

Master equation for non-Markovian quantum Brownian motion: The emergence of lateral coherences

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Understanding the behavior of a quantum system coupled to its environment is of fundamental interest in the general field of quantum technologies. It also has important repercussions on foundational problems in physics, such as the process of decoherence and the so-called quantum measurement problem. There have been many approaches to explore Markovian and non-Markovian dynamics within the framework of open quantum systems, but the richness of the ensuing dynamics is still not fully understood. In this paper we develop a non-Markovian extension of the standard Caldeira-Leggett model, based on expanding the dynamics of the reduced system at high temperature in inverse powers of the high-frequency cutoff of the Ohmic spectral density of the environment and derive a non-Markovian master equation for the reduced density matrix for the case of a general potential. We also obtain a fully analytical solution in the free particle case. While the short-time behavior of this solution does not diverge substantially from the Markovian behavior, at intermediate times we find a resurgence of coherence, which we name *lateral coherence*. We identify this with a corresponding transient negative entropy production rate, which is understood to be characteristic of non-Markovian dynamics. We also analyze the positivity of the reduced density matrix and derive the corresponding Fokker-Planck equation in the classical limit.

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I. INTRODUCTION

The study of quantum systems in interaction with their environment—so-called open quantum systems—touches upon a number of unsolved problems in fundamental physics. Decoherence due to the interaction of a quantum system with its environment must be understood or controlled in all applications where nonclassical dynamics need to be harnessed [1–3]. Much effort is thus currently devoted to either decoupling the system from its environment, or to engineering environmental features that protect quantum behavior [4,5].

Solving the dynamics of open quantum systems is, in general, a challenging task. A Markovian system is one in which the state of the system at time t_n is uniquely determined by its state at time t_{n-1} , and not by earlier times. In the Markovian case, many different techniques have been developed to analyze the dynamics (see for instance Ref. [6] for a review) and make a connection with the measurement problem [7]. In contrast, a non-Markovian system retains memory of the state at earlier times, and the analysis is usually more involved.

Non-Markovian behavior emerges naturally in both classical and quantum systems when the dynamics of environmental

degrees of freedom are projected out. Both the adoption of general projection techniques [8–10] and the explicit calculation of specific environmental models [11] lead to reduced dynamics for the system of interest characterized by memory kernels, which account in principle for non-Markovian behavior. However, if *ad hoc* assumptions about the relaxation timescales of such memory kernels are made, a Markov approximation can be performed. This leads to reduced quantum dynamics which are again Markovian and characterized by very fast (approximately exponential) decoherence of the coherent components [2], accounting thereby for the environment-induced measurement process [12,13], and the quantum-classical transition [14].

On the other hand, since the pioneering work of Hu, Paz, and Zhang [15], investigation of non-Markovian dynamics continues to gain increasing attention in an attempt to obtain a deeper insight into quantum behavior and in the study of the transition to classical dynamics [16–19]. One important implication of non-Markovianity has emerged recently in the context of the resulting quantum thermodynamics and on its implications for novel entropy dynamics. Recent work has identified unexpected behavior related to quantum coherences flowing back from the thermal bath into the system of interest [20,21], a phenomenon which has been associated with local negative entropy variations [22–26], and even heat-flow reversal [27–29]. This type of surprising behavior was recently given a thermodynamic and information-theoretical justification in Ref. [30]: the information backflow that characterizes non-Markovianity [31–34] is used by the system to “decode” lost information about its internal degrees of freedom and thereby, in principle, allow the system to perform more thermodynamic work than its Markovian counterpart.

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In this paper we address non-Markovian dynamics by building upon the path integral approach of Caldeira and Leggett [35]. We model the environment as a bath of oscillators at high temperature with a high-frequency cutoff Ω and develop a perturbative expansion in powers of Ω^{-1} . This method is similar to that applied in Ref. [36], in which the same expansion is applied to derive the classical Langevin equation. We obtain a quantum master equation with non-Markovian corrections perturbatively added to the standard Lindblad Markovian structure [37]. It is noticeable that the perturbative method introduced here converts the intrinsic time nonlocality of the non-Markovian regime [15,38] into a local structure defined in terms of higher-order spatial derivatives, thereby allowing an easier interpretation of the reduced dynamics. Our method is also valid for any general scalar potential felt by the system of interest.

We also present an exact analytical solution of the master equation, based on the method developed in Ref. [39]. Our solution shows the presence of an increase of coherence of the system of interest at intermediate times. After a fast decoherence process, not too dissimilar from that occurring in the standard Markovian system, our solution predicts a resurgence of quantum coherences, characterized by nonzero density-matrix elements away from the diagonal. The phenomenon is weakly observed on the decay of the initial coherent modes of the reduced density matrix, and quite pronounced when it emerges from the population structure. For this reason we name these *lateral coherences*. We demonstrate that there is a corresponding decrease in the von Neumann entropy of the system of interest, associated with a non-Markovian backflow of information into the system, in agreement with Refs. [25,26].

We also study the positivity of the non-Markovian master equation here derived. Positivity violation, with the consequent violation of the uncertainty principle, is a longstanding issue in the general analysis of open quantum systems, which manifests itself in the density-matrix spectrum characterized by at least one negative eigenvalue. While all open quantum systems falling in the Lindblad class can be shown to preserve positivity [37,40], other models, including the Caldeira and Leggett model, show violations of it, which have been analyzed in detail since the pioneering works of Dekker [41] and Diósi [42]. The master equation that we obtain is also expected to show some positivity violations [43] on general grounds, but we show that these do not occur in the parameter regime analyzed here. We adopt the approach of Ref. [44], which has highlighted that the purity of the system and the Robertson-Schrödinger uncertainty relation provide a necessary condition for positivity of the density matrix to be preserved. Numerical evaluation of these quantities for our non-Markovian system indeed show that these stay positive, giving us confidence in the physical reliability of our derivation and conclusions.

Finally, we derive the evolution equation of the Wigner function, and the corresponding Fokker-Planck equation in the classical limit. We find that there is a new term, corresponding to non-Markovian dynamics in the classical regime. We also find that our non-Markovian description bears similarity to known results obtained in a different contexts [15,43,45,46].

In Sec. II we review the Caldeira and Leggett model. In Sec. III we present our perturbative expansion in terms of inverse powers of Ω , in Sec. IV we derive the analytical solution of the system, and in Sec. V we illustrate the obtained reduced dynamics. We also show the implications of the new non-Markovian dynamics on the decoherence process and the emergence of lateral coherence at medium timescales. In Sec. VI we discuss the positivity of the reduced density matrix, and in Sec. VII we obtain the classical Fokker-Planck equation for the reduced system. Finally we draw conclusions in Sec. VIII.

II. THE CALDEIRA-LEGGETT MASTER EQUATION

In the Caldeira-Leggett Model [35] the dynamics of the system are described by the total Lagrangian

$$\mathcal{L} = \mathcal{L}_A + \mathcal{L}_B + \mathcal{L}_I, \quad (1)$$

where

$$\mathcal{L}_A = \frac{1}{2}M\dot{x}^2 - V(x), \quad (2)$$

$$\mathcal{L}_B = \sum_{k=1}^N \left(\frac{1}{2}m\dot{R}_k^2 - \frac{1}{2}m\omega_k^2 R_k^2 \right), \quad (3)$$

$$\mathcal{L}_I = -x \sum_{k=1}^N C_k R_k, \quad (4)$$

are the Lagrangians describing the system of interest, the bath oscillators, and their interaction, respectively.

By using the path integral formalism adopted in Refs. [11,35], the degrees of freedom of the bath can be integrated out, and the time evolution of the reduced density matrix of the system written in terms of the superpropagator:

$$J(x_f, y_f, t; x_i, y_i, 0) = \iint \mathcal{D}x \mathcal{D}y e^{i(S_A[x] - S_A[y])} \mathcal{F}[x, y], \quad (5)$$

where the above denote path integrals over all paths $x(t)$ [$y(t)$] from the initial point $x_i = x(0)$ [$y_i = y(0)$] to final point $x_f = x(t)$ [$y_f = y(t)$], S_A is the classical action of the system of interest, and $\mathcal{F}[x, y]$ is the influence functional, which describes the action of the environment on the system of interest. The explicit structure of the influence functional can be computed as in Ref. [35]:

$$\mathcal{F}[x, y] = \exp \left[-\frac{i}{\hbar} \int_0^t dt' f_-(t') I_I(t') - \frac{1}{\hbar} \int_0^t dt' f_-(t') I_R(t') \right], \quad (6)$$

where $I_I(t')$ and $I_R(t')$ are convolution integrals, given by

$$I_I(t') = \int_0^{t'} \alpha_I(t' - t'') f_+(t'') dt'', \quad (7)$$

$$I_R(t') = \int_0^{t'} \alpha_R(t' - t'') f_-(t'') dt'', \quad (8)$$

and $f_{\pm}(t) = x(t) \pm y(t)$. The integrals (7) and (8) are specified in terms of the memory kernels α_I and α_R , which in the

high-temperature limit $k_B T \gg \hbar \omega_k$ and under the assumption of an Ohmic environment read [35]

$$\alpha_I(t) = \frac{2M\gamma}{\pi} \frac{d}{dt} \left(\frac{\sin \Omega t}{t} \right), \quad (9)$$

$$\alpha_R(t) = \frac{4M\gamma k_B T}{\pi \hbar} \frac{\sin \Omega t}{t}. \quad (10)$$

Here, γ is the relaxation constant of the system of interest, and Ω is the cutoff frequency of the spectral density of the bath oscillators. This emerging non-Markovian structure, contained here by the convolution integrals, is a common occurrence in the reduction of both classical and quantum systems [47,48], stemming from projecting out the environmental degrees of freedom.

In the Markov limit ($\Omega \rightarrow \infty$), the memory kernels (9) and (10) become δ functions and a Markovian representation for the reduced dynamics of the system of interest is recovered. In this case, the reduced density matrix $\tilde{\rho}$ obeys the well-known master equation [2]

$$\frac{\partial \tilde{\rho}}{\partial t} = \hat{L}_M \tilde{\rho}, \quad (11)$$

where $\hat{L}_M$ is the Liouvillian operator in the Markov limit defined as

$$\hat{L}_M \tilde{\rho} = -\frac{i}{\hbar} [\hat{H}_R, \tilde{\rho}] - \gamma \left[(x-y) \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial y} \right) + \frac{2Mk_B T}{\hbar^2} (x-y)^2 \right] \tilde{\rho}. \quad (12)$$

$$C_\Omega(t) = \exp \left\{ \int_0^t d\tau \left[-\frac{i}{\hbar} \left(\frac{2M\gamma}{\pi} \right) \left(\frac{\pi}{2} \delta(\tau) f_-(\tau) f_+(\tau) - \frac{1}{\Omega} \delta'(\tau) f_-(\tau) f_+(0) - \frac{1}{\Omega} \delta'(\tau) f_-(\tau) f'_+(0) + O(\Omega^{-2}) \right) \right] \right\} \\ \times \exp \left\{ \int_0^t d\tau \left[-\frac{1}{\hbar} \left(\frac{4M\gamma k_B T}{\hbar \pi} \right) \left(\frac{1}{\Omega} \delta(\tau) f_-(\tau) f_-(0) + O(\Omega^{-2}) \right) \right] \right\}, \quad (15)$$

and the influence functional is now given by

$$\mathcal{F}^{(\Omega)}[x, y] = \exp \left[-\frac{i}{\hbar} \left(\frac{2M\gamma}{\pi} \right) \int_0^t dt' \left(-\frac{\pi}{2} f_-(t') \frac{df_+(t')}{dt'} - \Omega f_-(t') f_+(t') + \frac{1}{\Omega} f_-(t') f''_+(t') + O(\Omega^{-2}) \right) \right. \\ \left. - \frac{1}{\hbar} \left(\frac{4M\gamma k_B T}{\hbar \pi} \right) \int_0^t dt' \left(-\frac{\pi}{2} f_-^2(t') + \frac{1}{\Omega} f_-(t') f'_-(t') + O(\Omega^{-2}) \right) \right], \quad (16)$$

where $f_\pm(t) = x(t) \pm y(t)$ as before. To compute the time evolution of the reduced density matrix $\tilde{\rho}$, we now consider

$$\tilde{\rho}(x_f, y_f, t) = \iint dx_i dy_i J(x_f, y_f, t; x_i, y_i, 0) \tilde{\rho}(x_i, y_i, 0). \quad (17)$$

We follow the general principles of the standard approach, based on considering the propagation over a small interval of time ϵ . However, here we need to consider the propagation over two intervals to accommodate the effects of the second

Here, $\hat{H}_R$ is the renormalized Hamiltonian of the system. The second and third terms in (12) represent relaxation and decoherence dynamics, respectively.

III. A PERTURBATIVE NON-MARKOVIAN EXPANSION

We now develop a timescale expansion in inverse powers of Ω and identify perturbatively the contribution of terms in the neighborhood of $\Omega = \infty$. To obtain an expansion of the integrals (7) and (8) in powers of $1/\Omega$, we make use of Laplace transforms, which, for $\Omega \gg |s|$, allow us to write

$$\mathcal{L} \left\{ \frac{\sin(\Omega t)}{t} \right\} (s) = \sum_{n=0}^{\infty} \left(\frac{(-1)^n \left(\frac{\Omega}{s} \right)^{2n+1}}{(2n+1)} \right) \\ = \arctan \left(\frac{\Omega}{s} \right) = \frac{\pi}{2} - \arctan \left(\frac{s}{\Omega} \right) \\ = \frac{\pi}{2} - \sum_{n=0}^{\infty} \left(\frac{(-1)^n \left(\frac{s}{\Omega} \right)^{2n+1}}{(2n+1)} \right). \quad (13)$$

By using the convolution theorem for Laplace transforms, and after inverse transforming back to the time domain, we obtain the corresponding expression for the superpropagator, which, to first order in $1/\Omega$, reads

$$J(x_f, y_f, t; x_i, y_i, 0) \\ = \iint \mathcal{D}x \mathcal{D}y C_\Omega(t) e^{\frac{i}{\hbar} (S_A[x] - S_A[y])} \mathcal{F}^{(\Omega)}[x, y], \quad (14)$$

where $C_\Omega(t)$ is a prefactor collecting all boundary terms,

derivative in the superpropagator. Therefore we write

$$\tilde{\rho}(x, y, t + 2\epsilon) = \iint dx_0 dx_1 dy_0 dy_1 J(x, y, t + 2\epsilon; x_0, y_0, 0) \\ \times \tilde{\rho}(x_0, y_0, t), \quad (18)$$

where $x_1 = x(t + \epsilon)$, $y_1 = y(t + \epsilon)$ are new “intermediate” variables that must also be integrated out. The superpropagator separates into two contributions: The first, J_{QM} , is a “quasi-Markovian” part, which obeys the Chapman-Kolmogorov equation [49]—the integrand is separable in x_1 ,

x_0 and y_1, y_0 —and can therefore be integrated over each interval independently. The second contribution J_{NM} is a truly

non-Markovian part, which cannot be separated over the intervals:

$$\begin{aligned} \tilde{\rho}(x, y, t + 2\epsilon) = & \iint dx_1 dy_1 \left(J_{\text{QM}}(x, y, t + 2\epsilon; x_1, y_1, t + \epsilon) \iint dx_0 dy_0 J_{\text{QM}}(x_1, y_1, t + \epsilon; x_0, y_0, t) \tilde{\rho}(x_1, y_1, t + \epsilon) \right) \\ & + \iint dx_0 dx_1 dy_0 dy_1 J_{\text{NM}}(x, y, t + 2\epsilon; x_0, y_0, 0) \tilde{\rho}(x_0, y_0, 0). \end{aligned} \quad (19)$$

The quasi-Markovian and non-Markovian integrals are constructed by making the finite-difference substitutions

$$\dot{x} \approx \frac{x_1 - x_0}{\epsilon} = \frac{\beta_{x_0}}{\epsilon}, \quad (20)$$

$$\ddot{x} \approx \frac{(x_2 - x_1) - (x_1 - x_0)}{\epsilon^2} = \frac{\beta_{x_1} - \beta_{x_0}}{\epsilon^2}, \quad (21)$$

and the corresponding expressions for $\dot{y}, \ddot{y}$ into Eq. (16). Then, it is a straightforward but lengthy procedure to integrate over the β variables, truncating terms that are $O(\epsilon^2)$ or $O(1/\Omega^2)$. The resulting expression can be organized in powers of ϵ , and by collecting terms linear in ϵ we obtain a new master equation for the density matrix to leading order in our non-Markovian expansion in $1/\Omega$:

$$\frac{\partial \tilde{\rho}}{\partial t} = \left[\hat{L}_{\text{M}} + \frac{\gamma}{\pi\Omega} (\hat{L}_{\text{R}} + \hat{L}_{\text{V}} + \hat{L}_{\text{O}}) \right] \tilde{\rho}, \quad (22)$$

where $\hat{L}_{\text{M}}$ is the Markovian Liouvillian defined in Eq. (12) and the new, non-Markovian, terms are

$$\hat{L}_{\text{R}} = -2\gamma(x - y) \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial y} \right), \quad (23)$$

$$\hat{L}_{\text{V}} = -\frac{i}{2\hbar}(x - y) \left(\frac{dV}{dx} + \frac{dV}{dy} \right), \quad (24)$$

$$\hat{L}_{\text{O}} = -\frac{4ik_{\text{B}}T}{\hbar}(x - y) \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} \right). \quad (25)$$

In contrast to the Hu-Paz-Zhang master equation [15], equation (22) trades off explicit time dependencies of coefficients in favor of dynamical operators acting on the reduced density matrix and therefore offers a clearer physical interpretation of the ensuing time evolution of the system. The first Liouvillian operator, Eq. (23), is of the same form as the standard relaxation term in the Zurek equation, and therefore it only provides a cutoff-dependent renormalization of the related coefficient. The second Liouvillian, Eq. (24), is a potential-dependent term whose effect needs to be assessed case by case. The most interesting Liouvillian is the last one, Eq. (25). As we shall see, it corresponds to a term in the evolution equation for the Wigner function already identified in a different context [46]. It here emerges as a truly non-Markovian contribution to the evolution of the density matrix, and its effect is far from being trivial, as we show in the next section.

We note that Eq. (22) reduces to the Markovian master equation (11) for $\gamma/\Omega \rightarrow 0$. Because of the structure of Eq. (22), it is natural to define the dimensionless parameter $R_{\Omega} = \gamma/(\pi\Omega)$ as a measure of non-Markovianity of the system. If we identify $\tau_{\text{B}} \sim 1/\Omega$ as the timescale of the en-

vironment (corresponding to the period of the fastest, most dominant, bath oscillator) and $\tau_{\text{A}} = 1/\gamma$ as the relaxation timescale of the system of interest, the Markovian condition $\Omega \gg \gamma$ corresponds to $\tau_{\text{B}} \ll \tau_{\text{A}}$ and hence justifies the reduction of the dynamics obtained by integrating over the fast degrees of freedom of the environment. The case $\Omega \sim \gamma$ represents a situation in which system and environment become indistinguishable. The analysis of Ref. [50] thoroughly investigates the parameter space created by $\{T, \gamma, \Omega\}$ in the context of non-Markovian quantum Brownian motion by using the trace distance measure of non-Markovianity [51] to identify optimal combinations of parameter values. Although we are limited to $\Omega \gg \gamma$ by our perturbative expansion, this work suggests some natural choices of values for the parameters of interest.

IV. EXACT ANALYTICAL SOLUTION

We approach the derivation of the analytical solution by following the approach laid out in Ref. [39]. For the free particle, the master equation can be written as

$$\begin{aligned} \frac{\partial}{\partial t} \tilde{\rho}(x, y, t) = & \left[\frac{i\hbar}{2M} \left(\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} \right) \right. \\ & - \gamma(1 + 2R_{\Omega})(x - y) \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial y} \right) \\ & - \frac{2M\gamma k_{\text{B}}T}{\hbar^2} (x - y)^2 \\ & \left. - \frac{4iR_{\Omega}k_{\text{B}}T}{\hbar} (x - y) \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} \right) \right] \tilde{\rho}(x, y, t). \end{aligned} \quad (26)$$

This is cast in canonical form via the coordinate transformation

$$u = \frac{x + y}{2}, \quad v = x - y, \quad (27)$$

and the rotated coordinate u is then Fourier transformed into its associated momentum variable K , such that we can write the master equation as a first-order differential equation in one spatial dimension:

$$\begin{aligned} \frac{\partial}{\partial t} \tilde{\rho}(K, v, t) + & \left[\left(2\gamma_K v + \frac{\hbar K}{M} \right) \frac{\partial}{\partial v} \right. \\ & \left. + \frac{2M\gamma k_{\text{B}}T}{\hbar^2} v^2 - \frac{4R_{\Omega}k_{\text{B}}T}{\hbar} K v \right] \tilde{\rho}(K, v, t) = 0. \end{aligned} \quad (28)$$

For convenience, we have written $\gamma_R = \gamma(1 + 2R_\Omega)$. This equation can then be solved via the method of characteristics by introducing a parametrization variable s such that $v = v(s)$, $t = t(s)$, and $\tilde{\rho} = \tilde{\rho}(s)$ to give the characteristic equations

$$\begin{aligned} \frac{\partial t}{\partial s} &= 1, & \frac{\partial v}{\partial s} &= 2\gamma_R v + \frac{\hbar K}{M}, \\ \frac{d\tilde{\rho}}{ds} &= -\left(\frac{2M\gamma k_B T}{\hbar^2} v(s)^2 - \frac{4R_\Omega k_B T}{\hbar} K v(s)\right) \tilde{\rho}(s). \end{aligned} \quad (29)$$

By solving the equations for $t(s)$ and $v(s)$, with initial conditions $t(0) = 0$ and $v(0) = v_0$, we can substitute their solutions into the equation for $\tilde{\rho}(s)$, which is now a function of s only. After solving for $\tilde{\rho}(s)$, substituting $t = s$, and inserting the definition of

$$v_0 = -\frac{\hbar K}{2M\gamma_R} + \left(v(t) + \frac{\hbar K}{2M\gamma_R(1 + 2R_\Omega)}\right) e^{-2\gamma_R t}, \quad (30)$$

we finally obtain

$$\rho(K, v, t) = \rho_0(K, v_0) \exp \left\{ -\frac{k_B T}{2M\gamma_R(1 + 2R_\Omega)} \left[K^2(1 + 4R_\Omega + 8R_\Omega^2)t - \frac{MK}{\hbar}(1 + 4R_\Omega + 8R_\Omega^2)(v - v_0) + \frac{M^2\gamma_R}{\hbar^2}(v^2 - v_0^2) \right] \right\} \quad (31)$$

for the Fourier space solution. This is already a useful solution in its present form. The $R_\Omega \rightarrow 0$ limit of Eq. (31) is the final step presented in Ref. [39]. However, when the initial condition has a Gaussian form, we can exploit the convenient Fourier transform of a Gaussian function to recover the solution in coordinate space. This is done by collecting the coefficients of K in both the initial condition and the dynamical part of the Fourier-space solution and writing it as a sum of Gaussians in K that can be inverse Fourier transformed using the standard identity

$$\mathcal{F}_K^{-1}[e^{a_2 K^2 + a_1 K + a_0}](u) = \frac{e^{a_0}}{\sqrt{-2a_2}} \exp -\frac{(a_1 - iu)^2}{4a_2}. \quad (32)$$

For example, in Zurek's classic example of a "Schrödinger cat state" [2] (i.e., a double Gaussian with peak displacement $\pm b$ and variance $\frac{1}{2a}$), the solution is

$$\tilde{\rho}(x, y, t) = \frac{e^{E_0} \sum_i \exp[C_0^i - \frac{(E_1 + C_1^i - \frac{i(x+y)}{2})^2}{4(E_2 + C_2^i)}]}{(1 + e^{-2ab^2}) \sqrt{-16\pi(E_2 + C_2)}}, \quad (33)$$

with the index i counting over $c^\pm$ (the "coherences") and $p^\pm$ (the "populations"). The solution is a linear combination of the contributions coming from the evolution of the initial populations, or on-diagonal states, and the contributions from the evolution of the initial coherences, or off-diagonal states. The coefficients are

$$\begin{aligned} C_2 &= -\frac{1}{8a} - \frac{a\hbar^2}{8M^2\gamma_R^2}(1 - e^{-2\gamma_R t})^2, \\ C_1^{p\pm} &= \pm ib + \frac{a\hbar(x - y)}{2M\gamma_R} e^{-2\gamma_R t}(1 - e^{-2\gamma_R t}), \\ C_0^{p\pm} &= -\frac{1}{2}a(x - y)^2 e^{-4\gamma_R t}, \\ C_1^{c\pm} &= \frac{a\hbar(1 - e^{-2\gamma_R t})[(x - y)e^{-2\gamma_R t} \pm 2b]}{2M\gamma_R}, \\ C_0^{c\pm} &= -\frac{1}{2}a[(x - y)e^{-2\gamma_R t} \pm 2b]^2, \end{aligned}$$

$$\begin{aligned} E_2 &= -\frac{k_B T}{2M\gamma_R(1 + 2R_\Omega)} \left[(1 + 4R_\Omega + 8R_\Omega^2)t \right. \\ &\quad \left. + \frac{e^{-2\gamma_R t} - 1}{2\gamma_R}(1 + 4R_\Omega + 8R_\Omega^2) - \frac{(e^{-2\gamma_R t} - 1)^2}{4\gamma_R} \right], \end{aligned}$$

$$\begin{aligned} E_1 &= -\frac{k_B T}{2M\gamma_R(1 + 2R_\Omega)} \frac{M(x - y)}{\hbar} [e^{-2\gamma_R t}(1 - e^{-2\gamma_R t}) \\ &\quad - (1 - 4R_\Omega - 8R_\Omega^2)(1 - e^{-2\gamma_R t})], \\ E_0 &= -\frac{k_B T}{2M\gamma_R(1 + 2R_\Omega)} \frac{M^2\gamma_R(x - y)^2}{\hbar^2} (1 - e^{-4\gamma_R t}). \end{aligned}$$

From here, we can also deduce the steady-state solution, by taking the $t \rightarrow \infty$ limit of the coefficients. This gives

$$\tilde{\rho}_\infty(x, y) = \frac{1}{\sqrt{2\pi}} e^{-\frac{Mk_B T}{2(1 + 2R_\Omega)\hbar^2}(x - y)^2}. \quad (34)$$

In the $R_\Omega \rightarrow 0$ limit, this becomes the well-known Caldeira-Leggett steady-state solution [39,52].

V. ANALYSIS OF NON-MARKOVIAN DYNAMICS

To investigate the time evolution of the non-Markovian dynamics, we study numerically the free particle analytical solution (33), starting with a double Gaussian initial condition [2]. For convenience, we choose the particle to be a proton and the Gaussians to be separated by 5 Å. To ensure we are seeing true non-Markovian effects, we select parameters in the spirit of Ref. [50]. We also assume the environment to be at a temperature of $T = 320$ K, and take a value for the relaxation constant of $\gamma = k_B T / 20\hbar \approx 0.002$ fs, if not otherwise stated. The range of Ω we investigate is in the region highlighted in Ref. [50] as maximizing the effects of non-Markovianity. Our analysis is presented separately in the short- and long-time regimes.

A. Short-time behaviour

We focus on analyzing the non-Markovian behavior on short timescales first. In Fig. 1 we report the evolution of the density matrix up to 6 fs so as to highlight the initial stages of the decoherence process. While the behavior does not diverge substantially from the Markovian case, a slight delay in the decoherence process is visible.

To gain further insight into the decoherence process, we calculate the l_1 norm of coherence [53]. For some choice of discrete basis, $\{|i\rangle\}$, this is defined as $C_{l_1} = \sum_{i,j \neq i} |\rho_{i,j}|$. In the continuum limit of our configuration space basis, this

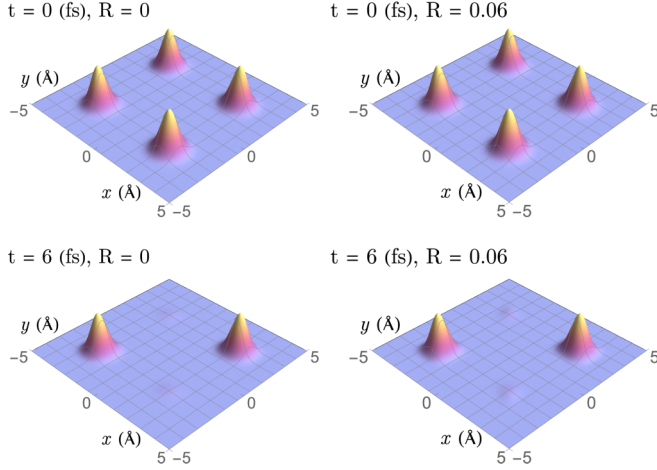


FIG. 1. Short-time evolution of the density matrix, via Eq. (33), in the Markovian case, $R_\Omega = 0$ (left column), and the non-Markovian case, $R_\Omega = 0.06$ (right column), for the free particle. Plots are over 6 fs to highlight the decoherence process. Parameters are $T = 320$ K, $\gamma = \frac{kT}{20\hbar}$ fs $^{-1}$. Only a slight deviation between the Markovian and non-Markovian cases is apparent.

becomes

$$C_{li}(t) = \int |\tilde{\rho}_{\text{off}}(x, y, t)| dx dy. \quad (35)$$

In a discrete system, it is straightforward to remove the $i = j$ diagonal elements from the sum. However, here the reduced density matrix is a continuous function of x and y , and defining the diagonal elements is less straightforward. We take a density matrix without initial off-diagonal peaks, evolving under the Markovian Caldeira-Leggett/Zurek master equation, to be an incoherent “baseline” which we can remove from our density matrix, such that $\tilde{\rho}_{\text{off}} = \tilde{\rho} - \tilde{\rho}_{\text{decoh}}^{\text{CL}}$.

In Fig. 2 we show that a slight delay in the decoherence process does indeed occur, which we ascribe to the presence of the Liouvillian $\hat{L}_O$ because the decoherence term in the Zurek equation is not modified by the finite cutoff frequency Ω . As we show in the next section, the action of $\hat{L}_O$ is to spread the peaks in an oscillatory manner. Although the effect is small, the off-diagonal peaks are spread along the line parallel to the diagonal, and this fractionally slows the decoherence process.

B. Long-time behaviour

We next analyze the non-Markovian behavior on medium-to-long timescales, as shown in Fig. 3. This regime allows us to identify a more interesting role played by the Liouvillian $\hat{L}_O$. Further to being responsible for a slower decoherence process at short times, it also causes the density matrix to spread away from the main diagonal after a longer time—a process we call *lateral coherences*. This phenomenon occurs long after the original off-diagonal coherences have decayed away and is a rather different signature of quantum behavior than the coherence of the initial state.

Figure 4 shows the long-time behavior of the coherences in the system. A resurgence of coherent behavior is evident and

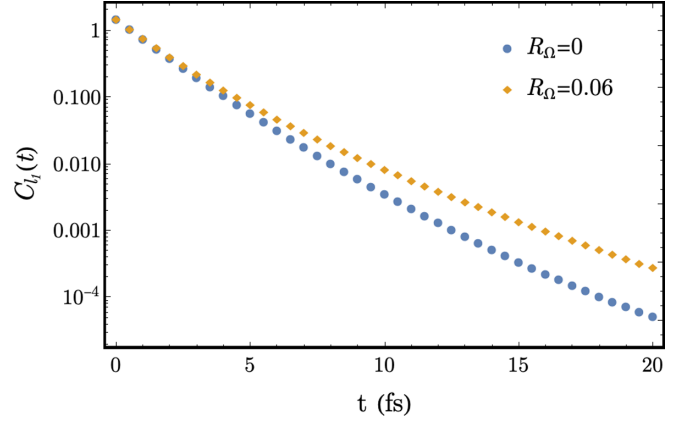


FIG. 2. The C_{li} norm of coherence, presented on a log-linear scale over 20 fs in the Markovian $R_\Omega = 0$ and non-Markovian $R_\Omega = 0.06$ case. Parameters are $T = 320$ K, $\gamma = \frac{kT}{20\hbar}$ fs $^{-1}$. The off-diagonals only have been isolated in this calculation to remove any effects from the diagonal. This calculation confirms that there is a slight but measurable deviation in the decoherence process.

can be associated with the emergence of lateral coherences identified in Fig. 3. A variety of values of R_Ω are shown to demonstrate the varying magnitude of the coherence resurgence. In the most extreme case ($R_\Omega = 0.06$) the coherence resurgence overshoots the initial value.

We interpret the resurgence of these lateral coherences as a signature of the non-Markovian character of the system. Coherence can be linked to entropy in quantum systems—for example, via the relative entropy of coherence [53]. The von Neumann entropy of a mixed Gaussian state is given in Ref. [54] as

$$\frac{S_{\text{vN}}(\tilde{\rho})}{k_B} = \frac{1 - \mu}{2\mu} \ln \frac{1 + \mu}{1 - \mu} - \ln \frac{2\mu}{1 + \mu}, \quad (36)$$

where $\mu = \text{Tr}(\tilde{\rho}^2)$ is the purity. Therefore, the corresponding von Neumann entropy production rate is

$$\frac{\dot{S}_{\text{vN}}(\tilde{\rho})}{k_B} = -\frac{\dot{\mu}}{2\mu^2} \ln \frac{1 + \mu}{1 - \mu}. \quad (37)$$

By calculating the purity directly from the analytic solution (as further discussed in the next section), we demonstrate in Fig. 5 that the coherence resurgence is associated with a transient negative entropy production rate. However, the second law is never violated because the global entropy production is always positive. It has been shown [55–57] that non-Markovian dynamics can never be used to create thermal engines which violate the second law because there is a thermodynamic cost associated with preparing states and removing the system from the bath.

We interpret the negative entropy production rate in the light of a growing body of literature [22–26] which characterizes this kind of behavior as characteristic of non-Markovianity. In the standard thermodynamic framework, entropy can be understood as measuring the amount of thermal energy within a system that is available to perform useful work. It has been shown that a so-called Szilárd engine [58] can use quantum information to do work, e.g., via a

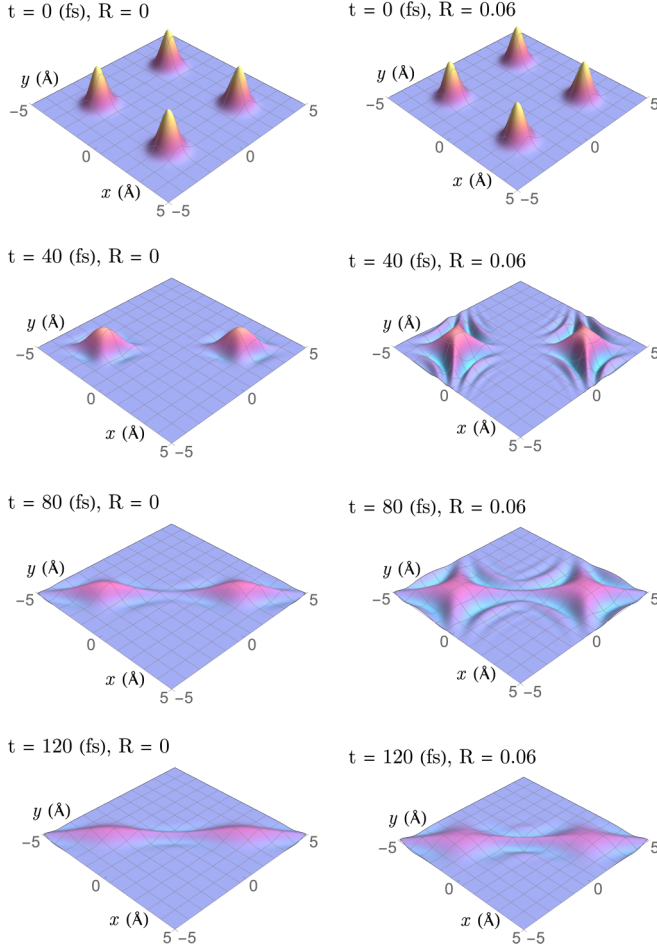


FIG. 3. Medium-time evolution of the density matrix, via Eq. (33) in the Markovian case, $R_\Omega = 0$ (left column), and the non-Markovian case, $R_\Omega = 0.06$ (right column), for the free particle. Plots are over 120 fs to highlight the emergence of lateral coherences. Parameters are $T = 320$ K, $\gamma = \frac{kT}{20\hbar}$ fs $^{-1}$. Significant deviation between the Markovian and non-Markovian cases is apparent in the emergence of lateral coherences, which spread the peaks of the density matrix orthogonal to the diagonal line.

Maxwell demon [59]. By combining the thermodynamic and information theoretic viewpoints as in Ref. [30], which is necessary for a full understanding of quantum thermodynamics, we see that increasing entropy can be understood as a loss of information about the system's internal degrees of freedom. However, information backflow is considered characteristic of non-Markovianity [31–34]. When information flows back into the system, it can be interpreted via Szilárd's framework as a new source of work. Therefore, there must be a corresponding entropy decrease to accommodate the system's increased thermodynamic capability. We further note that coherence itself can be a source for work [60], and therefore this interpretation can be understood in terms of either information or coherence.

VI. TESTS OF POSITIVITY

Density matrices must have spectra of positive eigenvalues summing to unity in order to consistently describe a quantum system. While it is well known that master equations in the

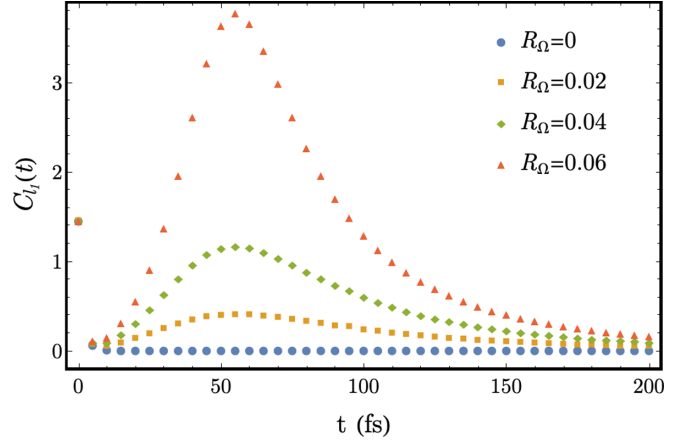


FIG. 4. The time evolution of the l_1 norm of coherence, via Eq. (35), over 200 fs. Parameters are $T = 320$ K, $\gamma = \frac{kT}{20\hbar}$ fs $^{-1}$. The resurgence of coherence, more pronounced with increasing R_Ω , is clearly observed. In the most extreme case ($R_\Omega = 0.06$), the coherence resurgence exceeds the initial value.

Lindblad form preserve the positivity of the density matrix [37], the same cannot be said for the Caldeira-Leggett master equation [44]. Our novel master equation deviates even further from the Lindblad form because of its non-Markovian dynamics. As a consequence it is natural to ask whether positivity is preserved in our model.

To check whether the density matrix violates positivity as it evolves in time, we would ideally need to compute its eigenvalues and verify that none of them becomes negative over time. Unfortunately this method is not straightforward when the density matrix represents a continuous system, as in our case, and so a different approach is taken. Thus we follow the approach presented in Ref. [44] and compute the purity of the system and the Robertson-Schrödinger uncer-

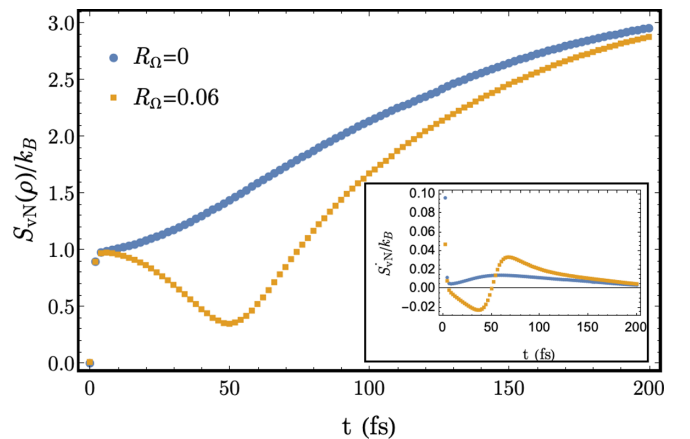


FIG. 5. The von Neumann entropy of the density matrix, in units of $1/k_B$, for the Markovian ($R = 0$) and non-Markovian ($R = 0.06$) cases. Parameters are $T = 320$ K, $\gamma = \frac{kT}{20\hbar}$ fs $^{-1}$. In the non-Markovian case, there is a transient reduction of the system entropy, corresponding negative entropy production rate. Inset shows entropy production rate. The non-Markovian entropy production rate clearly takes on negative values, whereas the Markovian entropy production is always positive.

tainty relation. Strictly speaking, these quantities only provide necessary, but not sufficient, conditions for the positivity of the density matrix, and therefore must be taken only as an indication of whether any violation of positivity is occurring in the system.

Let us consider first the purity of the density matrix, identified by the value $\text{Tr}(\rho^2)$ where, for a pure state, $\text{Tr}(\rho^2) = 1$. For a mixed state,

$$\rho = \sum_i \lambda_i |\phi_i\rangle\langle\phi_i|, \quad (38)$$

where λ_i are the eigenvalues of ρ , with $\sum_i \lambda_i = 1$ and $0 \leq \lambda_i \leq 1$ for all i . Positivity of the density matrix is thus satisfied if

$$\sum_i (\lambda_i^2 - \lambda_i) = \text{Tr}(\rho^2) - \text{Tr}(\rho) < 0. \quad (39)$$

Since $\text{Tr}(\rho) = 1$, positivity implies that $\text{Tr}(\rho^2) < 1$ at all times. Violation of this condition would imply violation of positivity of the density matrix, even though this condition being satisfied does not guarantee that the density matrix is positive definite [44].

Second, the Robertson-Schrödinger uncertainty relation reads

$$\sigma_x \sigma_p \geq \frac{\hbar}{2}, \quad (40)$$

where

$$\sigma_x \sigma_p = \sqrt{[\text{Tr}(\rho \hat{x}^2) - \text{Tr}^2(\rho \hat{x})][\text{Tr}(\rho \hat{p}^2) - \text{Tr}^2(\rho \hat{p})]}, \quad (41)$$

where $\hat{x}$ and $\hat{p}$ are the position and momentum operators. It has been shown [44] that the uncertainty relation (40) is based on the assumption of the density matrix being positive definite, and therefore violation of positivity may reflect in violation of Eq. (40).

The time evolution of the uncertainty, $\sigma_x \sigma_p$, and of the purity, $\text{Tr}(\rho^2)$, are shown in Fig. 6. We observe that both the Markovian and non-Markovian systems satisfy these necessary positivity criteria. We note that the purity undergoes a nonmonotonic decrease in the non-Markovian case, which corresponds to the coherence resurgence and transient negative entropy production. However, it never exceeds the initial value of 1. The uncertainty shows little disparity between the two cases during the resurgence, but tends to a slightly different final value.

VII. CLASSICAL LIMIT

The standard Markovian Caldeira and Leggett model can be shown to correspond in the classical limit to classical Brownian motion. By using the formalism of the Wigner transform, the Kramers equation [61] can be derived, which corresponds to a Fokker-Planck equation in (x, p) variables. Here, we derive the evolution equation for the Wigner function and the corresponding Fokker-Planck equation in the classical limit in the case of non-Markovian dynamics.

We begin by Wigner transforming the non-Markovian quantum master equation (22). The Wigner function W is

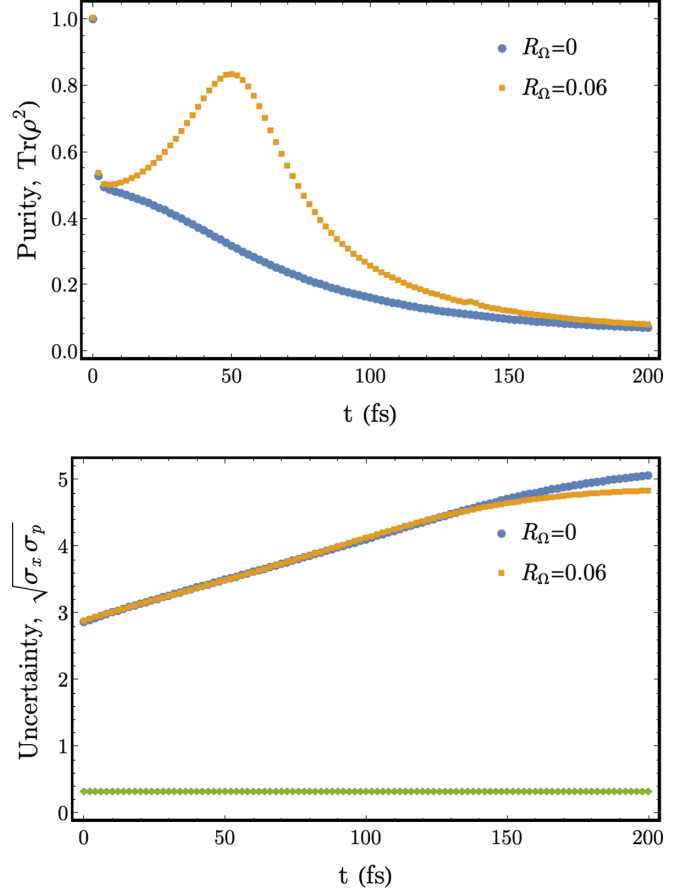


FIG. 6. Purity (top panel) and uncertainty (bottom panel) for Markovian $R_\Omega = 0$ (blue) and non-Markovian $R_\Omega = 0.06$ (orange) cases for the free particle. Parameters are $T = 320$ K, $\gamma = \frac{kT}{20\hbar}$ fs $^{-1}$.

defined as

$$W(x, p, t) = \frac{1}{2\pi\hbar} \int_{-\infty}^{\infty} e^{-ipy/\hbar} \left\langle x + \frac{y}{2} \left| \tilde{\rho} \left| x - \frac{y}{2} \right. \right. \right\rangle dy, \quad (42)$$

and allows us to rewrite Eq. (22) as

$$\frac{\partial W}{\partial t} = \left[\hat{L}_M^{(W)} + \frac{\gamma}{\pi\Omega} (\hat{L}_R^{(W)} + \hat{L}_V^{(W)} + \hat{L}_O^{(W)}) \right] W, \quad (43)$$

where

$$\begin{aligned} \hat{L}_M^{(W)} W &= -\frac{p}{M} \frac{\partial W}{\partial x} + \frac{dV}{dx} \frac{\partial W}{\partial p} \\ &\quad + 2\gamma \frac{\partial(pW)}{\partial p} + 2M\gamma k_B T \frac{\partial^2 W}{\partial p^2}, \end{aligned} \quad (44)$$

$$\hat{L}_R^{(W)} W = 4\gamma \frac{\partial}{\partial p} (pW), \quad (45)$$

$$\hat{L}_V^{(W)} W = -\sum_{n=0}^{\infty} \frac{(-\hbar^2)^n}{2^{2n} (2n)!} \frac{d^{2n+1} V(x)}{dx^{2n+1}} \frac{\partial^{2n+1} W}{\partial p^{2n+1}}, \quad (46)$$

$$\hat{L}_O^{(W)} W = 4k_B T \frac{\partial^2 W}{\partial p \partial x}. \quad (47)$$

Notice that $\hat{L}_M^{(W)}$ corresponds to the familiar Fokker-Planck operator describing classical Brownian motion.

By taking the classical limit $\hbar \rightarrow 0$ we obtain an expression containing the standard Markovian Fokker-Planck equation, but with additional non-Markovian corrections:

$$\begin{aligned} \frac{\partial W}{\partial t} = & -\frac{p}{M} \frac{\partial W}{\partial x} + (1 + R_\Omega) \frac{dV}{dx} \frac{\partial W}{\partial p} + 2\gamma(1 + 2R_\Omega) \frac{\partial}{\partial p}(pW) \\ & + 2M\gamma k_B T \frac{\partial^2 W}{\partial p^2} + 4k_B T R_\Omega \frac{\partial^2 W}{\partial x \partial p}. \end{aligned} \quad (48)$$

Here the first four terms correspond to the familiar Markovian form, while the last term is purely non-Markovian. Notice that the relaxation and the potential terms are modified by corrective factors for finite Ω , which allow us to reclaim the results of Ref. [35] in the Markov limit $\Omega \rightarrow \infty$.

The last, non-Markovian, term in the Fokker-Planck equation (48) describes novel dynamics brought about through non-Markovian memory effects. This term had already been identified in previous work [43], where it has been accounted for as part of the Markovian description of the damped harmonic oscillator.

It is interesting to note that our non-Markovian Fokker-Planck equation shares terms with the Markovian Fokker-Planck equation derived in Ref. [45], which describes the dissipative dynamics of a driven or damped harmonic oscillator. In particular, the term we label orthogonal which contains the mixed derivative $\partial_p \partial_q W$ and which is responsible for the novel non-Markovian dynamics is also present in the dissipative damped quantum harmonic-oscillator Fokker-Planck equation. However, it is not an exact equivalence: the sign and magnitude of the term in our equation are both different, and our equation is not restricted to a choice of potential.

VIII. CONCLUSIONS

In this paper we have introduced a perturbative expansion method to derive quantum master equations describing non-Markovian Brownian motion for any general potential acting on the system of interest. In contrast to Refs. [15,38], our derivation is not exact, as it is based on an expansion in $1/\Omega$, but it does provide a systematic way of addressing the time nonlocality of the system. It is legitimate to expect that the expansion captures the relevant dynamics at large enough values of Ω , while higher-order terms can in principle be consistently computed according to needs.

We have also identified an exact analytical solution of the master equation here derived, whose numerical evaluation has provided us with interesting interpretation for the new term. While on short timescales, the standard decoherence process prevails, and the decay of the l_1 norm of coherences is not too different from the standard Markovian case, in the long-time limit, dynamics seem to be much richer. A new phenomenon arises, which we name emergence of lateral coherences, by which a resurgence of coherent behavior emerges orthogonally from the population structure of the density matrix. This coherence resurgence is associated with a transient negative entropy production rate, a phenomenon that is becoming recognized as characteristic of non-Markovian dynamics. In the resource theory framework [62], coherence is a valuable tool to perform quantum operations. We speculate that the

coherence resurgence may promote purely quantum behavior, such as tunneling, in non-Markovian open systems.

We have also discussed the positivity of the reduced density matrix under the evolution of our novel set of Liouvillians (23)–(25). While the argument that we adopt here is not strictly conclusive, it bears the same strengths and weaknesses presented in Ref. [44]. On the one hand, analyzing purity and uncertainty as proxies for positivity of the reduced density matrix is questionable in that the identified conditions for positivity violations are necessary but not sufficient. On the other hand, however, direct checks of positivity by explicit calculations of the eigenvalues of the density matrix are difficult, if not impossible, especially in the coordinate representation. Our findings show nonetheless that positivity violations are not detected in the parameter regime analyzed here, despite being known to be present in general for this master equation [43].

Finally, we have derived the corresponding Fokker-Planck equation in the classical limit. This clearly describes novel non-Markovian dynamics, relying on our conversion of non-local dynamics expressed in terms of convolution integrals into series of higher-order spatial and momentum derivatives. While the methodology introduced here provides a clear advantage to effectively describe the non-Markovian dynamics involved in the system, it also has the drawback of deviating from more standard Langevin pictures for the corresponding classical system, leaving thereby questions open on the physical interpretation of the corresponding classical limit [43,63].

The non-Markovian framework introduced here is promising and worthy of further investigation. It potentially paves the way to possible applications in the broad area of quantum technologies, in all those cases in which harnessing quantum properties of the system of interest is desired and to possible further understanding of the quantum-classical transition when memory effects are retained in the system of interest.

ACKNOWLEDGMENTS

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APPENDIX A: EXACT SOLUTION OF THE FREE PARTICLE NON-MARKOVIAN MASTER EQUATION

We wish to find an exact solution for the following master equation:

$$\begin{aligned} \frac{\partial}{\partial t} \rho(x, y, t) = & \left[\frac{i\hbar}{2M} \left(\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} \right) - \gamma(x-y) \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial y} \right) \right. \\ & - \frac{2M\gamma k_B T}{\hbar^2} (x-y)^2 - \frac{4iRk_B T}{\hbar} (x-y) \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} \right) \\ & \left. - 2\gamma R(x-y) \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial y} \right) \right] \rho(x, y, t), \end{aligned} \quad (A1)$$

where here $R \equiv R_\Omega = \gamma/(\pi\Omega)$. We proceed by following the approach outlined by Roy and Venugopalan [39]. We first define the rotated variables,

$$u = \frac{x+y}{2}, \quad v = x-y. \quad (\text{A2})$$

This is a useful step because it removes the dependence on second derivatives (the second derivatives in u and v cancel out) and leaves us with a partial differential equation that is only first order in both coordinates.

In the rotated coordinate system, the master equation is

$$\frac{\partial}{\partial t} \rho(u, v, t) = \left(\frac{i\hbar}{M} \frac{\partial^2}{\partial u \partial v} - \frac{4RikT}{\hbar} v \frac{\partial}{\partial u} - 2\gamma(1+2R)v \frac{\partial}{\partial v} - \frac{2M\gamma kT}{\hbar^2} v^2 \right) \rho(u, v, t). \quad (\text{A3})$$

We then Fourier transform with respect to u to reduce the above equation to a first-order differential equation in v , with the Fourier transform defined such that

$$\rho(u, v, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dK e^{iKu} \bar{\rho}(K, v, t). \quad (\text{A4})$$

Note that here $K = p' - p$, with p and p' being the Fourier momentum variables associated with x and y , respectively. Therefore, $K = 0$ represents the diagonal values in momentum-space, and delocalized in position space, and $K \rightarrow \pm\infty$ represents a smearing out of the momentum, and the corresponding localization in position.

The Fourier transform gives us

$$\frac{\partial}{\partial t} \bar{\rho}(K, v, t) + \left\{ \left[2\gamma(1+2R)v + \frac{\hbar K}{M} \right] \frac{\partial}{\partial v} + \frac{2M\gamma kT}{\hbar^2} v^2 - \frac{4RkT}{\hbar} Kv \right\} \bar{\rho}(K, v, t) = 0. \quad (\text{A5})$$

We address this differential equation using the method of characteristics. By introducing a parametrization variable s , we can assign the following equalities:

$$\frac{\partial t}{\partial s} = 1, \quad \frac{\partial v}{\partial s} = 2\gamma(1+2R)v + \frac{\hbar K}{M}, \quad (\text{A6})$$

and solve to obtain

$$t(s) = s,$$

$$v(s) = -\frac{\hbar K}{2M\gamma(1+2R)} + \left(v_0 + \frac{\hbar K}{2M\gamma(1+2R)} \right) e^{2\gamma(1+2R)s}, \quad (\text{A7})$$

with $t(0) = 0$ and $v(0) = v_0$.

Because

$$\frac{d\bar{\rho}}{ds} = \frac{\partial \bar{\rho}}{\partial t} \frac{\partial t}{\partial s} + \frac{\partial \bar{\rho}}{\partial v} \frac{\partial v}{\partial s}, \quad (\text{A8})$$

we can write the equation to be solved as

$$\frac{d\bar{\rho}}{ds} + \left(\frac{2M\gamma kT}{\hbar^2} v^2 - \frac{4RkT}{\hbar} Kv \right) \bar{\rho} = 0. \quad (\text{A9})$$

By substituting in Eq. (A7), and considering the initial condition $\bar{\rho}_0(K, v_0)$ at $t = 0$, this has solution

$$\begin{aligned} \bar{\rho} = \bar{\rho}_0(K, v_0) \exp \left\{ -\frac{kT}{2M\gamma(1+2R)^2} \left[K^2(1+4R+8R^2)t - \frac{2M}{\hbar} K(1+2R+4R^2) \left(v_0 + \frac{\hbar K}{2M\gamma(1+2R)} \right) (e^{2\gamma(1+2R)t} - 1) \right. \right. \\ \left. \left. + \frac{M^2\gamma(1+2R)}{\hbar^2} \left(v_0 + \frac{\hbar K}{2M\gamma(1+2R)} \right)^2 (e^{4\gamma(1+2R)t} - 1) \right] \right\}, \end{aligned} \quad (\text{A10})$$

which, by inserting the definition of $v(t)$, can be simplified to

$$\begin{aligned} \bar{\rho}(K, v, t) = \bar{\rho}_0(K, v_0) \exp \left\{ -\frac{kT}{2M\gamma(1+2R)^2} \left[K^2(1+4R+8R^2)t - \frac{MK}{\hbar} (1+4R+8R^2)(v-v_0) \right. \right. \\ \left. \left. + \frac{M^2\gamma(1+2R)}{\hbar^2} (v^2 - v_0^2) \right] \right\}, \end{aligned} \quad (\text{A11})$$

where

$$v_0 = -\frac{\hbar K}{2M\gamma(1+2R)} + \left(v(t) + \frac{\hbar K}{2M\gamma(1+2R)} \right) e^{-2\gamma(1+2R)t}. \quad (\text{A12})$$

To accomplish the transform back into x and y , we need to write the solution in a form that can be easily inverse Fourier transformed. In particular, we can use the Fourier and inverse Fourier transform of a Gaussian function:

$$\begin{aligned} \mathcal{F}_u[\exp(a_2 u^2 + a_1 u + a_0)](K) &= \frac{e^{a_0}}{\sqrt{-2a_2}} \exp -\frac{(a_1 + iK)^2}{4a_2}, \\ \mathcal{F}_K^{-1}[\exp(a_2 K^2 + a_1 K + a_0)](u) &= \frac{e^{a_0}}{\sqrt{-2a_2}} \exp -\frac{(a_1 - iu)^2}{4a_2}. \end{aligned} \quad (\text{A13})$$

Because we are interested in x and y , not u , we can directly write the result of the inverse Fourier transform as

$$\frac{e^{a_0}}{\sqrt{-2a_2}} \exp -\frac{\left(a_1 - \frac{i(x+y)}{2} \right)^2}{4a_2}. \quad (\text{A14})$$

We can similarly substitute $v = x - y$ wherever we find it now that the equation has been solved.

The time-dependent part of Eq. (A11) is a Gaussian in K with time-dependent coefficients. We can write

$$e^{E_2 K^2 + E_1 K + E_0} = \exp \left\{ -\frac{kT}{2M\gamma(1+2R)^2} \left[K^2(1+4R+8R^2)t - \frac{MK}{\hbar}(1+4R+8R^2)(v-v_0) + \frac{M^2\gamma(1+2R)}{\hbar^2}(v^2-v_0^2) \right] \right\}, \quad (\text{A15})$$

with

$$\begin{aligned} E_2 &= -\frac{kT}{2M\gamma(1+2R)^2} \left((1+4R+8R^2)t + \frac{e^{-2\gamma(1+2R)t} - 1}{2\gamma(1+2R)}(1+4R+8R^2) - \frac{(e^{-2\gamma(1+2R)t} - 1)^2}{4\gamma(1+2R)} \right), \\ E_1 &= \frac{kT}{2M\gamma(1+2R)^2} \frac{M(x-y)}{\hbar} \left[(1-4R-8R^2)(1-e^{-2\gamma(1+2R)t}) - e^{-2\gamma(1+2R)t}(1-e^{-2\gamma(1+2R)t}) \right], \\ E_0 &= -\frac{kT}{2M\gamma(1+2R)^2} \frac{M^2\gamma(1+2R)(x-y)^2}{\hbar^2} (1-e^{-4\gamma(1+2R)t}). \end{aligned} \quad (\text{A16})$$

Now we address the initial condition. A generic normalized double Gaussian density matrix with width parameter b_1 , centered at $\pm b_2$, is given by

$$\begin{aligned} \rho_0(x, y) &= \frac{\sqrt{b_1}}{\sqrt{2\pi}(1+e^{-2b_1b_2^2})} (e^{-b_1(x+b_2)^2} + e^{-b_1(x-b_2)^2}) \\ &\times (e^{-b_1(y+b_2)^2} + e^{-b_1(y-b_2)^2}). \end{aligned} \quad (\text{A17})$$

In this expression, there are four distinct linear terms, two of which represent the on-diagonal, or “classical,” states, and two the off-diagonal, or “coherent,” states. These are constructed by multiplying out the brackets; the classical terms are those of the form $\exp[-b_1(x \pm b_2)^2 - b_1(y \pm b_2)^2]$ and the coherent terms are those of the form $\exp[-b_1(x \pm b_2)^2 - b_1(y \mp b_2)^2]$.

In terms of the rotated variables, Eq. (A17) is written as

$$\begin{aligned} \rho_0(u, v) &= \frac{\sqrt{b_1}}{\sqrt{2\pi}(1+e^{-2b_1b_2^2})} \\ &\times (e^{-\frac{b_1}{4}(2u-v-2b_2)^2} + e^{-\frac{b_1}{4}(2u-v+2b_2)^2}) \\ &\times (e^{-\frac{b_1}{4}(2u+v-2b_2)^2} + e^{-\frac{b_1}{4}(2u+v+2b_2)^2}). \end{aligned} \quad (\text{A18})$$

Each term in this sum is Gaussian in u , so can be Fourier transformed using the identities in Eq. (A13). This gives

$$\begin{aligned} \bar{\rho}_0(K, v) &= \frac{1}{2\sqrt{2\pi}(1+e^{-2b_1b_2^2})} \\ &\times \left(e^{-\frac{K^2}{8b_1} + ib_2K - \frac{b_1v^2}{2}} + e^{-\frac{K^2}{8b_1} - ib_2K - \frac{b_1v^2}{2}} \right. \\ &\left. + e^{-\frac{K^2}{8b_1} - \frac{b_1(2b_2+v)^2}{2}} + e^{-\frac{K^2}{8b_1} - \frac{b_1(2b_2-v)^2}{2}} \right). \end{aligned} \quad (\text{A19})$$

Here, the first two terms (containing a term linear in K) represent the classical states, or on-diagonal Gaussians, and the second two terms represent the coherent states, or off-diagonal Gaussians. Now, we replace v by v_0 in each of the four terms. We obtain a sum,

$$\rho_0(K, v_0) = \frac{1}{2\sqrt{2\pi}(1+e^{-2b_1b_2^2})} \sum_i e^{C_2^i K^2 + C_1^i K + C_0^i}, \quad (\text{A20})$$

where the index i takes values $p\pm$ to refer to the populations (classical states) and $c\pm$ to refer to the coherent states. The

values of the coefficients are

$$\begin{aligned} C_2^{p\pm} &= -\frac{1}{8b_1} - \frac{b_1\hbar^2}{8M^2\gamma^2(1+2R)^2} (1-e^{-2\gamma(1+2R)t})^2, \\ C_1^{p\pm} &= \pm ib_2 + \frac{b_1\hbar(x-y)}{2M\gamma(1+2R)} e^{-2\gamma(1+2R)t} (1-e^{-2\gamma(1+2R)t}), \\ C_0^{p\pm} &= -\frac{1}{2} b_1(x-y)^2 e^{-4\gamma(1+2R)t}, \\ C_2^{c\pm} &= -\frac{1}{8b_1} - \frac{b_1\hbar^2}{8M^2\gamma^2(1+2R)^2} (1-e^{-2\gamma(1+2R)t})^2, \\ C_1^{c\pm} &= \frac{b_1\hbar(1-e^{-2\gamma(1+2R)t})[(x-y)e^{-2\gamma(1+2R)t} \pm 2b_2]}{2M\gamma(1+2R)}, \\ C_0^{c\pm} &= -\frac{1}{2} b_1((x-y)e^{-2\gamma(1+2R)t} \pm 2b_2)^2, \end{aligned} \quad (\text{A21})$$

where as before we have replaced $v = x - y$.

Now, the exact time-dependent solution in Fourier space can be written as

$$\begin{aligned} \bar{\rho}(K, v, t) &= \frac{1}{2\sqrt{2\pi}(1+e^{-2b_1b_2^2})} \\ &\times \left(\sum_i e^{C_2^i K^2 + C_1^i K + C_0^i} \right) e^{E_2 K^2 + E_1 K + E_0} \\ &= \frac{1}{2\sqrt{2\pi}(1+e^{-2b_1b_2^2})} \\ &\times \sum_i e^{(E_2 + C_2^i)K^2 + (E_1 + C_1^i)K + (E_0 + C_0^i)}. \end{aligned} \quad (\text{A22})$$

This is a sum of Gaussians in K , with the coefficients formed by summing the coefficients of the time-dependent and initial condition parts of the solution. Therefore, taking the inverse Fourier transform is straightforward, and we obtain

$$\begin{aligned} \rho(x, y, t) &= \frac{e^{E_0}}{2\sqrt{2\pi}(1+e^{-2b_1b_2^2})} \sum_i \frac{1}{\sqrt{-2(E_2 + C_2^i)}} \\ &\times e^{C_0^i - \frac{(E_1 + C_1^i - \frac{i(x+y)}{2})^2}{4(E_2 + C_2^i)}}. \end{aligned} \quad (\text{A23})$$

Steady-state solution

In the limit $t = 0$, E_2 will diverge for all values of K not equal to 0. Therefore, we only need to consider the $K = 0$ terms: the density matrix is diagonalized in momentum at long times, with no preferred value of u . This effectively ensures it will become smoothed out in position, with no preferred value of u , in order to satisfy the uncertainty principle. The long-time solution ($t \rightarrow \infty$) and $K = 0$ is

$$\begin{aligned} \rho(0, v, \infty) &= \frac{e^{E_0(\infty)}}{2\sqrt{2\pi}(1 + e^{-2b_1 b_2^2})} \sum_i \exp C_0^i(\infty) \\ &= \frac{1}{\sqrt{2\pi}} e^{-\frac{MkT}{2(1+2R)\hbar^2} v^2}. \end{aligned} \quad (\text{A24})$$

We note that when $R \rightarrow 0$, this returns to the Caldeira-Leggett steady-state solution.

APPENDIX B: WIGNER TRANSFORM OF THE NON-MARKOVIAN MASTER EQUATION

We want to transform the following master equation:

$$\frac{\partial}{\partial t} \tilde{\rho}(x, y, t) = [\hat{L}_M + R_\Omega(\hat{L}_O + \hat{L}_V + \hat{L}_R)] \tilde{\rho}(x, y, t) \quad (\text{B1})$$

into the corresponding Wigner equation. In Eq. (B1), the Markovian Liouvillian is given by Caldeira and Leggett [35] as

$$\begin{aligned} \hat{L}_M &= \frac{i\hbar}{2M} \left(\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} \right) - \frac{i}{\hbar} [v_R(x) - v_R(y)] \\ &\quad - \gamma(x-y) \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial y} \right) - \frac{2M\gamma kT}{\hbar^2} (x-y)^2, \end{aligned} \quad (\text{B2})$$

and the new, non-Markovian Liouvillians are

$$\hat{L}_O = -\frac{4ikT}{\hbar} (x-y) \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} \right), \quad (\text{B3})$$

$$\hat{L}_V = -\frac{i}{2\hbar} (x-y) \left(\frac{\partial v_R}{\partial x} + \frac{\partial v_R}{\partial y} \right), \quad (\text{B4})$$

and

$$\hat{L}_R = -2\gamma(x-y) \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial y} \right). \quad (\text{B5})$$

We have also introduced the ratio

$$R_\Omega = \frac{\gamma}{\pi\Omega}. \quad (\text{B6})$$

The orthogonal term $\hat{L}_O$ is labeled as such due to the “+” sign between the derivatives, $\hat{L}_V$ is potential dependant, and $\hat{L}_R$ is a relaxation term to highlight its similarity to the corresponding term in the Caldeira-Leggett equation.

We first transform the Liouvillians into operator representation. The Markovian term [35] becomes

$$\hat{L}_M \tilde{\rho} = \frac{1}{i\hbar} [H_R, \tilde{\rho}] - \frac{i\gamma}{\hbar} [\hat{q}, \{\hat{p}, \tilde{\rho}\}] - \frac{2M\gamma kT}{\hbar^2} [q, [\hat{q}, \tilde{\rho}]] \quad (\text{B7})$$

and the non-Markovian terms

$$\hat{L}_O \tilde{\rho} = \frac{4kT}{\hbar^2} q, [\hat{p}, \tilde{\rho}] \quad (\text{B8})$$

$$\hat{L}_V \tilde{\rho} = \frac{1}{2\hbar^2} [q, \{\tilde{\rho}, [\hat{p}, \hat{v}_R]\}] \quad (\text{B9})$$

and

$$\hat{L}_R \tilde{\rho} = -\frac{2i\gamma}{\hbar} [\hat{q}, \{\hat{p}, \tilde{\rho}\}]. \quad (\text{B10})$$

The equalities presented above can be verified by taking the matrix elements of the commutators. This is performed by multiplying out the commutator, inserting several resolutions of identity, and using the identities

$$\langle x | \hat{q} | y \rangle = x \delta(x-y), \quad (\text{B11})$$

$$\langle x | \hat{v}_R | y \rangle = \hat{v}_R(x) \delta(x-y), \quad (\text{B12})$$

$$\langle x | \hat{p} | y \rangle = -i\hbar \frac{\partial}{\partial x} \delta(x-y) = i\hbar \frac{\partial}{\partial y} \delta(x-y). \quad (\text{B13})$$

Thus,

$$\begin{aligned} \langle x | \hat{q} \hat{p} \tilde{\rho} | y \rangle &= \int dA_1 dA_2 dA_3 \langle x | \hat{q} | A_1 \rangle \langle A_1 | \hat{p} | A_2 \rangle \langle A_2 | \tilde{\rho} | y \rangle \\ &= i\hbar \int dA_1 dA_2 dA_3 x \delta(x-A_1) \\ &\quad \times \frac{\partial}{\partial A_2} \delta(A_1-A_2) \langle A_2 | \tilde{\rho} | y \rangle \\ &= -i\hbar x \frac{\partial}{\partial x} \tilde{\rho}(x, y, t). \end{aligned} \quad (\text{B14})$$

Integration by parts has been used to move the derivative from the δ function to the density matrix. All operator representations can be verified in the same way.

To cast the master equation into phase space, we perform a Wigner transformation. The Wigner function at position q and momentum p takes the form

$$W(q, p, t) = \frac{1}{\hbar} \int du e^{-\frac{i}{\hbar} pu} \left\langle q - \frac{u}{2} \left| \tilde{\rho} \right| q + \frac{u}{2} \right\rangle. \quad (\text{B15})$$

The result of a Wigner transformation of the master equation, taken to the limit $\hbar \rightarrow 0$, is the Fokker-Planck equation

$$\frac{\partial W}{\partial t} = \lim_{\hbar \rightarrow 0} [\hat{L}_M \tilde{\rho} + R_\Omega(\hat{L}_O \tilde{\rho} + \hat{L}_V \tilde{\rho} + \hat{L}_R \tilde{\rho})]_W. \quad (\text{B16})$$

To explicitly calculate the equation of motion in phase space, we need to consider the Wigner transformation of various combinations of the operators $\hat{q}$, $\hat{p}$, $\hat{v}_R$, and $\tilde{\rho}$. We begin by establishing the useful derivative identity

$$\partial_{q \pm \frac{u}{2}} = \frac{1}{2} \partial_q \pm \partial_u. \quad (\text{B17})$$

We also take as given the Wigner transform of the Markovian terms [35]:

$$\begin{aligned} (\hat{L}_M \tilde{\rho})_W &= -\frac{p}{M} \frac{\partial W}{\partial q} + \frac{\partial v_R}{\partial q} \frac{\partial W}{\partial p} \\ &\quad + 2\gamma \frac{\partial}{\partial p} (pW) + 2M\gamma kT \frac{\partial^2 W}{\partial p^2}, \end{aligned} \quad (\text{B18})$$

and note that we can infer the Wigner transform of one of the non-Markovian terms which has the same form as the Caldeira-Leggett relaxation term,

$$(\hat{L}_R \tilde{\rho})_W = 4\gamma \frac{\partial}{\partial p}(pW). \quad (B19)$$

All that is left to do is find the Wigner transforms of the two remaining terms,

$$(\hat{L}_O)_W = \frac{4kT}{\hbar^2} [q, [\hat{p}, \tilde{\rho}]]_W \quad (B20)$$

and

$$(\hat{L}_V)_W = \frac{1}{2\hbar^2} [q, \{\tilde{\rho}, [\hat{p}, \hat{v}_R]\}]_W. \quad (B21)$$

1. Wigner transform of the orthogonal term

The commutator relation in the orthogonal term is

$$[q, [\hat{p}, \tilde{\rho}]] = \hat{q}\hat{p}\tilde{\rho} - \hat{q}\tilde{\rho}\hat{p} - \hat{p}\tilde{\rho}\hat{q} + \tilde{\rho}\hat{p}\hat{q}. \quad (B22)$$

We proceed term by term finding the Wigner transforms:

$$\begin{aligned} (\hat{q}\hat{p}\tilde{\rho})_W &= \frac{1}{\hbar} \int du e^{-\frac{i}{\hbar}pu} \int dA_1 dA_2 \left\langle q - \frac{u}{2} \right| \hat{q} | A_1 \rangle \langle A_1 | \hat{p} | A_2 \rangle \left\langle A_2 | \tilde{\rho} \left| q + \frac{u}{2} \right\rangle \right. \\ &= -\frac{1}{\hbar} (i\hbar) \int du e^{-\frac{i}{\hbar}pu} \int dA_1 dA_2 \left(q - \frac{u}{2} \right) \delta \left(A_1 - q + \frac{u}{2} \right) \frac{\partial}{\partial A_2} \delta(A_2 - A_1) \left\langle A_2 | \tilde{\rho} \left| q + \frac{u}{2} \right\rangle \right. \\ &= \frac{1}{\hbar} (i\hbar) \int du e^{-\frac{i}{\hbar}pu} \left(q - \frac{u}{2} \right) \left(\frac{1}{2} \partial_q - \partial_u \right) \left\langle q - \frac{u}{2} \right| \tilde{\rho} \left| q + \frac{u}{2} \right\rangle. \end{aligned} \quad (B23)$$

The commutator structure means we can write down a corresponding expression,

$$-(\hat{p}\tilde{\rho}\hat{q})_W = -\frac{1}{\hbar} (i\hbar) \int du e^{-\frac{i}{\hbar}pu} \left(q + \frac{u}{2} \right) \left(\frac{1}{2} \partial_q - \partial_u \right) \left\langle q - \frac{u}{2} \right| \tilde{\rho} \left| q + \frac{u}{2} \right\rangle, \quad (B24)$$

by making the replacement $(q - \frac{u}{2}) \rightarrow (q + \frac{u}{2})$. Similarly for the other “commutator pair,”

$$\begin{aligned} -(\hat{q}\tilde{\rho}\hat{p})_W &= \frac{1}{\hbar} \int du e^{-\frac{i}{\hbar}pu} \int dA_1 dA_2 \left\langle q - \frac{u}{2} \right| \hat{q} | A_1 \rangle \langle A_1 | \tilde{\rho} | A_2 \rangle \left\langle A_2 | \hat{p} \left| q + \frac{u}{2} \right\rangle \right. \\ &= -\frac{1}{\hbar} (i\hbar) \int du e^{-\frac{i}{\hbar}pu} \int dA_1 dA_2 \left(q - \frac{u}{2} \right) \delta \left(A_1 - q + \frac{u}{2} \right) \langle A_1 | \tilde{\rho} | A_2 \rangle \frac{\partial}{\partial A_2} \delta(A_2 - q - \frac{u}{2}) \\ &= \frac{1}{\hbar} (i\hbar) \int du e^{-\frac{i}{\hbar}pu} \left(q - \frac{u}{2} \right) \left(\frac{1}{2} \partial_q + \partial_u \right) \left\langle q - \frac{u}{2} \right| \tilde{\rho} \left| q + \frac{u}{2} \right\rangle, \end{aligned} \quad (B25)$$

and

$$(\tilde{\rho}\hat{p}\hat{q})_W = -\frac{1}{\hbar} (i\hbar) \int du e^{-\frac{i}{\hbar}pu} \left(q + \frac{u}{2} \right) \left(\frac{1}{2} \partial_q + \partial_u \right) \left\langle q - \frac{u}{2} \right| \tilde{\rho} \left| q + \frac{u}{2} \right\rangle. \quad (B26)$$

Summing these terms gives

$$[q, [\hat{p}, \tilde{\rho}]]_W = -\frac{1}{\hbar} (i\hbar) \int du e^{-\frac{i}{\hbar}pu} u \partial_q \left\langle q - \frac{u}{2} \right| \tilde{\rho} \left| q + \frac{u}{2} \right\rangle = -(i\hbar)^2 \frac{1}{\hbar} \int du \partial_p e^{-\frac{i}{\hbar}pu} \partial_q \left\langle q - \frac{u}{2} \right| \tilde{\rho} \left| q + \frac{u}{2} \right\rangle = \hbar^2 \partial_p \partial_q W. \quad (B27)$$

Therefore,

$$(\hat{L}_O)_W = \frac{4kT}{\hbar^2} (\hbar^2 \partial_p \partial_q W) = 4kT \partial_p \partial_q W. \quad (B28)$$

2. Wigner transform of the potential term

We use the same approach to calculate the Wigner transform of the potential term [Eq. (B9)]. The calculation is slightly more involved, but it results in

$$[q, \{\tilde{\rho}, [\hat{p}, \hat{v}_R]\}] = \hat{q}\rho\hat{p}\hat{v}_R - \hat{q}\rho\hat{v}_R\hat{p} + \hat{q}\hat{p}\hat{v}_R\rho - \hat{q}\hat{v}_R\hat{p}\rho - \rho\hat{p}\hat{v}_R\hat{q} + \rho\hat{v}_R\hat{p}\hat{q} - \hat{p}\hat{v}_R\rho\hat{q} + \hat{v}_R\hat{p}\rho\hat{q}. \quad (B29)$$

However, once again we can use the commutator structure to give

$$\begin{aligned} (\hat{q}\hat{p}\hat{v}_R)_W &= \frac{1}{\hbar} \int du e^{-i\frac{pu}{\hbar}} \int dA_1 dA_2 dA_3 \left\langle q - \frac{u}{2} \left| \hat{q} \right| A_1 \right\rangle \langle A_1 | \hat{p} | A_2 \rangle \langle A_2 | \hat{p} | A_3 \rangle \left\langle A_3 | \hat{v}_R | q + \frac{u}{2} \right\rangle \\ &= \frac{1}{\hbar} (i\hbar) \int du e^{-i\frac{pu}{\hbar}} v_R \left(q + \frac{u}{2} \right) \left(q - \frac{u}{2} \right) \left(\frac{1}{2} \partial_q + \partial_u \right) \left\langle q - \frac{u}{2} \left| \hat{p} \right| q + \frac{u}{2} \right\rangle, \end{aligned} \quad (\text{B30})$$

where we have used the same process as for the orthogonal term. The other half of the commutator pair is

$$-(\hat{p}\hat{v}_R\hat{q})_W = -\frac{1}{\hbar} (i\hbar) \int du e^{-i\frac{pu}{\hbar}} v_R \left(q + \frac{u}{2} \right) \left(q + \frac{u}{2} \right) \left(\frac{1}{2} \partial_q + \partial_u \right) \left\langle q - \frac{u}{2} \left| \hat{p} \right| q + \frac{u}{2} \right\rangle. \quad (\text{B31})$$

We proceed through the rest of the terms:

$$\begin{aligned} -(\hat{q}\hat{p}\hat{v}_R\hat{p})_W &= -\frac{1}{\hbar} \int du e^{-i\frac{pu}{\hbar}} \int dA_1 dA_2 dA_3 \left\langle q - \frac{u}{2} \left| \hat{q} \right| A_1 \right\rangle \langle A_1 | \hat{p} | A_2 \rangle \langle A_2 | \hat{v}_R | A_3 \rangle \left\langle A_3 | \hat{p} | q + \frac{u}{2} \right\rangle \\ &= -\frac{1}{\hbar} (i\hbar) \int du e^{-i\frac{pu}{\hbar}} \left(q - \frac{u}{2} \right) \left(\frac{1}{2} \partial_q + \partial_u \right) \left[v_R \left(q + \frac{u}{2} \right) \left\langle q - \frac{u}{2} \left| \hat{p} \right| q + \frac{u}{2} \right\rangle \right], \end{aligned} \quad (\text{B32})$$

$$(\hat{p}\hat{v}_R\hat{p}\hat{q})_W = \frac{1}{\hbar} (i\hbar) \int du e^{-i\frac{pu}{\hbar}} \left(q + \frac{u}{2} \right) \left(\frac{1}{2} \partial_q + \partial_u \right) \left[v_R \left(q + \frac{u}{2} \right) \left\langle q - \frac{u}{2} \left| \hat{p} \right| q + \frac{u}{2} \right\rangle \right], \quad (\text{B33})$$

$$\begin{aligned} (\hat{q}\hat{p}\hat{v}_R\hat{p})_W &= \frac{1}{\hbar} \int du e^{-i\frac{pu}{\hbar}} \int dA_1 dA_2 dA_3 \left\langle q - \frac{u}{2} \left| \hat{q} \right| A_1 \right\rangle \langle A_1 | \hat{p} | A_2 \rangle \langle A_2 | \hat{v}_R | A_3 \rangle \left\langle A_3 | \hat{p} | q + \frac{u}{2} \right\rangle \\ &= -\frac{1}{\hbar} (i\hbar) \int du e^{-i\frac{pu}{\hbar}} \left(q - \frac{u}{2} \right) \left(\frac{1}{2} \partial_q - \partial_u \right) \left[v_R \left(q - \frac{u}{2} \right) \left\langle q - \frac{u}{2} \left| \hat{p} \right| q + \frac{u}{2} \right\rangle \right], \end{aligned} \quad (\text{B34})$$

$$-(\hat{p}\hat{v}_R\hat{p}\hat{q})_W = \frac{1}{\hbar} (i\hbar) \int du e^{-i\frac{pu}{\hbar}} \left(q + \frac{u}{2} \right) \left(\frac{1}{2} \partial_q - \partial_u \right) \left[v_R \left(q - \frac{u}{2} \right) \left\langle q - \frac{u}{2} \left| \hat{p} \right| q + \frac{u}{2} \right\rangle \right], \quad (\text{B35})$$

$$\begin{aligned} (\hat{q}\hat{v}_R\hat{p}\hat{p})_W &= \frac{1}{\hbar} \int du e^{-i\frac{pu}{\hbar}} \int dA_1 dA_2 dA_3 \left\langle q - \frac{u}{2} \left| \hat{q} \right| A_1 \right\rangle \langle A_1 | \hat{v}_R | A_2 \rangle \langle A_2 | \hat{p} | A_3 \rangle \left\langle A_3 | \hat{p} | q + \frac{u}{2} \right\rangle \\ &= \frac{1}{\hbar} (i\hbar) \int du e^{-i\frac{pu}{\hbar}} v_R \left(q - \frac{u}{2} \right) \left(q - \frac{u}{2} \right) \left(\frac{1}{2} \partial_q + \partial_u \right) \left\langle q - \frac{u}{2} \left| \hat{p} \right| q + \frac{u}{2} \right\rangle, \end{aligned} \quad (\text{B36})$$

$$-(\hat{v}_R\hat{p}\hat{p}\hat{q})_W = -\frac{1}{\hbar} (i\hbar) \int du e^{-i\frac{pu}{\hbar}} v_R \left(q - \frac{u}{2} \right) \left(q + \frac{u}{2} \right) \left(\frac{1}{2} \partial_q + \partial_u \right) \left\langle q - \frac{u}{2} \left| \hat{p} \right| q + \frac{u}{2} \right\rangle. \quad (\text{B37})$$

When we sum these terms, they simplify into

$$[q, \{\hat{p}, [\hat{p}, \hat{v}_R]\}]_W = \frac{1}{\hbar} (i\hbar) \int du e^{-i\frac{pu}{\hbar}} u \left\{ \frac{1}{2} \partial_q \left[v_R \left(q + \frac{u}{2} \right) + v_R \left(q - \frac{u}{2} \right) \right] + \partial_u \left[v_R \left(q + \frac{u}{2} \right) - v_R \left(q - \frac{u}{2} \right) \right] \right\} \left\langle q - \frac{u}{2} \left| \hat{p} \right| q + \frac{u}{2} \right\rangle. \quad (\text{B38})$$

We use the Taylor series to address the potential terms, leading to an infinite sum in phase space:

$$v_R \left(q \pm \frac{u}{2} \right) = \sum_{n=0}^{\infty} \frac{1}{n!} \frac{\partial^n}{\partial q^n} v_R \left(\pm \frac{u}{2} \right)^n. \quad (\text{B39})$$

We can therefore calculate the derivatives,

$$\frac{1}{2} \partial_q \left[v_R \left(q + \frac{u}{2} \right) + v_R \left(q - \frac{u}{2} \right) \right] = \sum_{n=0}^{\infty} \frac{1}{(2n)!} v_R^{(2n+1)} \left(q \right) \left(\frac{u}{2} \right)^{2n} \quad (\text{B40})$$

and

$$\partial_u \left[v_R \left(q + \frac{u}{2} \right) - v_R \left(q - \frac{u}{2} \right) \right] = \sum_{n=0}^{\infty} \frac{1}{(2n)!} v_R^{(2n+1)} \left(q \right) \left(\frac{u}{2} \right)^{2n}. \quad (\text{B41})$$

Therefore, we have

$$\begin{aligned} [q, \{\hat{p}, [\hat{p}, \hat{v}]\}]_W &= \sum_{n=0}^{\infty} \frac{8}{(2n)!} v_R^{(2n+1)} \left(q \right) \frac{1}{\hbar} \left(\frac{i\hbar}{2} \right) \int du e^{-i\frac{pu}{\hbar}} \left(\frac{u}{2} \right)^{2n+1} \left\langle q - \frac{u}{2} \left| \hat{p} \right| q + \frac{u}{2} \right\rangle \\ &= 2\hbar^2 \sum_{n=0}^{\infty} \frac{(-\hbar^2)^n}{(2n)!} \left(\frac{1}{2} \right)^{2n} \frac{\partial^{2n+1} v_R(q)}{\partial q^{2n+1}} \frac{\partial^{2n+1} W}{\partial p^{2n+1}}, \end{aligned} \quad (\text{B42})$$

and the Wigner transform of the potential term is then

$$(\hat{L}_V)_W = \sum_{n=0} \frac{(-\hbar^2)^n}{(2n)!} \left(\frac{1}{2}\right)^{2n} \frac{\partial^{2n+1} v_R(q)}{\partial q^{2n+1}} \frac{\partial^{2n+1} W}{\partial p^{2n+1}}. \quad (\text{B43})$$

3. The Fokker-Planck equation

Collecting all the terms derived in the previous sections, we see that

$$(\hat{L}_M \tilde{\rho})_W = -\frac{p}{M} \frac{\partial W}{\partial q} + \frac{\partial v_R}{\partial q} \frac{\partial W}{\partial p} + 2\gamma \frac{\partial}{\partial p} (pW) + D \frac{\partial^2 W}{\partial p^2}, \quad (\text{B44})$$

$$(\hat{L}_O \tilde{\rho})_W = 4kT \frac{\partial^2 W}{\partial p \partial q}, \quad (\text{B45})$$

$$(\hat{L}_V \tilde{\rho})_W = \sum_{n=0} \frac{1}{(2n)!} (-\hbar^2)^n \left(\frac{1}{2}\right)^{2n} \frac{\partial^{2n+1} v_R(q)}{\partial q^{2n+1}} \frac{\partial^{2n+1} W}{\partial p^{2n+1}}, \quad (\text{B46})$$

$$(\hat{L}_R \tilde{\rho})_W = 4\gamma \frac{\partial}{\partial p} (pW). \quad (\text{B47})$$

In the limit $\hbar \rightarrow 0$, the potential term becomes

$$\lim_{\hbar \rightarrow 0} (\hat{L}_V \tilde{\rho})_W = \frac{\partial v_R(q)}{\partial q} \frac{\partial W}{\partial p}, \quad (\text{B48})$$

Thus, we have a Fokker-Planck equation of the form

$$\begin{aligned} \frac{\partial W}{\partial t} = & -\frac{p}{M} \frac{\partial}{\partial q} W + (1 + R_\Omega) \frac{\partial v_R(q)}{\partial q} \frac{\partial W}{\partial p} \\ & + 2\gamma(1 + 2R_\Omega) \frac{\partial}{\partial p} (pW) \\ & + 2M\gamma kT \frac{\partial^2 W}{\partial p^2} + 4kTR \frac{\partial^2 W}{\partial p \partial q}. \end{aligned} \quad (\text{B49})$$

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