

Partial quantization of non-Hermitian Berry phases in time-varying media

Calvin M. Hooper^{*}

School of Physics and Astronomy, [University of Exeter](#), Stocker Road, Exeter EX4 4QL, United Kingdom



(Received 12 May 2026; accepted 8 October 2026; published 20 October 2026)

A fundamental symmetry of the non-Hermitian operators describing wave propagation in time-varying media imbues such systems with nontrivial topology. This topology may be measured directly in a wide range of experimental settings as a quantized real part of the Berry phase, contrasting unconstrained geometric gain or loss. This topological invariant is provided explicitly for practical examples, including a non-Hermitian analog of the Su-Schrieffer-Heeger model.

DOI: [10.1103/92c1-dd6f](#)

I. INTRODUCTION

As a wave propagates through a material, sudden changes to material properties (faster than a period of the wave) affect it in ways impossible in slowly varying media [1], realizing temporal reflection [2,3], frequency conversion [4,5], or amplification [6,7]. Mathematically, this complexity is succinctly encoded by promoting scalar quantities such as the wave number to operators encoding coupling between different frequencies [8].

While the addition of operators results in all but the simplest problems becoming analytically intractable [9,10], this field is hardly the first to experience the difficulties of operator differential equations. Indeed, operators are the bane of many an undergraduate's introduction to quantum mechanics [11], where gaps in intuition host phenomena from the quantum Hall effect [12,13] to many-body scarring [14,15].

It thus seems natural to draw analogy between the study of classical waves in time-varying media, and quantum mechanics. Even in static media, such analogies form a basis for optical perturbation theory [16,17]. However, time-varying media strengthen this link hugely, naturally realizing [18] effects equivalent to $\mathcal{PT}$ symmetry [19,20], or being tuned to study field theories with synthetic dimensions [21,22].

However, overemphasizing this link hides an important difference between the operators of time-varying media and those of quantum mechanics: The former are typically non-Hermitian. Such non-Hermitian operators naturally exhibit phenomena neglected in standard quantum mechanics, but discussed frequently in its non-Hermitian generalizations [19,23]. Most notable of these effects, and the focus of this work, is the emergence of defective (nondiagonalizable) operators [24].

In this paper, we study topology in the inherently non-Hermitian setting of time-varying media, where boundaries

between adiabatically disconnected regions are no longer demarcated solely by degenerate operators, but also by defective operators [25] (serving intuitively as a non-Hermitian generalization of the former). However, adiabatically impassable boundaries provide topological results only if they cannot be circumvented. In Hermitian settings, such circumvention is prevented by imposing additional symmetries [26], with experimental observation of topological effects being limited by how accurately these symmetries are realized.

Symmetry is a similarly key component of topology in non-Hermitian settings [27], and our study is motivated by an inherent symmetry of such media: The operators describing them must map real-valued fields to real-valued fields, a feature referred to in the frequency domain as $\mathcal{RC}$ symmetry [18,28]. While this symmetry is spontaneously broken in static media, time-varying media allow it to remain unbroken. In this paper, we demonstrate that this unbroken symmetry equips time-varying media with a robust and nontrivial non-Hermitian topology, observed via a quantization in the real part of a complex Berry phase [29] (Fig. 1).

Such complex Berry phases have long been studied in the context of non-Hermitian physics [30], and similar symmetry-protected quantization results have even been achieved [31,32]. However, these results rely on heavily constrained systems (2×2 Hamiltonians in Ref. [31], and limitations to the adiabatic path in Ref. [32]). By contrast, our results generalize to arbitrarily high-dimensional systems, and require no external engineering to impose $\mathcal{RC}$ symmetry (especially relevant in high-dimensional systems, requiring precise, stable, and independent control of many parameters).

Just as the Berry phase was originally predicted [29] as a geometric phase evolved by a state as parameters of its system were varied arbitrarily slowly in time, we retrieve the complex Berry phase via arbitrarily slow spatial variations. Indeed, while mere slow parameter variations are insufficient to ensure adiabaticity in non-Hermitian systems [33–35], we show that the link between $\mathcal{RC}$ symmetry and gain [18] is sufficient to retrieve adiabaticity for the modes we consider.

Existing works studying topology in time-varying media typically either use time variations as an interesting way of breaking time-reversal symmetry and inducing chirality [36,37], or relate space-time structures directly to static

^{*}Contact author: ch1122@exeter.ac.uk

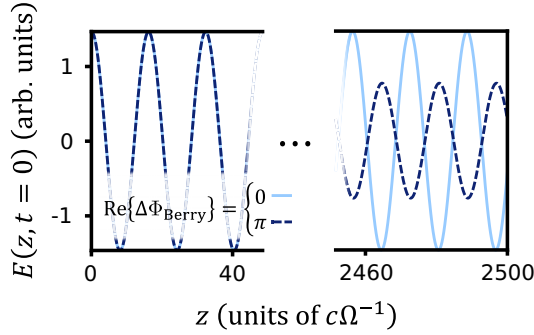


FIG. 1. Waves exhibiting different “partially quantized” complex Berry phases: Between two ends of a long time-varying cavity, waves may pick up a geometric phase (denoted $\text{Re}\{\Delta\Phi_{\text{Berry}}\}$) of either π (dashed) or 0 (undashed). In addition, such waves may experience an unquantized degree of geometric loss or gain, corresponding to a change in amplitude between ends of the cavity. The particular model used is the same as in Fig. 2.

analogous to obtain topological invariants [38–41]. By contrast, this work directly leverages a symmetry inherent and specific to time-varying systems (the natural $\mathcal{RC}$ symmetry) to observe uniquely temporal topological phenomena.

Furthermore, in constructing toy models to demonstrate this phenomenon, we will also provide, as a corollary, an alternative proof of Zak phase quantization in the Su-Schrieffer-Heeger (SSH) model [42–44], and the π phase shifts when encircling Dirac points in two-dimensional (2D) materials [45–47].

II. $\mathcal{RC}$ SYMMETRY AND ADIABATIC TOPOLOGY

As a sufficient example to demonstrate the inherent topology of time-varying media, and to prove the partial quantization condition in general, we consider the Berry phase [29,42] evolved between the ends of a long time-varying slab with some slow spatial variation.

In contrast to static media, such systems no longer preserve the frequencies of input waves. Thus, the Fourier transform no longer converts the 1 + 1-dimensional wave equation to a scalar differential equation. However, if one carries out the Fourier transform regardless, one obtains [8] the operator-valued differential equation

$$\frac{d^2 \tilde{E}(z)}{dz^2} = -\hat{K}^2(z/L) \tilde{E}(z), \quad (1)$$

where $\tilde{E}(z)$ is to be understood as an infinite-dimensional vector containing the frequency spectrum of the electric field at a one-dimensional position z . $\hat{K}(z/L)$ is an operator generalization of the wave number acting on this vector to convert between different frequencies as waves propagate through space. Its form is determined by an operator-valued dispersion relation, such as $\hat{K} = [\hat{\omega}^2(1 + \hat{\chi})]^{1/2}$, where $\hat{\omega}$ and $\hat{\chi}$ are frequency and susceptibility operators (see Refs. [8,18,28]), with the branch cut such that none of its eigenvalues lie in the lower half plane. It is further taken to be a (differentiable) function of z/L , thus describing a profile of material parameters stretched over $z \in [0, L]$ (we focus particularly on the adiabatic limit of $L \rightarrow \infty$). Finally, $\hat{K}$ is taken as equal for all $z \notin (0, L)$.

We now derive the key result of this paper: that topology arises from the fact that $\tilde{E}$ is the Fourier transform of a real-valued wave, implying $\tilde{E}(\omega) = \tilde{E}^*(-\omega)$. In the language of $\mathcal{PT}$ symmetry [19,20], this may be understood as a composite symmetry comprising two separate operations, frequency reflection $\mathcal{R}$ and complex conjugation $\mathcal{C}$, and will henceforth be referred to as $\mathcal{RC}$ symmetry.

Any operator (such as $\hat{K}^2$) mapping between the spectra of real fields thus obeys the symmetry

$$\mathcal{RC} \hat{K}^2 \mathcal{RC} = \hat{K}^2. \quad (2)$$

By applying this relation to the eigenvalue problem $k^2 |k^2\rangle = \hat{K}^2 |k^2\rangle$, eigenvectors and eigenvalues are found to fall into one of the following classes [18–20]:

(1) Symmetry-broken pairs: $\text{Im}\{k^2\} \neq 0$. There is a paired eigenvalue $\kappa^2 = (k^2)^*$ with eigenvector $\mathcal{RC}|k^2\rangle$.

(2) Symmetry-unbroken modes: $\text{Im}\{k^2\} = 0$, and k^2 has no degeneracy. $|k^2\rangle$ can be chosen such that $\mathcal{RC}|k^2\rangle = |k^2\rangle$, corresponding to a real-valued time-domain field. Such “ $\mathcal{RC}$ -symmetric” waves are unique to time-varying media [18].

(3) Degenerate modes: $\text{Im}\{k^2\} = 0$, and k^2 has an algebraic degeneracy greater than 1. These form the boundary between the preceding classes.

In the space of Hermitian operators, one may define a topology where two operators are connected if and only if an adiabatic path exists between them. By construction, the Berry phase is defined only for continuous paths in this adiabatic topology [29]. This is broken, however, when two eigenvalues collide, as in case (3). Similar eigenvalue collisions also cause the breakdown of the WKB approximation about points of zero wave number [48].

However, in general non-Hermitian systems, adiabatic evolution is physically protected only for the slowest-decaying eigenvalues [33–35]. With the above classification, it is clear that $\mathcal{RC}$ -symmetric waves must either be evanescent ($\text{Re}\{k\} = 0$) or extended ($\text{Im}\{k\} = 0$). With this in mind, we define two systems as being $\mathcal{RC}$ -adiabatically connected if and only if there exists a path between the two where all extended waves evolve adiabatically. It can be immediately seen that symmetry-breaking transitions cause regions to become disconnected.

The relevant adiabatic approximation is derived in Appendix A, following the adiabatic propagation of an extended $\mathcal{RC}$ -symmetric wave,

$$\tilde{E}_{\text{ad}}(z) = \sqrt{\frac{k(0)}{k(z)}} e^{i\Delta\phi_{\text{Berry,RC}}(z)} \cos[\phi_{\text{dyn}}(z)] |z, \text{RC}\rangle, \quad (3)$$

where

$$\phi_{\text{dyn}}(z) = \int_0^z k(z) dz, \quad (4)$$

$$\Delta\phi_{\text{Berry,RC}}(z) = \int_0^z \langle z, \text{RC} | i \frac{d}{dz} | z, \text{RC} \rangle dz. \quad (5)$$

Here, $|z, \text{RC}\rangle$ is taken as the (continuously varying) instantaneous $\mathcal{RC}$ -symmetric eigenvector of the operator $\hat{K}^2(z/L)$, with a corresponding positive eigenvalue $k^2(z)$, and a corresponding left eigenvector $\langle z, \text{RC} |$, normalized such that $\langle z, \text{RC} | z, \text{RC} \rangle = 1$.

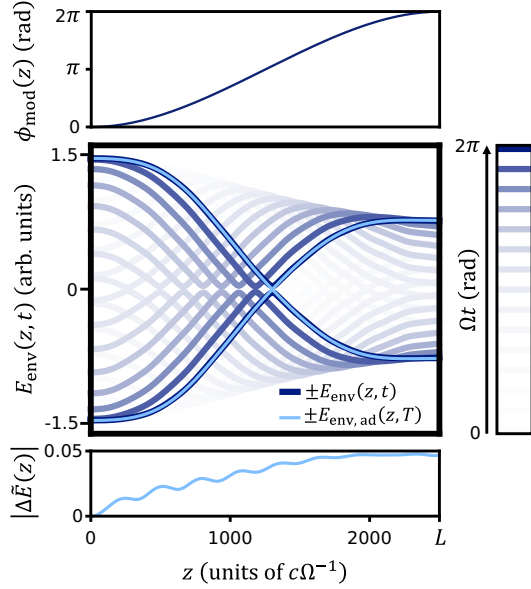


FIG. 2. The adiabatic approximation and numerical validation: This example considers waves evolving from the left-hand side of a time crystal with modulation frequency Ω , and a modulation phase $\phi_{\text{mod}}(z)$ varying slowly in space (top panel). The precise parameters are the same as in Refs. [18,28]. Middle panel: The numerically calculated envelope evolving over time (thick lines), and the corresponding adiabatic approximation at the final time (thinner line, overlaid) of an $\mathcal{RC}$ -symmetric wave in a large cavity, with time evolution pictured as a fading afterimage. Geometric loss is demonstrated by the reduction in the height of the wave across the cavity, while a π geometric phase is revealed by the node in the adiabatic approximation. This does not form a true node in the numerical solution, due to scattering to frequencies outside of the adiabatic approximation, decreasing to 0 as $L \rightarrow \infty$. Bottom panel: The total error in the spectra of the adiabatic approximation as a function of position, in the same units as the middle panel, demonstrating the accuracy of the adiabatic approximation.

This adiabatic evolution is intuitively simple, separating into three terms: $\sqrt{\frac{k(0)}{k(z)}}$ is the geometric term of the conventional WKB approximation; $\exp[i\Delta\phi_{\text{Berry,RC}}(z)]$ contains the Berry phase associated with the changing eigenvector $|z, \text{RC}\rangle$; and $\cos[\phi_{\text{dyn}}(z)]$ corresponds to the standing wave in space typically of $\mathcal{RC}$ -symmetric waves (as noted in Ref. [18]). Furthermore, one notes that the divergence at any $k(z) = 0$ ensures that an extended $\mathcal{RC}$ -symmetric wave is never adiabatically connected to an evanescent wave.

Both this approximation, and its numerical validation, are plotted in Fig. 2, verifying its accuracy for large L .

Although we present these results for $\mathcal{RC}$ -symmetric eigenvectors (as a convenient and natural generalization of the parallel transport gauge), the Berry phase maintains its gauge invariance from quantum mechanics, with the derivation of the geometric phase in Appendix A holding in any differentiable gauge.

Since preservation of $\mathcal{RC}$ symmetry is a fundamental property of wave propagation under Eq. (1), and with the adiabatic approximation becoming increasingly accurate for large L , it is necessary that $\tilde{E}_{\text{ad}}(z)$ must come arbitrarily close to

realizing this symmetry for large L . Thus inspired, we explicitly find the $\mathcal{RC}$ conjugate of (3):

$$\mathcal{RC}\tilde{E}_{\text{ad}}(z) = \sqrt{\frac{k(0)}{k(z)}} (e^{i\Delta\phi_{\text{Berry,RC}}(z)})^* \cos[\phi_{\text{dyn}}(z)] |z, \text{RC}\rangle. \quad (6)$$

Since $\Delta\phi_{\text{Berry,RC}}(z)$ is entirely independent of L [by Eq. (5)], this symmetry can only hold as $L \rightarrow \infty$ if $\text{Im}\{\exp[i\Delta\phi_{\text{Berry,RC}}(z)]\} = 0$. Furthermore, since $\Delta\phi_{\text{Berry,RC}}(0) = 0$, before varying continuously with z , this may be strengthened to $\text{Re}\{\Delta\phi_{\text{Berry,RC}}(z)\} = 0$.

However, this term does not encode the full phase difference between $\tilde{E}_{\text{ad}}(0)$ and $\tilde{E}_{\text{ad}}(L)$. Despite the fact that $|0, \text{RC}\rangle$ and $|L, \text{RC}\rangle$ are both eigenvectors of $\hat{K}^2(0) = \hat{K}^2(L)$ with the same eigenvalue, the requirement that $|z, \text{RC}\rangle$ be differentiable for all z may constrain them such that $\langle 0, \text{RC} | L, \text{RC} \rangle \neq 1$ [an example is later given by Eq. (16)]. However, even if this is the case, the fact that both correspond to real time-domain fields requires that $\text{Im}\{\langle 0, \text{RC} | L, \text{RC} \rangle\} = 0$.

Thus, the total geometric phase difference is given by

$$\Phi_{\text{Berry}} = \Delta\phi_{\text{Berry,RC}}(L) - i \ln(\langle 0, \text{RC} | L, \text{RC} \rangle), \quad (7)$$

resulting in the partial quantisation condition

$$\text{Re}\{\Phi_{\text{Berry}}\} = 0 \pmod{\pi}. \quad (8)$$

In contrast to the familiar Berry phase from quantum mechanics, Φ_{Berry} is no longer a real number. As a result, Eq. (8) no longer represents a complete quantisation of the Berry phase onto a discrete set of points, instead presenting an analogous non-Hermitian generalization: its partial quantization onto lines of constant real parts.

III. EXPLICIT EXAMPLES

A. 2×2 Matrices

As the simplest nontrivial example, consider a photonic time crystal with modulation frequency $\Omega > 0$, where coupling only occurs between the two frequencies $\pm \frac{\Omega}{2}$. Within this space, $\hat{K}^2(z)$ takes the form [18]

$$\hat{K}^2(z) = \left(\frac{\Omega}{2c}\right)^2 \begin{pmatrix} 1 + \chi_0^*(z) & \chi_1(z) \\ \chi_1^*(z) & 1 + \chi_0(z) \end{pmatrix}, \quad (9)$$

where χ_0 encodes the average susceptibility at a frequency $\Omega/2$, while χ_1 encodes the scale of the temporal modulation. A key feature of note in this system is that shifting the modulation in time by an arbitrarily phase results in a shift in the phase of χ_1 . This is in stark contrast to the phase of χ_0 , which has far more direct physical implications, encoding the intrinsic static gain or loss in the system.

A straightforward calculation predicts that collisions between eigenvalues occur when

$$\text{Im}\{\chi_0\} = \pm |\chi_1|. \quad (10)$$

Thus, the adiabatically distinct regions in the parameter space of $\hat{K}^2$ may be understood simply in terms of the variables $(\text{Re}\{\chi_1\} \text{ Im}\{\chi_1\} \text{ Im}\{\chi_0\}) \in \mathbb{R}^3$, forming the interior (exterior) of a double cone, corresponding to waves with spontaneously broken (unbroken) $\mathcal{RC}$ symmetry (see Fig. 3).

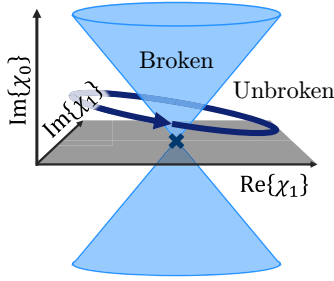


FIG. 3. Adiabatically distinct regions of broken and unbroken symmetry in the parameter space of a 2×2 $\mathcal{RC}$ -symmetric operator: Explicit analysis of Eq. (9) reveals that eigenvalue collisions in the 2×2 square wave-number operator are determined entirely by Fourier components of the susceptibility. In particular, symmetry-breaking occurs when the average loss or gain $\text{Im}\{\chi_0\}$ exceeds its time variation $|\chi_1|$, as is the case in all static media. Plotted, these regions form the interior of a double cone, in which all loops in parameter space are contractible. By contrast, there exist noncontractible loops in regions of unbroken symmetry. Quantization of $\text{Re}\{\Phi_{\text{Berry}}\}$ is thus determined solely by a winding number around the double cone.

Since a continuous deformation of an adiabatic loop in parameter space must result in a continuous change to the Berry phase, any contractible loop must obey $\text{Re}\{\Phi_{\text{Berry}}\} = 0$.

Following this continuity argument, determining the relationship between winding number and Berry phase requires only a single example for each winding number n . To this end, we consider the example of $\chi_0 = 0$, and $\chi_1 = \exp(2n\pi iz/L)$. $\hat{K}^2(z)$ is then Hermitian, with eigenvectors given by

$$|z\rangle_{\text{RC}} = \begin{pmatrix} \sqrt{\pm 1} \exp(n\pi iz/L) \\ \pm \sqrt{\pm 1} \exp(-n\pi iz/L) \end{pmatrix}. \quad (11)$$

Substituting these eigenvectors into Eq. (7), the direct relation between winding number and Berry phase becomes

$$\text{Re}\{\Phi_{\text{Berry},n}\} = n\pi \pmod{2\pi}. \quad (12)$$

B. Many dimensions

Explicit calculations are not limited to finite-dimensional examples. Instead, consider a time-varying system where the phase of some general periodic modulation with frequency Ω , slowly shifts in space, by a time $\Delta t = \frac{2n\pi}{\Omega} \frac{z}{L}$. Evaluating the Berry phase in this system is made tractable by noting that the result of shifting the modulation in time must result in equivalent shifts in eigenvectors.

To make this shift explicit, consider the $\mathcal{RC}$ -symmetric eigenvectors of $\hat{K}^2(z=0)$ for a particular eigenvalue k^2 , where $|\omega\rangle_{\mathcal{F}}$ denotes a Fourier component $e^{-i\omega t}$ in the time domain. These are naturally split into two categories: those containing frequencies at integer multiples of Ω (henceforth referred to as even eigenvectors)

$$|k^2; z=0\rangle_{\text{Even}} = A_0|0\rangle_{\mathcal{F}} + \sum_{m=1}^{\infty} (A_m|m\Omega\rangle_{\mathcal{F}} + A_m^*|-m\Omega\rangle_{\mathcal{F}}), \quad (13)$$

with $A_0 \in \mathbb{R}$, and those containing frequencies at odd-half-integer multiples of Ω (henceforth referred to as odd eigenvectors)

$$|k^2; z=0\rangle_{\text{Odd}} = \sum_{m=1}^{\infty} B_m|(m-1/2)\Omega\rangle_{\mathcal{F}} + \sum_{m=1}^{\infty} B_m^*|-(m-1/2)\Omega\rangle_{\mathcal{F}}. \quad (14)$$

With these *Ansätze*, the result of shifting the modulation by a time $\Delta t = \frac{2n\pi}{\Omega} \frac{z}{L}$ becomes explicit. In the even case,

$$|k^2; z\rangle_{\text{Even}} = A_0|0\rangle_{\mathcal{F}} + \sum_{m=1}^{\infty} A_m e^{2m\pi iz/L} |m\Omega\rangle_{\mathcal{F}} + \sum_{m=1}^{\infty} A_m^* e^{-2m\pi iz/L} |-m\Omega\rangle_{\mathcal{F}}, \quad (15)$$

and in the odd case,

$$|k^2; z\rangle_{\text{Odd}} = \sum_{m=1}^{\infty} B_m e^{(2m-1)\pi iz/L} |(m-1/2)\Omega\rangle_{\mathcal{F}} + \sum_{m=1}^{\infty} B_m^* e^{-(2m-1)\pi iz/L} |-(m-1/2)\Omega\rangle_{\mathcal{F}}. \quad (16)$$

One may then immediately verify that $|k^2; 0\rangle_{\text{Even}} = |k^2; L\rangle_{\text{Even}}$, and $|k^2; 0\rangle_{\text{Odd}} = (-1)^n |k^2; L\rangle_{\text{Odd}}$, which may then be substituted into Eq. (7), to find the general partial quantizations

$$\text{Re}\{\Phi_{\text{Berry,Even}}\} = 0 \pmod{2\pi}, \quad (17a)$$

$$\text{Re}\{\Phi_{\text{Berry,Odd}}\} = n\pi \pmod{2\pi}, \quad (17b)$$

which, as previously, characterize not only these particular trajectories, but any adiabatically connected to them.

IV. CONCLUSION

The loss in intuition as familiar quantities fail to commute is not a difficulty unique to time-varying media, and nor are its particular complexities disjoint from all other fields: Tools from general relativity [4] and quantum mechanics [8] have already been applied to solve time-varying problems. Thus, this paper considered how ideas of topology originating from quantum mechanics translate to the non-Hermitian setting of time-varying media. We found that the real-valued nature of classical waves guarantees that the operators describing their propagation possess sufficient symmetry to realize nontrivial topology, and we described how this topology may be characterized by a simple Berry phase measurement.

Nor are these results limited to the context of spatial propagation in which they were presented. Time-domain analyses [28,49–51] obey the same $\mathcal{RC}$ symmetry [28], raising the possibility of observing topological effects in even a single time-varying resonator [52], with a nearly periodic, but slowly drifting, modulation (see Appendix B). This generality should make these results far easier to apply in experimental settings.

Finally, we note that, while we have identified an important symmetry and a number of experimentally accessible topo-

logical consequences, we have not attempted to present an exhaustive characterization of this topology. Thus, a promising avenue for future studies is to consider the range of existing work on non-Hermitian topology [27,53–55], and to see which of those phenomena have analogs protected by $\mathcal{RC}$ symmetry, and may thus be realized in time-varying media.

ACKNOWLEDGMENT

C.M.H. thanks Simon Horsley, Ian Hooper, Oliver Breach, and James Walkling for useful discussions, and acknowledges financial support from the Engineering and Physical Sciences Research Council (EPSRC) of the UK via the Exeter University Physics DTP, and via the META4D Programme (Grant No. EP/Y015673/1).

DATA AVAILABILITY

The data that support the findings of this article are openly available [56].

APPENDIX A: ADIABATIC APPROXIMATION FOR LOSSY WAVE PROPAGATION

This appendix contains a quick derivation of Eq. (3). To begin, note that Eq. (1) may be written in matrix form as

$$\frac{d}{d\xi} \begin{pmatrix} \tilde{\Psi}(\xi) \\ \partial_\xi \tilde{\Psi}(\xi) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\hat{K}^2(\xi)L^2 & 0 \end{pmatrix} \begin{pmatrix} \tilde{\Psi}(\xi) \\ \partial_\xi \tilde{\Psi}(\xi) \end{pmatrix}, \quad (\text{A1})$$

with $\xi = z/L$.

Our focus in this appendix will then be directed towards approximating an evolution operator for the above equation, given by

$$\frac{d\hat{U}(\xi;0)}{d\xi} = \begin{pmatrix} 0 & 1 \\ -\hat{K}^2(\xi)L^2 & 0 \end{pmatrix} \hat{U}(\xi;0), \quad (\text{A2})$$

$$\hat{U}(0;0) = 1. \quad (\text{A3})$$

The adiabatic approximation will proceed by considering the instantaneous diagonalization

$$\begin{pmatrix} 0 & 1 \\ -\hat{K}^2(\xi)L^2 & 0 \end{pmatrix} = [\hat{X}_W(\xi)\hat{X}_K(\xi)] \begin{pmatrix} i\hat{D}_K L(\xi) & 0 \\ 0 & -i\hat{D}_K L(\xi) \end{pmatrix} \times [\hat{X}_K^{-1}(\xi)\hat{X}_W^{-1}(\xi)], \quad (\text{A4})$$

where $\hat{X}_K(\xi)\hat{D}_K(\xi)\hat{X}_K^{-1}(\xi) = \hat{K}(\xi)$, and

$$\hat{X}_W(\xi) = \begin{pmatrix} 1 & 0 \\ 0 & i\hat{K}(\xi)L \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}. \quad (\text{A5})$$

It is then straightforward to find that

$$\begin{aligned} \frac{d\tilde{U}(\xi;0)}{d\xi} &= \begin{pmatrix} i\hat{D}_K L(\xi) & 0 \\ 0 & -i\hat{D}_K L(\xi) \end{pmatrix} \tilde{U}(\xi;0) \\ &\quad - \hat{X}_K^{-1}(\xi)\hat{X}_W^{-1}(\xi) \frac{d\hat{X}_W(\xi)}{d\xi} \hat{X}_K(\xi) \tilde{U}(\xi;0) \\ &\quad - \hat{X}_K^{-1}(\xi) \frac{d\hat{X}_K(\xi)}{d\xi} \tilde{U}(\xi;0), \end{aligned} \quad (\text{A6})$$

where

$$\tilde{U}(\xi;0) = \hat{X}_K^{-1}(\xi)\hat{X}_W^{-1}(\xi)\hat{U}(\xi;0)\hat{X}_K(0)\hat{X}_W(0). \quad (\text{A7})$$

The adiabatic approximation then arises when one neglects the off-diagonal components on the coefficients of $\tilde{U}(\xi;0)$. To do so gives rise to the following equation,

$$\begin{aligned} \frac{d\tilde{U}_{\text{ad}}(\xi;0)}{d\xi} &= \begin{pmatrix} i\hat{D}_K L(\xi) & 0 \\ 0 & -i\hat{D}_K L(\xi) \end{pmatrix} \tilde{U}_{\text{ad}}(\xi;0) \\ &\quad + \hat{D}_{\text{WKB}}(\xi) \tilde{U}_{\text{ad}}(\xi;0) \\ &\quad + i\hat{D}_{\text{Berry}}(\xi) U_{\text{ad}}(\xi;0), \\ \tilde{U}_{\text{ad}}(0;0) &= 1, \end{aligned} \quad (\text{A8})$$

where

$$\hat{D}_{\text{WKB}}(\xi) = -\frac{1}{2} \frac{d \ln[\hat{D}_K(\xi)]}{d\xi}, \quad (\text{A9})$$

$$\hat{D}_{\text{Berry}}(\xi) = \text{ondiag} \left(\hat{X}_K^{-1}(\xi) i \frac{d}{d\xi} \hat{X}_K(\xi) \right). \quad (\text{A10})$$

This equation may be simply solved for $\tilde{U}_{\text{ad}}(\xi;0)$ to give

$$\tilde{U}_{\text{ad}}(\xi;0) = \tilde{U}_{\text{WKB}}(\xi;0) \tilde{U}_{\text{Berry}}(\xi;0) \quad (\text{A11})$$

$$\times \begin{pmatrix} \tilde{U}_{\text{dyn},+}(\xi;0) & 0 \\ 0 & \tilde{U}_{\text{dyn},-}(\xi;0) \end{pmatrix}, \quad (\text{A12})$$

where

$$\tilde{U}_{\text{dyn},\pm}(\xi;0) = \exp \left[\pm iL \int_0^\xi \hat{D}_K(\xi') d\xi' \right], \quad (\text{A13})$$

$$\tilde{U}_{\text{WKB}}(\xi;0) = [\hat{D}_K(0)]^{1/2} [\hat{D}_K(\xi)]^{-1/2}, \quad (\text{A14})$$

$$\tilde{U}_{\text{Berry}}(\xi;0) = \exp \left[i \int_0^\xi \hat{D}_{\text{Berry}}(\xi') d\xi' \right]. \quad (\text{A15})$$

Since $\tilde{U}_{\text{ad}}(\xi;0)$ approximates $\tilde{U}(\xi;0)$, by inverting Eq. (A7) one immediately finds the approximation $\hat{U}_{\text{ad}}(\xi;0)$ to $\hat{U}(\xi;0)$. The result presented in Eq. (3) is found by evaluating

$$\hat{\Psi}_{\text{ad}}(\xi;L) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \cdot \hat{U}_{\text{ad}}(\xi;0) \begin{pmatrix} 1 \\ 0 \end{pmatrix} \frac{1}{\sqrt{k(0)}} |0, \text{RC}\rangle. \quad (\text{A16})$$

APPENDIX B: ADIABATICITY IN FLOQUET QUASINORMAL MODE PROBLEMS

Consider the evolution of a wave within a cavity. In general, this may be phrased as

$$\left(\frac{d}{dt} + M(t, \tau) \right) \Psi(t) = 0, \quad (\text{B1})$$

where $\Psi(t)$ is a Hilbert space vector describing the mode of a cavity as a function of a time t , while $M(t, \tau)$ is an operator acting on this space to determine its time evolution, written explicitly in terms of a time variable t , and a slowly varying parameter $\tau = \epsilon t$, for $\epsilon \ll 1$.

Into this, one may substitute the *Ansatz*

$$\Psi(t) = \int_{-\infty}^{\infty} \tilde{\Psi}(\omega, \tau) e^{-i\omega t} \frac{d\omega}{2\pi}. \quad (\text{B2})$$

Noting that $\frac{d}{dt}$ may be written formally as $\frac{\partial}{\partial t} + \epsilon \frac{\partial}{\partial \tau}$, and following the techniques of Ref. [8], Eq. (B1) may be rewritten in the form

$$\left(\frac{\partial}{\partial \tau} + \epsilon^{-1} [-i\hat{\omega} + \hat{M}(\tau)] \right) |\tilde{\Psi}(\tau)\rangle = 0, \quad (\text{B3})$$

where $\hat{M}$ and $\hat{\omega}$ both act upon the spectrum vector $|\tilde{\Psi}(\tau)\rangle$. Note that, despite following the notation of multiple scale analysis, this rearrangement is exact.

Recalling the condition that $\epsilon \ll 1$, the adiabatic approximation then provides a good description of the evolution of the slowest-decaying instantaneous eigenmodes

$$[-i\hat{\omega} + \hat{M}(\tau)] |\tilde{\Psi}(\tau)\rangle_{\text{ins}} = i\omega_0 |\tilde{\Psi}(\tau)\rangle_{\text{ins}}, \quad (\text{B4})$$

which are precisely the modes of the system when τ is taken as a constant. Since $M(t, \tau)$ and $\frac{\partial}{\partial t}$ map between real-valued fields in the time domain, it follows that $\hat{M}$ and $i\hat{\omega}$ obey $\mathcal{RC}$ symmetry, and thus that the partial quantization derived in this paper applies whenever an $\mathcal{RC}$ -unbroken $|\tilde{\Psi}(\tau)\rangle_{\text{ins}}$ is inserted into the system.

In the case of nearly periodic media, $M(t, \tau) = M(t + \frac{2\pi}{\Omega}, \tau)$, and the instantaneous eigenmodes become precisely the Floquet quasinormal modes studied in [28,49–51]. Furthermore, it was demonstrated in Ref. [28] that $\mathcal{RC}$ -symmetric modes are ubiquitous in such systems, providing a practical alternative to achieving $\mathcal{RC}$ -unbroken modes in $\hat{K}^2$. Indeed, although $\mathcal{RC}$ symmetry does not immediately guarantee that an instantaneous eigenmode will be the slowest decaying, this is still true in many cases [28], resulting in the same Berry phase quantization.

-
- [1] E. Galiffi, R. Tirole, S. Yin, H. Li, S. Vezzoli, P. A. Huidobro, M. G. Silveirinha, R. Sapienza, A. Alù, and J. B. Pendry, Photonics of time-varying media, *Adv. Photonics* **4**, 014002 (2022).
 - [2] V. Bacot, M. Labousse, A. Eddi, M. Fink, and E. Fort, Time reversal and holography with spacetime transformations, *Nat. Phys.* **12**, 972 (2016).
 - [3] S. Vezzoli, V. Bruno, C. DeVault, T. Roger, V. M. Shalaev, A. Boltasseva, M. Ferrera, M. Clerici, A. Dubietis, and D. Faccio, Optical time reversal from time-dependent epsilon-near-zero media, *Phys. Rev. Lett.* **120**, 043902 (2018).
 - [4] S. A. R. Horsley and J. B. Pendry, Quantum electrodynamics of time-varying gratings, *Proc. Natl. Acad. Sci. USA* **120**, e2302652120 (2023).
 - [5] V. Pacheco-Peña and N. Engheta, Spatiotemporal cascading of dielectric waveguides [Invited], *Opt. Mater. Express* **14**, 1062 (2024).
 - [6] X. Wang, P. Garg, M. S. Mirmoosa, A. G. Lamprianidis, C. Rockstuhl, and V. S. Asadchy, Expanding momentum bandgaps in photonic time crystals through resonances, *Nat. Photonics* **19**, 149 (2025).
 - [7] A. Ganfornina-Andrades, J. E. Vázquez-Lozano, and I. Liberal, Quantum vacuum amplification in time-varying media with arbitrary temporal profiles, *Phys. Rev. Res.* **6**, 043320 (2024).
 - [8] S. A. R. Horsley, E. Galiffi, and Y.-T. Wang, Eigenpulses of dispersive time-varying media, *Phys. Rev. Lett.* **130**, 203803 (2023).
 - [9] J. J. Kovacic, An algorithm for solving second order linear homogeneous differential equations, *J. Symb. Comput.* **2**, 3 (1986).
 - [10] P. D. Lax, *Hyperbolic Partial Differential Equations*, Courant Lecture Notes (American Mathematical Society, Providence, RI, 2006).
 - [11] C. Singh, Student understanding of quantum mechanics, *Am. J. Phys.* **69**, 885 (2001).
 - [12] K. von Klitzing, Developments in the quantum Hall effect, *Philos. Trans. R. Soc. A* **363**, 2203 (2005).
 - [13] D. Tong, Lectures on the quantum Hall effect, *arXiv:1606.06687*.
 - [14] S. Moudgalya, B. A. Bernevig, and N. Regnault, Quantum many-body scars and Hilbert space fragmentation: A review of exact results, *Rep. Prog. Phys.* **85**, 086501 (2022).
 - [15] A. Chandran, T. Iadecola, V. Khemani, and R. Moessner, Quantum many-body scars: A quasiparticle perspective, *Annu. Rev. Condens. Matter Phys.* **14**, 443 (2023).
 - [16] M. Testorf, Perturbation theory as a unified approach to describe diffractive optical elements, *J. Opt. Soc. Am. A* **16**, 1115 (1999).
 - [17] J. L. Remo, Second-order perturbation theory for Fabry–Perot resonators, *Opt. Lett.* **3**, 193 (1978).
 - [18] C. M. Hooper, J. R. Capers, I. R. Hooper, and S. A. R. Horsley, Symmetry-protected lossless modes in dispersive time-varying media, *Phys. Rev. A* **111**, 033507 (2025).
 - [19] C. M. Bender and S. Boettcher, Real spectra in non-Hermitian Hamiltonians having $\mathcal{PT}$ symmetry, *Phys. Rev. Lett.* **80**, 5243 (1998).
 - [20] C. M. Bender, Making sense of non-Hermitian Hamiltonians, *Rep. Prog. Phys.* **70**, 947 (2007).
 - [21] D. Cheng, K. Wang, C. Roques-Carnes, E. Lustig, O. Y. Long, H. Wang, and S. Fan, Non-Abelian lattice gauge fields in photonic synthetic frequency dimensions, *Nature (London)* **637**, 52 (2025).
 - [22] L. Yuan, Q. Lin, M. Xiao, and S. Fan, Synthetic dimension in photonics, *Optica* **5**, 1396 (2018).
 - [23] Y. Ashida, Z. Gong, and M. Ueda, Non-Hermitian physics, *Adv. Phys.* **69**, 249 (2020).
 - [24] E. D. Nering, *Linear Algebra and Matrix theory* (Wiley, New York, 1970).
 - [25] E. J. Bergholtz, J. C. Budich, and F. K. Kunst, Exceptional topology of non-Hermitian systems, *Rev. Mod. Phys.* **93**, 015005 (2021).
 - [26] S. Ryu, A. P. Schnyder, A. Furusaki, and A. W. W. Ludwig, Topological insulators and superconductors: Tenfold way and dimensional hierarchy, *New J. Phys.* **12**, 065010 (2010).
 - [27] K. Kawabata and S. Ryu, Non-Hermitian disordered systems, *arXiv:2603.20393*.
 - [28] C. M. Hooper, I. R. Hooper, and S. A. R. Horsley, Quasinormal modes in dispersive photonic time crystals, *Phys. Rev. A* **113**, 043526 (2026).
 - [29] M. V. Berry, Quantal phase factors accompanying adiabatic changes, *Proc. R. Soc. London Ser. A* **392**, 45 (1984).
 - [30] J. Garrison and E. Wright, Complex geometrical phases for dissipative systems, *Phys. Lett. A* **128**, 177 (1988).

- [31] M. Wagner, F. Dangel, H. Cartarius, J. Main, and G. Wunner, Numerical calculation of the complex Berry phase in non-Hermitian systems, *Acta Polytech.* **57**, 470 (2017).
- [32] S. Tsubota, H. Yang, Y. Akagi, and H. Katsura, Symmetry-protected quantization of complex Berry phases in non-Hermitian many-body systems, *Phys. Rev. B* **105**, L201113 (2022).
- [33] G. Nenciu and G. Rasche, On the adiabatic theorem for nonself-adjoint Hamiltonians, *J. Phys. A: Math. Gen.* **25**, 5741 (1992).
- [34] S. Ibáñez and J. G. Muga, Adiabaticity condition for non-Hermitian Hamiltonians, *Phys. Rev. A* **89**, 033403 (2014).
- [35] E.-M. Graefe, A. A. Mailybaev, and N. Moiseyev, Breakdown of adiabatic transfer of light in waveguides in the presence of absorption, *Phys. Rev. A* **88**, 033842 (2013).
- [36] Q. Ma, J. W. You, L. Chen, Z. Gu, S. L. Qin, Z. Lan, and T. J. Cui, Floquet topological states in time-varying metasurfaces, *Sci. Adv.* **11**, eadx9025 (2025).
- [37] J. C. Serra and M. G. Silveirinha, Engineering topological phases with a traveling-wave spacetime modulation, *Laser Photonics Rev.* **20**, e00560 (2026).
- [38] E. Lustig, Y. Sharabi, and M. Segev, Topological aspects of photonic time crystals, *Optica* **5**, 1390 (2018).
- [39] X. Ni, S. Yin, H. Li, and A. Alù, Topological wave phenomena in photonic time quasicrystals, *Phys. Rev. B* **111**, 125421 (2025).
- [40] O. Segal, Y. Plotnik, E. Lustig, Y. Sharabi, M.-I. Cohen, A. Dikopoltsev, and M. Segev, Two-dimensional topological edge states in periodic space-time interfaces, *Phys. Rev. Lett.* **135**, 163801 (2025).
- [41] A. Caballero, T. F. Allard, and P. A. Huidobro, Interface states in space-time photonic crystals: Topological origin, propagation, and amplification, *ACS Photonics* **13**, 3012 (2026).
- [42] J. K. Asbóth, L. Oroszlány, and A. Pályi, *A Short Course on Topological Insulators: Band Structure and Edge States in One and Two Dimensions* (Springer International Publishing, Cham, 2016), pp. 1–53.
- [43] J. Zak, Berry's phase for energy bands in solids, *Phys. Rev. Lett.* **62**, 2747 (1989).
- [44] W. P. Su, J. R. Schrieffer, and A. J. Heeger, Solitons in polyacetylene, *Phys. Rev. Lett.* **42**, 1698 (1979).
- [45] A. H. Castro Neto, F. Guinea, N. M. R. Peres, K. S. Novoselov, and A. K. Geim, The electronic properties of graphene, *Rev. Mod. Phys.* **81**, 109 (2009).
- [46] F. Ghahari, D. Walkup, C. Gutiérrez, J. F. Rodríguez-Nieva, Y. Zhao, J. Wyrick, F. D. Natterer, W. G. Cullen, K. Watanabe, T. Taniguchi, L. S. Levitov, N. B. Zhitenev, and J. A. Stroscio, An on/off Berry phase switch in circular graphene resonators, *Science* **356**, 845 (2017).
- [47] O. Breach, R.-J. Slager, and F. N. Ünal, Interferometry of non-Abelian band singularities and Euler class topology, *Phys. Rev. Lett.* **133**, 093404 (2024).
- [48] C. M. Bender and S. A. Orszag, WKB theory, in *Advanced Mathematical Methods for Scientists and Engineers I: Asymptotic Methods and Perturbation Theory* (Springer, New York, NY, 1999), pp. 484–543.
- [49] A. Canós Valero, S. Gladyshev, D. Globosits, S. Rotter, E. A. Muljarov, and T. Weiss, Revealing the resonant physics of open photonic time crystals, *Laser Photonics Rev.* e71318 (2025).
- [50] B. Vial and R. V. Craster, Quasinormal modes of Floquet media slabs, *Phys. Rev. B* **112**, 214201 (2025).
- [51] M. M. Sabaté, B. Vial, R. Wiltshaw, S. Guenneau, and R. V. Craster, Fabry-Pérot quasinormal modes for topological edge states, *Phys. Rev. B* **113**, L140102 (2026).
- [52] I. R. Hooper, D. B. Phillips, and S. A. R. Horsley, Harnessing the frequency eigenchannels of ultrafast time-varying media, *arXiv:2508.12753*.
- [53] N. Okuma and M. Sato, Non-Hermitian topological phenomena: A review, *Annu. Rev. Condens. Matter Phys.* **14**, 83 (2023).
- [54] Z. Gong, Y. Ashida, K. Kawabata, K. Takasan, S. Higashikawa, and M. Ueda, Topological phases of non-Hermitian systems, *Phys. Rev. X* **8**, 031079 (2018).
- [55] D. S. Borgnia, A. J. Kruchkov, and R.-J. Slager, Non-Hermitian boundary modes and topology, *Phys. Rev. Lett.* **124**, 056802 (2020).
- [56] C. Hooper, Time-varying adiabaticity figure generation code [Computer software], Zenodo, 2026, <https://doi.org/10.5281/zenodo.23243040>.