

Enhanced scattering from an almost-periodic optical temporal slabStefanos Fr. Koufidis^{1,*}, Theodoros T. Koutserimpas^{1,2,†}, Francesco Monticone², and Martin W. McCall¹¹*Blackett Laboratory, Department of Physics, Imperial College of Science,**Technology and Medicine, Prince Consort Road, London SW7 2AZ, United Kingdom*²*School of Electrical and Computer Engineering, Cornell University, Ithaca, New York 14853, United States of America*

(Received 13 June 2024; accepted 16 September 2024; published 1 October 2024)

We showcase the impact of almost-periodicity on the parametric amplification associated with the first-order momentum gap in photonic time-crystals with time-varying permittivity. Utilizing a vectorial coupled-wave theory approach, we rigorously analyze the scattering by a temporal slab of the considered medium. We pinpoint a critical regime wherein flaws in material tuning paradoxically enhance amplification due to the coupling of fewer, broader modes, resulting in a higher and broader pulselike amplification envelope. Additionally, we demonstrate that the intensity reflectances of time-reversed waves corresponding to secondary “Bragg” resonances achieve remarkably high levels of subharmonic parametric amplification, with the epsilon-near-zero regime serving as a preferred candidate for experimental implementation. Our counterintuitive findings highlight the potential of intentionally leveraging modulation desynchronization and impurities in the temporal unit cell of photonic time-crystals to enhance both the level and the bandwidth of amplification.

DOI: [10.1103/PhysRevA.110.L041501](https://doi.org/10.1103/PhysRevA.110.L041501)

One of the most intriguing features of temporal media [1], whose parameters such as the permittivity vary periodically over time rather than space, is the potential for achieving parametric amplification [2–4]. Other compelling applications include dynamic spherical scattering [5], enhanced absorption [6], and even synthetic axion response [7], to name a few. Analogous to their spatial counterparts, these temporal gratings have finite durations containing multiple periods [4], with recent experimental advancements paving the path towards furthering their number [8]. However, unlike spatial gratings, where the refractive index is created via a diffraction pattern from a phase mask illuminated by UV light, temporal gratings rely on the temporally tunable response of certain materials, such as ITO, probed by a suitable beam [9]. Consequently, achieving perfectly-periodic modulation is particularly challenging. Therefore, a more realistic approach to potential desynchronizations in material tuning necessitates examining a scheme whereby the time-varying permittivity is an almost-periodic function.

H. A. Bohr’s seminal publication [10] marked the beginning of significant attention to almost-periodic functions over a century ago. Their most remarkable aspect is that upon comparison with their direct subset, periodic functions, even slight deviations in their frequencies cause their graphical depictions to diverge significantly after a few periods [11].

Hence, exploring such structures within the realm of time-varying media is valuable as they offer a more realistic model than perfectly-periodic ones. Naturally, small perturbations continuously disturb the ideal cellular structure of a device consisting of any medium, including those with amplitude or phase fluctuations, structural defects, or impurities [12]. Almost-periodicity mirrors this scenario by acknowledging such flaws. These analogies are directly translated to imperfections in photonic time-crystals and their temporal unit cell, where the dimension of periodicity is deemed that of time [13]. Our temporal medium, modeled as an almost-periodic structure, constitutes the temporal analog of *quasicrystals* [14], which are indeed the most eminent physical example of almost periodicity in Nature [15].

In the weak dispersion limit [16,17], the constitutive relations of a medium with a time-dependent relative permittivity, ϵ_r , read [18–20]

$$\mathbf{D} = \epsilon_r(t)\epsilon_0\mathbf{E} \quad \text{and} \quad \mathbf{B} = \mu_r\mu_0\mathbf{H}. \quad (1)$$

Here, $\mathbf{E}$, $\mathbf{B}$ are the primitive electromagnetic fields, and $\mathbf{D}$, $\mathbf{H}$ their excitations; ϵ_0 and μ_0 are the free-space permittivity and permeability, respectively, and μ_r is the relative permeability. Recent experimental demonstrations of ultrafast single-cycle modulation in the visible spectrum [8,21], as well as the meta-surface implementation of a photonic time-crystal in [22], support the approximation of Eq. (1), provided that the fields oscillate well below the resonances of the material medium [17,23].

Combining Maxwell’s macroscopic source-free curl relations with the connections of Eq. (1), we can arrive at the wave equation for the dielectric displacement field derived in [2–4]. The ansatz of the displacement field is $\mathbf{D}(z, t; k) = \mathbf{D}_s(z; k)D_t(t; k)$, where $\mathbf{D}_s \propto \hat{\mathbf{n}}e^{ikz}$ and D_t denote the spatial and temporal part of the field, respectively; $\hat{\mathbf{n}}$ accounts for the

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polarization. Whence the corresponding equation is

$$\left[\frac{d^2}{dt^2} + k^2 c^2 \theta_r(t) \right] D_t(t) = 0, \quad (2)$$

where k is the wave number associated with the momentum content imparted by the lightwave that is being conserved [24], $c = (\epsilon_0 \mu_0)^{-1/2}$ is the phase velocity of light in vacuum, and $\theta_r(t) = \epsilon_r^{-1}(t)$. For simplicity, $\mu_r = 1$.

Formally, a function f is called *almost-periodic* if for every $\varepsilon > 0$ there exists a translational number $\tau(\varepsilon)$ such that the set $R(\tau) = \{\tau \mid |f(t + \tau) - f(t)| < \varepsilon, \forall t \in \mathbb{R}\}$ is relatively dense [10]. This implies that for any positive constant T , every interval $[a, a + T]$, $a \in \mathbb{R}$, intersects $R(\tau)$, ensuring the periodiclike behavior of $f(t)$ within ε neighborhoods for all t . If for every $\varepsilon > 0$ the translational number $\tau(\varepsilon)$ can be chosen as T , ensuring $|f(t + T) - f(t)| = 0$, then f is *perfectly-periodic*.

Henceforth, the θ_r term in Eq. (2) is treated as a periodic or almost-periodic function of time, and the implications of this subtle distinction are explored. To bypass the aforementioned technical definition, we begin by considering the Fourier expansion of Eq. (2), namely,

$$\theta_r(t) = \bar{\theta} + \sum_{n=1}^N \delta\theta_n \cos(\Omega_n t + \phi_n), \quad (3)$$

where $\bar{\theta}$ is the dc term of the modulation, $\delta\theta_n$, $n \geq 1$, stands for the modulation depth of the n th tone, and ϕ_n is a phase term. Moreover, $\Omega_n = \Omega_0 + \Delta\Omega_n$, where Ω_0 denotes the unperturbed fundamental frequency of the temporal grating and $\Delta\Omega_n$ controls the tone spacing. Thus, when at least two of the modulation frequencies, Ω_n , are noncommensurable, Eq. (3) becomes almost-periodic. Otherwise, a period T can be found, and an incident wave with an angular frequency $\omega_0 \approx \pi/T$ falls within the first-order momentum gap and is *parametrically* amplified [3,4] (*parameters*: modulation depths).

If θ_r is periodic, Eq. (2) transforms into a Hill equation [2]. The corresponding solutions require a cumbersome stability analysis of the associated hypergeometric equation. These analytical challenges have been circumvented by employing a more physical solution exploiting a proper Bloch–Floquet ansatz [3].

On the other hand, if θ_r is almost-periodic, additional difficulties arise as there is no Hill’s determinant [11], which further underscores the necessity for a suitable alternative physical solution. Ergo, inspired by the coupled-wave theory description of time-periodic media [3,4] and the almost space-periodic analysis [25], we elect a suitable ansatz for Eq. (2) as

$$D_t(t) = D_t^+ e^{-i\omega_0 t} + \sum_{n=1}^N [D_t^-]_n e^{i(\omega_0 + \Delta\Omega_n)t}, \quad (4)$$

where $\omega_0 \approx \Omega_0/2$. Pursuant to Eq. (4), a wave propagating in the positive direction will generate a sum of counterpropagating plane waves. Under an e^{ikz} spatial convention, negative time, or equivalently, negative frequency, $e^{i(-\omega_0)(-t)}$, is akin to negative refraction or phase conjugation [26]. Certainly, this is merely an apt mathematical treatment of backward-propagating waves.

Given the presuppositions of weak modulation strengths, $|\delta\theta_n| \ll 1$, and mild envelope variation within a wavelength, $|d^2 D_t/dt^2| \ll \Omega_0 |dD_t/dt|$, we substitute Eqs. (3) and (4) into Eq. (2). Thenceforth, we may phase-match synchronous terms by time averaging over several optical cycles. If $\mathbf{D}_t^+ = (D_t^+)$ is functionally equivalent to a scalar and $\mathbf{D}_t^- = (D_{t,1}^-, D_{t,2}^-, \dots, D_{t,N}^-)^\top$, with $^\top$ denoting transpose, the vectorial coupled-wave equations are concisely expressed as

$$\frac{d}{dt} \begin{pmatrix} \mathbf{D}_t^+ \\ \mathbf{D}_t^- \end{pmatrix} = i \begin{pmatrix} \Delta_{++} & \kappa \\ -\kappa^\dagger & -\Delta_{--} \end{pmatrix} \begin{pmatrix} \mathbf{D}_t^+ \\ \mathbf{D}_t^- \end{pmatrix}. \quad (5)$$

In this notation, $\Delta_{++} = (\Delta_0)$ is also functionally equivalent to a scalar and $\Delta_{--} = \text{diag}(\Delta_1, \Delta_2, \dots, \Delta_N)$, with

$$\Delta_0 = kc\sqrt{\bar{\theta}} - \frac{\Omega_0}{2} \quad \text{and} \quad \Delta_n = \Delta_0 - \Delta\Omega_n. \quad (6)$$

Additionally, $\kappa = (\kappa_1, \kappa_2, \dots, \kappa_N)$, with the coupling coefficient of each tone being given by

$$\kappa_n = \frac{k^2 c^2 \delta\theta_n}{4\omega_0} e^{i\phi_n}, \quad (7)$$

while the dagger † denotes Hermitian conjugation.

The coupled-wave system of Eq. (5) resembles that of scalar Bragg gratings [27] or structurally chiral media [28]. Since the linear system has constant coefficients, there exists an explicit matrix exponential, defined via a convergent series, that can be calculated through the linear superposition of the eigenvectors. Preceding studies on almost-periodic spatially modulated media [11,25,29] employed the so-called invariant embedding method to reduce the coupled-wave system to a vectorial Riccati equation [27]. Nevertheless, in the temporal case, causality dictates that time-reversed waves are generated upon the onset of modulation, *not* prior [17]. Hence, harnessing the invariant embedding method does not yield the scattering coefficients. The proper treatment for identifying the scattering coefficients of the temporal slab under discussion follows.

Let us assume a temporal slab of the considered medium, whereof the relative permittivity is modulated over the time interval between a moment, say, t_0 , and t , as in [30], but with its inverse following our Eq. (3). If $\mathbf{T}$ is a transfer matrix that relates the amplitudes of the displacement fields at t to those at t_0 ,

$$\begin{pmatrix} \mathbf{D}_t^+ \\ \mathbf{D}_t^- \end{pmatrix}_t = \begin{pmatrix} \mathbf{T}_{11}^{1 \times 1} & \mathbf{T}_{12}^{1 \times N} \\ \mathbf{T}_{21}^{N \times 1} & \mathbf{T}_{22}^{N \times N} \end{pmatrix} \begin{pmatrix} \mathbf{D}_{t_0}^+ \\ \mathbf{D}_{t_0}^- \end{pmatrix}, \quad (8)$$

then the initial condition is taken as $\mathbf{D}_{t_0}^- = \mathbf{0}$. We may then define the forward and backward reflection vectors as $\mathbf{R}_F = \mathbf{T}_{11}^{1 \times 1}$ and $\mathbf{R}_B = \mathbf{T}_{21}^{N \times 1}$, respectively. We observe that the latter is an $N \times 1$ vector, wherein each component R_B^j , $j = 1 \dots N$, corresponds to the reflection of the j th mode when the slab is impinged by the forward-propagating wave [i.e., the first term in Eq. (4)].

Defining the auxiliary variable $\boldsymbol{\rho} = \mathbf{R}_B \cdot \mathbf{R}_F^{-1}$, the combination of Eqs. (5) and (8) readily demonstrates the evolution of $\boldsymbol{\rho}$ as per the Riccati equation [29]:

$$\frac{d\boldsymbol{\rho}}{dt} = -i\kappa^\dagger - i(\Delta_{--} \cdot \boldsymbol{\rho} + \boldsymbol{\rho} \cdot \Delta_{++}) - i\boldsymbol{\rho} \cdot \kappa \cdot \boldsymbol{\rho}. \quad (9)$$

It is highlighted that ρ is *not* the local reflectivity but rather the ratio of the amplitude reflection to the amplitude transmission. Due to the nonconservation of energy, both reflection coefficients may show exponential growth [20]. The scattering from these dynamic media, combined with their adherence to $\mathcal{PT}$ symmetry, has resulted in applications such as bidirectional lasing and invisibility (see [31]).

Assuming spatial translational invariance, a direct consequence of Minkowski momentum conservation, which can be also proven via Noether's theorem [24], states that

$$|\mathbf{R}_F|^2 - |\mathbf{R}_B|^2 = \frac{\eta_1}{\eta_2}, \quad (10)$$

where η_1 (respectively, η_2) is the impedance of the pre- (respectively, post-) modulation medium; in this instance $\eta_1 = \eta_2 = \eta_0$, with $\eta_0 = (\mu_0/\epsilon_0)^{1/2}$ being the free-space impedance. Crucially, Eq. (10) is valid regardless of the temporal modulation profile of the system, thus generalizing the conclusions of [32] to include almost-periodicity.

Integration of Eq. (9) yields ρ , and via Eq. (10) we can obtain the forward and backward intensity reflectances

$$|\mathbf{R}_F|^2 = \frac{1}{1 - |\rho|^2} \quad \text{and} \quad |\mathbf{R}_B|^2 = \frac{|\rho|^2}{1 - |\rho|^2}, \quad (11)$$

respectively. In this instance, $|\mathbf{R}_F|^2$ is the forward-scattered intensity, i.e., the ratio of transmitted energy to incident energy, whereas $|\mathbf{R}_B|^2$ is the backward-scattered intensity, i.e., the ratio of reflected energy to incident energy. Of course, impedance- (or index-) matching in a temporal slab is not a trivial task. Thus, strictly speaking, the Fresnel-type coefficients, as given in Eqs. (2) and (4) of [33], ought to be considered. These coefficients offer the opportunity to eliminate backscattering by impedance matching through additional modulation of the “surrounding” medium, i.e., before and after the duration of the temporal slab. Nonetheless, for small modulation depths, the expressions in Eq. (11) reasonably estimate the *total* intensity scattering coefficients. For high modulation depths, where coupled-wave theory tends to overestimate the bandwidth and amplification factor of first-order parametric amplification interactions (cf. Fig. 2 in [3]), exact intensity reflectances may be computed upon resorting to the brute force of the plane-to-line mapping adaptation of Floquet theory [11].

Hitherto, our analysis has been general; next, we focus on the two-tone scenario under which Eq. (3) reduces to

$$\theta_r(t) = \bar{\theta} + \delta\theta_1 \cos(\Omega_1 t) + \delta\theta_2 \cos(\Omega_2 t). \quad (12)$$

Hereafter, $\phi_1 = \phi_2 = 0$ and $\Delta\Omega_2 = -\Delta\Omega_1 = \Delta\Omega$, so that $\Omega_1 = \Omega_0 - \Delta\Omega$ and $\Omega_2 = \Omega_0 + \Delta\Omega$. Setting $\rho = (\rho_1, \rho_2)^T$, Eq. (9) leads to the coupled system

$$\frac{d\rho_1}{dt} = -i\kappa_1(1 + \rho_1^2) - i(\Delta_0 + \Delta_1)\rho_1 - i\kappa_2\rho_1\rho_2, \quad (13)$$

$$\frac{d\rho_2}{dt} = -i\kappa_2(1 + \rho_2^2) - i(\Delta_0 + \Delta_2)\rho_2 - i\kappa_1\rho_1\rho_2, \quad (14)$$

whose solution's substitution in Eq. (11) yields the spectra of almost-periodic media subsequently shown. For perfectly-periodic media, standard techniques apply [11,29].

The first-order momentum gap “Bragg” condition for the first tone is $\omega_1 = \Omega_1/2$, with an associated bandwidth $\Delta\omega_1 =$

$(\Omega_1/2)(\delta\theta_1/\bar{\theta})$, whereas for the second tone, the respective parameters are $\omega_2 = \Omega_2/2$ and $\Delta\omega_2 = (\Omega_2/2)(\delta\theta_2/\bar{\theta})$ [4]. We may then define the *critical* tone spacing as $\Delta\Omega_c = \max(\Delta\omega_1, \Delta\omega_2)$, which, for equal tone contribution, approximates to

$$\Delta\Omega_c = \frac{\Omega_0 - \Delta_c\Omega}{2} \frac{\delta\theta_1}{\bar{\theta}} \Rightarrow \Delta\Omega_c \approx \frac{\Omega_0}{2} \frac{\delta\theta_1}{\bar{\theta}}. \quad (15)$$

Thence, we can forthwith distinguish between the following scenarios: the *subcritical* regime ($\Delta\Omega < \Delta\Omega_c$) and the *supercritical* regime ($\Delta\Omega > \Delta\Omega_c$).

For demonstration purposes, let us examine a temporal slab of duration $\Delta t = 40\pi/\Omega_0$, where $\Omega_0 = 68.51 \times 10^{14}$ rad/s. We may then revisit Eq. (12) and assume tones with equal contribution by setting $\bar{\theta} = 1$ and $\delta\theta_1 = \delta\theta_2 = 0.1$. For these parameters, Eq. (15) yields a critical tone-spacing of $\Delta\Omega_c = 3.42 \times 10^{14}$ rad/s so that $\Omega_1 = 71.93 \times 10^{14}$ and $\Omega_2 = 65.09 \times 10^{14}$.

Up to double precision, modulation frequencies appear indistinguishable whether commensurable or noncommensurable. However, the distinction between almost-periodic and perfectly-periodic media, delineated by Mickelson [12] and subsequent studies [11,29], is encoded in Eq. (9). Indeed, perfectly-periodic media require numerous spatial harmonics as frequency spacing decreases, resulting in a complex harmonic structure and multiple bandgaps. Conversely, as derived in Appendix A of [29], almost-periodic media maintain simplicity even with closely spaced frequencies, typically featuring few spatial harmonics and up to two bandgaps that can smoothly merge. For the optical response of almost-periodic media, we thus numerically integrate Eqs. (13) and (14) with $\phi_1 = \phi_2 = 0$ using the detunings and coupling coefficients from Eqs. (6) and (7), respectively. For the perfectly-periodic case, we calculate the spectrum by numerical integration of Eq. (6) from [29], employing detunings and coupling coefficients given by Eqs. (15) and (16) of the same reference. In both scenarios, backward intensity reflectances are computed as per Eq. (11).

At precisely the *critical* tone spacing, and for the aforementioned parameters, two parametric amplification regimes emerge at $kc\sqrt{\bar{\theta}}/\Omega_0 \approx 0.48$ and 0.53 . For a modulation duration $\Delta t = 40\pi/\Omega_0$, Fig. 1 compares the intensity reflectances of the time-reversed waves of almost-periodic modulation to those of perfectly-periodic modulation. The backward scattering coefficient of the ω_1 resonance for the former case is $|\mathbf{R}_B^{\text{AP}}|^2 = 3.93$, whereas for the latter it is $|\mathbf{R}_B^{\text{P}}|^2 = 3.91$ (n.b., slightly detuned from the almost-periodic). This indicates a marginal difference between the two modulations at the critical regime, with the disparity diminishing to levels below 0.51% for first-order perturbations. The same situation occurs at the ω_2 -resonance peak, where the discrepancy between almost-periodic and perfectly-periodic cases is similar to the earlier case. Notably, the ω_2 resonance peak is significantly higher than the ω_1 resonance peak due to the dependence of the coupling coefficient on the wave number, k [cf. Eq. (7)].

As anticipated, at the *critical* tone spacing, each periodic pattern appears to independently amplify electromagnetic waves. Indeed, when an incident lightwave with suitable frequency enters the first-order momentum gap, it undergoes significant amplification, provided there is ample time for

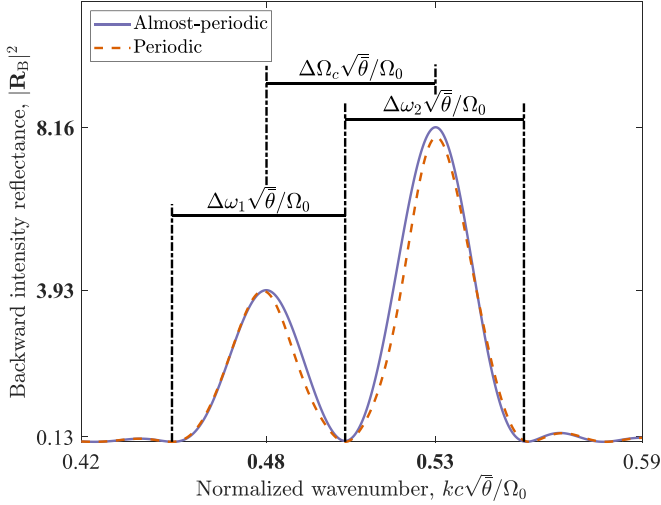


FIG. 1. Intensity reflectances of the time-reversed waves generated by a two-tone temporal slab of duration $\Delta t = 40\pi/\Omega_0$. The inverse of the permittivity follows Eq. (12), with $\bar{\theta} = 1$ and $\delta\theta_1 = \delta\theta_2 = 0.1$. In the illustrated *critical* tone spacing, $\Omega_1 = \Omega_0 - \Delta\Omega$ and $\Omega_2 = \Omega_0 + \Delta\Omega$; $\Omega_0 = 68.51 \times 10^{14}$ rad/s, $\Delta\Omega = \Delta\Omega_c = 3.42 \times 10^{14}$ rad/s. The solid line represents the optical spectrum of almost-periodic media, obtained by numerically integrating the coupled-wave system described in Eqs. (13) and (14), with $\phi_1 = \phi_2 = 0$, and the detunings and coupling coefficients given by Eqs. (6) and (7), respectively. The dashed line indicates the corresponding spectrum of perfectly-periodic media, obtained by numerically integrating Eq. (6) from [29], with the detunings and coupling coefficients given by Eqs. (15) and (16) of the same paper. In all cases, the backward intensity reflectances are calculated using Eq. (11). For first-order perturbations, the two curves are virtually indistinguishable (discrepancy $< 0.51\%$). The asymmetry between the two peaks is caused by the dependence of the coupling coefficients on the wave number.

the amplification process to be enacted. This lessens the significance of nonamplified interfering waves with frequencies falling outside the first “Bragg” zone. Such a principle also applies in the *supercritical* regime, where the tones are sufficiently spaced to avoid meeting the “Bragg” condition simultaneously.

On the contrary, in the *subcritical* regime, the almost-periodic amplifier outperforms the corresponding perfectly-periodic one, as clearly illustrated in Fig. 2 (same parameters as Fig. 1, but $\Delta\Omega = \Delta\Omega_c/2$). Specifically, at around $kc\sqrt{\bar{\theta}}/\Omega_0 \approx 0.489$, the amplification achieved by the almost-periodic temporal slab is $|\mathbf{R}_B^{\text{AP}}|^2 = 5.456$. By contrast, at nearly the same wave number but slightly shifted towards the red, the perfectly-periodic slab attains a $|\mathbf{R}_B^{\text{P}}|^2 = 4.952$. Counterintuitively, the maximum amplification level of the almost-periodic slab is 9.68% greater than that of an otherwise similar periodic slab, thereby reinforcing our assertion that imperfections can lead to greater amplifications. Notably, in the vicinity of $kc\sqrt{\bar{\theta}}/\Omega_0 \approx 0.5$, Fig. 2 reveals that the spectral hole in the almost-periodic curve is less pronounced compared to that of the periodic one, thus elucidating another salient feature: broadband amplification within an envelope resembling a trapezoid pulse of width $\Omega_0\bar{\theta}^{-1/2}|0.515 - 0.486| \approx 1.98 \times 10^{14}$ rad/s (cf. the shaded area in Fig. 2).

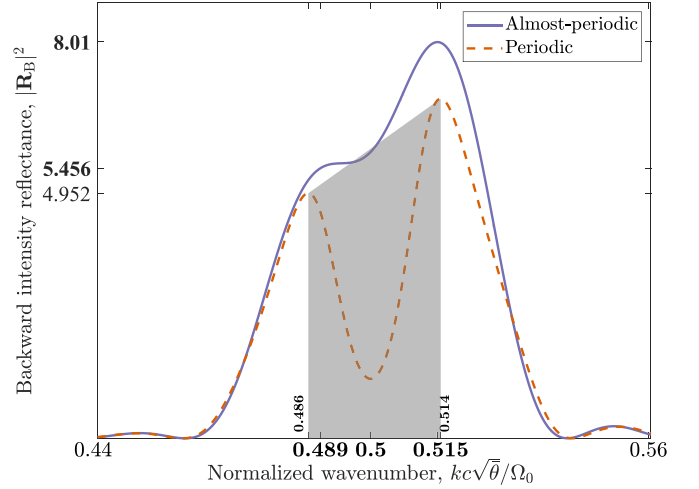


FIG. 2. Optical response of the temporal slab in the *subcritical* regime ($\Delta\Omega = \Delta\Omega_c/2$). The remaining parameters are those of Fig. 1. At $kc\sqrt{\bar{\theta}}/\Omega_0 \approx 0.489$, the maximum attainable amplification by the almost-periodic modulation is 9.68% higher compared to that achieved by the perfectly-periodic one. The shaded window marks the region wherein a trapezoid amplification envelope is accessible due to the system’s almost-periodicity.

Such a comparison is alternatively conveyed in Fig. 3. There the difference between almost-periodic and periodic backward intensity reflectances, $|\mathbf{R}_B^{\text{AP}}|^2 - |\mathbf{R}_B^{\text{P}}|^2$, is plotted against the normalized wave number for various regimes of tone separation, with each situation distinguished by a different color (other parameters same as Fig. 1). In the *subcritical* region, the designated red curve consistently exceeds zero over an exceptionally broad frequency

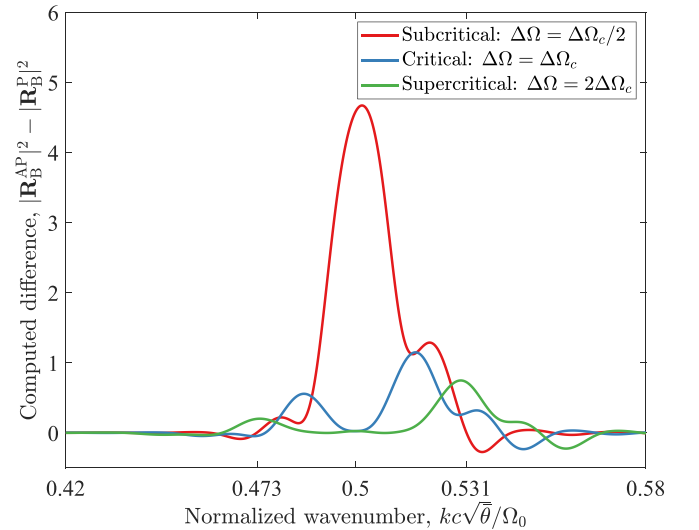


FIG. 3. Differences between almost- and perfectly-periodic backward intensity reflectances for various tone spacings. Red curve: *subcritical* regime ($\Delta\Omega = \Delta\Omega_c/2$), blue curve: *critical* regime ($\Delta\Omega = \Delta\Omega_c$), and green curve: *supercritical* regime ($\Delta\Omega = 2\Delta\Omega_c$). The parameters are those of Fig. 1. The *subcritical* almost-periodic amplifier surpasses the perfectly-periodic one at a broad frequency range ($\approx 3.97 \times 10^{14}$ rad/s).

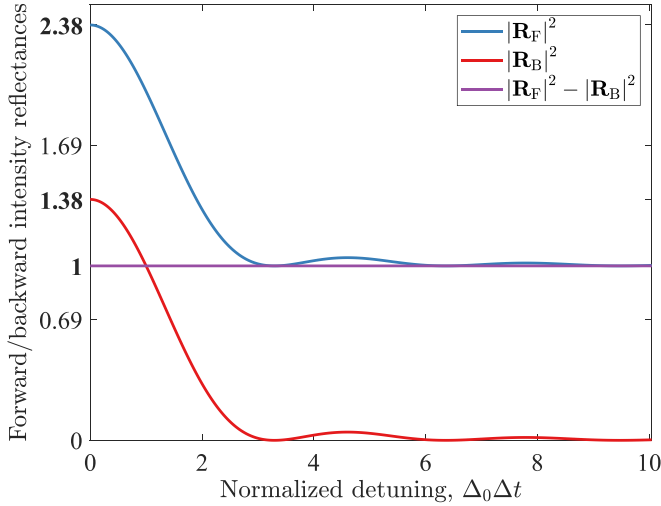


FIG. 4. Subharmonic optical response, as per Eq. (16). The design specifications are $\kappa \Delta t = 1$, $\omega_0 = 34.25 \times 10^{14}$ rad/s, and the modulation parameters are those of Fig. 1, albeit $\Delta\Omega = 0.1\Omega_0$. Despite lower amplification compared to first-order resonances, the prominent subharmonic scattering underscores the significance of such secondary resonances.

range $(\Omega_0 \bar{\theta}^{-1/2} |0.531 - 0.473| \approx 3.97 \times 10^{14}$ rad/s), signifying the almost-periodic system's superior amplification capabilities over those of the periodic one. Reminiscent to the spatial case [29], the disparity between the two increases with longer slab durations, narrower tone intervals, or higher modulation depths.

In a two-tone almost-periodic spatial medium, subharmonic “Bragg” resonances occur [34]. Revisiting Eq. (12), the temporal beat frequency is $\Delta\Omega = (\Omega_2 - \Omega_1)/2$. At $\omega_0 \approx \Delta\Omega$, momentum conservation dictates that the eigenfrequencies of the medium $\pm(\Omega_0 \pm \Delta\Omega)$ match $\pm\omega_0$ to $\mp\omega_0$ and vice versa. Wherefore, a suitable ansatz is

$$D_t(t) = D_{t,1}^+ e^{i\Omega_0 t} + D_{t,1}^- e^{-i\Omega_0 t} + D_{t,2}^+ e^{i\Delta\Omega t} + D_{t,2}^- e^{-i\Delta\Omega t}.$$

Combining this ansatz with Eqs. (2) and (12), phase-matching potentially synchronous terms leads to

$$\frac{d}{dt} \begin{pmatrix} D_{t,1}^+ \\ D_{t,1}^- \end{pmatrix} = i \begin{pmatrix} \Delta_0 & \kappa \\ -\kappa & -\Delta_0 \end{pmatrix} \begin{pmatrix} D_{t,1}^+ \\ D_{t,1}^- \end{pmatrix},$$

where the detuning and the coupling coefficient are

$$\Delta_0 = \omega_0 - [\Delta\Omega + \mathcal{O}(\delta\theta_1^2, \delta\theta_2^2)] \quad \text{and} \quad \kappa = \frac{\delta\theta_1 \delta\theta_2}{\bar{\theta}^2} \frac{\Delta\Omega^3}{8\Omega_0^2},$$

respectively. In this instance, higher-order terms are grouped in $\mathcal{O}(\delta\theta_1^2, \delta\theta_2^2) = [(\delta\theta_1^2 + \delta\theta_2^2)/4\bar{\theta}^2](\Delta\Omega^3/\Omega_0^2)$.

The backward intensity reflectance can be at once cast in a closed analytic form [27], viz.,

$$|R_B|^2 = \frac{\kappa^2}{|\kappa^2 - \Delta_0^2|} \left| \sinh[(\kappa^2 - \Delta_0^2)^{1/2} \Delta t] \right|^2. \quad (16)$$

The forward intensity reflectance is retrievable via the pseudoconservation relation: $|R_F|^2 = 1 + |R_B|^2$ [4]. The optical response of Fig. 4 accentuates the significance of higher-order “Bragg” resonances, despite their lower amplification factors in comparison to first-order resonances. Achieving subharmonic backward intensity reflectances exceeding unity requires multicycle modulation or strong modulation depths. Half the refractive index modulation amplitudes have been experimentally achieved in epsilon-near-zero frequencies [8], rendering it a candidate for subharmonic parametric amplification.

In summary, our exploration of almost-periodic temporal modulations illuminates the intricate interplay between imperfections and amplification. Via a vectorial coupled-wave theory, we overcome traditional shortcomings of Bloch–Floquet theory, deepening our understanding of these dynamic phenomena. Paradoxically, defects in photonic time-crystals can enhance amplification, especially in the identified *subcritical* regime wherein the almost-periodic medium clearly outperforms its periodic counterpart. Unlike perfectly-periodic media, almost-periodic media couple fewer, broader modes, leading to more effective coupling and thus to a higher and broader pulselike amplification envelope. Furthermore, almost-periodicity enhances scattering coefficients for subharmonic resonances. Recent experimental advances in augmenting modulation depths in epsilon-near-zero frequencies are promising for implementing a subharmonic parametric amplifier. Our study underscores the potential of tailoring temporal defects in photonic time-crystals to boost their parametric amplification capabilities.

S.F.K. is supported by the Bodossaki Foundation. T.T.K. is supported by the Bodossaki Foundation through the “Stamatis G. Mantzavinos” scholarship. F.M. is supported by the Air Force Office of Scientific Research (AFOSR) with Grant No. FA9550-22-1-0204 through Dr. Arje Nachman. The authors thank the anonymous reviewers for their constructive comments, which have improved this study.

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