

Operational interpretation of the Choi rank through exclusion tasks

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The Choi state is an indispensable tool in the study and analysis of quantum channels. Considering a channel in terms of its associated Choi state can greatly simplify problems. It also offers an alternative approach to the characterization of a channel, with properties of the Choi state providing novel insight into a channel's behavior. The rank of a Choi state, termed the Choi rank, has proven to be an important characterizing property, and here, its significance is further elucidated through an operational interpretation. The Choi rank is shown to provide a universal bound on how successfully two agents, Alice and Bob, can perform an entanglement-assisted exclusion task. The task can be considered an extension of superdense coding, where Bob can only output information about Alice's encoded bit string with certainty. Conclusive state exclusion, in place of state discrimination, is therefore considered at the culmination of the superdense coding protocol. In order to prove this result, a necessary condition for conclusive k -state exclusion of a set of states is presented in order to achieve this result, and the notions of weak and strong exclusion are introduced.

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Introduction. States give us only a snapshot in time. To model how systems evolve, interact with other systems, and respond to external stimuli, it is essential to understand dynamics. Such understanding then enables the prediction of a system's future state, facilitates the design and implementation of controls for on-demand manipulation, and allows for the characterization of a system's response to external influences, such as noise.

In closed quantum systems, dynamics is modeled by unitary operators; the evolution of a state ρ to ρ' is given by $\rho' = U\rho U^\dagger$ for some unitary U . A more general notion of quantum dynamics is captured by quantum channels, or simply channels, which are completely positive trace-preserving (CPTP) linear maps [1,2]. Operationally, channels can be thought of as modeling the dynamics of open quantum systems, where a system evolves while interacting with an environment. Any channel $\mathcal{N}$ acting on a system S has a Stinespring dilation [3] given by $\mathcal{N}(\rho) = \text{tr}_E[U_{SE}(\rho_S \otimes \tau_E)U_{SE}^\dagger]$, where S, E represent the system and environment, respectively, τ_E is some environment state, and U_{SE} is a unitary acting on SE . All quantum dynamics can therefore be modeled as unitary with respect to some higher-dimensional space, and can be described by a single state (τ_E) and unitary (U_{SE}). While the Stinespring dilation gives a physically motivated description of quantum dynamics, a more mathematically mo-

tivated description is given through a Kraus decomposition [1].¹ For a channel $\mathcal{N}$ there always exists a set of M operators $\{K_x\}_{x=1}^M$, where $\sum_{x=1}^M K_x^\dagger K_x = \mathbb{I}$, such that the action of $\mathcal{N}$ on a state ρ is given by $\mathcal{N}(\rho) = \sum_{x=1}^M K_x \rho K_x^\dagger$. The Kraus decomposition is useful when applying a channel since one only needs to consider the input state and not the environment. A given channel can have infinitely many Stinespring dilations and Kraus representations, all sets of operators that are unitarily equivalent to $\{K_x\}_{x=1}^M$ define the same channel, for example. However, and perhaps surprisingly, a channel can be uniquely described through its action on a single quantum state. The Choi-Jamiołkowski isomorphism [4,5] is a linear mapping between quantum channels and bipartite quantum states. For a channel $\mathcal{N}_A$ acting on a system A with dimension d , its Choi state $\mathcal{J}_{AB}^{\mathcal{N}}$ is a bipartite state in AB (where B also has dimension d) defined by

$$\mathcal{J}_{AB}^{\mathcal{N}} = (\mathcal{N}_A \otimes \mathcal{I}_B)(|\Phi^+\rangle\langle\Phi^+|_{AB}), \quad (1)$$

where $|\Phi^+\rangle_{AB} := \sum_{i=0}^{d-1} |ii\rangle_{AB}/\sqrt{d}$ is a maximally entangled state in AB ² ($|i\rangle_A$ and $|i\rangle_B$ are elements of fixed orthonormal bases of A, B , respectively), and $\mathcal{I}_B$ is an identity channel acting on B . Subscripts will be used to explicitly denote the corresponding (sub)systems if needed. From the Choi state, the action of $\mathcal{N}_A$ on a state ρ can be recovered as $\mathcal{N}_A(\rho) = d \text{tr}_B[(\mathbb{I}_A \otimes \rho^t) \mathcal{J}_{AB}^{\mathcal{N}}]$, where $(\cdot)^t$ is the transpose operation in

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¹Note, other methods of modeling the dynamics of open quantum systems beyond quantum channels do exist, one could also consider microscopically derived master equations or the von Neumann equation, for example.

²In general $|\Phi\rangle_{AB}$ can be any full-Schmidt-rank pure state.

the given fixed basis. Hence, remarkably, one can fully characterize the action of $\mathcal{N}$ on arbitrary states by a single action on $|\Phi^+\rangle_{AB}$.

Choi states have proved to be one of the most powerful tools for understanding and characterizing channels, both analytically and numerically. For instance, a linear map $\mathcal{N}_A$ acting on A is a channel if and only if its Choi state is positive semidefinite, $\mathcal{J}_{AB}^{\mathcal{N}} \geq 0$, and maximally mixed in the B subspace, $\text{tr}_A[\mathcal{J}_{AB}^{\mathcal{N}}] = \mathbb{I}_B/d$ [4,5]. By applying further operations to Choi states, channels can be categorized into relevant subsets—a task often challenging when relying on Stinespring dilation or Kraus decomposition [6–9]. For example, a channel is entanglement breaking [10] if its Choi state is separable; it is steering breaking if its Choi state is steerable [11]. Also, nonsignaling conditions can be described by simple formulas of Choi states [12–18]. Generally, Choi states allow problems concerning quantum dynamics to be reformulated as numerically feasible convex optimizations [19,20], making them efficiently solvable. To date, Choi states are widely recognized as an essential tool in characterizing and quantifying dynamical quantum signatures—they provide an alternative, often simpler, approach to dealing with channels [7–39].

An important property of the Choi state is its rank, $r_c^{\mathcal{N}}$, termed Choi rank. This is a key characterizing property that provides insight into the structure of quantum dynamics; it has been shown to place mathematical bounds on the channel's description. For example, it is one, $r_c^{\mathcal{N}} = 1$, if and only if the channel is unitary [40]; it serves as a lower bound on the number of operators needed in the Kraus decomposition of the channel, $r_c^{\mathcal{N}} \leq M$ [1]; it equals the minimum dimension of the environment of a channel's Stinespring dilation [41]; and, when considering mixed unitary channels, is used to bound the number of unitaries needed to define the channel [40].

To date, the Choi rank has been used solely as a mathematical tool, lacking a clear operational interpretation. In this work, we provide the Choi rank with an operational interpretation, further cementing its importance as a characterizing property. To this end, we introduce the task of entanglement-assisted subchannel exclusion, phrasing them as communication tasks that resemble superdense coding [42], but where state exclusion [43–45] replaces state discrimination [46]. We show that the Choi rank establishes a tight, fundamental upper bound on one's ability to succeed in a task of this nature. Hence, this bound can be used to provide the Choi rank with a physical and operational meaning.

Informally, in the task of state exclusion, a referee gives a player a state from a set of N possible predetermined states. The player then performs a measurement on the state and aims to exclude a set of k states that were not given to them. This leaves the player with $N - k$ possible states that they were sent. In comparison, if performing the task of state discrimination, after the measurement the player would aim to declare a single state that they were sent. For some sets of states, the player can perform conclusive state exclusion, excluding k states with unit probability, even when they can say nothing deterministically about what state they do have. Exclusionary information [47]—knowledge about what state the player does not have—can, therefore, be the only certain knowledge about the state that it is possible for the player to obtain. Hence, when using states to encode messages, one

may be able to deterministically say what message was not encoded, while only being able to probabilistically say what message was encoded. The importance of exclusionary information has already been demonstrated in the foundations of quantum theory [43,45,48] and in the quantification of quantum resources [47,49,50]. Here, we build on its ongoing significance in quantum information theory [44,51,52,54–58] by establishing a connection between exclusionary information and the Choi state.

State exclusion tasks. Formally, in a state exclusion task [43–45], a referee has a set of states $\{\rho_x\}_{x=1}^N$ and sends one state from the set, with probability p_x , to a player. The player performs a general N -outcome measurement described by a positive operator-valued measure (POVM) [1] $\{T_a\}_{a=1}^N$, where $T_a \geq 0 \forall a$ and $\sum_{a=1}^N T_a = \mathbb{I}$, on the state and outputs a label $g \in \{1, \dots, N\}$. They win if $g \neq x$ and fail if $g = x$. Namely, the player wins if they successfully exclude the state by outputting a label that was not associated to the sent state; they fail if they output the label associated to the sent state.

If the player outputs a single label g such that $g \neq x$ with certainty, this is conclusive 1-state exclusion. This occurs if the player is able to find a POVM such that

$$\text{tr}[T_x \rho_x] = 0 \quad \forall x \in \{1, \dots, N\}. \quad (2)$$

If the player gets the measurement outcome associated to T_g , they output g knowing with certainty the referee could not have sent ρ_g .

If the player outputs a set of k labels, $\{g_i\}_{i=1}^k$, such that $x \notin \{g_i\}_{i=1}^k$ with certainty, this is conclusive k -state exclusion. There are $\binom{N}{k}$ different sets of k labels the player could exclude, corresponding to all the different subsets of $\{1, \dots, N\}$ of length k . Therefore, when performing k -state exclusion, the player aims to find a POVM with $\binom{N}{k}$ elements such that each measurement outcome allows the player to exclude a subset of states from $\{\rho_x\}_{x=1}^N$ of length k .

Whilst the notion of state exclusion is widely understood, some of the nuances in the definition are not agreed upon. In addition to Eq. (2) being a condition for conclusive state exclusion, the following additional condition,

$$\sum_{x=1}^N \text{tr}[T_a \rho_x] \neq 0 \quad \forall a \in \{1, \dots, N\}, \quad (3)$$

has also been implicitly or explicitly enforced on occasion [44,51], while on other occasions it has not [52]. This additional condition ensures that all outcomes of the POVM $\{T_a\}_{a=1}^N$ have some probability of occurring. By enforcing both Eq. (2) and Eq. (3), conclusive 1-state exclusion on the set $\{\rho_x\}_{x=1}^N$ is defined to be the existence of an N element POVM where each element excludes a different state from $\{\rho_x\}_{x=1}^N$ with certainty, as seen in Fig. 1(a). We define this to be strong state exclusion.

On the other hand, by only enforcing Eq. (2), conclusive 1-state exclusion on $\{\rho_x\}_{x=1}^N$ is defined to be the existence of a POVM with L nonzero elements, where $L \leq N$, such that each conclusively exclude a different state from a subset of $\{\rho_x\}_{x=1}^N$ of size L , as seen in Fig. 1(b). We define this to be weak state exclusion.

When extended to k -state exclusion, strong exclusion means that there exists a POVM that can exclude all possible

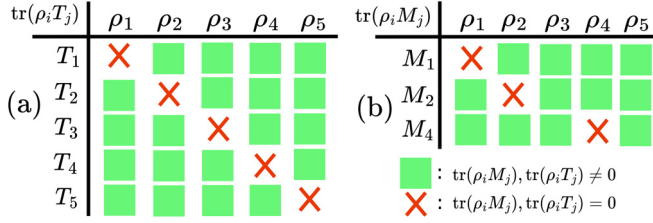


FIG. 1. A graphical depiction of weak versus strong state exclusion. (a) Strong state exclusion: each outcome of the POVM $\{T_a\}_{a=1}^5$ excludes a different state from the set $\{\rho_x\}_{x=1}^5$. (b) Weak state exclusion: each outcome of the POVM $\{M_a\}_{a \in \{1,2,4\}}$ excludes one state from the set $\{\rho_x\}_{x=1}^5$. However, only some states in $\{\rho_x\}_{x=1}^5$ are ever excluded. There exists no POVM element in $\{M_a\}_{a \in \{1,2,4\}}$ to exclude ρ_3 or ρ_5 .

subsets of $\{\rho_x\}_{x=1}^N$ of length k . Weak exclusion then means that there exists a POVM that can only exclude only some subsets of $\{\rho_x\}_{x=1}^N$ of length k . More details on weak and strong state exclusion can be found in Supplemental Material, Sec. A [53].

The task of state exclusion is reminiscent of state discrimination, where the player instead tries to output a label g such that $g = x$. It can be seen that conclusive state discrimination, where a player outputs a label $g = x$ with certainty, is a special case of strong conclusive k -state exclusion where $k = N - 1$. Outputting $N - 1$ labels of states that were definitely not sent is equal to outputting one label of the state that definitely was sent. It is a well-known result that conclusive state discrimination, and hence conclusive $(N - 1)$ -state exclusion, is only possible if all states in $\{\rho_x\}_{x=1}^N$ are orthogonal [1].

A closely related task is subchannel exclusion. Consider a collection of completely positive trace-nonincreasing linear maps, $\Psi = \{\Psi_x\}_{x=1}^N$, such that $\sum_{x=1}^N \Psi_x$ is a channel [2]. This collection is called an instrument, and each map Ψ_x is called a subchannel. In subchannel exclusion, a player has a reference state ρ that they send to the referee. The referee then measures ρ using the instrument and returns the postmeasurement state to the player. The player measures a POVM on the state and outputs a label $g \in \{1, \dots, N\}$. They succeed if they output a label of a subchannel that was not applied. As before, the player can output the label of a subchannel not applied with certainty, they can output k labels, $\{g_i\}_{i=1}^k$, or they can output k labels with certainty.

Necessary condition for k -state exclusion. It has previously been shown that all k -state exclusion tasks can be recast as one-state exclusion tasks by reformulating the set $\{\rho_x\}_{x=1}^N$ (see Appendix I of Ref. [44]). Conceptually, this means that all k -state exclusion tasks are mathematically equivalent to a 1-state exclusion task—solving one task means you have solved the other. Consequently, all state exclusion tasks can be studied within the 1-state exclusion framework through this equivalent task. This has led to a consensus that only the task of 1-state exclusion needed to be studied, and hence, all feasibility conditions in the literature for both weak and strong state exclusion tasks have been for conclusive 1-state exclusion [43,44,51,52,54,55]. However, when using the reformulation method for accessing k -state exclusion tasks, the size of the reformulated sets can get very large for particular values of N and k , potentially making the equivalent task

computationally infeasible to access. In addition, scenarios may exist where one wants to consider the original task rather than the equivalent task; this may happen, for instance, if the set of states upon which exclusion is being performed holds some operational significance. By reformulating the set into the equivalent task, the states in the set will differ, potentially significantly, from the physical states considered in the original task. Hence, a condition for k -state exclusion that is dependent only on the original set $\{\rho_x\}_{x=1}^N$ is valuable—it bypasses the need to consider the equivalent task. Here, a necessary condition of this form is presented as our first main result. It allows for a feasibility test of conclusive k -state exclusion where the number of conditions to be checked is always linear in N .

Lemma 1. A referee has a set of N d -dimensional quantum states $\{\rho_x\}_{x=1}^N$. A necessary condition for the existence of a POVM such that the player can perform conclusive strong or weak k -state exclusion is

$$\sum_{x=1}^N \Pi_x \leq (N - k)\mathbb{I}, \quad (4)$$

where Π_x is the projector onto the support of ρ_x for all x . See Appendix A for the proof. Note, given that every subchannel exclusion task induces an effective state exclusion task, Lemma 1 can be applied to both state and subchannel exclusion tasks. The proof of Lemma 1 does not enforce Eq. (3) as a condition, and hence is a provably necessary condition for strong state exclusion.³ It is left open as to whether Lemma 1 is a sufficient condition for weak k -state exclusion. However, if the inequality is saturated, then Lemma 1 is sufficient for both weak and strong k -state exclusion. In this case, measuring the POVM $\{(\mathbb{I} - \Pi_x)/k\}_{x=1}^N$ would perform k -state exclusion.

As an application of Lemma 1, consider $\{\rho_x\}_{x=1}^N$ to be a set of N orthogonal states. It follows that $\sum_{x=1}^N \Pi_x \leq \mathbb{I}$, and, hence, the largest value of k such that Lemma 1 is satisfied is $k = N - 1$. Lemma 1 therefore implies the ability to perform conclusive state discrimination on a set of N orthogonal states, as expected. This also shows that there exists a set of states for all values of N and d for which Eq. (4) is saturated; namely, Lemma 1 is tight. In addition, if considering rank r states of dimension d , one can always find a weak exclusion task for which Eq. (4) is saturated if $N = \binom{d}{r}$. First, let $\{|i\rangle\}_{i=0}^{d-1}$ be a basis in the d -dimensional space. One can then consider $\{\rho_x\}_{x=1}^N$ such that each Π_x is a projector onto the basis elements contained in each subset of $\{|i\rangle\}_{i=0}^{d-1}$ of length r , of which there are N of them. By measuring the POVM $\{|i\rangle\langle i|\}_{i=0}^{d-1}$, conclusive weak k -state exclusion can be performed with $k = \binom{d-1}{r-1}$. This is as predicted by Lemma 1, as $\sum_{x=1}^N \Pi_x = \binom{d-1}{r-1}\mathbb{I}$, with $\binom{d}{r} - \binom{d-1}{r} = \binom{d-1}{r-1}$. Finally, Lemma 1 also leads to the following corollary on the maximum value of k .

³Consider the set of states with projectors onto their supports of $\{|00\rangle\langle 00|, |01\rangle\langle 01|, |00\rangle\langle 00| + |10\rangle\langle 10|, |01\rangle\langle 01| + |11\rangle\langle 11|\}$. These satisfy Lemma 1 for $k = 2$, but by considering the equivalent task it can be seen that strong state exclusion is never possible as there is a full-rank state in the reformulated set.

Corollary 1. When performing conclusive state exclusion on $\{\rho_x\}_{x=1}^N$, an upper bound on the number, k , of states that can be excluded is given by

$$k \leq N - (2^{D_{\max}(\omega \| \mathbb{I}/d)} \alpha) / d \leq N(d-1)/d, \quad (5)$$

where $\alpha := \text{tr}[\sum_{x=1}^N \Pi_x]$, $\omega := \sum_{x=1}^N \Pi_x / \alpha$, and $D_{\max}(\psi \| \sigma) := \log_2 \min\{\lambda \geq 1 : \psi \leq \lambda \sigma\}$ is the max-relative entropy [59]. See Appendix B for the proof.

Corollary 1 sets a fundamental limit on the number of states that can be excluded, and, interestingly, gives the max-relative entropy a novel operational meaning in terms of state exclusion tasks. Specifically, given a set of states, the max-relative entropy establishes an upper bound on the number of states that can be conclusively excluded. Below, as another main result, the Choi rank is given a novel operational interpretation; it sets a universal upper bound on the number of states that be excluded in a communication task.

Operational interpretation of Choi rank. An operational interpretation of the Choi rank of a channel $\mathcal{N}$ is now presented through an entanglement-assisted subchannel exclusion task. The task is defined as a communication task between two spatially separated parties, Alice (A) and Bob (B). Alice aims to use a preshared entangled state to increase the amount of classical information she can send to Bob through a single use of the channel $\mathcal{N}$, as in superdense coding [42]. It is assumed that the message Alice is sending is of the utmost importance, meaning Bob chooses to only output information about the encoded message that he is certain of.

Task 1. Alice and Bob share a maximally entangled state of local dimension d . Alice encodes x , one of N bit strings, that she wants to send to Bob by applying one of the unitary channels from $\{U_x\}_{x=1}^N$ to her half of the maximally entangled state. She then sends her half of the maximally entangled state to Bob via the channel $\mathcal{N}$. Bob performs a joint measurement and aims to output a set of k bit strings that he is certain Alice did not encode.

Task 1 aims to operationally quantify the information about Alice's encoded message (communicated via $\mathcal{N}$) that Bob can obtain with certainty. Its significance can be understood by first considering the case with $k = N - 1$ and $N = d^2$. If Bob achieves this, he can exclude all bit strings but one, effectively outputting a single bit string, l , such that $l = x$ with certainty. Hence, Bob knows with certainty what message Alice encoded, with Task 1 becoming equivalent to (conclusive) superdense coding. However, as stated above, Bob can only achieve $k = N - 1$ if the states after encoding and sending are orthogonal. This is, in general, not true when $\mathcal{N}$ introduces noise (see, e.g., Ref. [60]). Bob can therefore instead attempt to say something with certainty about Alice's encoded message by outputting a set of k bit strings, where $k \leq N - 1$, which he is certain does not contain x .

We will focus on Bob's ability to maximize the value of k , measuring his success in Task 1 by the maximum number of bit strings that it is possible for him to exclude. The larger the value of k , the more Bob knows about which bit string Alice encoded. This culminates in Bob performing conclusive state discrimination if $k = N - 1$ and hence knowing Alice's encoded bit string with certainty. The following result upper

bounds k via the Choi rank of $\mathcal{N}$ and holds for all possible unitary encoding and decoding (POVMs) strategies.

Result 1. The maximum number of bit strings, k , that Bob can exclude in Task 1 is

$$k \leq N(d^2 - r_c^{\mathcal{N}}) / d^2. \quad (6)$$

See Appendix C for the proof.

Note that Eq. (6) is tight, as the equality can be saturated. To see this, applying Result 1, it can immediately be seen that if $\mathcal{N}$ is a depolarizing channel, $\mathcal{D}_p^{\text{pol}}(\rho) := p\rho + (1-p)\text{tr}[\rho]\mathbb{I}/d$, then $k = 0$. This is because the Choi states of depolarizing channels are full rank, $r_c^{\mathcal{N}} = d^2$, for all p . Hence, Bob can say nothing with certainty about the message encoded by Alice when $\mathcal{N}$ is a depolarizing channel. Consider instead that Alice is trying to perform superdense coding, encoding one of d^2 bit strings into a maximally entangled state with local dimension d using the Heisenberg-Weyl operators [61]. She then sends her half of the state to Bob via a dephasing channel, $\mathcal{D}_p^{\text{ph}}(\rho) := p\rho + (1-p)\sum_{n=0}^{d-1} |n\rangle\langle n|\rho|n\rangle\langle n|$, which has $r_c^{\mathcal{N}} = d$. Result 1 then implies that k -state exclusion is possible for $k \leq d^2 - d$. If Bob measures the POVM that projects into the Bell basis, Eq. (6) is saturated in this instance, since Bob can perform conclusive weak $(d^2 - d)$ -exclusion. Hence, Result 1 is tight. See Supplemental Material, Sec. B [53] for details. Result 1 is derived under minimal assumptions about dynamics, namely, that they are modeled by quantum channels, making it broadly applicable. Moreover, it is fundamental because, while tight, one can never improve on it. This fundamental bound on an operational task (Task 1) can then be used to provide the Choi rank with an operational interpretation: it is the quantity that limits the number of bit strings that could ever be excluded in Task 1. A natural extension to Result 1 is to ask if it also holds for encodings via general quantum channels in Task 1, rather than just unitary channels. By noting that unital channels are rank nondecreasing (see Supplemental Material, Sec. C [53]) and that the transpose of a unital channel is still a unital channel [21], Result 1 can initially be expanded to include all encodings via unital channels. We then conjecture that Result 1 generalizes to include encodings via general (nonunital) channels, leaving the proof for future work.

Discussions. We give the Choi rank a novel operational interpretation as the fundamental limit on an entanglement-assisted exclusion tasks. To drive this result, a necessary condition for conclusive k -state exclusion has been presented, and the notion of weak and strong state exclusion has been introduced. This condition allows the viability of conclusive k -state exclusion to be assessed without the need to first reformulate the set and apply the conditions for one-state exclusion. Although, by considering $k = 1$, this also adds to the conditions for conclusive one-state exclusion already present in the literature [43,44,51,52,54,55]. While it is known that this condition is not sufficient for strong state exclusion, it would be interesting to know if it is sufficient for weak state exclusion.

There are several directions in which Result 1 could be generalized. First, while Result 1 holds for all possible unital encoding and (general) decoding strategies, it is unknown if it holds for all initial states shared between Alice and Bob.

It follows from the definition of the Choi state that Result 1 holds for any full-Schmidt-rank state shared between Alice and Bob. And, intuitively, one would imagine that using a less entangled initial state could only reduce one's ability to succeed at the task. This intuition arises from the knowledge that entanglement is a resource for superdense coding, which is a special case of Task 1. Understanding this would enable us to determine the underlying resources of Task 1. Second, as stated in the above conjecture, Result 1 could be generalizing to include general (nonunitary) encoding strategies.

Understanding how these extensions affect one's ability to succeed in the task will help assess the boundaries of the limitations imposed by a channel's Choi rank. Moreover, it will allow the significance of this task in quantifying resources to be assessed [47–50,62–64].

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Appendix A: Proof of Lemma 1. The following Lemma is first proved.

Lemma 2. If $\sigma \geq 0$ is some state and $0 \leq Q \leq \mathbb{I}$ some operator such that $\text{tr}[\sigma Q] = 0$, then $\Pi_\sigma \leq \mathbb{I} - Q$, where Π_σ is the projector onto the support of σ .

Proof. First, note that $\sigma \geq \mu_{\min}(\sigma)\Pi_\sigma$, where $\mu_{\min}(\sigma) > 0$ is the minimal positive eigenvalue of σ . Therefore, $0 = \text{tr}[\sigma Q] \geq \mu_{\min}(\sigma)\text{tr}[\Pi_\sigma Q\Pi_\sigma]$. Given $\mu_{\min}(\sigma) > 0$ and $\Pi_\sigma Q\Pi_\sigma \geq 0$, it can be seen that $\Pi_\sigma Q\Pi_\sigma = 0$. Hence, $\Pi_\sigma \leq \Pi_{\ker(Q)} \leq \mathbb{I} - \Pi_{\text{supp}(Q)}$, where $\Pi_{\ker(Q)}$ and $\Pi_{\text{supp}(Q)}$ are the projectors onto the kernel and support of Q , respectively. Finally, given that $Q \leq \Pi_{\text{supp}(Q)}$, we have that $Q \leq \Pi_{\text{supp}(Q)} \leq \mathbb{I} - \Pi_\sigma$, completing the proof. ■

The proof of Lemma 1 is now given, employing Lemma 2.

Proof. A referee has a set of states $\{\rho_x\}_{x=1}^N$. Let $\mathcal{Y}_{(N,k)}$ be the set of all subsets of length k of the set $\{1, \dots, N\}$ [44]. During each round of the task, the referee randomly generates a label x and sends the state ρ_x to the player. The player applies a POVM on the state ρ_x and aims to output a set of k labels $Y \in \mathcal{Y}_{(N,k)}$ such that $x \notin Y$. Such a measurement will be a POVM with $\binom{N}{k}$ elements, denoted by $\mathbf{S} := \{S_Y\}_{Y \in \mathcal{Y}_{(N,k)}}$. The player is able to perform conclusive k -state exclusion if for all $Y \in \mathcal{Y}_{(N,k)}$ there exists a S_Y such that $\text{tr}[S_Y \rho_y] = 0 \forall y \in Y$. If the player gets the measurement outcome associated to S_Y , they can output the set Y knowing with certainty the referee could not have sent any of the states in the set $\{\rho_y\}_{y \in Y}$. By defining the operator $R_Y := \sum_{y \in Y} \rho_y$, the conclusive k -state exclusion task can then be succinctly expressed as

$$\text{tr}[S_Y R_Y] = 0 \quad \forall Y \in \mathcal{Y}_{(N,k)}. \quad (\text{A1})$$

Letting $L := \binom{N}{k}$, one can order $\mathbf{S}$'s elements and write $\mathbf{S} := \{S_l\}_{l=1}^L$. Similarly, we also order the operators R_Y 's by the same label and write $\{R_l\}_{l=1}^L$. Now, it can be seen that a given

$x \in \{1, \dots, N\}$ will appear in $\binom{N-1}{k-1} = Lk/N$ many subsets in $\mathcal{Y}_{(N,k)}$. This means that, for each state ρ_x , there are Lk/N many operators R_l 's that contain it. For each $x \in \{1, \dots, N\}$, let X_x denote the set of all labels l corresponding to these R_l 's. Each X_x thus contain Lk/N many labels, and we have $\text{tr}(S_l \rho_x) = 0 \forall l \in X_x$. Equation (A1) thus implies

$$\text{tr} \left[\rho_x \left(\sum_{l \in X_x} S_l \right) \right] = 0 \quad \forall x \in \{1, \dots, N\}. \quad (\text{A2})$$

Using Lemma 2, the above N equations implies

$$\Pi_x \leq \mathbb{I} - \sum_{l \in X_x} S_l \quad \forall x \in \{1, \dots, N\}, \quad (\text{A3})$$

where Π_x is the projector onto the support of the state ρ_x . Summing over the N individual conditions in Eq. (A3) gives

$$\sum_{x=1}^N \Pi_x \leq N\mathbb{I} - \sum_{x=1}^N \sum_{l \in X_x} S_l. \quad (\text{A4})$$

By Eq. (A1), for each l , there are exactly k many possible labels x 's such that $l \in X_x$. Hence, each POVM element S_l appears exactly k times, meaning that $\sum_{x=1}^N \sum_{l \in X_x} S_l = \sum_{l=1}^L k S_l = k\mathbb{I}$ and thus $\sum_{x=1}^N \Pi_x \leq (N - k)\mathbb{I}$, as desired. ■

Appendix B: Proof of Corollary 1. By comparison to Lemma 1, it can be seen that λ used in the definition of $D_{\max}$ can be related to $N - k$ when considering the largest possible k . Rearranging gives the first inequality. By taking the trace of both sides of Lemma 1, the second inequality can be shown. The trace of the left-hand side is lower bounded by N , which is achieved when all ρ_x 's are rank-one projectors, i.e., all of them are pure states. In this best-case scenario where all ρ_x 's are rank-one projectors, we have $\alpha = N$. It can then be seen that the second inequality always upper bounds the first given that $0 \leq D_{\max}(\psi \parallel \sigma) \forall \psi, \sigma$. ■

Appendix C: Proof of Result 1. Alice and Bob share a maximally entangled state $|\Phi^+\rangle_{AB} = \sum_{i=0}^{d-1} |ii\rangle_{AB}/\sqrt{d}$ with an equal local dimension d (here, A, B denotes Alice's and Bob's systems). If Alice encodes the bit string x (via a unitary channel $\mathcal{U}_{x,A}(\cdot) := U_{x,A}(\cdot)U_{x,A}^\dagger$ on A) and then sends her half of the state to Bob via the channel $\mathcal{N}_A$, Bob has the state

$$\begin{aligned} \rho_{AB}^{x|\mathcal{N}} &:= (\mathcal{N}_A \otimes \mathcal{I}_B) \circ (\mathcal{U}_{x,A} \otimes \mathcal{I}_B)(|\Phi^+\rangle\langle\Phi^+|_{AB}) \\ &= (\mathcal{I}_A \otimes \mathcal{U}_{x,A})(\mathcal{J}_{AB}^{\mathcal{N}}), \end{aligned} \quad (\text{C1})$$

where we define $\mathcal{U}_{x,B}^t(\cdot) := U_{x,B}^t(\cdot)U_{x,B}^{t,\dagger}$. Note that, after the channel $\mathcal{N}_A$, Bob has the whole bipartite state. From Bob's point of view, he, therefore, has a state from the set $\{\rho_{AB}^{x|\mathcal{N}}\}_{x=1}^N$. Note that all elements of this set have the same rank: the Choi rank, $r_c^{\mathcal{N}}$, of the channel $\mathcal{N}_A$. This is due to the rank of states being invariant under unitary channels. Bob now aims to perform conclusive state exclusion on this set and hence Lemma 1 can be applied. Given all states in the set are of rank $r_c^{\mathcal{N}}$, all projectors onto the support of those states are of rank $r_c^{\mathcal{N}}$. Taking the trace of both sides of Eq. (4) in Lemma 1 therefore gives $Nr_c^{\mathcal{N}} \leq (N - k)d^2$. Rearranging completes the proof. ■

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