

Coherence fraction in Grover's search algorithm

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The question of which resources drive the advantages in quantum algorithms has long been a fundamental challenge. While entanglement and coherence are critical to many quantum algorithms, our results indicate that they do not fully explain the quantum advantage achieved by the Grover search algorithm. By introducing a generalized Grover search algorithm, we demonstrate that the success probability depends not only on the querying number of oracles but also on the coherence fraction, which quantifies the fidelity between an arbitrary initial quantum state and the equal superposition state. Additionally, we explore the role of the coherence fraction in the quantum minimization algorithm, which offers a framework for solving complex problems in quantum machine learning. These findings offer insights into the origins of quantum advantage and open pathways for the development of new quantum algorithms.

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I. INTRODUCTION

Quantum computers, grounded in the principles of quantum mechanics, have attracted great attention over the past few decades. These devices are theoretically more efficient than the most advanced classical ones for certain problems [1], such as large integer factoring [2], unstructured search [3,4], linear equation solving [5], and quantum system simulation [6,7]. Leveraging these quantum advantages, quantum computers hold promise for various applications, from advancing artificial intelligence [8,9] to developing next-generation materials and pharmaceuticals [10,11]. Additionally, the quest for quantum computing has inspired the development of new efficient classical algorithms [12]. While several small-scale proof-of-concept quantum computers have been realized [13],

ongoing research is focused on building larger quantum computers with the potential to surpass the classical ones in specific tasks [14–16].

Quantum computation devices have their advantages in addressing specialized problems [17,18]. One of the key challenges is investigating which inherent quantum properties of quantum states contribute to the benefits of quantum computation. It has been shown that entanglement and coherence play crucial roles in quantum algorithms [19–25]. Our findings reveal that the average success probability of the generalized Grover search algorithm does not depend on the entanglement or coherence of the initial state. Instead, it depends on the number of oracle queries and the coherence fraction of the initial state, defined by the fidelity between the initial state and the equal superposition state (i.e., maximally coherent state). Once the optimal number of oracle queries is reached, the optimal average success probability depends solely on the coherence fraction of the initial state. A sketch diagram is provided in Fig. 1. In quantum teleportation, there have been similar results regarding the fully entangled fraction, which measures the optimal fidelity of quantum teleportation [26–31].

We focus on the specific quantum algorithm with a fixed structure: the Grover search algorithm (GSA) [3,4]. While there have been several proposed generalizations of the GSA [32–36], we revisit the GSA from the source of its quantum advantage. In this work, we consider a generalized version

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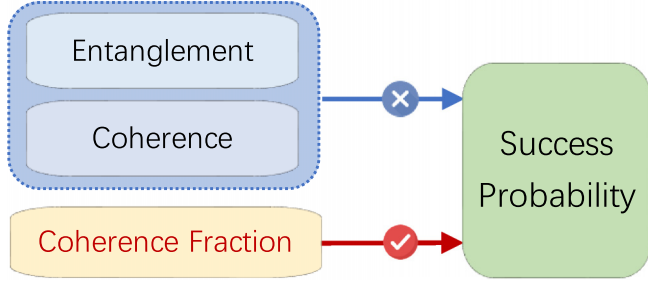


FIG. 1. Diagrammatic sketch of coherence fraction in Grover search algorithm. The upper bound of the average success probability for the generalized Grover search algorithm depends solely on the coherence fraction, rather than quantum entanglement or coherence.

of the Grover search algorithm (GGSA), where the Hadamard gate before the Grover operator is replaced by an arbitrary unitary gate. We provide an exact formulation for the average success probability of GGSA. Our analysis reveals that the upper bound of the average success probability depends solely on the coherence fraction of the initial state. We define the coherence fraction of a state ρ as the Uhlmann's fidelity F between ρ and $|\eta\rangle$,

$$f_c(\rho) := F(|\eta\rangle, \rho) = \langle \eta | \rho | \eta \rangle, \quad (1)$$

where $|\eta\rangle$ is the equal superposition state or the maximally coherent state. The coherence fraction quantifies how close a quantum state is to a maximally coherent state [37,38]. The purpose of this work is to connect the average success probability of GGSA with the coherence fraction $f_c(\rho)$. In the limiting case, the equal superposition state achieves the highest success probability. However, it is important to note that $f_c(\rho)$ is neither an entanglement measure nor a coherence measure of ρ . Only under specific constraints does it turn out to be a coherence measure. Our findings explain why the Hadamard gate is typically used to generate the equal superposition state in GSA, rather than an arbitrary quantum unitary gate.

II. GENERALIZED GROVER SEARCH ALGORITHM

GSA is a quantum algorithm designed for unstructured search tasks [3,4,39]. It efficiently locates r elements within a data set of size N by searching through the indices. GSA achieves a quadratic speedup compared to classical search algorithms, reducing the computational complexity required for large data sets. For convenience, assume that the size of the data set is $N = 2^n$. Consider a function $f(x) : \{0, 1\}^n \rightarrow \{0, 1\}$, where $f(x) = 1$ if x is a solution to the search problem, and $f(x) = 0$ otherwise.

The implementation of GSA involves three main steps: (i) Apply the Hadamard gate $H^{\otimes n}$ to the input state $|0^n\rangle$ of the register to obtain $|\eta\rangle := \sum_{x=0}^{N-1} |x\rangle / \sqrt{N}$; (ii) Perform the Grover operator $\mathcal{G}$ approximately $O(\sqrt{N})$ times; (iii) Measure the final state. The Grover operator $\mathcal{G}$ consists of two quantum subroutines: the quantum oracle $\mathcal{O}$ and the diffusion transform. The quantum oracle $\mathcal{O}$ rotates the phase of the marked states by π radians. It corresponds to a unitary operation on the computational basis states of the register

ALGORITHM 1. GGSA

Consider an oracle $\mathcal{O}$, a register of n qubits initially in the state $|0^n\rangle$, and an ancilla qubit $|1\rangle_q$.

1. Apply an arbitrary unitary quantum gate $\mathcal{U}$ to the initial state $|0^n\rangle$ of the register, and the Hadamard gate H to the ancilla qubit initially in state $|1\rangle_q$. The resulting state becomes $|\psi\rangle|-\rangle$, where $|\psi\rangle := \sum_{x=0}^{N-1} a_x |x\rangle$.
2. Perform the Grover iteration $\mathcal{G} = \mathcal{D}\mathcal{O}\tau$ times, where $\mathcal{D} = H^{\otimes n}(2|0^n\rangle\langle 0^n| - I^{\otimes n})H^{\otimes n}$ represents the diffusion transform acting on the register state, and $\mathcal{O}|x\rangle|-\rangle_q = (-1)^{f(x)}|x\rangle|-\rangle_q$ denotes the oracle.
3. Measure the resulting state.

$|x\rangle$, and the ancilla $|q\rangle$: $\mathcal{O}|x\rangle|q\rangle = |x\rangle|q \oplus f(x)\rangle$. The ancilla qubit is initially set to the state $|-\rangle_q = (|0\rangle - |1\rangle)/\sqrt{2}$. With this choice, the oracle $\mathcal{O}$ can be described as $\mathcal{O}|x\rangle|-\rangle_q = (-1)^{f(x)}|x\rangle|-\rangle_q$. This implies that $|x\rangle$ is flipped if $f(x) = 1$, and remains unchanged otherwise. The diffusion transform $\mathcal{D} = H^{\otimes n}(2|0^n\rangle\langle 0^n| - I^{\otimes n})H^{\otimes n} = 2|\eta\rangle\langle\eta| - I^{\otimes n}$ applies to the state in the register.

We consider a generalized Grover search algorithm (GGSA) detailed in Algorithm 1, with its corresponding circuit depicted in Fig. 2. In contrast to the Hadamard gate used in GSA, GGSA implements an arbitrary unitary gate $\mathcal{U}$ before the Grover operator $\mathcal{G}$ to obtain the pure initial state $|\psi\rangle$. Additionally, it provides a direct generalization to the case in which the initial state is a mixed state ρ . We are interested in how the initial state impacts the algorithm's search performance and which intrinsic quantum properties contribute to the advantages of GGSA.

III. THE AVERAGE SUCCESS PROBABILITY OF GGSA AND THE COHERENCE FRACTION

We investigate the performance of GGSA in terms of average success probability. Given r marked states, the number of possible sets is represented by the binomial coefficient $\binom{N}{r}$ [33,34]. Denote all possible sets of marked states as $\{\mathcal{M}_i\}_{i \in I}$, with I being the indicator set encompassing all possible sets.

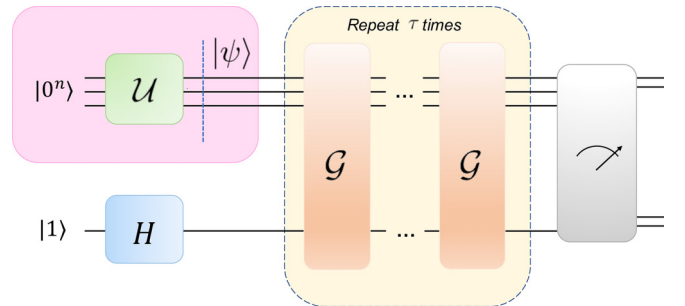


FIG. 2. Quantum Circuit for GGSA. An arbitrary unitary quantum gate $\mathcal{U}$ is applied to the input state $|0^n\rangle$ of the register, while the Hadamard gate H is applied to the ancilla qubit $|1\rangle_q$. The initial state is $|\psi\rangle := \mathcal{U}|0^n\rangle = \sum_{x=0}^{N-1} a_x |x\rangle$, where a_x is the amplitude of $|x\rangle$. The resulting state is $|\psi\rangle|-\rangle$. Subsequently, the Grover operator $\mathcal{G}$ is applied τ times, and the final state is measured on the computational basis.

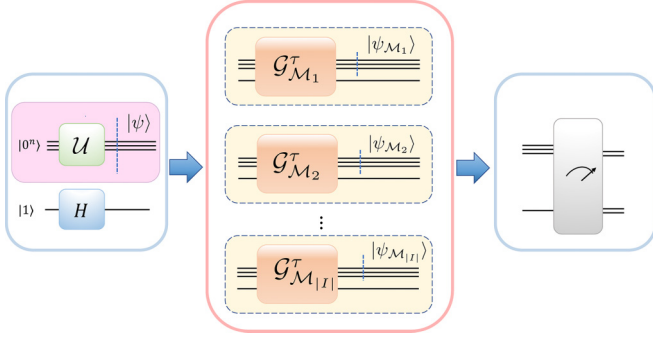


FIG. 3. Grover operator $\mathcal{G}_{\mathcal{M}_i}$. The algorithm is depicted in three outer boxes, each representing a different step. The first box initializes the state as $|\psi\rangle$. In the second box, the Grover operator $\mathcal{G}_{\mathcal{M}_i}$ is applied, resulting in the final state $|\psi_{\mathcal{M}_i}\rangle$. The small boxes inside signify the selections of a specific marked state each time the algorithm proceeds. The final box represents the measurement.

The cardinality $|I|$ signifies the total number of sets, i.e., $|I| = \binom{N}{r}$. The oracle $\mathcal{O}|x\rangle|-\rangle_q = (-1)^{f(x)}|x\rangle|-\rangle_q$ in the Grover operator $\mathcal{G}$ is crucial for marking states. Consequently, in the second step, $\mathcal{G}$ is replaced by $\mathcal{G}_{\mathcal{M}_i}$ tailored for each set $\mathcal{M}_i$. After applying the Grover operator $\mathcal{G}_{\mathcal{M}_i}$ τ times to the state $|\psi\rangle$, we obtain $|\psi_{\mathcal{M}_i}\rangle$, as demonstrated in Fig. 3. Each set $\{\mathcal{M}_i\}_{i \in I}$ corresponds to an operator $\mathcal{G}_{\mathcal{M}_i}$ and a specific final state $|\psi_{\mathcal{M}_i}\rangle$. Once a set is chosen, both the operator and the resulting final state are fixed.

The variability in the second step of Algorithm 1 depends on the selection of marked states, influencing the diversity of possible outcomes. Let $P_{\text{succ}}^{\mathcal{M}_i}(|\psi\rangle, \tau)$ denote the success probability for the set $\mathcal{M}_i$. For any initial state $|\psi\rangle$, the average success probability $P_{\text{ave}}(|\psi\rangle, \tau)$ of GGSA over all possible sets after τ iterations is given by

$$P_{\text{ave}}(|\psi\rangle, \tau) := \frac{1}{|I|} \sum_i P_{\text{succ}}^{\mathcal{M}_i}(|\psi\rangle, \tau), \quad (2)$$

where $|I|$ represents the total number of possible sets.

Theorem 1. For an initial state $|\psi\rangle$, the average success probability (2) of GGSA is given by the following formula:

$$P_{\text{ave}}(|\psi\rangle, \tau) = \frac{N \sin^2 \vartheta - r}{N - 1} f_c(|\psi\rangle) + \frac{r - \sin^2 \vartheta}{N - 1}, \quad (3)$$

where $\vartheta := \theta(\tau + 1/2)$, $\theta := \arccos(1 - 2r/N)$ and r is the number of marked elements in an unsorted database of size N .

The proof of Theorem 1 is presented in Appendix A. From Eq. (3), we see that the average success probability $P_{\text{ave}}(|\psi\rangle, \tau)$ is a function of variables τ and the quantity $f_c(|\psi\rangle)$, where $f_c(|\psi\rangle)$ is the coherence fraction of the initial state defined by the fidelity between $|\psi\rangle$ and the equal superposition state $|\eta\rangle = \sum_{x=0}^{N-1} |x\rangle/\sqrt{N}$.

For an optimal number of iterations $\tau_{\text{opt}} = \lfloor \frac{\pi}{4} \sqrt{\frac{N}{r}} \rfloor$ [39], the optimal average success probability after performing τ_{opt} iterations is

$$P_{\text{ave}}^{\text{opt}}(|\psi\rangle) = \frac{N - r}{N - 1} f_c(|\psi\rangle) + \frac{r - 1}{N - 1}, \quad (4)$$

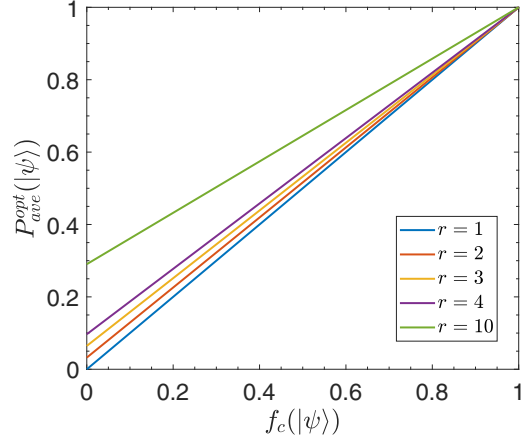


FIG. 4. The optimal average success probability as a function of the coherence fraction. We set $N = 2^5$ and the plot follows the Eq. (4).

where $\lfloor \cdot \rfloor$ is the floor function. Based on Eq. (4), the optimal average success probability $P_{\text{ave}}^{\text{opt}}(|\psi\rangle)$ is completely characterized by the coherence fraction $f_c(|\psi\rangle)$. Figure 4 shows the relation between the optimal average success probability and the coherence fraction for $N = 2^5$ and $r = 1, 2, 3, 4, 10$. Note that $P_{\text{ave}}^{\text{opt}}(|\psi\rangle) \in [(r-1)/(N-1), 1]$ since $f_c(|\psi\rangle) \in [0, 1]$. The lower bound of $P_{\text{ave}}^{\text{opt}}(|\psi\rangle)$ is saturated when the initial state is $|\psi\rangle = \sum_{x=0}^{N-1} (-1)^x |x\rangle/\sqrt{N} = |-\rangle^{\otimes n}$. The upper bound of $P_{\text{ave}}^{\text{opt}}(|\psi\rangle)$ is achieved when the initial state is $|\psi\rangle = \sum_{x=0}^{N-1} |x\rangle/\sqrt{N} = |+\rangle^{\otimes n} = |\eta\rangle$, which corresponds to the original Grover search algorithm [39]. It indicates that the Hadamard gate achieves the best performance in terms of maximum success probability. It is important to note that our previous analysis focused on the pure states. Similar results for mixed states are presented in Appendix B.

We have shown that the maximum success probability of GGSA is determined exclusively by the coherence fraction of the initial state. This discovery establishes a connection between the success probability and the coherence fraction. Here the coherence fraction is similar to the fully entangled fraction, which measures the fidelity of optimal quantum teleportation [26–31].

To explore the full potential of quantum computing, it is vital to understand which quantum properties offer the computational advantage. This insight will also facilitate the design of future quantum algorithms [19,23,24,40–43]. Various quantum resources, such as entanglement [44,45], coherence [46–55] and discord [56–58], have played significant roles in quantum information processing. Here, we emphasize that the coherence fraction $f_c(\rho) = F(|\eta\rangle, \rho)$ is neither an entanglement measure nor a coherence measure of ρ , and only under specific constraint does it turn out to be a coherence measure, see detailed proof in Appendix C.

IV. EXAMPLES

We consider a special case in which the unitary quantum gate we applied in GGSA is a product of arbitrary local operations, $\mathcal{U} = (\mathcal{U}(\alpha, \beta, \theta))^{\otimes n}$. The unitary gate applied to each

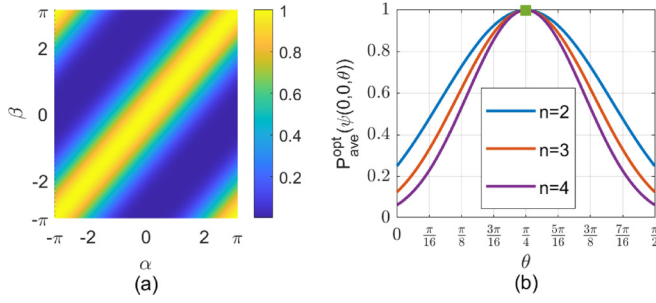


FIG. 5. The link between parameters (α, β, θ) and the optimal average success probability $P_{\text{ave}}^{\text{opt}}(|\psi(\alpha, \beta, \theta)\rangle)$. Example (a). Fixed the parameter $\theta = \pi/4$, the success probability $P_{\text{ave}}^{\text{opt}}(|\psi(\alpha, \beta, \pi/4)\rangle) = 1$ when $\alpha = \beta + 2k\pi$, ($k = 0, \pm 1, \dots$). Here is the case of $n = 2$. Example (b). Fixed parameters $\alpha = \beta = 0$, the upper bound of $P_{\text{ave}}^{\text{opt}}(|\psi(0, 0, \theta)\rangle) = 1$ is achieved when $\theta = \pi/4$. Three lines in the figure demonstrate 2, 3, and 4 qubits from top to bottom.

qubit in the register is

$$\mathcal{U}(\alpha, \beta, \theta) = \begin{pmatrix} e^{i\alpha} \cos \theta & e^{-i\beta} \sin \theta \\ e^{i\beta} \sin \theta & -e^{-i\alpha} \cos \theta \end{pmatrix}, \quad (5)$$

where α and β are parameters, $\theta \in [0, \pi/2]$. Especially, $\mathcal{U}(0, 0, \pi/4)$ is the Hadamard gate. Apply the unitary gate to the single qubit $|0^n\rangle$ and obtain the initial state

$$|\psi(\alpha, \beta, \theta)\rangle = \sum_{j=0}^{N-1} (e^{i\alpha} \cos \theta)^{k_j} (e^{i\beta} \sin \theta)^{n-k_j} |j\rangle, \quad (6)$$

where $N = 2^n$ and k_j is the number of zero numbers in binary representation $j = j_1 j_2 \dots j_n$. Thus, the coherence fraction of $|\psi(\alpha, \beta, \theta)\rangle$ is

$$f_c(|\psi(\alpha, \beta, \theta)\rangle) = \frac{1}{2^n} |(e^{i\alpha} \cos \theta + e^{i\beta} \sin \theta)^n|^2. \quad (7)$$

For one marked state (i.e., $r = 1$), the optimal average success probability is given by

$$P_{\text{ave}}^{\text{opt}}(|\psi(\alpha, \beta, \theta)\rangle) = f_c(|\psi(\alpha, \beta, \theta)\rangle). \quad (8)$$

We consider two examples:

- (a). $P_{\text{ave}}^{\text{opt}}(|\psi(\alpha, \beta, \pi/4)\rangle) = \frac{1}{2^n} |(e^{i\alpha} + e^{i\beta})^n|^2$;
- (b). $P_{\text{ave}}^{\text{opt}}(|\psi(0, 0, \theta)\rangle) = \frac{1}{2^n} (\cos \theta + \sin \theta)^{2n}$.

The link between parameters (α, β, θ) and the optimal average success probability $P_{\text{ave}}^{\text{opt}}(|\psi(\alpha, \beta, \theta)\rangle)$ of these two examples are shown in Fig. 5. See more details in Appendix D.

V. QUANTUM MINIMIZATION ALGORITHM

We further discuss the role of coherence fraction in the Dürr and Høyer's quantum minimization algorithm (QMA) [59]. The QMA offers a groundbreaking way to tackle complex issues in artificial intelligence. It is an important subroutine in unsupervised and supervised quantum machine learning [8, 60–62]. In particular, QMA can be integrated into quantum versions of k -means and k -median algorithms for unsupervised learning, as well as quantum support vector machines

ALGORITHM 2. GQMA

1. Randomly choose an input x_1 and set $d_1 = f(x_1)$.
2. Repeat the following steps until convergence is attained:
 - (1) Initialize the first register with n input qubits and the second register with m qubits to hold the threshold value d_i .
 - (2) Apply an arbitrary unitary quantum gate $\mathcal{U}$ to the first register to prepare an arbitrary state. Initialize the memory as $|\psi\rangle \otimes |d_i\rangle$. Mark each item x where $f(x) < d_i$.
 - (3) Apply the Grover operator $\mathcal{G}$ approximately τ_i times. Let the resulting state be $|x\rangle$, and evaluate the function value $d = f(x)$.
 - (4) Update the threshold: If $d < d_i$, set $x_{i+1} = x$ and $d_{i+1} = d$. Otherwise, retain the previous values by setting $x_{i+1} = x_i$ and $d_{i+1} = d_i$.
3. Return x .

for supervised learning. The QMA solves general optimization problems by performing the GSA multiple times.

We here present a generalized version of the quantum minimization algorithm (GQMA). The main subroutine is a generalization of GSA called quantum exponential searching algorithm [63]. Consider an unsorted table of N items, $0, 1, \dots, N-1$, each holding a value from an ordered set. The algorithm requires the objective function $f(x) : \{0, 1\}^n \rightarrow \mathbb{R}$. The goal is to find the index x such that $f(x)$ is minimized, theoretically identifying the global optimum $x_{\min}$ at the end. Building on QMA, we consider a search using GQMA for one minimum among r marked states within a space of $N = 2^n$ computational basis states, where n is the number of qubits in the first register. Instead of $|+\rangle^{\otimes n}$, we apply an arbitrary unitary quantum gate $\mathcal{U}$ to $|0\rangle^{\otimes n}$ to obtain an arbitrary pure state. The detailed steps of GQMA are explained below in Algorithm 2, and the corresponding quantum circuit is presented in Fig. 6.

To analyze the probability of success for the aforementioned scenario, assume that the GQMA algorithm runs sufficiently long to find the global minimum. The algorithm finding the global minimum $x_{\min}$ implies that $r = 1$. Suppose that convergence is attained after s iterations. In the limit of a large search space and with an arbitrary initial state $|\psi\rangle$, the global minimum $x_{\min}$ can be determined after τ_s iterations, with the success probability given by

$$P_{\text{ave}}(|\psi\rangle, \tau_s) = \frac{N \sin^2 \vartheta - 1}{N - 1} f_c(|\psi\rangle) + \frac{1 - \sin^2 \vartheta}{N - 1}, \quad (9)$$

where $\vartheta := \theta(\tau_s + 1/2)$ and $\theta := \arccos(1 - 2/N)$. After $\tau_s^{\text{opt}} = \lceil \pi \sqrt{N}/4 \rceil$ iterations, the success probability of obtaining the correct measurement is upper bounded by

$$P_{\text{ave}}^{\text{opt}}(|\psi\rangle) = f_c(|\psi\rangle). \quad (10)$$

Thus, there is a monotonic relationship between the coherence fraction $f_c(|\psi\rangle)$ and the optimal average success probability.

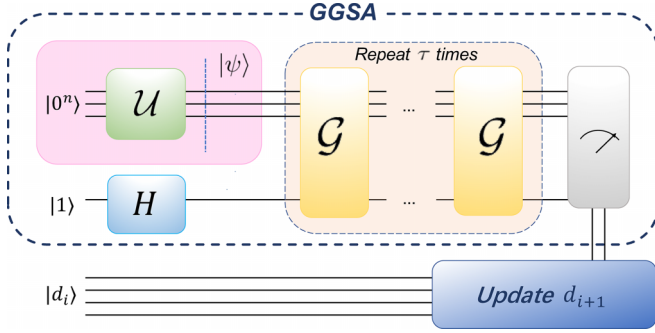


FIG. 6. Quantum Circuit for GQMA. The main subroutine of GQMA is the GGSA, as indicated by the outermost dashed box. In contrast to GGSA, here it involves three registers. Initialize the first register with n input qubits and the second register with m qubits to store the threshold d_i . Apply an arbitrary unitary quantum gate $\mathcal{U}$ to the first register to prepare an arbitrary state. Set the initial state to $|\psi\rangle \otimes |d_i\rangle$. Mark all items x for which $f(x) < d_i$. Execute the Grover operator $\mathcal{G}$ for approximately τ_i iterations. Measure the output state $|x\rangle$ and compute the function value $d = f(x)$. Update the threshold: if $d < d_i$, then set $x_{i+1} = x$ and $d_{i+1} = d$. Otherwise, keep the previous values by setting $x_{i+1} = x_i$ and $d_{i+1} = d_i$.

VI. DISCUSSIONS

We have systematically explored the relationship between the intrinsic quantum properties of the initial quantum states and the efficacy of quantum algorithms. Our analysis reveals a clear connection between the success probability and the coherence fraction of the initial state. Notably, the maximum coherence fraction corresponds to the highest success probability in the Grover search algorithm. These findings emphasize the crucial significance of the equal superposition state in quantum computation, which implies why the Hadamard gate is frequently employed to generate such states in the design of quantum algorithms.

While there have been exciting discoveries in quantum algorithms, it is important to note that not all quantum algorithms have been found to outperform their classical counterparts [12,64,65]. However, Shor's algorithm and Grover's search algorithm are two quantum algorithms that offer significant quantum advantages. In particular, Shor's factoring algorithm provides a super-polynomial speedup compared to all currently known classical factoring algorithms [2]. On the other hand, Grover's search algorithm has been proven to exhibit a quadratic speedup compared to any classical algorithms [3,4,66,67]. Recent research has highlighted the crucial role of quantum coherence in achieving the advantage of Shor's factoring algorithm with a fixed structure [24]. These findings indicate that the quantum advantage of the Grover search algorithm is attributed to the coherence fraction of quantum states, rather than quantum coherence and entanglement. While our focus is primarily on the Grover search algorithm, these insights can be applied to other algorithms based on Grover's principles as well [68,69]. Our results provide a new perspective on understanding the quantum advantages of different quantum algorithms and may enable the design of algorithms that truly achieve quantum advantages.

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APPENDIX A: PROOF OF THEOREM 1

Consider a search space containing $N = 2^n$ elements in which a subset of r elements are marked. The initial state is given by in the basis $|x\rangle$,

$$\begin{aligned} |\psi\rangle &= \sum_{x=0}^{N-1} a_x |x\rangle \\ &= \sum_{m \in \mathcal{M}} a_m |m\rangle + \sum_{u \in \mathcal{U}} a_u |u\rangle \\ &= \sqrt{P_0} |\phi_M\rangle + \sqrt{1 - P_0} |\phi_U\rangle, \end{aligned} \quad (\text{A1})$$

where $|\phi_M\rangle = \frac{1}{\sqrt{P_0}} \sum_{m \in \mathcal{M}} a_m |m\rangle$, $|\phi_U\rangle = \frac{1}{\sqrt{1-P_0}} \sum_{u \in \mathcal{U}} a_u |u\rangle$, $P_0 = \sum_{m \in \mathcal{M}} |a_m|^2$, $\mathcal{M}$ and $\mathcal{U}$ denote the marked and unmarked sets, respectively. The equally superposed state can be correspondingly written as

$$\begin{aligned} |\eta\rangle &= \frac{1}{\sqrt{N}} \sum_{x=0}^{N-1} |x\rangle \\ &= \sqrt{\frac{r}{N}} |\eta_M\rangle + \sqrt{\frac{N-r}{N}} |\eta_U\rangle. \end{aligned} \quad (\text{A2})$$

Using the Gram-Schmidt procedure we construct two orthogonal states

$$\begin{aligned} |\psi_M\rangle &= \frac{\sqrt{P_0} |\phi_M\rangle - \sqrt{r} \bar{a}_M |\eta_M\rangle}{\sqrt{P_0 - r |\bar{a}_M|^2}}, \\ |\psi_U\rangle &= \frac{\sqrt{1-P_0} |\phi_U\rangle - \sqrt{N-r} \bar{a}_U |\eta_U\rangle}{\sqrt{1-P_0 - (N-r) |\bar{a}_U|^2}}, \end{aligned} \quad (\text{A3})$$

where $\bar{a}_M = \frac{1}{r} \sum_{m \in \mathcal{M}} a_m$ and $\bar{a}_U = \frac{1}{N-r} \sum_{u \in \mathcal{U}} a_u$. The initial state can then be expressed as

$$\begin{aligned} |\psi\rangle &= \sqrt{P_0 - r |\bar{a}_M|^2} |\psi_M\rangle + \sqrt{N-r} \bar{a}_U |\eta_U\rangle \\ &\quad + \sqrt{1-P_0 - (N-r) |\bar{a}_U|^2} |\psi_U\rangle + \sqrt{r} \bar{a}_M |\eta_M\rangle, \end{aligned} \quad (\text{A4})$$

which is spanned by $|\psi_M\rangle$, $|\eta_M\rangle$ and $|\psi_U\rangle$, $|\eta_U\rangle$. For convenience, we use the following vector representations:

$$|\psi_M\rangle \equiv \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad |\psi_U\rangle \equiv \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad (\text{A5})$$

$$|\eta_U\rangle \equiv \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad |\eta_M\rangle \equiv \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}. \quad (\text{A6})$$

Thus the Grover iteration $\mathcal{G}$ acts as a linear transformation within a four-dimensional vector space. Its matrix representation $\mathcal{Q}$ is given by

$$\mathcal{Q} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & \cos \theta & -\sin \theta \\ 0 & 0 & \sin \theta & \cos \theta \end{pmatrix}. \quad (\text{A7})$$

After τ iterations the state is of the form,

$$\begin{aligned} |\psi(\tau)\rangle &= \mathcal{Q}^\tau |\psi\rangle \\ &= \sqrt{P_0 - r|\bar{a}_M|^2} |\psi_M\rangle \\ &\quad + (-1)^\tau \sqrt{1 - P_0 - (N-r)|\bar{a}_U|^2} |\psi_U\rangle \\ &\quad + (\sqrt{N-r}\bar{a}_U \cos(\theta\tau) - \sqrt{r}\bar{a}_M \sin(\theta\tau)) |\eta_U\rangle \\ &\quad + (\sqrt{N-r}\bar{a}_U \sin(\theta\tau) + \sqrt{r}\bar{a}_M \cos(\theta\tau)) |\eta_M\rangle, \end{aligned} \quad (\text{A8})$$

where $\cos \theta = 1 - \frac{2r}{N}$. The success probability $P_{succ}(\tau)$ of a measurement is given via the projection of $|\psi(\tau)\rangle$ on the plane of marked states:

$$\begin{aligned} P_{succ}(\tau) &= P_0 + \frac{1}{2}[(N-r)|\bar{a}_U|^2 - r|\bar{a}_M|^2] \\ &\quad - \frac{1}{2}[(N-r)|\bar{a}_U|^2 - r|\bar{a}_M|^2] \cos(2\theta\tau) \\ &\quad + \frac{1}{2}\sqrt{r(N-r)}(\bar{a}_U^* \bar{a}_M + \bar{a}_U \bar{a}_M^*) \sin(2\theta\tau). \end{aligned} \quad (\text{A9})$$

This provides an upper bound P_{max} of the success probability $P_{succ}(\tau)$,

$$P_{max} = P_0 + (N-r)|\bar{a}_U|^2. \quad (\text{A10})$$

The average of P_0 over all possible choices of the r marked states is given by

$$\langle P_0 \rangle = \left\langle \sum_m |a_m|^2 \right\rangle = \frac{r}{N} \sum_{j=0}^{N-1} |a_j|^2 = \frac{r}{N}. \quad (\text{A11})$$

Since

$$\begin{aligned} \langle |\bar{a}_M|^2 \rangle &= \frac{r-1}{r(N-r)} f_c(|\psi\rangle) + \frac{N-r}{Nr(N-1)}, \\ \langle |\bar{a}_U|^2 \rangle &= \frac{N-r-1}{(N-1)(N-r)} f_c(|\psi\rangle) + \frac{r}{N(N-1)(N-r)}, \\ \langle \bar{a}_U^* \bar{a}_M \rangle &= \langle \bar{a}_U \bar{a}_M^* \rangle = \frac{1}{N-1} f_c(|\psi\rangle) + \frac{1}{N(N-1)}, \end{aligned} \quad (\text{A12})$$

where $f_c(|\psi\rangle) = F(|\eta\rangle, |\psi\rangle)$ is the Uhlmann's fidelity F between quantum states $|\psi\rangle$ and $|\eta\rangle$, we obtain the average of $P_{succ}(\tau)$ over all possible choices of the set of r

marked states,

$$\begin{aligned} P_{ave}(|\psi\rangle, \tau) &= \frac{N f_c(|\psi\rangle) - 1}{N-1} \sin^2[\theta(\tau + 1/2)] \\ &\quad - \frac{r f_c(|\psi\rangle)}{N-1} + \frac{r}{N-1}. \end{aligned} \quad (\text{A13})$$

It is clear that $P_{ave}(|\psi\rangle, \tau)$ depends on the coherence fraction $f_c(|\psi\rangle)$ and the quantity $\sin^2[\theta(\tau + 1/2)]$. For a fixed $f_c(|\psi\rangle)$, the average success probability $P_{ave}(|\psi\rangle, \tau)$ is upper bounded by the following inequality:

$$P_{ave}(|\psi\rangle, \tau) \leq P_{ave}^{\text{opt}}(|\psi\rangle) = \frac{N-r}{N-1} f_c(|\psi\rangle) + \frac{r-1}{N-1}, \quad (\text{A14})$$

where we use the fact that $\sin^2[\theta(\tau + 1/2)] \in [0, 1]$. The equality is attained at $\theta(\tau + 1/2) \approx \frac{\pi}{2}$. Thus, we obtain $\tau = \lfloor \frac{\pi}{2\theta} - \frac{1}{2} \rfloor$. As $\cos \theta = 1 - \frac{2r}{N}$, namely, $\sin \theta = \frac{2}{N} \sqrt{r(N-r)}$, for large N , $\theta \approx \frac{2}{N} \sqrt{r(N-r)}$. As a result, we get $\tau = \lfloor \frac{N\pi\sqrt{r(N-r)}}{4r(N-r)} \rfloor$. For $r \ll N$, $\tau = \lfloor \frac{N\pi\sqrt{rN}}{4rN} \rfloor = \lfloor \frac{\pi}{4} \sqrt{\frac{N}{r}} \rfloor$. The optimal number of iterations is $\tau_{\text{opt}} = \lfloor \frac{\pi}{4} \sqrt{\frac{N}{r}} \rfloor$ [39].

APPENDIX B: THE AVERAGE SUCCESS PROBABILITY OF GGSA FOR MIXED INITIAL STATES

Assume the initial state of the register is an arbitrary n -qubit mixed state,

$$\rho = \sum_{\mu} p_{\mu} |\psi_{\mu}\rangle \langle \psi_{\mu}|, \quad (\text{B1})$$

where $\sum_{\mu} p_{\mu} = 1$, $|\psi_{\mu}\rangle$ has the form

$$|\psi_{\mu}\rangle = \sum_{i=0}^{N-1} a_{\mu i} |i\rangle. \quad (\text{B2})$$

After τ Grover iterations, the pure state $|\psi_{\mu}\rangle$ is changed to $\mathcal{G}^{\tau} |\psi_{\mu}\rangle$, with the corresponding density operator given by

$$\rho(\tau) = \sum_{\mu} p_{\mu} \mathcal{G}^{\tau} |\psi_{\mu}\rangle \langle \psi_{\mu}| \mathcal{G}^{\tau\dagger}. \quad (\text{B3})$$

Averaging the probability of success over all possible choices of the set of r ($r \ll N$) marked states, we have

$$\begin{aligned} P_{ave}(\rho, \tau) &= \frac{N \sin^2 \vartheta - r}{N-1} \sum_{\mu} p_{\mu} f_c(|\psi_{\mu}\rangle) + \frac{r - \sin^2 \vartheta}{N-1} \\ &= \frac{N \sin^2 \vartheta - r}{N-1} f_c(\rho) + \frac{r - \sin^2 \vartheta}{N-1}. \end{aligned} \quad (\text{B4})$$

where $\vartheta := \theta(\tau + 1/2)$ and $\theta = \arccos(1 - 2r/N)$, $f_c(\rho) = \langle \eta | \rho | \eta \rangle$ is the coherence fraction of the initial state ρ . Thus, after $\tau_{\text{opt}} = \lfloor \pi \sqrt{N/4} \rfloor$ iterations, the upper bound of the average success probability is given as follows:

$$\begin{aligned} P_{ave}^{\text{opt}}(\rho) &= \frac{N-r}{N-1} \sum_{\mu} p_{\mu} f_c(|\psi_{\mu}\rangle) + \frac{r-1}{N-1} \\ &= \frac{N-r}{N-1} f_c(\rho) + \frac{r-1}{N-1}. \end{aligned} \quad (\text{B5})$$

APPENDIX C: ABOUT THE MEASURE PROPERTIES OF COHERENCE FRACTION

1. Coherence fraction is not a measure in general

We show that the coherence fraction $f_c(\rho) = F(|\eta\rangle, \rho)$ is neither an entanglement measure nor a coherence measure of ρ .

(1) Consider a separable state $|\phi_0\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |10\rangle) = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \otimes |0\rangle$ and an entangled state $|\phi_1\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$. Obviously, $f_c(\phi_0) = f_c(\phi_1) = \frac{1}{2}$, i.e., f_c cannot distinguish a separable state from an entangled one.

(2) Consider the incoherent state ρ_0 and the coherent state ρ_1 ,

$$\rho_0 = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}, \quad \rho_1 = \begin{pmatrix} \frac{1}{3} & -\frac{1}{3}i \\ \frac{1}{3}i & \frac{2}{3} \end{pmatrix}. \quad (\text{C1})$$

We have $f_c(\rho_0) = f_c(\rho_1) = \frac{1}{2}$, namely, f_c cannot distinguish an incoherent state from a coherent one.

2. Coherence fraction as a coherence measure under specific conditions

For states ρ such that $\langle i|\rho|j\rangle \geq 0$ for all i and j , $C(\rho) := f_c(\rho) - 1/N = F(|\eta\rangle, \rho) - 1/N$ is just the l_1 norm of coherence [48]. Therefore, in this case, $f_c(\rho)$ quantifies the coherence, up to a constant factor $1/N$. Moreover, the optimal average success probability can be rewritten as $P_{\text{ave}}^{\text{opt}}(\rho) = (N-r)C(\rho)/(N-1) + r/N$.

APPENDIX D: DETAILED DERIVATIONS FOR THE EXAMPLE

Consider the following unitary quantum gate $\mathcal{U}$,

$$\mathcal{U} = (\mathcal{U}(\alpha, \beta, \theta))^{\otimes n}, \quad (\text{D1})$$

where the unitary gate $\mathcal{U}(\alpha, \beta, \theta)$ applied to each qubit in the register is given by

$$\mathcal{U}(\alpha, \beta, \theta) = \begin{bmatrix} e^{i\alpha} \cos \theta & e^{-i\beta} \sin \theta \\ e^{i\beta} \sin \theta & -e^{-i\alpha} \cos \theta \end{bmatrix} \quad (\text{D2})$$

with parameters α, β and $\theta \in [0, \pi/2]$. Especially, $\mathcal{U}(0, 0, \pi/4)$ is the Hadamard operator.

Applying the unitary gate to the single qubit state $|0\rangle$, we obtain

$$|\phi(\alpha, \beta, \theta)\rangle = \mathcal{U}(\alpha, \beta, \theta)|0\rangle \quad (\text{D3})$$

$$= e^{i\alpha} \cos \theta |0\rangle + e^{i\beta} \sin \theta |1\rangle. \quad (\text{D4})$$

The initial state is then given by

$$|\psi(\alpha, \beta, \theta)\rangle = |\phi(\alpha, \beta, \theta)\rangle^{\otimes n} \quad (\text{D5})$$

$$= \sum_{j=0}^{N-1} (e^{i\alpha} \cos \theta)^{k_j} (e^{i\beta} \sin \theta)^{n-k_j} |j\rangle, \quad (\text{D6})$$

where $N = 2^n$ and k_j is the number of zeros in the binary representation of $j = j_1 j_2 \cdots j_N$. The coherence fraction of $|\psi(\alpha, \beta, \theta)\rangle$ is

$$f_c(|\psi(\alpha, \beta, \theta)\rangle) = \frac{1}{2^n} |(e^{i\alpha} \cos \theta + e^{i\beta} \sin \theta)^n|^2. \quad (\text{D7})$$

The optimal average success probability after τ_{opt} iterations is given by

$$P_{\text{ave}}^{\text{opt}}(|\psi\rangle) = \frac{N-r}{N-1} f_c(|\psi\rangle) + \frac{r-1}{N-1}, \quad (\text{D8})$$

For one marked state (i.e., $r = 1$), the optimal average success probability is given by

$$P_{\text{ave}}^{\text{opt}}(|\psi(\alpha, \beta, \theta)\rangle) = f_c(|\psi(\alpha, \beta, \theta)\rangle) \quad (\text{D9})$$

$$= \frac{1}{2^n} |(e^{i\alpha} \cos \theta + e^{i\beta} \sin \theta)^n|^2. \quad (\text{D10})$$

In particular, we have

$$P_{\text{ave}}^{\text{opt}}(|\psi(\alpha, \beta, \pi/4)\rangle) = \frac{1}{4^n} |(e^{i\alpha} + e^{i\beta})^n|^2, \quad (\text{D11})$$

$$P_{\text{ave}}^{\text{opt}}(|\psi(0, 0, \theta)\rangle) = \frac{1}{2^n} (\cos \theta + \sin \theta)^{2n}. \quad (\text{D12})$$

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- [1] C. H. Bennett and D. P. DiVincenzo, *Nature (London)* **404**, 247 (2000).
 - [2] P. W. Shor, in *Proceedings of the 35th Annual Symposium on Foundations of Computer Science* (ACM, New York, 1994), pp. 124–134.
 - [3] L. K. Grover, in *Proceedings of the Twenty-Eighth Annual ACM Symposium on Theory of Computing* (ACM, New York, 1996), pp. 212–219.
 - [4] L. K. Grover, *Phys. Rev. Lett.* **79**, 325 (1997).
 - [5] A. W. Harrow, A. Hassidim, and S. Lloyd, *Phys. Rev. Lett.* **103**, 150502 (2009).
 - [6] R. P. Feynman, in *Feynman and Computation* (CRC Press, Boca Raton, FL, 2018), pp. 133–153.
 - [7] I. M. Georgescu, S. Ashhab, and F. Nori, *Rev. Mod. Phys.* **86**, 153 (2014).
 - [8] J. Biamonte, P. Wittek, N. Pancotti, P. Rebentrost, N. Wiebe, and S. Lloyd, *Nature (London)* **549**, 195 (2017).
 - [9] M. T. West, S.-L. Tsang, J. S. Low, C. D. Hill, C. Leckie, L. C. Hollenberg, S. M. Erfani, and M. Usman, *Nat. Mach. Intell.* **5**, 581 (2023).
 - [10] Y. Cao, J. Romero, and A. Aspuru-Guzik, *IBM J. Res. Dev.* **62**, 6:1 (2018).
 - [11] A. Robert, P. K. Barkoutsos, S. Woerner, and I. Tavernelli, *npj Quantum Inf.* **7**, 38 (2021).
 - [12] E. Tang, in *Proceedings of the 51st Annual ACM SIGACT Symposium on Theory of Computing*, STOC 2019 (Association for Computing Machinery, New York, NY, 2019), pp. 217–228.
 - [13] T. D. Ladd, F. Jelezko, R. Laflamme, Y. Nakamura, C. Monroe, and J. L. O'Brien, *Nature (London)* **464**, 45 (2010).
 - [14] A. W. Harrow and A. Montanaro, *Nature (London)* **549**, 203 (2017).
 - [15] C. Neill, P. Roushan, K. Kechedzhi, S. Boixo, S. V. Isakov, V. Smelyanskiy, A. Megrant, B. Chiaro, A. Dunsworth, K. Arya et al., *Science* **360**, 195 (2018).

- [16] M.-H. Yung, *Natl. Sci. Rev.* **6**, 22 (2019).
- [17] F. Arute, K. Arya, R. Babbush, D. Bacon, J. C. Bardin, R. Barends, R. Biswas, S. Boixo, F. G. Brandao, D. A. Buell *et al.*, *Nature (London)* **574**, 505 (2019).
- [18] H.-S. Zhong, H. Wang, Y.-H. Deng, M.-C. Chen, L.-C. Peng, Y.-H. Luo, J. Qin, D. Wu, X. Ding, Y. Hu *et al.*, *Science* **370**, 1460 (2020).
- [19] H.-L. Shi, S.-Y. Liu, X.-H. Wang, W.-L. Yang, Z.-Y. Yang, and H. Fan, *Phys. Rev. A* **95**, 032307 (2017).
- [20] R. Jozsa, *Philos. Trans. R. Soc. A* **454**, 323 (1998).
- [21] A. Ekert and R. Jozsa, *Philos. Trans. R. Soc. A* **356**, 1769 (1998).
- [22] M. Pan and D. Qiu, *Phys. Rev. A* **100**, 012349 (2019).
- [23] R. Jozsa and N. Linden, *Philos. Trans. R. Soc. A* **459**, 2011 (2003).
- [24] F. Ahnfeldt, T. Theurer, D. Egloff, J. M. Matera, and M. B. Plenio, *Phys. Rev. Lett.* **129**, 120501 (2022).
- [25] M. Pan, H. Situ, and S. Zheng, *Europhys. Lett.* **138**, 48002 (2022).
- [26] C. H. Bennett, D. P. DiVincenzo, J. A. Smolin, and W. K. Wootters, *Phys. Rev. A* **54**, 3824 (1996).
- [27] R. Horodecki and M. Horodecki, *Phys. Rev. A* **54**, 1838 (1996).
- [28] M. Horodecki, P. Horodecki, and R. Horodecki, *Phys. Rev. A* **60**, 1888 (1999).
- [29] P. Badziag, M. Horodecki, P. Horodecki, and R. Horodecki, *Phys. Rev. A* **62**, 012311 (2000).
- [30] S. Albeverio, S.-M. Fei, and W.-L. Yang, *Phys. Rev. A* **66**, 012301 (2002).
- [31] M.-J. Zhao, *Phys. Rev. A* **91**, 012310 (2015).
- [32] G. L. Long, *Phys. Rev. A* **64**, 022307 (2001).
- [33] O. Biham, D. Shapira, and Y. Shimoni, *Phys. Rev. A* **68**, 022326 (2003).
- [34] D. Shapira, Y. Shimoni, and O. Biham, *Phys. Rev. A* **71**, 042320 (2005).
- [35] G. Li and L. Li, *Info. Comput.* **292**, 105042 (2023).
- [36] X. Zhou, X. Xu, S. Zheng, and L. Luo, *arXiv:2405.06963*.
- [37] Y. Yao, D. Li, and C. P. Sun, *Phys. Rev. A* **100**, 032324 (2019).
- [38] S. Karmakar, A. Sen, I. Chattopadhyay, A. Bhar, and D. Sarkar, *Quantum Info. Proc.* **18**, 275 (2019).
- [39] L. K. Grover, *Phys. Rev. Lett.* **80**, 4329 (1998).
- [40] O. Biham, M. A. Nielsen, and T. J. Osborne, *Phys. Rev. A* **65**, 062312 (2002).
- [41] M. Hillery, *Phys. Rev. A* **93**, 012111 (2016).
- [42] N. Anand and A. K. Pati, *arXiv:1611.04542*.
- [43] Y.-C. Liu, J. Shang, and X. Zhang, *Entropy* **21**, 260 (2019).
- [44] R. Horodecki, P. Horodecki, M. Horodecki, and K. Horodecki, *Rev. Mod. Phys.* **81**, 865 (2009).
- [45] O. Gühne and G. Tóth, *Phys. Rep.* **474**, 1 (2009).
- [46] G. Gour and R. W. Spekkens, *New J. Phys.* **10**, 033023 (2008).
- [47] I. Marvian and R. W. Spekkens, *Nat. Commun.* **5**, 3821 (2014).
- [48] T. Baumgratz, M. Cramer, and M. B. Plenio, *Phys. Rev. Lett.* **113**, 140401 (2014).
- [49] Y.-T. Wang, J.-S. Tang, Z.-Y. Wei, S. Yu, Z.-J. Ke, X.-Y. Xu, C.-F. Li, and G.-C. Guo, *Phys. Rev. Lett.* **118**, 020403 (2017).
- [50] A. Streltsov, G. Adesso, and M. B. Plenio, *Rev. Mod. Phys.* **89**, 041003 (2017).
- [51] W. Zheng, Z. Ma, H. Wang, S.-M. Fei, and X. Peng, *Phys. Rev. Lett.* **120**, 230504 (2018).
- [52] M.-L. Hu, X. Hu, J. Wang, Y. Peng, Y.-R. Zhang, and H. Fan, *Phys. Rep.* **762**, 1 (2018).
- [53] H. Xu, F. Xu, T. Theurer, D. Egloff, Z.-W. Liu, N. Yu, M. B. Plenio, and L. Zhang, *Phys. Rev. Lett.* **125**, 060404 (2020).
- [54] K.-D. Wu, A. Streltsov, B. Regula, G.-Y. Xiang, C.-F. Li, and G.-C. Guo, *Adv. Quantum Technol.* **4**, 2100040 (2021).
- [55] Y.-H. Yang, X. Yang, X.-Z. Zheng, and M.-X. Luo, *Cell Rep. Phys. Sci.* **4**, 101725 (2023).
- [56] L. Henderson and V. Vedral, *J. Phys. A* **34**, 6899 (2001).
- [57] H. Ollivier and W. H. Zurek, *Phys. Rev. Lett.* **88**, 017901 (2001).
- [58] A. Datta, A. Shaji, and C. M. Caves, *Phys. Rev. Lett.* **100**, 050502 (2008).
- [59] C. Durr and P. Hoyer, *arXiv:quant-ph/9607014*.
- [60] P. Wittek, *Quantum Machine Learning: What Quantum Computing Means to Data Mining* (Academic, New York, 2014).
- [61] P. Reberntrost, M. Mohseni, and S. Lloyd, *Phys. Rev. Lett.* **113**, 130503 (2014).
- [62] M. Schuld, I. Sinayskiy, and F. Petruccione, *Contemp. Phys.* **56**, 172 (2015).
- [63] M. Boyer, G. Brassard, P. Høyer, and A. Tapp, *Fortschr. Phys.* **46**, 493 (1998).
- [64] E. Tang, *Phys. Rev. Lett.* **127**, 060503 (2021).
- [65] Y. Shao, F. Wei, S. Cheng, and Z. Liu, *Phys. Rev. Lett.* **133**, 120603 (2024).
- [66] C. H. Bennett, E. Bernstein, G. Brassard, and U. Vazirani, *SIAM J. Comput.* **26**, 1510 (1997).
- [67] C. Zalka, *Phys. Rev. A* **60**, 2746 (1999).
- [68] A. Gilliam, S. Woerner, and C. Gondiulea, *Quantum* **5**, 428 (2021).
- [69] H. Ohno, *Quan. Machine Intell.* **6**, 12 (2024).