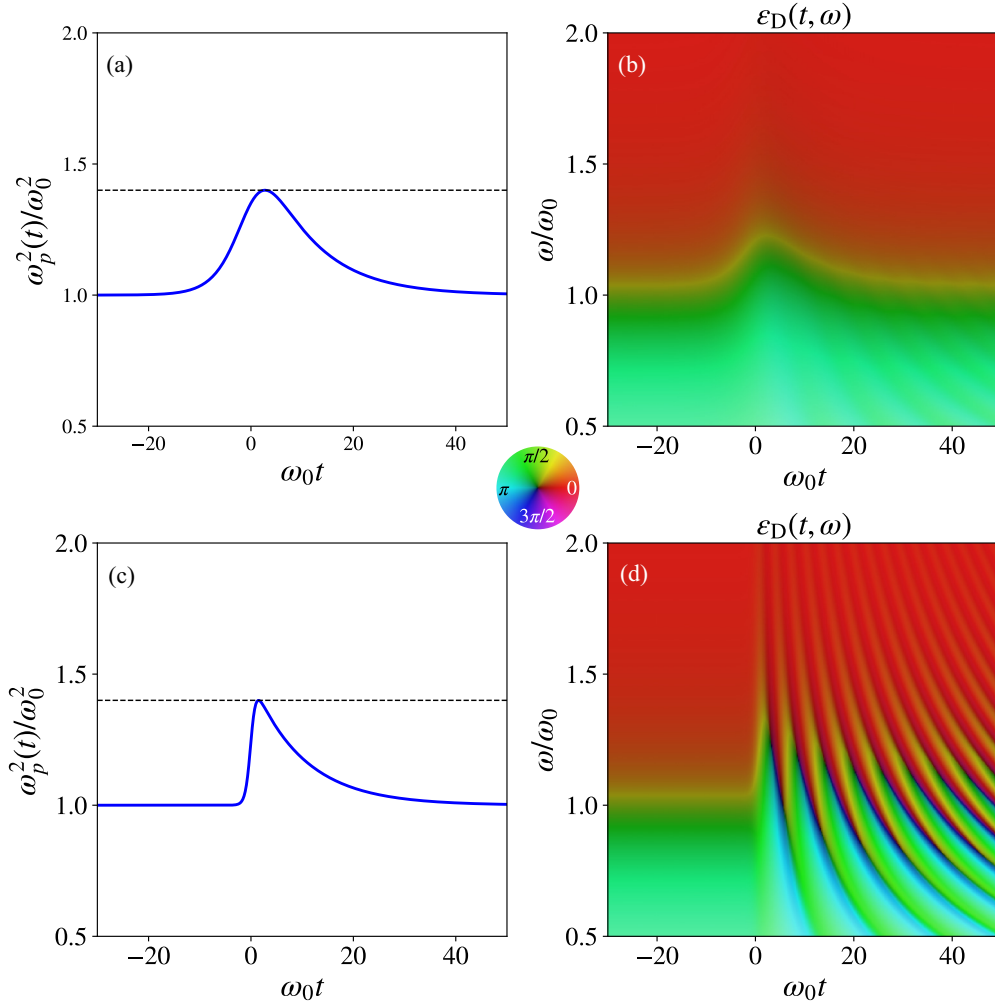


FIG. 3. Stepanov model: Parameters identical to Fig. 2, but with the permittivity equal to the Stepanov model defined in Eq. (9). As anticipated, when the relative change in the plasma frequency is small (a) and (b), the permittivity of the Stepanov model approaches that of the approximate Drude model (6), meanwhile for rapid changes (c) and (d) the frequency-time permittivity develops oscillations.

memory term by parts, we can see that it can only be neglected when $\omega^{-n} d_t^n \omega_p^2 \ll \omega_p^2$ for all $n \geq 1$, i.e., to use Eq. (6), the relative change in the plasma frequency must be small over the time scale ω^{-1} . As Fig. 3 shows, this memory term leads to significant oscillations in the permittivity when the carrier density is changed rapidly. Despite first appearances, these oscillations do not indicate complex material dynamics, but represent the phase accumulated since the time of the modulation. We compare two choices of time-modulated Lagrangian with this classical Stepanov model.

II. TWO CANDIDATE LAGRANGIANS

Our first challenge is to find a field plus reservoir Lagrangian to represent something close to Stepanov's time-varying Drude model. We now give two possible MQED Lagrangians. Both reproduce the approximate permittivity (6) when the relative change in the carrier density is slow, $\dot{N}/N \ll \omega$ (i.e., taking the limit shown in the upper panels of Figs. 2 and 3). However, as we shall see, the “obvious” modification to MQED produces a permittivity that—like the approximate expression (6)—becomes discontinuous when the

carrier density is changed discontinuously. This discontinuity implies an infinite current, which is not physical given that both the number of carriers and their energy remains finite.

A. The “modulated coupling” Lagrangian

The obvious route to the theory of MQED in a time-dependent Drude metal is to simply let the carrier density vary within the interaction Lagrangian (4). As discussed in the previous section, for the unmodulated system, the field-reservoir coupling is given by

$$\alpha(\omega) = \sqrt{\frac{2\omega\epsilon_0 \text{Im}[\epsilon(\omega)]}{\pi}} = \sqrt{\frac{2\gamma e^2 N}{\pi m^*(\omega^2 + \gamma^2)}}. \quad (10)$$

Therefore, we might expect that a time-varying carrier density would modify the interaction Lagrangian as follows:

$$\begin{aligned} \mathcal{L}_1 &\rightarrow -\frac{\partial A}{\partial t} \int_0^\infty d\omega \alpha(\omega, t) X_\omega \\ &= -\frac{\partial A}{\partial t} \int_0^\infty d\omega \sqrt{\frac{2\gamma e^2 N(t)}{\pi m^*(\omega^2 + \gamma^2)}} X_\omega. \end{aligned} \quad (11)$$

Applying Lagrange's equations to the sum of (2), (3), and (11), we find each oscillator in the reservoir acts as a driven, undamped simple harmonic oscillator,

$$\left(\frac{\partial^2}{\partial t^2} + \omega^2\right)X_\omega = -\alpha(t, \omega)\frac{\partial A}{\partial t}, \quad (12)$$

whereas the vector potential obeys the one-dimensional wave equation with the reservoir acting as a source of waves, $j = \partial_t P$,

$$\begin{aligned} \left(\epsilon_\infty \frac{\partial^2}{\partial t^2} - c^2 \frac{\partial^2}{\partial x^2}\right)A &= \frac{1}{\epsilon_0} \frac{\partial P}{\partial t} \\ &= \frac{1}{\epsilon_0} \frac{\partial}{\partial t} \int_0^\infty d\omega \alpha(t, \omega) X_\omega, \end{aligned} \quad (13)$$

where P is the polarization density. The first of these coupled equations, (12), is simply that for a continuum of undamped, driven simple harmonic oscillators. This has a general solution in terms of the retarded Green function $G_\omega(t - t') = \Theta(t - t')\omega^{-1} \sin[\omega(t - t')]$, given by

$$\begin{aligned} X_\omega &= - \int_{-\infty}^t dt' G_\omega(t - t') \alpha(\omega, t') \frac{\partial A}{\partial t'} \\ &\quad + C_\omega e^{-i\omega t} + C_\omega^* e^{i\omega t}, \end{aligned} \quad (14)$$

where C_ω are arbitrary complex amplitudes that impose the $t \rightarrow -\infty$ boundary conditions on the oscillators. Inserting the solution (14) into the definition of the polarization density (13), we find

$$\begin{aligned} P &= - \int_{-\infty}^t dt' \int_0^\infty d\omega \alpha(t, \omega) G_\omega(t - t') \alpha(t', \omega) \frac{\partial A}{\partial t'} \\ &\quad + \int_0^\infty d\omega \alpha(t, \omega) (C_\omega e^{-i\omega t} + C_\omega^* e^{i\omega t}) \\ &= - \int_{-\infty}^t dt' \chi_C(t, t') \frac{\partial A}{\partial t'} + P_0. \end{aligned} \quad (15)$$

In the final line of Eq. (15) we have separated the polarization into a part induced by the electric field, with the associated susceptibility $\chi_C(t, t')$, plus the additional contribution P_0 due to the undriven motion of the reservoir. Although classically we can set $P_0 = 0$, equivalent to having the reservoir at rest when $t \rightarrow -\infty$, quantum mechanically this would violate the uncertainty principle, meaning that in general we must retain P_0 . This so-called “noise polarization” is responsible for zero point and thermal radiation, which we return to in Sec. III.

The relative permittivity is given in terms of the above two-time susceptibility as $\epsilon_C(t, t') = \epsilon_\infty + \chi_C(t, t')$. To compare this with the earlier results for the time-dependent Drude model (9), we perform a Fourier transform of the permittivity in the second time argument t' , giving

$$\epsilon_C(t, \omega) = \epsilon_\infty - \frac{1}{\omega(\omega + i\gamma)} [\omega_p^2(t) - \sigma_1(t, \omega)], \quad (16)$$

where the memory term σ_1 is here defined as

$$\begin{aligned} \sigma_1 &= \frac{i\omega_p(t)}{\gamma} \int_{-\infty}^t dt' \frac{d\omega_p(t')}{dt'} e^{i\omega(t-t')} \\ &\quad \times [\omega(e^{-\gamma(t-t')} - 1) - i\gamma]. \end{aligned} \quad (17)$$

The effective permittivity (16) derived from the Lagrangian of MQED (12) and (13) has the same overall form as the earlier time-dependent Drude model (9), with the same limiting form (6), when the carrier density changes slowly. However, there are some extremely important differences between this permittivity and the model given in Ref. [51] [i.e., Eq. (9)]. Most importantly the permittivity in Eq. (16) changes as rapidly as the function $N(t)$, becoming discontinuous when the carrier density undergoes a temporal discontinuity (see Fig. 4). By implication, in this limit this leads to a discontinuous polarization, and thus an *infinite* current density, $\partial_t P$. Clearly this does not reflect the expected behavior from a sudden but *finite* increase in carrier density in the Drude model, which we want to represent with our Lagrangian. The singular current comes directly from our introduction of the time-dependent coupling in Eq. (11): while the reservoir equation of motion (12) ensures continuity of both X_ω and $\partial_t X_\omega$, neither the polarization $P = \int d\omega \alpha(\omega, t) X_\omega$ nor its time derivative inherit this continuity when α is changed abruptly. Figure 4 shows significant differences between the permittivity arising from this modulated coupling Lagrangian and the Stepanov model. One notable additional difference evident from this figure is the π phase shift of the oscillations between the two forms of $\epsilon(\omega, t)$. This difference arises from the fact that the time dependence of the material response enters in a different order when integrating the equations of motion.

Just as importantly, the noise polarization shows the same discontinuity as a function of time,

$$\begin{aligned} P_0 &= \int_0^\infty d\omega \alpha(t, \omega) (C_\omega e^{-i\omega t} + C_\omega^* e^{i\omega t}) \\ &= \omega_p(t) \int_0^\infty d\omega \sqrt{\frac{2\gamma\epsilon_0}{\pi(\omega^2 + \gamma^2)}} (C_\omega e^{-i\omega t} + C_\omega^* e^{i\omega t}), \end{aligned} \quad (18)$$

meaning that when we quantize the time-varying Drude model with the modulated coupling (11), we would predict arbitrarily large fluctuating thermal currents, constrained only by the time scale over which we can modulate the carrier density (as shown in Fig. 6).

B. The “modulated reservoir” Lagrangian

It thus appears that introducing a time modulation within the field-reservoir coupling in MQED does not accurately capture the effect of changing the carrier density in a metal. As we have seen, this leads to spuriously large transient currents.

We now remove the carrier density from the field-reservoir coupling

$$\alpha(\omega) \rightarrow \sqrt{\frac{2\gamma e^2}{\pi m^*(\omega^2 + \gamma^2)}} \quad (19)$$

and instead introduce it as a scaling of the reservoir Lagrangian,

$$\mathcal{L}_R \rightarrow \frac{1}{2} \int_0^\infty d\omega \frac{1}{N(t)} \left[\left(\frac{\partial X_\omega}{\partial t} \right)^2 - \omega^2 X_\omega^2 \right]. \quad (20)$$

This makes a modulation of the carrier density equivalent to a change in the mass of every oscillator. For, e.g., a large

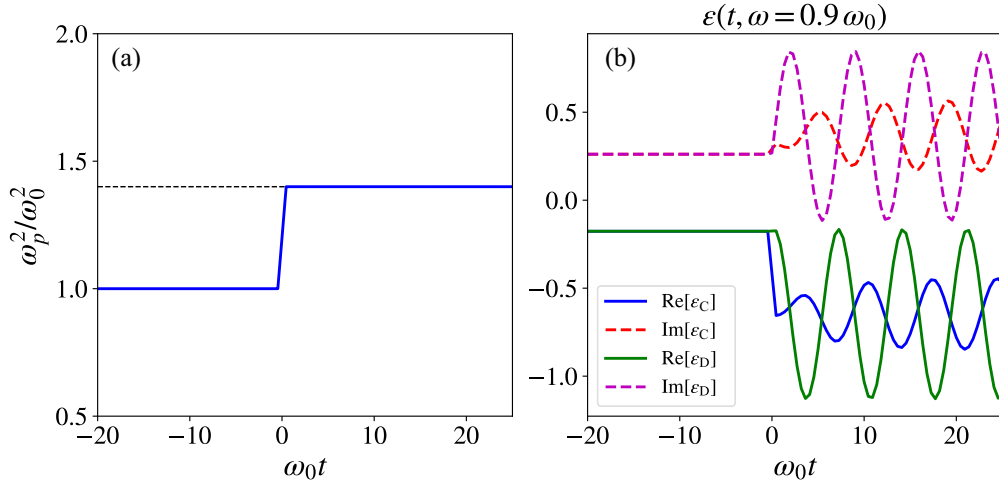


FIG. 4. Stepanov vs modulated coupling Lagrangian: (a) the plasma frequency changes abruptly as $\omega_p^2(t) = \omega_0^2 + \delta\omega_p^2 \frac{1}{2}[1 + \tanh(t/t_r)]$ with $\omega_0 t_r = 0.05$ and $\delta\omega_p^2 = 0.4\omega_0^2$. (b) Comparison between the permittivities $\varepsilon(\omega, t)$ given by the Stepanov model (9) and calculated from the modulated coupling Lagrangian (16), for the fixed frequency $\omega = 0.9\omega_0$. Note that the permittivity (16) changes extremely rapidly around $t = 0$, whereas (9) remains continuous. Both permittivity functions exhibit the oscillations noted earlier in Fig. 3.

concentration of carriers this effective mass is small, indicating that then—as expected—it is easy to polarize the material.

Applying Lagrange's equations to the sum of (2), (4), and (20), the electromagnetic wave equation takes the same form as earlier, given in Eq. (13). The dynamics of the reservoir, however, is now governed by

$$N \frac{\partial}{\partial t} \left(\frac{1}{N} \frac{\partial X_\omega}{\partial t} \right) + \omega^2 X_\omega = -N(t) \alpha(\omega) \frac{\partial A}{\partial t}, \quad (21)$$

which is very similar to the differential equation for the polarization (7). Indeed, multiplying Eq. (21) by the coupling constant $\alpha(\omega)$ and integrating over ω , Eq. (21) becomes

$$N \frac{\partial}{\partial t} \left(\frac{1}{N} \frac{\partial P}{\partial t} \right) + \int_0^\infty d\omega \omega^2 \alpha(\omega) X_\omega = -\frac{e^2 N(t)}{m^*} \frac{\partial A}{\partial t}, \quad (22)$$

which besides the damping term, is the same as the differential equation governing the time-varying Drude model (7). Indeed, given that the plasma frequency and the damping arise from coupling to the same system (here the reservoir), it is questionable whether the damping should be assumed constant as it is in Eq. (7) and Ref. [51].

For a general variation of the carrier density, it is difficult to solve (21). We take the special case where there is an abrupt change at $t = 0$: $N(t) = N_0 \Theta(-t) + N_1 \Theta(t)$ (the plasma frequency accordingly changing from ω_0 to ω_1 at $t = 0$). In this case the solution to Eq. (21) is

$$X_\omega = - \int_{-\infty}^t dt' \frac{\alpha(\omega)}{\omega} \frac{\partial A}{\partial t'} \{ \sin[\omega(t-t')] N(t') + \Theta(t) \Theta(-t') N_0 \Delta \sin(\omega t) \cos(\omega t') \} + X_{0\omega}, \quad (23)$$

where $\Delta = 1 - N_1/N_0$. The final quantity $X_{0\omega}$ is the homogeneous solution to Eq. (21), defined—as in the previous section—in terms of a set of complex amplitudes C_ω ,

$$\begin{aligned} X_{0\omega} &= C_\omega [e^{-i\omega t} + i\Delta \Theta(t) \sin(\omega t)] + \text{c.c.} \\ &= C_\omega \xi_\omega(t) + C_\omega^* \xi_\omega^*(t). \end{aligned} \quad (24)$$

Inserting the solution (23) for the reservoir dynamics into the definition of the polarization and performing a Fourier transform, we find the effective permittivity is again given by

$$\varepsilon_R(t, \omega) = \varepsilon_\infty - \frac{1}{\omega(\omega + i\gamma)} [\omega_p^2(t) - \sigma_2(t, \omega)], \quad (25)$$

where the memory term is now given by

$$\sigma_2(t, \omega) = \Theta(t) \frac{\omega_{p0}^2 \Delta}{1 - \frac{i\gamma}{\omega}} \left[\frac{i\gamma}{\omega} e^{2i\omega t} - e^{(i\omega - \gamma)t} \right]. \quad (26)$$

When $t \rightarrow 0^+$ the above expression reduces to $\sigma_2 = \omega_1^2 - \omega_0^2$, showing that the permittivity (25) is continuous across the jump in carrier density at $t = 0$. Note that, although the memory term (26) appears to have a pole in the upper half of the complex frequency plane at $\omega = +i\gamma$, which would violate the Kramers-Kronig relations [52], the residue of this pole is zero.

Figure 5 shows a comparison between the permittivity $\varepsilon(t, \omega)$ computed from the Stepanov model (9) and that we just derived from the Lagrangian density with the modified reservoir (20). Unlike the earlier permittivity (16), derived from modulating the field-reservoir coupling, ε_C is continuous at $t = 0$ and becomes equal to the permittivity of the Stepanov model in the limit of small damping, $\gamma/\omega \ll 1$. In this small-damping limit we have therefore found a Lagrangian description of a time-varying Drude metal that we can quantize.

As shown above, the homogeneous solutions to Eq. (21) are given by (24), which are not simple complex exponentials. Substituting these expressions into the definition of the noise polarization, $P_0 = \int_0^\infty d\omega \alpha(\omega) X_{0\omega}$, we find this equals

$$\begin{aligned} P_0 &= \int_0^\infty d\omega \alpha(\omega) [C_\omega \xi_\omega(t) + C_\omega^* \xi_\omega^*(t)] \\ &= \int_0^\infty d\omega \sqrt{\frac{2\gamma e^2}{\pi m^*}} \frac{1}{\omega^2 + \gamma^2} [C_\omega \xi_\omega(t) + C_\omega^* \xi_\omega^*(t)]. \end{aligned} \quad (27)$$

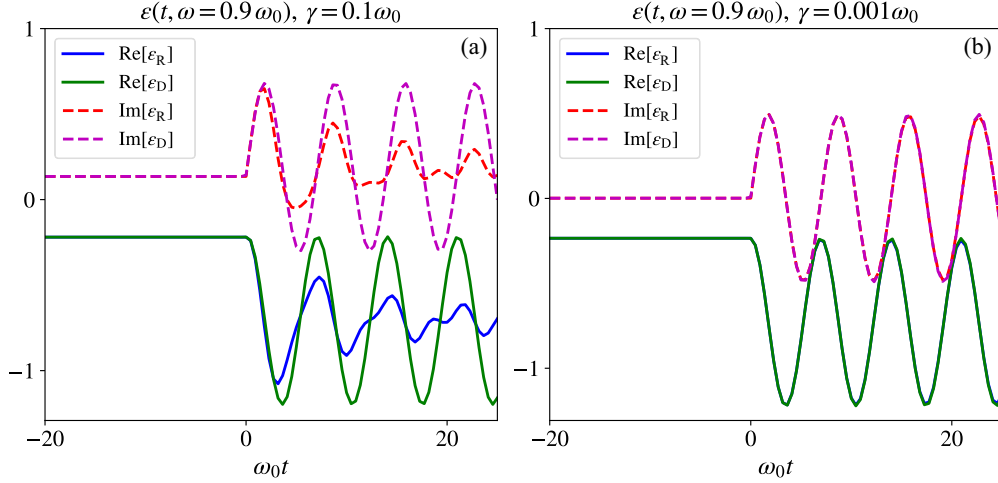


FIG. 5. Stepanov vs modulated reservoir Lagrangian: Comparison between the permittivity (9) derived from the Stepanov model and that derived from the “mature Drude” Lagrangian (25), for the modulation shown in Fig. 4(a) and the fixed frequency $\omega = 0.9\omega_0$. (a) The case of large damping, $\gamma = 0.1\omega_0$, showing that ϵ_R and ϵ_D oscillate in phase, with the reservoir approach yielding an additional harmonic and an exponential decay. (b) In the limit of small damping, $\gamma = 0.001\omega_0$, the two permittivity functions, ϵ_D and ϵ_R , become equal.

Unlike the noise polarization (18), which is discontinuous in time when the plasma frequency is changed abruptly, we now have a noise polarization (27) that is continuous, although its first derivative is not (see Fig. 6). This rapid change in the derivative of the noise current $\partial_t j = \partial_t^2 P_0$ will lead to a “flash” of radiation at $t = 0$.

Note that although the results of this section are only valid for a step change in the carrier density, we can approximate any continuous change in density as a series of such steps, using, e.g., a transfer matrix to generally compute the time evolution of the oscillator amplitudes X_ω .

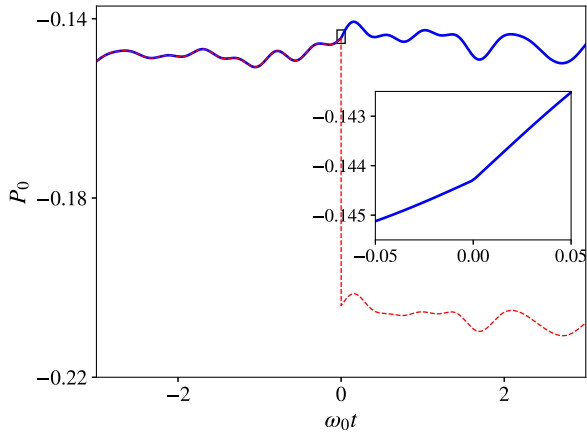


FIG. 6. Visualizing noise polarization dynamics: Taking $N = 3500$ frequency points evenly spaced between $\omega = 0$ and $15\omega_0$, the magnitude and phase of the complex amplitudes C_ω were chosen randomly to compute the classical noise polarization arising from the modulated coupling (18) (red dashed) and modulated reservoir (27) (blue solid) approaches, for a large, abrupt change $\omega_p^2 = \omega_0^2 \rightarrow 2\omega_0^2$ in plasma frequency at $t = 0$ ($\gamma = 0.001\omega_0$). While the modulated coupling Lagrangian predicts a discontinuous noise polarization, the modulated reservoir approach predicts a discontinuous time derivative (shown inset).

III. QUANTIZATION

We have thus found a Lagrangian density that—through its equation of motion—leads to Maxwell’s equations with a permittivity that tends to Stepanov’s expression (9) in the limit of small damping. Although we make no claim that this particular model generally represents the dynamics of a real metal, it is clear the predictions of MQED very much depend on exactly how the time modulation is included within the Lagrangian. While a modulated field-reservoir coupling (11) can yield extremely large polarization currents, modulating the effective mass of the reservoir oscillators (20) does not suffer this defect. We shall now compare the predictions of the quantum counterparts of these theories, giving the counterpart of the fluctuation-dissipation theorem for the two Lagrangians described above.

The first step is to derive the Hamiltonian density, written in terms of a set of field variables and their conjugate momenta. For the reservoir variables the canonical momentum density, $\Pi_\omega = \delta\mathcal{L}/\delta\dot{X}_\omega$ is given by, in the cases of modulated coupling (C) and modulated reservoir (R),

$$\Pi_\omega = \begin{cases} \frac{\partial X_\omega}{\partial t} & \text{(C)} \\ \frac{1}{N(t)} \frac{\partial X_\omega}{\partial t} & \text{(R)}, \end{cases} \quad (28)$$

and similarly for the vector potential of the EM field, where the canonical momentum density, $\Pi_A = \delta\mathcal{L}/\delta\dot{A}$, equals

$$\Pi_A = \begin{cases} \epsilon_0 \epsilon_\infty \frac{\partial A}{\partial t} - \int_0^\infty d\omega \alpha(\omega, t) X_\omega & \text{(C)} \\ \epsilon_0 \epsilon_\infty \frac{\partial A}{\partial t} - \int_0^\infty d\omega \alpha(\omega) X_\omega & \text{(R)}, \end{cases} \quad (29)$$

where the field-reservoir coupling is defined in Eqs. (11) and (19), respectively. Forming the classical Hamiltonian density via the usual expression and separating the result into three

terms,

$$\mathcal{H} = \Pi_A \partial_t A + \int_0^\infty d\omega \Pi_\omega \partial_t X_\omega - \mathcal{L} \quad (30)$$

$$= \mathcal{H}_E + \mathcal{H}_B + \mathcal{H}_R, \quad (31)$$

we find that, in the two cases described above, we have the reservoir energy

$$\mathcal{H}_R = \begin{cases} \frac{1}{2} \int_0^\infty d\omega (\Pi_\omega^2 + \omega^2 X_\omega^2) & \text{(C)} \\ \frac{1}{2} \int_0^\infty d\omega (N(t) \Pi_\omega^2 + \frac{\omega^2}{N(t)} X_\omega^2) & \text{(R)} \end{cases} \quad (32)$$

and the energy stored in the electric field

$$\mathcal{H}_E = \frac{1}{2\epsilon_0 \epsilon_\infty} \times \begin{cases} (\Pi_A + \int_0^\infty d\omega \alpha(\omega, t) X_\omega)^2 & \text{(C)} \\ (\Pi_A + \int_0^\infty d\omega \alpha(\omega) X_\omega)^2 & \text{(R)}. \end{cases} \quad (33)$$

In both cases the magnetic energy equals $\mathcal{H}_B = \frac{1}{2\mu_0} (\partial_x A)^2$. Promoting these classical field and momentum variables to operators with the equal-time commutation relations, $[\hat{X}_\omega(x, t), \hat{\Pi}_{\omega'}(x', t)] = i\hbar \delta(\omega - \omega') \delta(x - x')$ and $[\hat{A}(x, t), \hat{\Pi}_A(x', t)] = i\hbar \delta(x - x')$, the quantum mechanical Hamiltonian is then given by $\hat{H} = \int \hat{\mathcal{H}} dx$.

Due to the time dependence of the material, it might be questioned whether we can use this Hamiltonian operator to usefully compute thermal averages or vacuum radiation, given the lack of thermal equilibrium. We can use the Heisenberg picture to sidestep this difficulty, working with the states $|\psi\rangle$ of the system defined at $t = -\infty$ and placing all the time evolution within the expressions for the operators, any operator $\hat{O}$ obeying $d_t \hat{O} = \frac{i}{\hbar} [\hat{H}, \hat{O}] + \partial_t \hat{O}$. Evaluating these operator equations of motion, we find they are formally identical to the classical equations of motion, e.g., (12), (13), and (21), as expected due to the quadratic form of the Hamiltonian.

This formal equivalence between classical and operator equations of motion means we can solve the quantum mechanical evolution through simply promoting our classical solutions to operator expressions. For instance, the homogeneous solutions (14) and (24) to the operator dynamics of the reservoir are

$$\hat{X}_{\omega, \omega} = \begin{cases} \sqrt{\frac{\hbar}{2\omega}} (\hat{C}_\omega e^{-i\omega t} + \hat{C}_\omega^\dagger e^{i\omega t}) & \text{(C)} \\ \sqrt{\frac{\hbar N_0}{2\omega}} [\hat{C}_\omega \xi_\omega(t) + \hat{C}_\omega^\dagger \xi_\omega^*(t)] & \text{(R)}, \end{cases} \quad (34)$$

where ξ_ω is defined in Eq. (24) and the prefactors of $\sqrt{\hbar/2\omega}$ and $\sqrt{\hbar N_0/2\omega}$ are chosen to enforce the canonical commutation relations, $[\hat{X}_{\omega, \omega}(x, t), \hat{\Pi}_{0, \omega'}(x', t)] = i\hbar \delta(x - x') \delta(\omega - \omega')$. We have promoted the complex classical amplitudes C_ω to bosonic operators satisfying the continuum commutation relations

$$[\hat{C}_\omega(x), \hat{C}_{\omega'}^\dagger(x')] = \delta(\omega - \omega') \delta(x - x'), \quad (35)$$

with $\hat{C}_\omega^\dagger$ and $\hat{C}_\omega$ respectively interpreted as polariton creation and annihilation operators. As in the classical discussion above, the noise polarization arising from our two Lagrangians—defined in (15) and (27)—are given as integrals

over the homogenous solutions to the reservoir dynamics,

$$\hat{P}_0 = \begin{cases} \int_0^\infty d\omega \sqrt{\frac{\epsilon_0 \hbar \gamma \omega_p^2(t)}{\pi \omega (\omega^2 + \gamma^2)}} \hat{C}_\omega e^{-i\omega t} + \text{H.c.} & \text{(C)} \\ \int_0^\infty d\omega \sqrt{\frac{\epsilon_0 \hbar \gamma \omega_0^2}{\pi \omega (\omega^2 + \gamma^2)}} \hat{C}_\omega \xi_\omega(t) + \text{H.c.} & \text{(R)}, \end{cases} \quad (36)$$

where we have replaced the coupling functions $\alpha(\omega)$ and $\alpha(\omega, t)$ with their definitions in (11) and (19). Remember that to compare the two operators in Eq. (36) we must consider an abrupt change in plasma frequency, from ω_0 to ω_1 at $t = 0$.

In terms of the noise polarization (36), the noise current is given by $\hat{j}_0 = \partial \hat{P}_0 / \partial t$. This noise current is the source of the quantum fluctuations in the electromagnetic field indicated in Fig. 1, usually predicted from the fluctuation-dissipation theorem. As a concrete example we consider these fluctuating currents when the system is initially in its vacuum state $|0\rangle$, defined as the zero polariton state,

$$\hat{C}_\omega(x)|0\rangle = 0. \quad (37)$$

To understand the noise currents within our two models, here we compare their vacuum current-current correlation functions,

$$\begin{aligned} \frac{1}{2} \langle 0 | \hat{j}_0(x, t) \hat{j}_0(x', t') + \hat{j}_0(x', t') \hat{j}_0(x, t) | 0 \rangle \\ = \delta(x - x') K(t, t'), \end{aligned} \quad (38)$$

where we define $K(t, t')$ as the time autocorrelation of the noise current at a fixed point in space. Computing this autocorrelation function for the modulated coupling theory, using both Eqs. (36) and (37), we have the autocorrelation

$$\begin{aligned} K(t, t') = \int_0^\infty \frac{d\omega}{2\pi} \frac{\epsilon_0 \hbar \gamma}{\omega (\omega^2 + \gamma^2)} [\delta(t) \omega_0 \Delta + i \omega \omega_p(t)] \\ \times [\delta(t') \omega_0 \Delta - i \omega \omega_p(t')] e^{-i\omega(t-t')} + \text{c.c.}, \end{aligned} \quad (39)$$

whereas for the modulated reservoir theory the autocorrelation is

$$\begin{aligned} K(t, t') = \int_0^\infty \frac{d\omega}{2\pi} \frac{\epsilon_0 \hbar \gamma \omega \omega_0^2}{\omega^2 + \gamma^2} [e^{-i\omega t} - \Delta \Theta(t) \cos(\omega t)] \\ \times [e^{i\omega t'} - \Delta \Theta(t') \cos(\omega t')] + \text{c.c.} \end{aligned} \quad \text{(R)} \quad (40)$$

Taking the limit of an unmodulated carrier density, $N_1 = N_0$, both (39) and (40) reduce to the same expression,

$$\begin{aligned} K(t - t') = \epsilon_0 \hbar \int_0^\infty \frac{d\omega}{2\pi} \frac{\gamma \omega \omega_p^2}{\omega^2 + \gamma^2} e^{-i\omega(t-t')} + \text{c.c.} \\ = \epsilon_0 \hbar \int_0^\infty \frac{d\omega}{2\pi} \omega^2 \text{Im}[\epsilon(\omega)] e^{-i\omega(t-t')} + \text{c.c.}, \end{aligned} \quad (41)$$

the Fourier transform of which equals

$$\tilde{K}(\omega) = \epsilon_0 \hbar \omega^2 \text{sign}(\omega) \text{Im}[\epsilon(\omega)]. \quad (42)$$

Equation (42) is the zero-temperature limit of the fluctuation-dissipation theorem [5,6], a well-known relation between the spectrum of the noise current fluctuations (here $\tilde{K}$) and the imaginary part of the susceptibility.

Figure 7 compares the two current autocorrelation functions (39) and (40), derived above. As anticipated from our

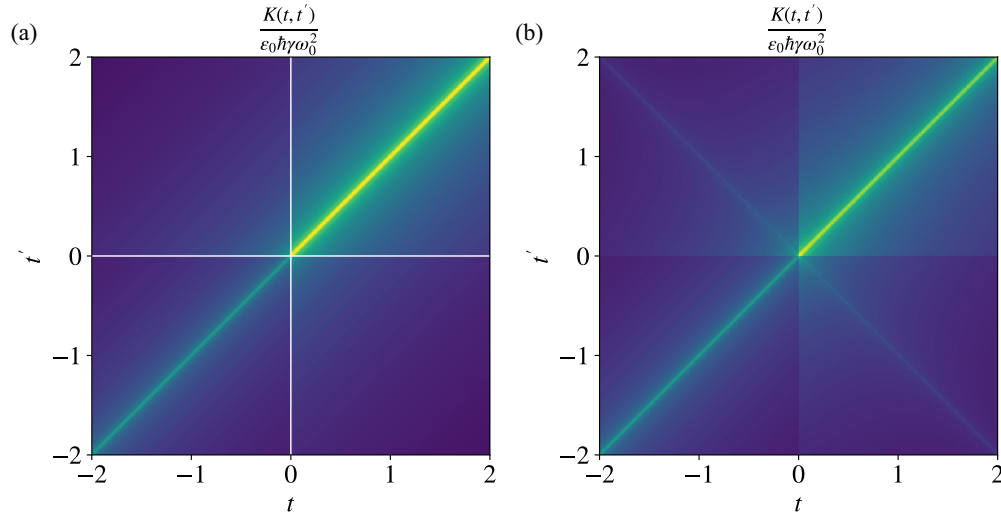


FIG. 7. Noise current autocorrelation: Normalized in units of $\epsilon_0 \hbar \gamma \omega_0^2$ and for the parameters $N_1 = 1.4N_0$ and $\gamma = 0.1$. (a) Computed for the modulated coupling theory (39), the white lines indicating where the autocorrelation is singular, and (b) for the modulated reservoir theory (40), which is discontinuous but finite. Note the presence of correlations along $t = -t'$.

classical analysis, in the case of the modulated coupling theory, Fig. 7(a) shows that the autocorrelation of the current is singular along the lines $t = 0$ and $t' = 0$, due to the abrupt change in the carrier density. Moreover, as a simple analysis shows, away from these lines, where we can neglect the delta functions in Eq. (39), the correlation is simply that given by the fluctuation-dissipation theorem (41), scaled by the product of the plasma frequencies at the two times, t and t' , divided by ω_0^2 .

By contrast, the modulated reservoir theory yields a nonsingular autocorrelation function, albeit one that is discontinuous across the lines $t = 0$ and $t' = 0$. An important feature shown in Fig. 7(b) is that this theory develops additional correlations in the noise current along the lines $t = \pm t'$. These arise because in this case we are modulating the properties of the reservoir, which leads to the temporal analog of reflection in the oscillator dynamics. Modulating the field-reservoir coupling cannot ever lead to such correlations.

IV. SUMMARY AND CONCLUSIONS

Although MQED is a well-established tool for predicting quantum behaviour in real-world materials, here we have shown that it is nontrivial to extend this theory to the case of time-varying media.

To include the effects of dispersion and dissipation, MQED makes use of a fictitious reservoir of harmonic oscillators, the linear response of which perfectly matches that of the real medium, specifying the coupling between the field and the reservoir in terms of the imaginary part of the susceptibilities. When we quantize this theory we are thus quantizing electromagnetism coupled to a system with a linear response that exactly matches that of the material, as measured in the laboratory. We have shown that the usual simple relationship between the system-reservoir coupling and the permittivity is broken when the material response is modulated in time.

Using the example of a Drude model with a time-varying carrier density, we have shown that a naive replacement of $\text{Im}[\epsilon(\omega)] \rightarrow \text{Im}[\epsilon(\omega, t)]$ in the standard macroscopic QED theory leads to a polarization of the material that can change as rapidly as we can modulate the carrier density, the polarization current becoming singular in the limit of a discontinuous change (which is nonphysical, given that both the total charge and energy of the reservoir remain finite). As an alternative we considered a theory where the change in carrier density is incorporated through modifying the reservoir dynamics. This alternative model leads to a permittivity that reproduces the classical time-varying Drude model of Stepanov [51] in the limit of small damping, with the polarization current remaining finite.

To determine the exact form of the Lagrangian required to model any particular system requires some experimental input. Considering, for example, the experiments on modulated ITO described in Ref. [50], the form of $\epsilon(\omega, t)$ is observable through the reflected spectrum of a probe pulse, with some controllable delay with respect to the modulation. As we can see from Figs. 4 and 5, if the initial change in the permittivity occurs over a time scale that is small compared to an oscillation cycle, then there are characteristic oscillations in the function $\epsilon(\omega, t)$ that can be used to distinguish between the different models presented here. Although these short time-scale modulations may be difficult to realize at optical frequencies, this approach could also be applied to other experimental platforms that operate at lower frequencies (e.g., modulation at terahertz frequencies as described in, e.g., Ref. [53]).

One of the central predictions of MQED is the presence of noise currents within dispersive, dissipative media. The presence of these currents is necessary to retain thermal equilibrium, and they are missed in both classical theories and quantum theories that neglect dispersion. Again, using the example of a time-modulated Drude model with an abrupt change in the carrier density, we have shown that the

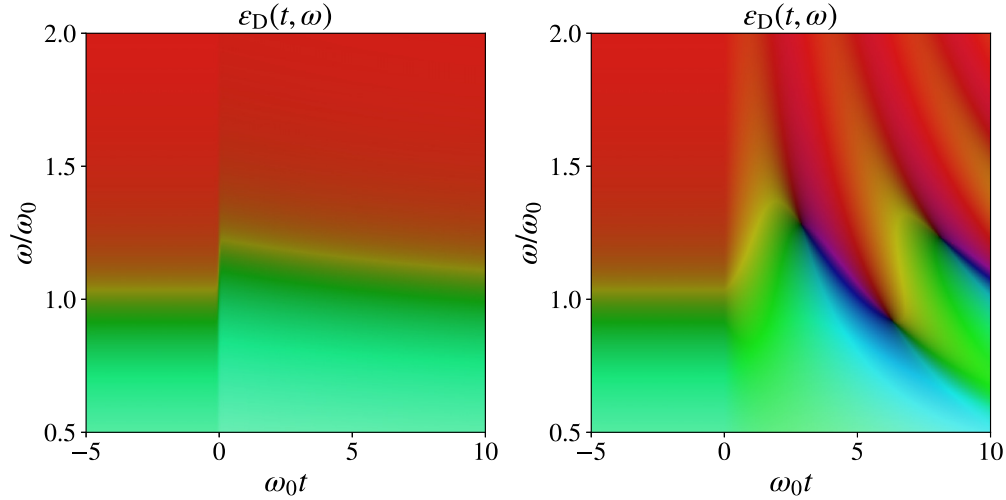


FIG. 8. Continuity of the Stepanov permittivity: Comparison between (left) the approximate Drude model (6) and (right) the Stepanov model (9) for an abrupt change in carrier density $\omega_0 t_r = 0.05$ and $\omega_0 t_d = 10$ (see Fig. 2).

discontinuous polarization of the modulated coupling theory of MQED yields an infinite noise current. Meanwhile, our alternative modulated reservoir theory predicts a finite noise current, plus additional correlations that arise from “time reflection” within the reservoir dynamics, due to the modulated parameters.

We have shown that a theory of MQED with modulated reservoir parameters more closely models the dynamics of a metal with a modulated carrier density, compared with a naive modification of the standard theory. This modulated reservoir theory leads to new phenomena, such as additional correlations within the noise currents, which although difficult to observe in the vacuum state, may be observable in higher temperature thermal correlations. More generally, it also remains an interesting question as to how to incorporate the precise linear response of any particular time-varying material within a Lagrangian description.

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APPENDIX: CONTINUITY OF THE PERMITTIVITY

Here, we give an example illustrating the continuity of the Stepanov permittivity (9) in the case of an abrupt change in the carrier density. Figure 8 compares the complex functions (6) and (9), showing that the approximate permittivity tends toward being discontinuous.

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