

Detecting quantum many-body states with imperfect measuring devices

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We study a coarse-graining map arising from incomplete and imperfect addressing of particles in a multipartite quantum system. In its simplest form, corresponding to a two-qubit state, the resulting channel produces a convex mixture of the two partial traces. We derive the probability density of obtaining a given coarse-grained state, using geometric arguments for two qubits coarse-grained to one, and random-matrix methods for larger systems. As the number of qubits increases, the probability density sharply concentrates around the maximally mixed state, making nearly pure coarse-grained states increasingly unlikely. For two qubits, we also compute the inverse state needed to characterize the effective dynamics under coarse-graining and find that the average preimage of the maximally mixed state contains a finite singlet component. Finally, we validate the analytical predictions by inferring the underlying probabilities from Monte Carlo-generated coarse-grained statistics.

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I. INTRODUCTION

Our knowledge of the natural laws governing a physical system fundamentally relies on our measurement capabilities. Bridging the gap between a system's intrinsic nature and our understanding of it depends crucially on both the measurement devices we employ and the theoretical frameworks we use to interpret the obtained information. In this regard, Einstein famously remarked to Heisenberg during an informal conversation: “It is the theory which decides what we can observe. You must appreciate that observation is a very complicated process” [1]. Years later, Heisenberg echoed this insight, stating: “We have to remember that what we observe is not nature herself, but nature exposed to our method of questioning” [2]. Thus, recognizing the limits of our measurements is crucial to understanding what the quantum theory can actually tell us.

These considerations motivate us to address specific problems in the detection of quantum many-body states. Detecting and characterizing quantum states is essential for the validation, control, and optimization of quantum devices, becoming increasingly critical in modern technological advancements [3–5]. However, reconstructing states in quantum many-body systems presents significant challenges. One of the main issues stems from intrinsic measurement uncertainties. Measurement devices are susceptible to systematic errors even

under optimal experimental conditions, significantly affecting the accurate characterization of states in complex many-body quantum systems [6]. Therefore, such measurement imperfections cannot be ignored and must be carefully managed.

Several proposals have been put forward to address the issue of imperfect measurements, both from a fundamental perspective and in terms of their impact on applications. Among them are approaches based on generalized measurements, characterized by parameters such as bias and sharpness, which enable the modeling of various effects contributing to measurement nonideality [7]. Compatibility criteria between fuzzy measurements have also been investigated [8]. More recently, methods involving neural networks—trained to recognize quantum states in a manner analogous to blurred image recognition—have been proposed, along with active learning techniques designed to optimize data collection during the characterization process [9–11]. While these strategies have enabled significant progress in state reconstruction and the identification of noise-resilient patterns, they face notable limitations. These include their reliance on system-specific training data, the challenges associated with extending them in a controlled way to multipartite systems, and the lack of a systematic framework that directly links model parameters to the operational features of the measurement devices.

A systematic framework based on a coarse-graining map for studying quantum many-body systems under imperfect measurements was proposed in Ref. [12]. This map, denoted by $\mathcal{C}$, applies to systems composed of N particles and is constructed from a fuzzy measurement followed by a reduction in the system's degrees of freedom. The resulting procedure defines a quantum channel designed to model experimental errors stemming from particle-indexing ambiguities and from the intrinsic limitations in the ability to precisely identify

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the number of particles composing the system. The map $\mathcal{C}$ complies with Uhlmann's theorem and predicts a doubly exponential reduction in the volume of distinguishable states as the number of particles increases, which aligns with the challenges faced in quantum tomography.

Coarse-graining maps use well-defined rules to produce a simpler description of a physical system from a more complete one, accounting for the different ways in which information can be lost [13]. This makes it possible to describe not only the system at a given moment, but also how its effective dynamics relate to the fine dynamics of the full description. If the initial state is obtained through measurements modeled by a coarse-graining map, it is called a granular state. After the system evolves, measuring it again gives its granular or effective dynamics. This dynamics is related to the fine dynamics—which link fine states in the complete description—through a map that acts as a bridge between the two [14,15]. Therefore, knowing this map is essential for fully addressing the problem of imperfect measurements.

In this work, we adopt the $\mathcal{C}$ map to analyze the many-body (fine-grained) states that give rise to the detection of coarse-grained states under imperfect measurements. We systematically characterize the set of fine-grained states that are mapped to a small neighborhood of a given monopartite state. This leads to the definition of the preimage set, whose volume determines the probability density function (PDF) associated with a target single-qubit state obtained from an N -qubit system. The PDF serves as a fundamental tool for analyzing how observable properties, quantum coherence, and correlations are affected by the coarse-graining process. In particular, it allows us to identify quantum features that remain robust under imperfect measurements described by this map.

Building on the characterization of preimage sets, we define an “inverse” map by establishing a rule to assign a preimage state to each target state. By definition, the coarse-graining (CG) map is not one-to-one and thus cannot be strictly inverted. Nevertheless, physically motivated criteria can be introduced to select a specific fine-grained preimage for each target state. For example, in Ref. [16], the authors propose choosing the fine-grained state of maximum entropy compatible with the given target state. Motivated by quantum state tomography procedures—where reconstructed states naturally correspond to averages over multiple experimental realizations—we assign, as the preimage of a target state, the average of all fine-grained states that the coarse-graining map $\mathcal{C}$ sends within a distance ϵ of the target state. We refer to this rule as the average assignment and denote it by $\overline{\mathcal{C}^{-1}}$. We then compute and analyze the average state in the case where the system is coarse-grained from two particles to one.

We focus on pure states of systems composed of N qubits, analyzing the scenario in which imperfections in the measurement device lead to the tomographic reconstruction of coarse-grained states describing a single qubit. Within this framework, we address two fundamental questions: What is the most representative fine-grained state corresponding to the observed coarse-grained state, given that it was obtained through a device subject to indexing errors and limited in its ability to resolve all degrees of freedom of the system? And how do such imperfections impact our knowledge of

the quantum properties of the original system? To address these questions, we employ geometric methods and random matrix theory (RMT) to compute the volume of the preimage set when the coarse-graining map $\mathcal{C}$ reduces the system from N qubits to one, and we compute the average state when $\mathcal{C}$ reduces a two-qubit system to a single qubit.

The article is organized as follows: Section II reviews the structure and main properties of the coarse-graining map $\mathcal{C}$. In Sec. III, we compute and analyze the PDF for a two-qubit system reduced to a single qubit, using geometrically motivated parametrizations and analytical methods from RMT. These RMT-based techniques are then generalized to obtain the PDF for the more general case of an N -qubit system reduced to a single qubit. Section IV focuses on the computation and characterization of the average state in the two-to-one qubit case, emphasizing its symmetries and structural features. In Sec. V, we compare our theoretical predictions with numerical results obtained via Monte Carlo simulations and propose practical strategies for identifying coarse-graining effects in experimental scenarios. Finally, Sec. VI presents our conclusions and outlines possible directions for future research.

II. COARSE-GRAINING MAP

In this section, we introduce the coarse-graining (CG) map, which forms the basis of our study [12]. We define how the $\mathcal{C}$ map is constructed and describe its role in our approach. We also point out a symmetry of the map that will be important in later sections.

The imperfect detections considered in this study arise from two main limitations of the measurement device. First, it is not possible to identify each particle in the system with complete accuracy. For example, in two-dimensional ultracold atomic systems, atoms are detected by fluorescence imaging with a high-resolution microscope. An algorithm determines the presence of an atom based on its brightness and proximity to predefined grid points on the image [17–19]. Because the grid is discrete and has finite resolution—typically on the order of hundreds of nanometers—there is an inherent indexing error in identifying particles. These errors reduce the precision of the measurement and are referred to as fuzzy measurements (FMs).

The simplest case is where we deal with a bipartite system. Taking into account that the measurement device has some probability p of mistaking the two particles that determine the state ϱ . This situation is described as a fuzzy measurement of the form

$$\mathcal{F}[\varrho] = (1 - p)\varrho + pP[\varrho], \quad (1)$$

where the brackets denote the action of the permutation operator on the density matrix, $P[\varrho] = P\varrho P^\dagger$. Then, in the general situation where we have an N -partite system, our imperfect measurement device can detect any possible permutation of the N particles with some probability $p_P \geq 0$, so the FM takes the form

$$\mathcal{F}[\varrho] = \sum_{P \in S_N} p_P P[\varrho], \quad \sum_{P \in S_N} p_P = 1, \quad (2)$$

where P belongs to a subset of the symmetric group S_N of N particles.

The second type of imperfection we consider arises from the measurement device's inability to resolve the fine details of the whole system, leading to errors in identifying the number of particles composing the system. As a consequence, groups of particles are collectively represented within effective states corresponding to a reduced number of particles. This coarse-grained approach to observing the system gives rise to the term coarse-grained measurement. Both types of errors are deeply related, as illustrated by the example of a two-dimensional quantum system, where two or more atoms can appear indistinguishable behind a single bright spot. Their presence implies that the way particles are grouped collectively depends on p_P , meaning that such groups are not sharply determined. It is worth noting, however, that coarse-graining of the system can arise either from the limited sensitivity of the measurement device—as in the cited example—or because only a few degrees of freedom are relevant to the specific property under study.

Suppose a detector is capable of identifying only n particles among the N comprising a system. Since it cannot distinguish the remaining $m = N - n$ particles, in each measurement it randomly selects a specific subset of n particles and disregards the information about the others. This procedure is mathematically equivalent to performing a partial trace over a fuzzy measurement of the entire system, thus defining the coarse-graining (CG) map

$$\mathcal{C} : \mathcal{B}(\mathbb{H}_d^{\otimes N}) \longrightarrow \mathcal{B}(\mathbb{H}_d^{\otimes n}), \quad (3)$$

$$\mathcal{C}[\varrho] = \text{Tr}_{\bar{n}} \mathcal{F}[\varrho], \quad (4)$$

where d is the number of levels for each particle in the system, $\bar{n}$ denotes the complementary subset to the identified n particles (such that $|\bar{n}| = m$), and $\mathcal{B}(\mathbb{H})$ represents the space of bounded linear operators acting on $\mathbb{H}$.

We consider a system of qubits ($d = 2$) measured by a device that can detect only one particle ($n = 1$). Without loss of generality, we label the detectable particle as particle 1. As a result, the measurement can only exchange this particle with any of the others, and only the swap operations $S_{1,i}$ are relevant, since all other permutations vanish once the nonmeasured particles are traced out. For simplicity, we restrict the discussion to pure states $\varrho = |\psi\rangle\langle\psi|$ with $|\psi\rangle \in \mathbb{H}_2^{\otimes N}$. Under these assumptions, the coarse-graining map becomes

$$\mathcal{C}[|\psi\rangle\langle\psi|] = \text{Tr}_1 \sum_{i=1}^N p_{1i} S_{1,i} [|\psi\rangle\langle\psi|]. \quad (5)$$

For each swap, the particle brought to the detector determines the outcome of the partial trace, which becomes its reduced density matrix. Thus, we can express the map (5) in the simplified form,

$$\mathcal{C}[|\psi\rangle\langle\psi|] = \sum_{i=1}^N p_i \rho_i, \quad (6)$$

with

$$\rho_i = \text{Tr}_1 |\psi\rangle\langle\psi|, \quad \sum_{i=1}^N p_i = 1, \quad (7)$$

where ρ_i denotes the reduced density matrix associated with the i th particle, and p_i is the probability of detecting that particle.

The form given by (6) and (7) highlights a symmetry property of the CG map. Specifically, the compatibility of the partial trace with unitary transformations [20], expressed as

$$U |\psi_i\rangle\langle\psi_i| U^\dagger = \text{Tr}_1 U^{\otimes N} |\psi\rangle\langle\psi| U^{\dagger \otimes N}, \quad (8)$$

where $|\psi\rangle = \bigotimes_{i=1}^N |\psi_i\rangle$ directly leads to the following unitary covariance relation for the map,

$$\mathcal{C}[U^{\otimes N} |\psi\rangle\langle\psi| U^{\dagger \otimes N}] = U \mathcal{C}[|\psi\rangle\langle\psi|] U^\dagger, \quad (9)$$

where U is a unitary operator in $\mathcal{B}(\mathbb{H}_2)$. Later on, we leverage this property to reveal characteristics of certain average states.

III. PROBABILITY OF A TARGET STATE

Consider the physical situation in which measurements are performed on a system of N qubits using imperfect devices that can only resolve measurements on one of the particles, described by a channel like in Eq. (6). Since it is not possible to access the full description of the system, we assume that an ensemble of random N -partite states is available.

The question then arises, what is the probability that the observed state falls within a small volume V_ϵ surrounding a given target state $\rho = (I + \mathbf{r}_{\text{ts}} \cdot \boldsymbol{\sigma})/2$? Here, $\boldsymbol{\sigma}$ is a vector whose entries are the Pauli matrices, $\mathbf{r}_{\text{ts}}$ is a vector in the Bloch ball B representing ρ , and the volume V_ϵ corresponds to the region defined by the usual measurement uncertainty bars — the region within which the observed state can be guaranteed to lie.

To answer this question, it is necessary to establish the probability distribution followed by the fine-grained states. In this section, we address the $N = 2$ case in detail, considering both the situation where the fine-grained states are uniformly distributed with respect to the Fubini–Study measure on the full space, and the situation where this distribution is restricted to the space of separable states. For the N -to-one case, only the first scenario will be analyzed.

To this end, we describe the probability measure in the pure state manifold. We start with the set of N -partite pure states which, after fixing the normalization and discarding the overall phase, correspond to the complex projective space $\mathbb{C}P^D \equiv \mathbb{P}$, with $D = 2^N - 1$, whose points represent $2^N \times 2^N$ pure state density matrices. An immediate result obtained by analyzing the right-hand side of Eq. (6) for separable and maximally entangled states, is that the image of the map $\mathcal{C}$ covers the entire Bloch ball B for any choice of the probabilities p_i . In fact, a stronger result is valid: B can be covered by the image, under $\mathcal{C}$, of $\mathbb{P}_{\text{symm}}$, the subset of $\mathbb{P}$ consisting of the totally symmetric (under subsystem permutations) multipartite states. Indeed, for symmetric states, all partial traces ρ_i in (6) are equal, $\rho_i = \bar{\rho}$, and, clearly, $\mathcal{C}[|\psi\rangle\langle\psi|] = \bar{\rho}$ for all possible values of the probabilities p_i . Consider now the curve $\gamma(\alpha) = \cos \alpha |+\rangle^{\otimes N} + \sin \alpha |-\rangle^{\otimes N}$, $0 \leq \alpha \leq \pi/4$, which lies in $\mathbb{P}_{\text{symm}}$. Its image $\mathcal{C}(\gamma(\alpha))$ starts, at $\alpha = 0$, at the surface of B ($|+\rangle^{\otimes N}$ being a separable state) and ends, for $\alpha = \pi/4$, at its center ($(|+\rangle^{\otimes N} + |-\rangle^{\otimes N})/\sqrt{2}$ being a maximally entangled state). Now take into account that when the original multipartite state $\gamma(\alpha)$ is rotated in space, its $\mathcal{C}$ image is similarly

rotated (a particular case of the unitary covariance expressed in (9))—using these rotations, the above curve $\mathcal{C}(\gamma(\alpha))$ can be made to fill B (since it has points at every radius r , $0 \leq r \leq 1$, which can be moved to any other point with the same radius by rotations), hence the $\mathcal{C}$ image of $\mathbb{P}_{\text{symm}}$ fills the entire Bloch ball.

The projective space $\mathbb{P}$ is equipped with a metric inherited from the structure of the Hilbert space $\mathcal{H} \simeq \mathbb{C}^{2^N}$, known as the Fubini–Study (FS) metric [21]. This metric is characterized by being the only one—up to a multiplicative constant—in $\mathbb{P}$ that remains invariant under the action of the unitary group $U(2^N)$. We denote by $d\mu$ the normalized volume measure associated with the FS metric, which can also be described as the measure induced on $\mathbb{P}$ by the Haar measure on the group $U(2^N)$.

We denote by $\Omega_{\mathbf{R}}^N$ the preimage of $\mathbf{R} \in B$ under $\mathcal{C}$,

$$\Omega_{\mathbf{R}}^N = \{\varrho : \mathcal{C}[\varrho] = \frac{1}{2}(I + \mathbf{R} \cdot \boldsymbol{\sigma})\}, \quad (10)$$

and by Ω_{ϵ}^N the preimage of V_{ϵ} ,

$$\Omega_{\epsilon}^N = \left\{ \bigcup_{\mathbf{R}} \Omega_{\mathbf{R}}^N : \mathbf{R} \in V_{\epsilon} \right\}. \quad (11)$$

Since $\int_{\mathbb{P}} d\mu = 1$, the volume measure $d\mu$ defines the probability measure invariant under unitary transformations on $\mathbb{P}$. Then, the probability of observing a coarse-grained state within V_{ϵ} is obtained by computing the volume of the preimage set Ω_{ϵ}^N with respect to $d\mu$:

$$V(\Omega_{\epsilon}^N) = \int_{\Omega_{\epsilon}^N} d\mu. \quad (12)$$

In what follows, we also consider the case of fine-grained product states in order to highlight the role of entanglement. We therefore define the sets

$$\Omega_{\mathbf{R}}^{N\otimes} = \left\{ \varrho = \bigotimes_i \rho_i : \mathcal{C}[\varrho] = \frac{1}{2}(I + \mathbf{R} \cdot \boldsymbol{\sigma}) \right\}, \quad (13)$$

$$\Omega_{\epsilon}^{N\otimes} = \left\{ \bigcup_{\mathbf{R}} \Omega_{\mathbf{R}}^{N\otimes} : \mathbf{R} \in V_{\epsilon} \right\}, \quad (14)$$

and use $d\mu^{\otimes}$ to denote the FS measure restricted to the subspace of product states in $\mathbb{P}$. Other variants of the above problem can also be considered; for example, when dealing with states that possess particular symmetries, the integration in Eq. (12) can be restricted to the appropriate subspace.

In the rest of this section, we first carry out the integration in (12) for the case $N = 2$. We give an appealing geometric description that allows us to visualize the sets Ω_{ϵ}^2 and $\Omega_{\epsilon}^{2\otimes}$ and compute the corresponding integrals. To keep the notation simple, we omit the superscript 2 when referring to this case.

For $N > 2$, the geometric approach becomes unmanageable, and we resort to random matrix techniques to obtain an exact result. As preparation, we first study the case $N = 2$ and then address the general case. We conclude by discussing several physical consequences of the results obtained.

A. Bipartite system

1. Parametrization of pure two-qubit states

We now turn to a convenient parametrization of two-qubit pure states. This parametrization is defined in terms of (i) the coordinates of the reduced density matrices of each qubit, and (ii) two nonlocal parameters: one quantifying the degree of entanglement between the qubits, and another representing a relative phase, which will not play a role in our present analysis.

We now work through the technical details of such parametrization. The Schmidt decomposition of a two-qubit state involves a rotation of a three-dimensional coordinate system associated with each particle in such a way that an axis of each coordinate system lines up with the Bloch vector of the corresponding particle, while the correlation tensor between the two qubits, $C_{kl} = \text{Tr}[(\sigma_k \otimes \sigma_l) |\psi\rangle\langle\psi|]$, becomes diagonal [22,23]. As a result, a general normalized two-qubit pure state $|\psi\rangle$ can be written in the form

$$|\psi\rangle = \sqrt{\lambda}|e_1, f_1\rangle + \sqrt{1-\lambda}|e_2, f_2\rangle, \quad (15)$$

where λ is real and non-negative, and $\{|e_i\rangle\}$, $\{|f_i\rangle\}$ are particular orthonormal bases in the single-qubit Hilbert space that depend on $|\psi\rangle$. Taking $\{|e_1\rangle, |e_2\rangle\} = \{|\mathbf{n}_1\rangle, |-\mathbf{n}_1\rangle\}$, $\{|f_1\rangle, |f_2\rangle\} = \{|\mathbf{n}_2\rangle, e^{i\gamma}|\mathbf{n}_2\rangle\}$, where $|\mathbf{n}_i\rangle = \cos \frac{\eta_i}{2} e^{-i\frac{\phi_i}{2}}|0\rangle + \sin \frac{\eta_i}{2} e^{i\frac{\phi_i}{2}}|1\rangle$, and putting $\sqrt{\lambda} = \cos \frac{\eta}{2}$, $0 \leq \eta \leq \pi$, we get

$$\begin{aligned} |\psi\rangle &= \cos \frac{\eta}{2} |\mathbf{n}_1, \mathbf{n}_2\rangle + \sin \frac{\eta}{2} e^{i\gamma} |-\mathbf{n}_1, -\mathbf{n}_2\rangle, \\ &= e^{-\frac{1}{2}i(\phi_1+\phi_2)} \left(\cos \frac{\eta}{2} \cos \frac{\theta_1}{2} \cos \frac{\theta_2}{2} \right. \\ &\quad \left. + e^{i\gamma} \sin \frac{\eta}{2} \sin \frac{\theta_1}{2} \sin \frac{\theta_2}{2} \right) |0, 0\rangle \\ &\quad + e^{-\frac{1}{2}i(\phi_1-\phi_2)} \left(\cos \frac{\eta}{2} \cos \frac{\theta_1}{2} \sin \frac{\theta_2}{2} \right. \\ &\quad \left. - e^{i\gamma} \sin \frac{\eta}{2} \sin \frac{\theta_1}{2} \cos \frac{\theta_2}{2} \right) |0, 1\rangle \\ &\quad + e^{\frac{1}{2}i(\phi_1-\phi_2)} \left(\cos \frac{\eta}{2} \sin \frac{\theta_1}{2} \cos \frac{\theta_2}{2} \right. \\ &\quad \left. - e^{i\gamma} \sin \frac{\eta}{2} \cos \frac{\theta_1}{2} \sin \frac{\theta_2}{2} \right) |1, 0\rangle \\ &\quad + e^{\frac{1}{2}i(\phi_1+\phi_2)} \left(\cos \frac{\eta}{2} \sin \frac{\theta_1}{2} \sin \frac{\theta_2}{2} \right. \\ &\quad \left. + e^{i\gamma} \sin \frac{\eta}{2} \cos \frac{\theta_1}{2} \cos \frac{\theta_2}{2} \right) |1, 1\rangle. \end{aligned} \quad (16)$$

The reduced density matrices are now easily computed,

$$\rho_i = \frac{1}{2} \begin{pmatrix} 1 + \cos \eta \cos \theta_i & e^{-i\phi_i} \cos \eta \sin \theta_i \\ e^{i\phi_i} \cos \eta \sin \theta_i & 1 - \cos \eta \cos \theta_i \end{pmatrix}, \quad (17)$$

giving for the corresponding Bloch vectors,

$$\mathbf{r}_i = \text{Tr}(\boldsymbol{\sigma} \rho_i) = r(\cos \phi_i \sin \theta_i, \sin \phi_i \sin \theta_i, \cos \theta_i), \quad (18)$$

the radius of which is $r = \cos \eta$. Note that θ_i is the polar angle and ϕ_i is the azimuthal angle of the Bloch vector $\mathbf{r}_i$. r and

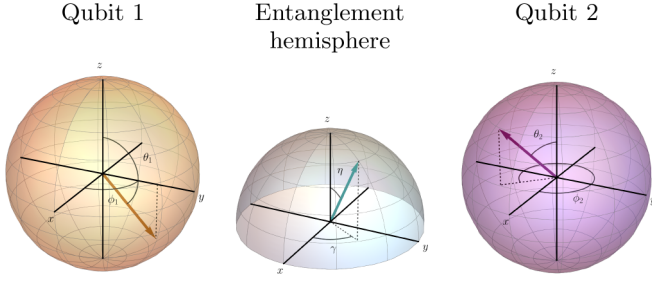


FIG. 1. Visualization of a pure state formed by two qubits in the parametrization (16). The state shown corresponds to the values $\eta = \frac{\pi}{6}$, $\gamma = \frac{\pi}{3}$, $\theta_1 = \frac{3\pi}{4}$, $\phi_1 = \frac{\pi}{2}$, $\theta_2 = \frac{\pi}{4}$, and $\phi_2 = \frac{7\pi}{4}$. The orange and purple spheres correspond to the Bloch spheres of the reduced states of qubit 1 and 2, respectively. The Bloch vectors in both cases have a length $\cos \eta = \sqrt{3}/2$. The gray hemisphere between the spheres is the entanglement hemisphere, it shows the concurrence angle η and the relative phase γ . Note that the reduced states can only be pure when the entanglement arrow points to the north pole, i.e., when $\eta = 0$.

η are related to the entanglement between the two qubits. Using the concurrence $\mathcal{C}$ to measure entanglement [24], we have that $\mathcal{C} = \sin \eta = (1 - r^2)^{1/2}$. Hence, we refer to η as the concurrence angle. Note that the overall phase factor of $|\psi\rangle$ has been fixed so that the coefficient of $|\mathbf{n}_1, \mathbf{n}_2\rangle$ be real and non-negative.

As a result, the state $|\psi\rangle$ depends on six parameters, which provide coordinates on the corresponding projective space $\mathbb{P}$. A visualization of these coordinates involves the Bloch spheres of each qubit in the system and the entanglement hemisphere between them. The position of the Bloch vectors is determined by the parameters (θ_1, ϕ_1) and (θ_2, ϕ_2) and both have magnitude $\cos \eta$. In the entanglement hemisphere, the unit vector is oriented by the concurrence and relative phase angles (η, γ) . Figure 1 shows an example visualization of a bipartite state in this parametrization.

2. Coarse-graining statistics via preimage volume

Since the CG map in Eq. (6) treats all subsystems equally, in the case $N = 2$ we assume that $p_1 < p_2$ without loss of generality. In this way, we write the coarse-grained state as $\rho = p\rho_1 + (1 - p)\rho_2$ with $0 \leq p \leq \frac{1}{2}$. In terms of Bloch vectors, this is expressed as

$$\mathbf{r}_{\text{ts}} = \frac{1 - h}{2}\mathbf{r}_1 + \frac{1 + h}{2}\mathbf{r}_2, \quad (19)$$

where $\mathbf{r}_1$ and $\mathbf{r}_2$ are the Bloch vectors corresponding to the reduced density matrices ρ_1 and ρ_2 and the symmetric parameter $h \equiv 1 - 2p$ is introduced. Notice that h quantifies the degree of precision in the detection process. The value $h = 0$ represents maximum uncertainty, where the measuring apparatus provides no information about which particle was detected, whereas $h = 1$ corresponds to the identification of particle 2, occurring with unit probability.

The requirement of invariance under the action of the unitary group on $\mathbb{P}$, specifies, up to an overall scale factor, the invariant Fubini-Study metric [21]

$$(ds)^2 = \langle \psi | \psi \rangle^{-2} (\langle \psi | \psi \rangle \langle d\psi | d\psi \rangle - \langle \psi | d\psi \rangle \langle d\psi | \psi \rangle), \quad (20)$$

where $d = dx^i \frac{\partial}{\partial x^i}$ is the exterior derivative on $\mathbb{P}$, $\{x^i\}$ are the coordinates $\{r, \gamma, \theta_1, \phi_1, \theta_2, \phi_2\}$, and $|d\psi\rangle \equiv d|\psi\rangle$, $\langle d\psi| \equiv d\langle\psi|$. The corresponding invariant measure is given by $d\mu = \mathcal{N} \sqrt{\det g} d^6x$ where g is the metric corresponding to the line element $(ds)^2 = g_{\mu\nu} dx^\mu dx^\nu$ in (20), and with $\mathcal{N}$ a normalization factor determined by the requirement $\int_{\mathbb{P}} d\mu = 1$. Substituting $|\psi\rangle$ from Eq. (16) into (20), we find for the probability measure

$$d\mu = \frac{3}{2\pi} r^2 dr d\gamma d\omega_1 d\omega_2, \quad (21)$$

where $d\omega_i = \sin \theta_i d\theta_i d\phi_i / 4\pi$. Restricted to product states, the probability measure, computed similarly, takes the form

$$d\mu^\otimes = d\omega_1 d\omega_2, \quad (22)$$

i.e., it just factorizes into the product of the measures of two Bloch spheres. A geometric depiction of the set $\Omega_{\mathbf{r}_{\text{ts}}}$ provides mathematical and physical intuition. This is given by the locus of all pairs of vectors $\mathbf{r}_1$ and $\mathbf{r}_2$ that lead to the same $\mathbf{r}_{\text{ts}}$ in Eq. (19) as two spheres within the Bloch sphere. Notice that since the bipartite state is pure, the purity of the two reduced states is the same, which implies that $\|\mathbf{r}_1\| = \|\mathbf{r}_2\| = r$. Replacing $\mathbf{r}_1$ and $\mathbf{r}_2$ as in Eq. (19) in the expression for purity equality, and completing the squares, we arrive at $\|\mathbf{r}_1 - \mathbf{c}_1\| = R_1$ and $\|\mathbf{r}_2 - \mathbf{c}_2\| = R_2$ where $\mathbf{c}_1 = -\mathbf{r}_{\text{ts}}(1 - h)/2h$ and $\mathbf{c}_2 = \mathbf{r}_{\text{ts}}(1 + h)/2h$ are vectors indicating the position of the centers of the spheres, while $R_1 = r_{\text{ts}}(1 + h)/2h$ and $R_2 = r_{\text{ts}}(1 - h)/2h$ correspond to their radii, where $r_{\text{ts}} \equiv \|\mathbf{r}_{\text{ts}}\|$. The two spheres touch in the target state, since $\mathbf{c}_1 + R_1 \mathbf{r}_{\text{ts}} / \|\mathbf{r}_{\text{ts}}\| = \mathbf{r}_{\text{ts}}$ and $\mathbf{c}_2 - R_2 \mathbf{r}_{\text{ts}} / \|\mathbf{r}_{\text{ts}}\| = \mathbf{r}_{\text{ts}}$. A geometrically more intuitive description of vectors $\mathbf{r}_i$ is described in Appendix A.

Moreover, the radii of the spheres can vary from zero (for $h \rightarrow 1$) to infinity (for $h \rightarrow 0$); the spheres can be totally contained in the Bloch sphere, or be partially outside of it. Since a vector corresponding to a state must have norm less than one, this means that the sphere will be incomplete. This transition occurs when $r_{\text{ts}} = h$. Separable states arise as solutions only when $r_{\text{ts}} \geq h$. In Fig. 2, we illustrate the above discussion.

To continue, we calculate the volume $V(\Omega_\epsilon) \equiv V_{\Omega_\epsilon}$ by integrating the FS measure over Ω_ϵ . In the limit where V_ϵ is an infinitesimal neighborhood centered at $\mathbf{r}_{\text{ts}}$, we find that

$$V_{\Omega_\epsilon} = \frac{3V_\epsilon}{2\pi(1+h)} \begin{cases} \frac{1}{h} & \text{if } 0 \leq r_{\text{ts}} < h \\ \frac{1}{1-h} \frac{1-r_{\text{ts}}}{r_{\text{ts}}} & \text{if } h \leq r_{\text{ts}} \leq 1. \end{cases} \quad (23)$$

Note that the preimage volume is constant when $0 \leq r_{\text{ts}} < h$ (which corresponds to the two subspheres in Fig. 2 being entirely contained in the outer Bloch sphere) and then decreases to zero, as r_{ts} tends to 1. Thus, as h approaches 1 (i.e., as p tends to zero), the preimage volume becomes constant, independent of the magnitude of r_{ts} .

In the case of separable states, we find that

$$V_{\Omega_\epsilon}^\otimes = \frac{V_\epsilon}{2\pi(1-h^2)} \begin{cases} 0 & \text{if } 0 \leq r_{\text{ts}} < h \\ \frac{1}{r_{\text{ts}}} & \text{if } h \leq r_{\text{ts}} \leq 1. \end{cases} \quad (24)$$

Note that in the interval $0 \leq r_{\text{ts}} < h$, the subspheres do not intersect the Bloch sphere and $\Omega_\epsilon^\otimes$ is the empty set. The details of the above volume calculations can be found in Appendix B. As expected, due to the covariance of the CG map established in Eq. (9) and the unitary invariance of the ensemble

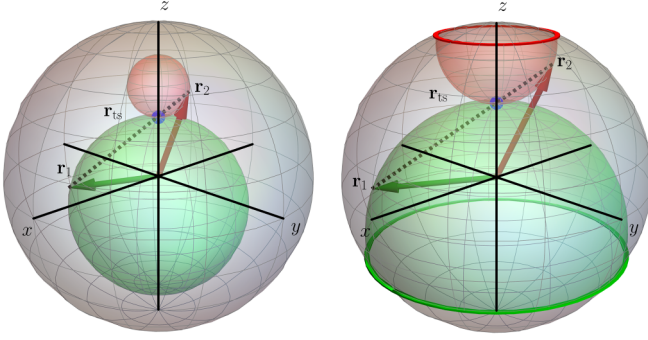


FIG. 2. Green and red subspheres corresponding to the locus of states ρ_1 and ρ_2 , respectively. Each state in the green subsphere has a corresponding state in the red subsphere, so that the sum of their Bloch vectors is equal to r_{ts} (chosen here along the z axis). The gray sphere is the Bloch sphere associated with the target state ρ , which can be seen at the blue dot. In both cases shown $h = 0.4$. In the left figure, $r_{ts} = 0.3$ was used, so the subspheres are contained within the Bloch sphere; while in the right, $r_{ts} = 0.5$. Separable states, represented by bold curves, are available only when $r_{ts} \geq h$ as in the right figure. See the main text for details.

of bipartite states, in both cases the preimage volume does not depend on the orientation of the target state. Then, the relevant quantities are the probability densities P_2 and $P_2^\otimes$ of r_{ts} , which are obtained from the above results, respectively, by integrating the density $V_{\Omega_\epsilon}/V_\epsilon$ over the surface of constant r_{ts} in the Bloch sphere

$$P_2(h; r_{ts}) = \frac{6}{1+h} \begin{cases} \frac{1}{h} r_{ts}^2 & \text{if } 0 \leq r_{ts} < h \\ \frac{1}{1-h} r_{ts} (1 - r_{ts}) & \text{if } h \leq r_{ts} \leq 1, \end{cases} \quad (25)$$

$$P_2^\otimes(h; r_{ts}) = \begin{cases} 0 & \text{if } 0 \leq r_{ts} < h \\ \frac{2}{1-h^2} r_{ts} & \text{if } h \leq r_{ts} \leq 1. \end{cases} \quad (26)$$

A plot of $P_2(h; r_{ts})$, for various values of h , appears in Fig. 3.

A couple of remarks are due at this point. The first one is on Eq. (23), given that as h goes to zero, it appears that

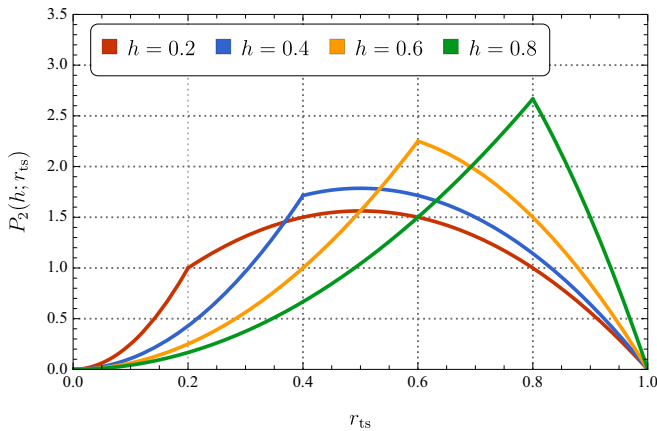


FIG. 3. Probability density $P_2(h; r_{ts})$ vs r_{ts} for various values of h . In all cases, P_2 tends to zero for both $r_{ts} \rightarrow 0$ (maximally mixed state) and $r_{ts} \rightarrow 1$ (pure states). The cusp at $r_{ts} = h$ corresponds to the two subspheres in Fig. 2 touching the outer Bloch sphere.

V_{Ω_ϵ} increases boundlessly as $r_{ts} < h$. This would be problematic, as V_{Ω_ϵ} is a subset of $\mathbb{P}$, and the volume of the latter is normalized to 1. However, what really happens here is the following: V_{Ω_ϵ} is a well-defined function of the volume V_ϵ regardless its geometry (i.e., spherical, cubic, etc.). Since $V_{\Omega_\epsilon}(V_\epsilon = 0) = 0$, the Taylor expansion of $V_{\Omega_\epsilon}(V_\epsilon)$, starts with the linear term, $V_{\Omega_\epsilon}(V_\epsilon) = \partial V_{\Omega_\epsilon}/\partial V_\epsilon|_{V_\epsilon=0} V_\epsilon + O(V_\epsilon^2)$, and it is exactly this truncated Taylor expansion what is provided by Eq. (23). When h tends to zero, the above Taylor expansion breaks down (just like, e.g., that of $\sqrt{x}$ at $x = 0$), and Eq. (23) is no longer a good approximation to $V_{\Omega_\epsilon}(V_\epsilon)$, which is always finite. In Appendix B, we explicitly compute the preimage volume of the origin $r_{ts} = 0$ in the limit $h \rightarrow 0$.

The second remark is that Eq. (24) has been derived assuming that separable states are produced following the distribution in Eq. (22), which is only defined on the subset of product states.

3. Coarse-graining statistics via RMT methods

The methods developed in the previous sections allowed us to gain deeper physical insight into the problem. However, when considering systems composed of a larger number of particles, the number of parameters involved in the parametrization grows exponentially, which increases computational complexity. For this reason, we resorted to random matrix theory, which will subsequently enable us to analyze the case in which an N -partite system is reduced to a single one. To introduce the RMT framework, in this section we compute the probability density of ρ for the bipartite case and compare our results with those obtained using geometric methods.

To establish the connection between the statistical properties of ρ and those of its matrix elements, we first recall the derivative principle for unitarily invariant ensembles of matrices of dimension $D \times D$. This theorem relates the joint probability density function (PDF) f_ρ of its eigenvalues λ_i and the joint PDF Ψ_ρ of its diagonal elements. Subsequently, we employ this relationship in the case where the target state is obtained from pure bipartite states through the coarse-graining mapping. Since the eigenvalues of a single-qubit state depend only on r_{ts} , the PDF of the eigenvalues directly gives us the PDF of r_{ts} . We express the joint PDF of the diagonal elements of the target state in integral form, considering the conditions that these elements must satisfy. The resulting integrals are then evaluated in Laplace space, which naturally facilitates the generalization of the expressions to N -partite systems. Finally, the Laplace transforms are inverted, and Ψ_ρ is substituted into the derivative principle theorem to obtain f_ρ .

The derivative principle was developed in Ref. [25], where it was used to calculate the eigenvalue PDFs of reduced density matrices of multipartite entangled states. Using the standard language of random matrix theory, it is enounced as follows [26,27]:

Let ρ be a random matrix drawn from a unitarily invariant random matrix ensemble, f_ρ the joint PDF of the eigenvalues of ρ , and Ψ_ρ the joint PDF of the diagonal elements of ρ .

Then,

$$f_\rho(\vec{\lambda}) = \left(\prod_{n=1}^D n! \right)^{-1} \Delta(\vec{\lambda}) \Delta(-\partial_{\vec{\lambda}}) \Psi_\rho(\vec{\lambda}), \quad (27)$$

where $\vec{\lambda} = (\lambda_1, \lambda_2, \dots, \lambda_D)$ are the eigenvalues of ρ , $\Delta(\vec{\lambda}) = \prod_{i < j} (\lambda_j - \lambda_i)$ is the Vandermonde determinant, and $\Delta(-\partial_{\vec{\lambda}})$ is the differential operator $\prod_{i < j} (\frac{\partial}{\partial \lambda_i} - \frac{\partial}{\partial \lambda_j})$.

In this article, we have restricted ourselves to target states that are monopartite, so in all cases we have $D = 2$. We identify the diagonal elements of ρ as ρ_{00} and ρ_{11} . Since ρ has unit trace, if the PDF of ρ_{00} is known, the joint PDF of the diagonal elements of the target state can be expressed as $\Psi_\rho(\rho_{00}, \rho_{11}) = \Psi(\rho_{00})\delta(\rho_{00} + \rho_{11} - 1)$. Substituting $\Psi_\rho(\rho_{00}, \rho_{11})$ into Eq. (27), it follows that $f_\rho(\lambda_1, \lambda_2) = \frac{1}{2}(\lambda_2 - \lambda_1)\Psi'(\lambda_1)\delta(\lambda_1 + \lambda_2 - 1)$. Integrating the PDF of the eigenvalues over λ_2 , we obtain the distribution of the single independent eigenvalue $f_\rho(\lambda_1) = \frac{1}{2}(1 - 2\lambda_1)\Psi'(\lambda_1)$. Moreover, the eigenvalues of the target state ρ are $\frac{1}{2}(1 \pm r_{ts})$, so there are two values of λ_1 that correspond to the same r_{ts} . Since both branches contribute equally to the probability, performing the change of variable $\lambda_1 = \frac{1}{2}(1 + r_{ts})$, we finally obtain

$$f_\rho(r_{ts}) = -r_{ts} \frac{d\Psi(r_{ts})}{dr_{ts}}. \quad (28)$$

We now use the relation (28) in the simplest case, when $N = 2$. Given the structure of the coarse-grained map $\mathcal{C}$, when applied to the pure bipartite state $|\psi\rangle = \sum_{i,j} c_{ij}|i, j\rangle$, the diagonal elements of the target state have the form

$$\rho_{00} = x_{00} + p_1 x_{01} + p_2 x_{10}, \quad (29)$$

$$\rho_{11} = x_{11} + p_2 x_{01} + p_1 x_{10} = 1 - \rho_{00}, \quad (30)$$

where it holds that $p_1 + p_2 = 1$ and $x_{ij} = |c_{ij}|^2 \in [0, 1]$. Additionally, since the state $|\psi\rangle$ is normalized, the condition $\sum_{i,j} x_{ij} = 1$ is satisfied, which implies that $\rho_{00} + \rho_{11} = 1$.

The equations (29) and (30) define a mapping from the three-simplex in the space of the x'_{ij} s to the one-simplex in the space of the ρ'_{ii} s. It can be shown that, for density matrices distributed according to the FS measure, the variables x_{ij} follow a uniform distribution on the corresponding three-simplex $S_0 \subset \mathbb{R}^4$ [28]. Therefore, the probability measure on S_0 can be expressed as $d\nu = 3!dx_{00}dx_{01}dx_{10}$, using coordinates (x_{00}, x_{01}, x_{10}) with $0 \leq x_{ij}$ and the constraint $0 \leq 1 - x_{00} - x_{01} - x_{10}$.

The PDF induced in the space of the ρ'_{ii} s is obtained by integrating the induced measure over the preimage of a point in the one-simplex. To perform this integration, we take ρ_{00} as the coordinate in both the one-simplex and the three-simplex, since ρ_{00} is defined in terms of the coordinates x_{ij} , as shown in Eq. (29). In this way, the preimage of ρ_{00} is parametrized by x_{01} and x_{10} , then

$$\begin{aligned} \Psi(\rho_{00}) &= 3! \int_H dx_{01} dx_{10} \\ &= 3! \int_{\mathbb{R}_+^2} \Theta(\rho_{00} - \tilde{\rho}_{00}) \Theta(\rho_{11} - \tilde{\rho}_{11}) dx_{01} dx_{10} \\ &\equiv 3!A(\rho_{00}, \rho_{11}), \end{aligned} \quad (31)$$

where H represents the preimage of ρ_{00} , defined as the region where the inequalities $0 \leq x_{01}$, $0 \leq x_{10}$, $0 \leq \rho_{00} - \tilde{\rho}_{00} = x_{00}$, and $0 \leq \rho_{11} - \tilde{\rho}_{11} = x_{11}$ hold, with

$$\tilde{\rho}_{00} = p_1 x_{01} + p_2 x_{10}, \quad (32)$$

$$\tilde{\rho}_{11} = p_2 x_{01} + p_1 x_{10}. \quad (33)$$

There are several ways to calculate the area $A(\rho_{00}, \rho_{11})$ bounded by the lines $\rho_{ii} = \tilde{\rho}_{ii}$ in the x_{01} - x_{10} space. However, with the aim of simplifying the calculations and generalizing the procedure to N -partite systems, we propose reformulating the problem in the Laplace space. For this, we consider the variables ρ_{ii} as independent in the domain $\rho_{ii} \in [0, \infty)$, then we can calculate the Laplace transform of $A(\rho_{00}, \rho_{11})$ and finally perform the integrals in x_{01} and x_{10} . As a result, we obtain

$$\begin{aligned} \tilde{A}(s_0, s_1) &= \mathcal{L}_{s_0, s_1}[A(\rho_{00}, \rho_{11})] \\ &= \frac{1}{s_0 s_1 (s_0 p_1 + s_1 p_2)(s_0 p_2 + s_1 p_1)}. \end{aligned} \quad (34)$$

Next, we calculate the inverse Laplace transform of the previous equation to obtain the PDF of ρ_{00} :

$$\begin{aligned} \Psi(\rho_{00}) &= 3!A(\rho_{00}, \rho_{11} = 1 - \rho_{00}) \\ &= \frac{3}{(p_2 - p_1)p_1 p_2} \left[(p_2 - p_1)(1 - \rho_{00})^2 \right. \\ &\quad \left. + (p_1 - \rho_{00})^2 \Theta\left(1 - \frac{\rho_{00}}{p_1}\right) \right. \\ &\quad \left. - (p_2 - \rho_{00})^2 \Theta\left(1 - \frac{\rho_{00}}{p_2}\right) \right]. \end{aligned} \quad (35)$$

Evaluating the function in (35) at $\frac{1}{2}(1 + r_{ts})$ and substituting into Eq. (28) with $p_1 = \frac{1}{2}(1 - h)$, $p_2 = \frac{1}{2}(1 + h)$, and $h \in [0, 1]$, we obtain

$$f_\rho(r_{ts}) = \frac{6r_{ts}}{h(1 - h^2)} [h(1 - r_{ts}) + (r_{ts} - h)\Theta(h - r_{ts})], \quad (36)$$

which recovers the PDF of r_{ts} obtained in Eq. (25) from the previous section.

4. A geometric interlude

The techniques used in the previous section are both powerful and intuitively opaque (at least to the authors). We try here to gain an intuitive, geometric understanding of some of the previous results.

Consider $\mathbb{R}^4$ with coordinates $\{x_{00}, x_{01}, x_{10}, x_{11}\}$, then the fact the x'_{ij} s are non-negative and sum to one means that the locus of x is the simplex S_0 with vertices on the standard basis vectors, and $a \equiv \rho_{00} = r'_1 \cdot x$, with $r'_1 = (1, p_1, p_2, 0)$, see Eq. (29). It will prove convenient to switch to coordinates (y_1, y_2, y_3, y_4) , defined by $y = Mx$, with $M \in SO(4)$ given by

$$M = \frac{1}{2} \begin{pmatrix} 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & 1 & 1 & 1 \end{pmatrix}. \quad (37)$$

Then $a = r_1 \cdot y$, with $r_1 = Mr'_1 = (p_2, 0, p_1, 1)$ —note that we denote vectors in the x basis by a prime, and those in the y

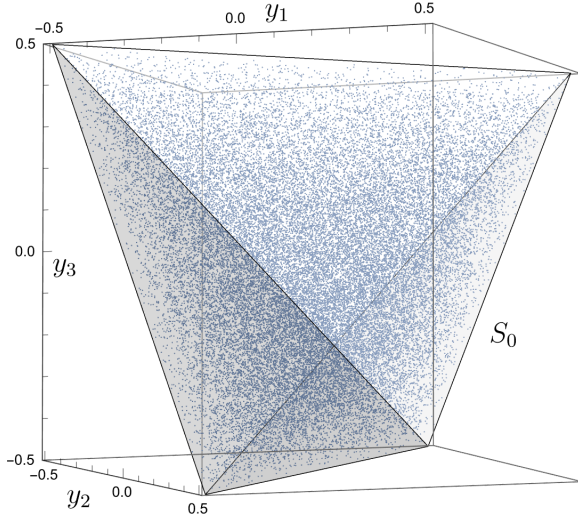


FIG. 4. The simplex S_0 and the image of 40 000 bipartite states in it.

basis without. The hyperplane Π_0 of S_0 has the equation $x \cdot n' = 1/2$, with $n' = (1, 1, 1)/2$ being the unit normal vector of Π_0 . In the y coordinates that equation becomes $y \cdot n = 1/2$, with $n = Mn' = (0, 0, 0, 1)$, i.e., all points on S_0 have $y_4 = 1/2$, so we may truncate y to its first three components when dealing with S_0 , $y = (\vec{y}, 1/2)$ on Π_0 . The vertices of S_0 , in the $\vec{y}$ coordinates, are given by the first three entries of the columns of M ,

$$S_0 = \left\{ \frac{1}{2}(1, 1, 1), \frac{1}{2}(-1, -1, 1), \frac{1}{2}(1, -1, -1), \frac{1}{2}(-1, 1, -1) \right\}. \quad (38)$$

The preimage of a on S_0 consists of those points $\vec{y}$ satisfying

$$\vec{r}_1 \cdot \vec{y} = a - \frac{1}{2}, \quad (39)$$

with $\vec{r}_1 = (p_2, 0, p_1)$. The above equation defines a 2-plane $\Pi_{p_1 a}$ in $\vec{y}$ space, which, generically, intersects S_0 in a convex polygon $H_{p_1 a}$. The value of p_1 determines the orientation of $H_{p_1 a}$, while varying a shifts $H_{p_1 a}$ parallel to itself. As mentioned before, when ρ_ψ follows the FS distribution, the corresponding distribution of $\vec{y}$ is uniform on S_0 . In Fig. 4, we plot S_0 and the image of 40 000 bipartite states in it, generated with the FS measure.

Then the PDF of a , for given p_1 , is proportional to the area of $H_{p_1 a}$, since, due to linearity, the “thickness” of the preimage of an infinitesimal segment da is constant, independent of a . In Fig. 5, we plot histograms of the distribution of a for various values of p_1 , using the above-mentioned set of states. In Fig. 6, we plot $H_{p_1 a}$ for two values of p_1 , and, for each of them, for six values of a . As explained in the caption to the figure, the evolution of the area of $H_{p_1 a}$ as a varies and reproduces the characteristics of the histograms in Fig. 5. To check on our results of the previous section, we compare with the expression (35), verifying the (piecewise) quadratic dependence on a , as expected from the geometric considerations presented above. A (suitably scaled) plot of $\Psi(\rho_{00})$, is superimposed on the corresponding histogram in Fig. 7.

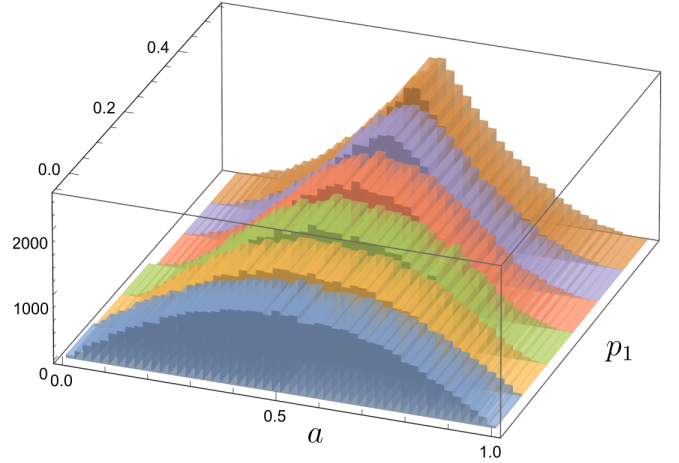


FIG. 5. Histogram of the distribution of $a \equiv \rho_{00}$, for various values of p_1 . Note that, for $p_1 = 0.5$ (orange-ochre histogram in the back), there is a cusp at $a = 0.5$, which is smoothed out for the other values of p_1 .

B. N -partite systems

A geometric description of systems with $N > 2$ qubits becomes increasingly complex because the number of parameters needed for a complete parametrization grows like $2(2^N - 1)$. Instead, we resort to the derivative principle theorem of RMT [25], which establishes a relationship, shown in Eq. (27), between the joint PDF Ψ_ρ of the diagonal elements of a state ρ and f_ρ the joint PDF of its eigenvalues, provided that ρ belongs to a unitarily invariant ensemble of matrices of dimension $D \times D$.

Let us express pure N -partite states as $|\psi\rangle = \sum_l c_l |l\rangle$ with $|l\rangle$ an element of the computational basis. We can then calculate $\Psi_\rho(r_{ts})$, and using Eq. (27), we find that the PDF of finding a target state, defined by the Bloch vector $\mathbf{r}_{ts}$, when

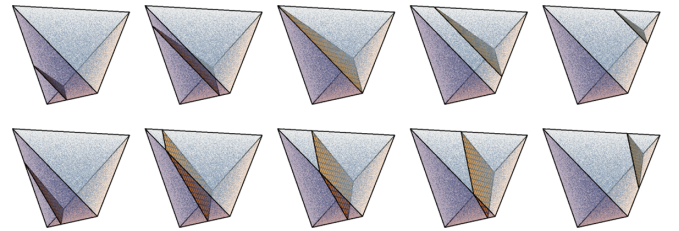


FIG. 6. (top row) The preimage $H_{p_1 a}$ of a in S_0 , for $p_1 = 0.5$ and $a = 0.2, 0.4, 0.5, 0.6, 0.8$ (from left to right). Compare with the orange-ochre histogram in Fig. 5. Because of the particular orientation of $H_{p_1 a}$ in this case, its size increases linearly with a , which implies that the “tails” on either side of the cusp in Fig. 5 are parabolic, while the cusp itself corresponds to the frame in the middle. (bottom row) Same as above, for $p_1 = 0.3$. Because the orientation of $H_{p_1 a}$ in this case, with respect to S_0 , is generic, the cusp observed for $p_1 = 0.5$ is smoothed out—nonanalyticity still creeps in when, e.g., $H_{p_1 a}$ meets (or leaves) the “front” edge of S_0 .

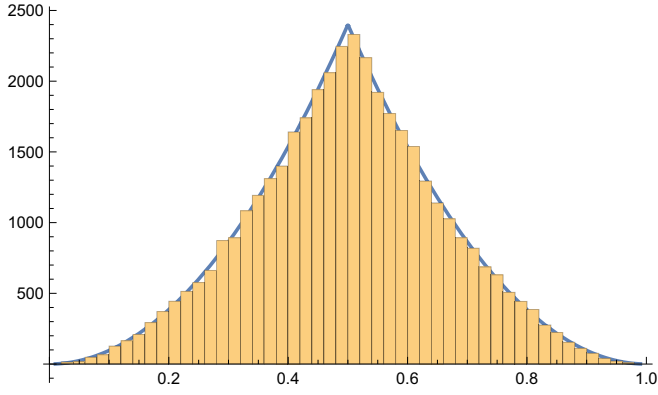


FIG. 7. PDF of a (appropriately scaled), for $p_1 = 0.499$, superimposed on the corresponding histogram (orange-ochre histogram in the back in Fig. 5).

applying the CG map to an N -partite state, is

$$P_N(p, r_{ts}) = c_N r_{ts} \sum_{l \neq (0, \dots, 0)} \frac{(\sum_i \tilde{p}_i^{(l)} - r_{ts})^{2^N - 3} \Theta(\sum_i \tilde{p}_i^{(l)} - r_{ts})}{\prod_{l' \neq (0, \dots, 0)} l' \cdot \tilde{p}^{(l)}}, \quad (40)$$

where $c_N = (2^N - 1)(2^N - 2)/2^{2^N - 2}$ is a normalization constant, $l = (l_1, \dots, l_N)$ is a multi-index with $l_i = 0, 1$, $p = (p_1, \dots, p_N)$ is the probability vector of the CG map, and $\tilde{p}_i^{(l)} = (2l_i - 1)p_i$ are the components of the probability vector with signs flipped according to l . The explicit calculation can be found in Appendix C. Equation (40) is piecewise continuous and, as a function of p , has poles at those points where two or more probabilities p_i are equal to each other. Furthermore, Eq. (40) reduces to Eq. (23) when $N = 2$, which supports its validity.

Figure 8 displays the corresponding PDF for the radius r_{ts} for 2, 3, 4, and 5 qubits for some specific values of the probabilities p_i . Note that as the number of qubits in the system

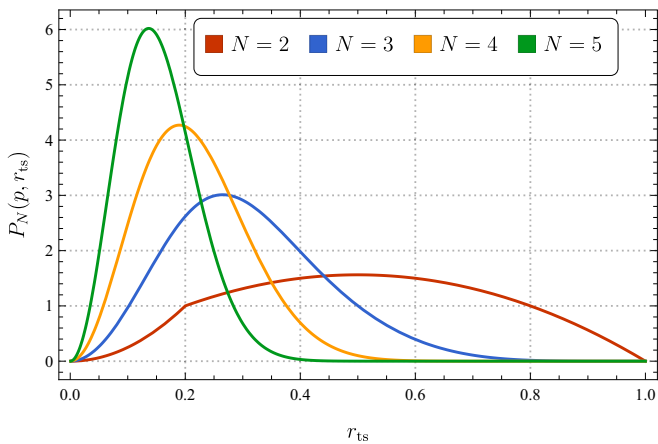


FIG. 8. PDF for $N = 2, 3, 4$, and 5 . Each of the functions has been normalized to the Bloch sphere of the target state. The probabilities p used are For $N = 2$, $\{0.4, 0.6\}$; for $N = 3$, $\{0.4, 0.35, 0.25\}$; for $N = 4$, $\{0.4, 0.35, 0.15, 0.1\}$; and for $N = 5$, $\{0.4, 0.35, 0.15, 0.08, 0.02\}$. Note that, as the number N of qubits in the system increases, the volume tends to take the form of a Gaussian.

increases, the probability density progressively resembles a Gaussian distribution, with mean value and variance that tend to zero, so that, for N large enough, the observed state is the maximally mixed one.

IV. AVERAGE STATE

Now we proceed to calculate the average states of the sets Ω_ϵ and $\Omega_\epsilon^\otimes$, describe their symmetries, and discuss some special cases. Let us recall that, within our framework, these average states serve as the preimages of the target state in the sense that $\overline{\mathcal{C}}^{-1}(\rho) = \varrho^{\text{as}}$. This average state plays a central role in mediating the connection between the emergent coarse-grained dynamics and the fine-grained unitary dynamics. Furthermore, we examine how the purity and coherence of the average states depend on the target radius r_{ts} . Finally, we comment on the difficulties involved in computing average states for N -partite systems and outline the strategy we adopt to address them.

The average state ϱ^{as} of the set Ω_ϵ^N is defined as

$$\varrho^{\text{as}} = \frac{1}{V(\Omega_\epsilon^N)} \int_{\Omega_\epsilon^N} d\mu \varrho, \quad (41)$$

and may be expanded in tensor products of Pauli σ_μ matrices, $\mu = 0, 1, 2, 3$ (with $\sigma_0 = I$),

$$\varrho^{\text{as}} = \frac{1}{2^N} \sum_v \varrho_v^{\text{as}} \sigma_v, \quad (42)$$

where $v = (v_1 \dots v_N)$ is a multi-index, and $\sigma_v \equiv \sigma_{v_1} \otimes \dots \otimes \sigma_{v_N}$. Since the Pauli matrices satisfy $\text{Tr}(\sigma_{\mu_1} \sigma_{\mu_2}) = 2\delta_{\mu_1 \mu_2}$, the coefficients ϱ_v can be obtained as

$$\varrho_v^{\text{as}} = \text{Tr}(\varrho^{\text{as}} \sigma_v) = \frac{1}{V(\Omega_\epsilon^N)} \int_{\Omega_\epsilon^N} d\mu \langle \sigma_v \rangle_\varrho. \quad (43)$$

For bipartite states ($N = 2$), this calculation involves the 16 elements of the density matrix, but only six elements are nonzero, with $\varrho_{0,0}^{\text{as}} = 1$. Additionally, we find that $\varrho_{2,2}^{\text{as}} = \varrho_{1,1}^{\text{as}}$, so only four elements are needed to fully determine the average state. This simplification arises from the symmetry of the system, as ρ is polarized in the z direction. Detailed calculations of the average state components for the set Ω_ϵ and a visualization of these components are provided in Appendix D.

For the average state of the set Ω_ϵ , when $r_{ts} \leq h$, we find that also $\varrho_{3,3}^{\text{as}} = \varrho_{1,1}^{\text{as}}$, meaning that in reality, knowing three average expected values is sufficient to completely determine ϱ^{as} . When $r_{ts} > h$, the four average expected values are distinct from each other, but it can be identified that there exists a symmetry $\varrho_{0,3}^{\text{as}}(h, r_{ts}) = \varrho_{3,0}^{\text{as}}(-h, r_{ts})$, which is also valid for the average state of the set of product bipartite states.

To gain physical insight about the properties of the average state, we consider the two extreme cases: the maximally mixed and pure target states. If the target state is $\rho = \frac{1}{2}I$ (equivalently $r_{ts} = 0$), the state satisfies $U\rho U^\dagger = \rho$ for any unitary U . Then, the symmetry property Eq. (9) of the CG map ensures that the average state will satisfy the condition $(U \otimes U)\varrho^{\text{as}}(U^\dagger \otimes U^\dagger) = \varrho^{\text{as}}$. This implies that the average

state will be isotropic and thus is of the form

$$\varrho^{\text{as}} = \alpha |\phi_1\rangle\langle\phi_1| + \frac{1-\alpha}{4} I \otimes I, \quad (44)$$

where $|\phi_1\rangle$ is the singlet state [29]. It turns out (using Eqs. (D11) and (D12), see Appendix D) that $\alpha = \frac{1-h}{3(1+h)}$. Note that the preimage of the single-qubit maximally mixed state is, by definition, the set of maximally entangled two-qubit states. On the other hand, the pure target state $|\hat{\mathbf{n}}\rangle$ has as its preimage (for generic h) the coherent state $|\hat{\mathbf{n}}\rangle \otimes |\hat{\mathbf{n}}\rangle$.

We now turn our attention to the set $\Omega_\epsilon^\otimes$. This case is particularly interesting because the convexity of the set of separable mixed states ensures that the average of $\Omega_\epsilon^{N\otimes}$ is separable, meaning that it does not exhibit entanglement between the qubits that compose it. In this case, we find that the components of the average state satisfy $\varrho_{3,3}^{\text{as}} = \varrho_{3,0}^{\text{as}}\varrho_{0,3}^{\text{as}}$, showing that the correlation along the z direction factorizes into the local polarizations of each qubit. This average state only exists when $r_{\text{ts}} \geq h$, as in the opposite case, the set $\Omega_\epsilon^\otimes$ is empty. Notably, when $r_{\text{ts}} = h$, the first element of the set appears, specifically $\varrho^{\text{as}} = |1, 0\rangle\langle 1, 0|$. In this scenario, the average state associated with a target state is the same for generic h . Additionally, when $r_{\text{ts}} = 1$, the preimage of the target state ρ is the same for both the complete and separable cases, i.e., $\Omega_R^\otimes = \Omega_R$.

In Fig. 9, the purity of the average state as a function of r_{ts} is depicted, along with the behavior of the coherences. Notice that the density matrix of the average state contains only two nonzero coherences. In the computational basis, these are $\varrho_{01,10}^{\text{as}} \equiv \varrho_{2,3}^{\text{as}}$ and $\varrho_{10,01}^{\text{as}} \equiv \varrho_{3,2}^{\text{as}}$ —not to be confused with the $(2, 3)$ and $(3, 2)$ elements in the Pauli basis, which are zero—and they are equal $\varrho_{2,3}^{\text{as}} = \varrho_{3,2}^{\text{as}}$. We observe that the average state derived from the set Ω_ϵ is pure only when the target state is pure; however, the probability density of measuring this state is zero. Furthermore, the coherences of this average state remain constant in the region where $r_{\text{ts}} \leq h$, and then they tend toward the line $\frac{1}{4}(1 - r_{\text{ts}})$; in fact, they are bounded: $0 \leq |\varrho_{2,3}^{\text{as}}| \leq \frac{1}{4}(1 - r_{\text{ts}})$. Regarding the average state obtained from the set $\Omega_\epsilon^\otimes$, there are two values of r_{ts} for which the average state is pure, both with a finite probability density of being observed. When $r_{\text{ts}} = h$ the average state is $|\psi\rangle = |1, 0\rangle$ and when $r_{\text{ts}} = 1$, it becomes $|\psi\rangle = |0, 0\rangle$. In both cases, the coherences vanish. Accordingly, the off-diagonal term satisfies the bound $0 \leq \varrho_{2,3}^{\text{as}} \leq \frac{1}{2}(1 - r_{\text{ts}})$.

Calculating the average state for $N > 2$ with geometrical arguments is again difficult, for identical reasons as exposed in Sec. III B. Additionally, since the sets of average expected values $\varrho_{i,j}^{\text{as}}$ are not unitarily invariant ensembles, the RMT methods used for the volume of N -partite systems are not applicable for calculating the corresponding expected values. Nevertheless, an analysis of the average state of the set Ω_ϵ has revealed that the symmetry $\mathcal{C}[(U \otimes U)|\psi\rangle\langle\psi|(U^\dagger \otimes U^\dagger)] = U\mathcal{C}[|\psi\rangle\langle\psi|]U^\dagger$ provides insight into the general structure of that state. Consequently, the analysis of the symmetry in Eq. (9) represents a promising approach for understanding the average state of a set composed of N -partite states.

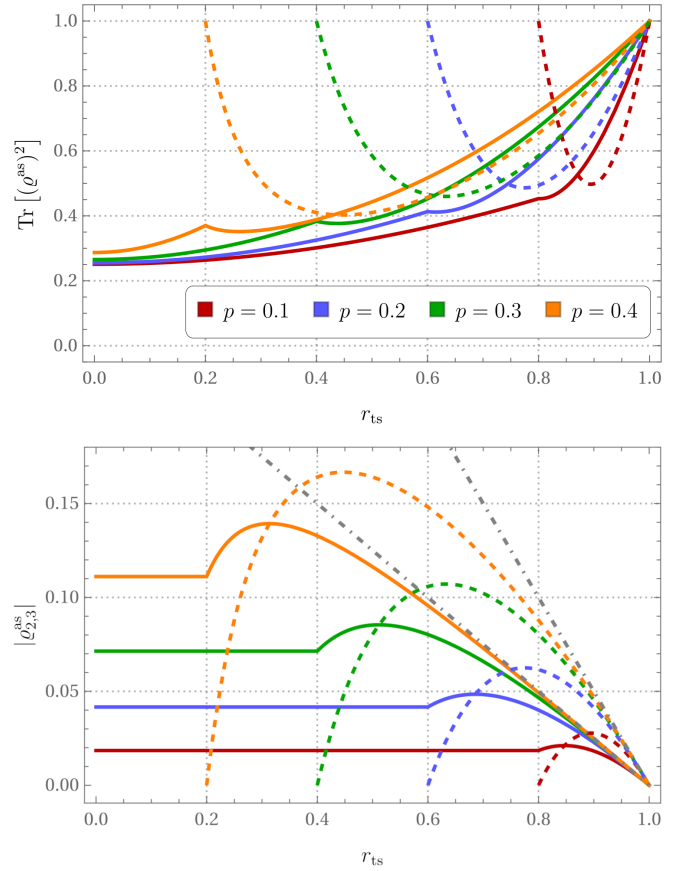


FIG. 9. The top plot shows the behavior of purity of the average state ϱ^{as} as a function of r_{ts} for different values of $p = 1/2(1 - h)$. The bottom plot shows how $\varrho_{2,3}^{\text{as}} = \varrho_{3,2}^{\text{as}}$ (the only coherences in the computational basis that are not null) change with r_{ts} . Both plots display results for Ω_ϵ , represented by solid lines, and $\Omega_\epsilon^\otimes$, shown with dashed lines. The dotted gray lines correspond, from left to right, to $\frac{1}{4}(1 - r_{\text{ts}})$ and $\frac{1}{2}(1 - r_{\text{ts}})$.

V. STATISTICS

A. Numerical preimage volumes

As discussed before, under a CG map, bipartite systems of two qubits are mapped to single-qubit mixed states $\rho \in \mathcal{M}_2$. In this particular case, the manifold $\mathcal{M}_2$ is diffeomorphic to the Euclidean ball B^3 , which allows us to use the standard Euclidean metric to specify a neighborhood V_ϵ around the target state ρ parametrized by the vector $\mathbf{r}_{\text{ts}}$. We calculate the volume $V(\Omega_\epsilon)$ of the preimage—lying in the space of pure bipartite states $\mathbb{C}P^3$ —of $\mathbf{r}_{\text{ts}}$.

In the more general case where the CG map sends N -partite pure states to m -partite mixed states $\rho \in \mathcal{M}_D$, with $\mathcal{M}_D$ the space of Hermitian positive matrices of size $D = 2^m$, the map is given by

$$C_m[\varrho] = \sum_{i=1}^b p_i \text{Tr}_{c(i)} \varrho, \quad (45)$$

where $b = \binom{N}{m'}$, $c(i)$ is the i th element of the set of m' -combinations from N elements, and $m + m' = N$. We notice that the target space is no longer diffeomorphic to any sub-

space of the Euclidean space; furthermore, the dimensions of both spaces $\mathbb{C}P^{2^N-1}$, and $\mathcal{C}_m(\mathbb{C}P^{2^N-1}) \subset \mathcal{M}_D$ where the target state lives, grow large at a very quick rate.

An alternative for exploring the behavior of the CG map in such cases is to calculate the normalized expected value of the composite spin operator, which corresponds to the natural action of the elements of the Lie algebra $\mathfrak{su}(2)$ on the tensor product of spin states

$$\mathcal{S}_\alpha = \sum_{i=1}^m I_1 \otimes \cdots \otimes \sigma_{\alpha_i} \otimes \cdots \otimes I_m, \quad (46)$$

where the subindex in the operators labels the Hilbert space where the i th qubit lives, and σ_α , $\alpha = 1, 2, 3$ are the three Pauli matrices. Then, given a target state $\rho \in \mathcal{M}_D$, its normalized spin expectation value (SEV) can be computed as

$$\langle \mathcal{S} \rangle = \frac{1}{m} \text{Tr}(\rho \mathcal{S}) \equiv \mathbf{r}_{\text{ts}} \in B^3, \quad (47)$$

which defines a map $\langle \mathcal{S} \rangle : \mathcal{C}_m(\mathbb{C}P^{2^N-1}) \rightarrow B^3$, that composed with the CG map gives us a mapping that sends a pure N -partite state to a physical vector in $\mathbb{R}^3$.

Due to the spherical symmetry of the SEV probability distribution, we are only interested in the distribution of its modulus r_{ts} . Then we ask for the probability of an N -partite state to be mapped to a spherical shell of radius r_{ts} and thickness ϵ . This question is equivalent to the one made before in the bipartite case, when we asked for the probability of a two-qubit system to be mapped to a sphere $V_\epsilon \subset B^3$ of radius ϵ , since the volume $V(\Omega_\epsilon^{N,m})$ of the preimage of some neighborhood around $\mathbf{r}_{\text{ts}}$ does not depend on the geometry of V_ϵ , but only on its volume.

As it has been mentioned before in this work, to provide an analytic answer in the general case is complicated due to the large amount of parameters needed to specify both the original pure states of N qubits and their images under the CG map. However, such an analytic result is provided in Sec. III B for the cases $N \rightarrow 1$, and here we use Monte Carlo simulations with samples of $n = 10^4$ random N -partite systems with $N = 2, 3, 4, 5$ to compare such results, and confirm that this statistical analysis can be used in the pursue of understanding the more general $N \rightarrow m$ case.

Then, we may call W the set of all the SEV calculated as in Eq. (47) from the sample of random target states. The preimage volume $V(\Omega_\epsilon^N)$ can be calculated through a Monte Carlo simulation as

$$V(\Omega_\epsilon^N)_{\text{MC}} = \frac{1}{n} \sum_{r \in W} \chi_{V_\epsilon}(r), \quad (48)$$

where $\chi_{V_\epsilon}(r)$ is the characteristic function of V_ϵ that, in this case, is taken as a spherical shell of thickness ϵ centered in r_{ts} ,

$$\chi_{V_\epsilon}(r) = \begin{cases} 1 & |r - r_{\text{ts}}| \leq \epsilon/2 \\ 0 & \text{else,} \end{cases} \quad (49)$$

the superindex in Ω_ϵ^N denotes the number of qubits that constitute the original pure state, and the superindex $m = 1$ has been dismissed for simplicity in the notation. Then, we calculate the

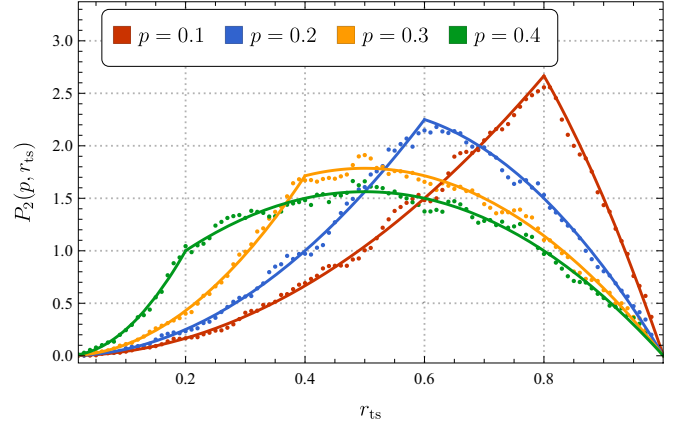


FIG. 10. Plot of the PDF function $P_2(p, r_{\text{ts}})$ taking different values of $p = 1/2(1 - h)$ (solid lines), compared with the results obtained numerically using a Monte Carlo simulation (dots), with $\epsilon = 0.04$.

PDF $P_N(p, r_{\text{ts}})$ as

$$P_N(p, r_{\text{ts}}) = \frac{V(\Omega_\epsilon^N)}{V_\epsilon} 4\pi r_{\text{ts}}^2, \quad (50)$$

where the factor $4\pi r_{\text{ts}}^2$ is due that $V(\Omega_\epsilon^N)/V_\epsilon$ represents the PDF of a target state specified only by the module of its SEV, then, since this probability does not depend on the orientation of the target state, we can obtain the PDF of the complete set of all states with the same r_{ts} , which is the quantity given by $P_N(p, r_{\text{ts}})$, just by multiplying by the area of the sphere with such radius.

As expected, when we performed the numerical experiment in the case when $N = 2$, we recovered the same results as with Eq. (23) and the corresponding case of Eq. (40), shown in Fig. 10. Similarly, for the cases when $N = 2, 3, 4, 5$, as shown in Fig. 11 the results are consistent with the analytic expression obtained in Eq. (40) via RMT methods.

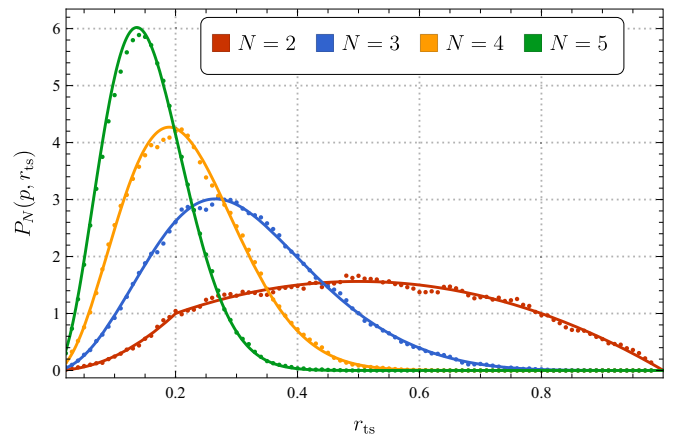


FIG. 11. Plot of the PDF, $P_N(p, r_{\text{ts}})$ for the cases when $N = 2$ to 5 (solid lines), compared with the results obtained numerically using a Monte Carlo simulation (dots), with $\epsilon = 0.04$.

B. CG detection and optimal volumes for numerical simulations

In a real physical situation, we would like to know the amount of uncertainty we must take into account due to our imperfect instruments. In other words, when an experiment is performed, we would like to know the coarse-graining we should consider in our measurements.

In the bipartite case, for example, this is easy to calculate from a set of experimental data using the least squares fitting method to compare the statistical analysis of the experimental results with the PDF given by Eq. (25).

We confirmed this by performing a numerical Monte Carlo experiment with a sample of the same amount n of data as in the previous simulations and a value of $p = 1/2(1 - h)$ chosen randomly. Using the method of least squares, we recover an approximate value of p_{fit} whose difference with respect to the original p will depend on the amount of data used for the fitting, and the choice of ϵ to define the volume of the neighborhood around the target state, in this case, the thickness of the spherical shell within B^3 .

On the other hand, while performing our simulations, we would like to have a method to choose an optimal volume V_ϵ for estimating the probability density $P_N(p, r_{\text{ts}})$; if ϵ is too large, the numeric simulations will return only averaged values of the volumes for a large range of values of r_{ts} , as expected, this leads to a PDF of an ensemble of random states which provides no information of the system. However, if ϵ is too small, the results of the simulations will contain considerably more noise and the data will be too scattered away from the model it is supposed to fit. Thus, the choice of an optimal value of ϵ while performing our numerical simulations is another problem we would like to address, and to which we can approach with this same procedure of looking for the best possible approximation for p .

For example, let us consider $p_{\text{test}} = 0.26$, and a sample of $n = 10^4$ bipartite states that have been mapped to single qubit mixed states via CG map, then we may use the least squares method to look for the value of p that best fits our data to the theoretical model $P_2(p, r_{\text{ts}})$, using two extreme cases as mentioned above $\epsilon = 0.001, 0.3$, and $\epsilon = 0.04$, obtaining $p_{\text{fit}}(\epsilon) = 0.2664, 0.4998, 0.2625$, respectively. With these results, we can conclude that the optimal ϵ in this case is $\epsilon = 0.04$ as it is shown in Fig. 12.

Then, with this method is possible to detect not only the presence of CG in some experiment but also a very precise estimate of the amount of it, as for our numerical experiments, it is possible to estimate an optimal value for the parameter ϵ that defines the volume of the neighborhood around the target state and with which we can calculate the volume of its preimage.

VI. CONCLUSIONS AND OUTLOOK

In this paper, we have considered the problem of coarse-graining in quantum many-body systems, focusing on the framework proposed in Ref. [12], which models detectors with addressing errors that cannot perfectly resolve individual particles. Our analysis centered on two fundamental questions: (i) the characterization of the probability distribution over N -partite fine-grained states compatible with a coarse-

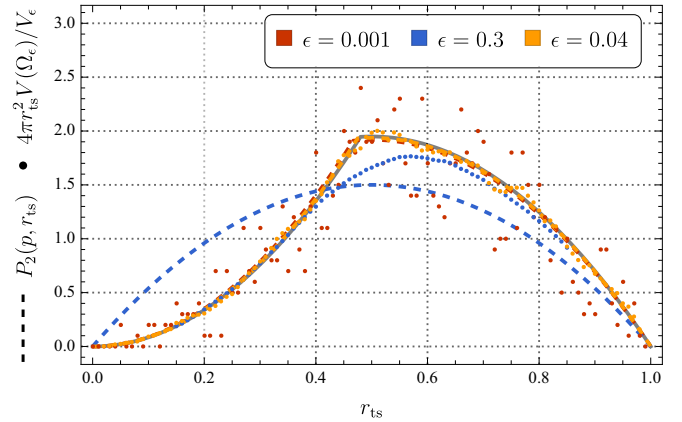


FIG. 12. Approximation to $P_2(p, r_{\text{ts}})$ with $p = 0.26$ (solid gray) using the least squares method (dashed lines), from the results of Monte Carlo simulations using spherical shells of thickness ϵ in the Bloch ball, taking two extreme cases of $\epsilon = 0.001$ (red) and $\epsilon = 0.3$ (blue) as described in the text, and one optimal value $\epsilon = 0.04$. The numeric Monte Carlo simulations are displayed as scattered dots.

grained observation, and (ii) the structure of the average state, particularly in the case where two particles are coarse-grained into a single subsystem.

The probability density function of a target state was obtained by computing the volume of the set of fine-grained states associated with an infinitesimal neighborhood of the target state and distributed according to the appropriate Fubini-Study measure. This PDF also provides insight into the distribution of physical properties in quantum many-body systems. In fact, it identifies the most likely states under this hypothesis; a sharply peaked distribution suggests the emergence of potentially classical-like behavior. In the scenario under study, we derived this distribution analytically for the bipartite case using geometric arguments, and for larger systems using tools from random matrix theory. Both results are exact and analytical. For bipartite systems, the preimage volume of an infinitesimal neighborhood around the target state is constant for target states within a certain distance from the maximally mixed state, and vanishes at the surface of the Bloch sphere (i.e., for pure states). For larger systems, the probability density behaves like a Gaussian centered at the maximally mixed state, implying an exponential difficulty in observing pure states with imperfect measuring devices.

Regarding the average state, we obtained an exact analytical expression in the bipartite case. The resulting state exhibits, in general, quantum correlations.

A particularly relevant finding is that the average preimage of the maximally mixed single-qubit state is not the maximally mixed two-qubit state, but rather an isotropic mixture with a singlet component. We also showed that, for a generic value of h , a pure target state can only arise from a coherent preimage state, which explains why the probability density vanishes in such cases.

Although the explicit computation of the average state was restricted to the case $N = 2$, the unitary covariance of the coarse-graining map provides a pathway to generalize these results. This symmetry can be exploited to simplify the characterization of preimages in more complex systems. All

analytical findings were supported by numerical simulations, which also serve as a practical tool for estimating the uncertainty introduced by measurement imperfections.

Important future research directions include analyzing preimage structures under more general coarse-graining schemes, such as mappings to multiple effective particles, or systems with local dimension $d > 2$. Using the structure of the average state to study the effective dynamics of coarse-grained systems constitutes an immediate next step. Furthermore, it would be highly desirable to explore connections between our approach and entropy-based reconstruction methods, as these could simplify the exploration of emergent dynamics under imperfect detection.

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DATA AVAILABILITY

The data that support the findings of this article are openly available [30].

APPENDIX A: FACTORIZATION OF THE FUBINI-STUDY MEASURE VIA A CG-ADAPTED PARAMETRIZATION

For the two-qubit case, a specific target state ρ is represented by a vector $\mathbf{r}_{ts}$ in the Bloch sphere. From Fig. 2, it can be observed that multiple states in $\mathbb{P}$ are mapped to the same target state ρ . Then, it is convenient to rewrite the FS measure in coordinates that are adapted to our problem of determining the inverse image of the CG map—we use it later in Appendix D. We factorize the FS measure as the product of a measure defined on the preimage $\Omega_{\mathbf{R}}$ of a target state $\mathbf{R}$ in the Bloch sphere and a measure on the Bloch ball itself.

As discussed in Sec. III A 2, the Bloch vectors of the reduced states $\mathbf{r}_1$ and $\mathbf{r}_2$, lie on spheres of radii $R_1 = r_{ts}(1 + h)/2h$ and $R_2 = r_{ts}(1 - h)/2h$, with centers $\mathbf{c}_1 = -\mathbf{r}_{ts}(1 - h)/2h$ and $\mathbf{c}_2 = \mathbf{r}_{ts}(1 + h)/2h$, respectively. A convenient way to parametrize the reduced states is to place both spheres along the z axis, and then rotate the z axis onto the axis of a target state $\mathbf{R}$. This parametrization can be written as

$$\begin{aligned} \mathbf{r}_i &= R \mathcal{R}_{\Theta\Phi} \mathbf{a}_i(h, u, v), \\ \mathbf{a}_1(h, u, v) &= \frac{1}{2h} ((1 + h)\hat{\mathbf{m}}_{uv} - (1 - h)\hat{\mathbf{k}}), \\ \mathbf{a}_2 &= \mathbf{a}_1(-h, u, v), \end{aligned} \quad (\text{A1})$$

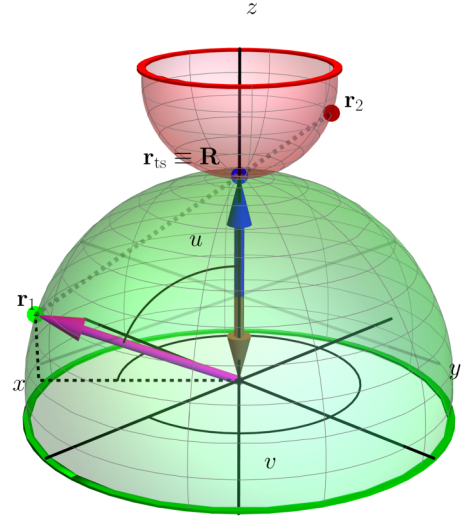


FIG. 13. Subspheres corresponding to the reduced states that form a target state $\mathbf{R}$ aligned along the z axis. The Bloch vectors $\mathbf{r}_1$ and $\mathbf{r}_2$ are shown as the green and red points, respectively, while the vector of the target state $\mathbf{R}$ is represented in blue. The vector $\mathbf{r}_1$ points in the direction of $\frac{1+h}{2h}\hat{\mathbf{m}}_{uv}$ (magenta arrow) from the center of the green sphere $-\frac{1-h}{2h}\hat{\mathbf{k}}$ (brown arrow). If $\mathbf{R}$ is fixed, once $\mathbf{r}_1$ is chosen, $\mathbf{r}_2$ is completely determined. Hence, (u, v) serve as coordinates in the preimage $\Omega_{\mathbf{R}}$.

where $\hat{\mathbf{m}}_{uv}$ is a unit vector with angular coordinates (u, v) , which parametrizes the sphere outside the origin where each of the vectors $\mathbf{r}_i$ lies, see Fig. 13. The matrix $\mathcal{R}_{\Theta\Phi}$ represents a rotation by an angle Θ along the axis $(-\sin \Phi, \cos \Phi, 0)$, which maps the z axis to the (Θ, Φ) position on the unit sphere along a geodesic. Notice that when $\Theta = 0$, $\mathbf{r}_i = R\mathbf{a}_i$.

In this way, the Bloch vectors satisfy Eq. (19) where $\mathbf{r}_{ts} \equiv \mathbf{R}$ has spherical coordinates (R, Θ, Φ) . Now, we write the surface elements $d\omega_i$ in Eq. (21) as

$$d\omega_i = r^{-3}(x_i dy_i dz_i + \text{cyclic}), \quad (\text{A2})$$

and, using Eq. (A1), we express them in terms of the variables (u, v, R, Θ, Φ) . This gives

$$d\omega_1 d\omega_2 = h^{-2} r^{-4} R^4 \sin u \sin \Theta du dv d\Theta d\Phi, \quad (\text{A3})$$

with r directly calculated from Eq. (A1) as

$$r^2 = r_1^2 = r_2^2 = \frac{R^2}{2h^2} [1 + h^2 - (1 - h^2) \cos u]. \quad (\text{A4})$$

Furthermore, (A4) can be rewritten as

$$2h^2 \kappa^2 = 1 + h^2 - (1 - h^2) \cos u, \quad (\text{A5})$$

with $\kappa = r/R$ denoting the ratio between the radius of the reduced states and that of the target state. Differentiating the above expressions allows us to write the differentials du and dr in Eqs. (A3) and (21) in terms of the differentials $d\kappa$ and dR . Substituting these into Eq. (21), we obtain

$$d\mu = \frac{3}{8\pi^3(1 - h^2)} d\kappa dv d\gamma dV \equiv dV_{\Omega} dV, \quad (\text{A6})$$

with $dV = R^2 \sin \Theta dR d\Theta d\Phi$ being the Euclidean volume measure on the Bloch sphere. The parameters (R, Θ, Φ) are

coordinates on the CG map image, while (κ, v, γ) are coordinates on the preimage $\Omega_{\mathbf{R}}$ of a fixed-target state $\mathbf{R}$, and dV_{Ω} is a measure on such preimage. Since $\cos u \in [-1, 1]$, $\kappa \in [1, h^{-1}]$, and constrained by $r \leq 1$, we have $1 \leq \kappa \leq \kappa_f$, with $\kappa_f = h^{-1}$ if $R \leq h$, and $\kappa_f = R^{-1}$ if $h < R \leq 1$. The coordinates v and γ range from 0 to 2π .

In the case of separable states, all states are of the form $|\Psi\rangle = |\mathbf{n}_1\rangle|\mathbf{n}_2\rangle$, so $(\theta_1, \phi_1, \theta_2, \phi_2)$ are coordinates in this space. The Fubini-Study metric in the bipartite state space induces an invariant metric in the space of separable states with measure $d\mu^{\otimes} = d\omega_1 d\omega_2 / 16\pi^2$, where $16\pi^2$ is the total volume of separable states. Taking into account that separable states satisfy $r = 1$, by differentiating Eq. (A4), we can express du in terms of dR and substitute it into Eq. (A3) to obtain

$$d\mu^{\otimes} = \frac{dv dV}{4\pi^2(1-h^2)R}, \quad (\text{A7})$$

where the range of κ now imposes that $h \leq R \leq 1$, while the other variables span their entire range.

APPENDIX B: VOLUME INTEGRALS FOR TWO-QUBIT SYSTEM

We calculate the volume $V(\Omega_{\epsilon})$ of the preimage of a neighborhood V_{ϵ} in the Bloch ball by integrating the measure in (A6), first over the preimage $\Omega_{\mathbf{R}}$ of each $\mathbf{R}$ in V_{ϵ} , and then over the entire V_{ϵ} . For this, we consider that, in the most general case, there is a region V_{in} in V_{ϵ} within a sphere of radius h , where $\kappa \in [1, h^{-1}]$, and another region V_{out} outside the sphere of radius h , where $\kappa \in [1, R^{-1}]$. This results in

$$\begin{aligned} V(\Omega_{\epsilon}) &= \int_{V_{\epsilon}} dV \int_{\Omega_{\mathbf{R}}} dV_{\Omega} \\ &= \frac{3}{8\pi^3(1-h^2)} \int_{V_{\epsilon}} dV \int_1^{\kappa_f} d\kappa \int_0^{2\pi} dv \int_0^{2\pi} d\gamma \\ &= \frac{3}{2\pi(1-h^2)} \left[\int_{V_{\text{in}}} dV (h^{-1} - 1) + \int_{V_{\text{out}}} dV (R^{-1} - 1) \right] \\ &= \frac{3}{2\pi h(1+h)} \int_{V_{\text{in}}} dV \\ &\quad + \frac{3}{2\pi(1-h^2)} \int_{V_{\text{out}}} dV (R^{-1} - 1), \end{aligned} \quad (\text{B1})$$

where the integral over κ in the second line is performed up to $\kappa_f = h^{-1}$ for $\mathbf{R} \in V_{\text{in}}$ and up to $\kappa_f = R^{-1}$ for $\mathbf{R} \in V_{\text{out}}$. Expression (B1) is not subject to any approximation and can be integrated numerically for any V_{ϵ} within the Bloch ball. However, we associate the neighborhood V_{ϵ} with the experimental error bars of the measurements determining the state ρ . As a first approximation, we assume that these error bars define an infinitesimal neighborhood centered at the Bloch vector $\mathbf{r}_{\text{ts}}$. In this case, we can disregard the transition in which V_{ϵ} changes from being entirely within the sphere of radius h to being entirely outside, as only these two cases are relevant. We also take the integrands as constants, equal to their value at $\mathbf{r}_{\text{ts}}$. The volume then reduces to

$$V(\Omega_{\epsilon}(\mathbf{r}_{\text{ts}})) = \frac{3V_{\epsilon}}{2\pi(1+h)} \begin{cases} \frac{1}{h} & \text{if } 0 \leq r_{\text{ts}} < h \\ \frac{1-r_{\text{ts}}}{(1-h)r_{\text{ts}}} & \text{if } h < r_{\text{ts}} < 1, \end{cases} \quad (\text{B2})$$

where $\Omega_{\epsilon}(\mathbf{r}_{\text{ts}})$ is the preimage of an infinitesimal neighborhood V_{ϵ} at a distance r_{ts} from the origin. From now on, we use V_{ϵ} to refer both to the neighborhood and its volume.

It is important to mention that expression (B2) does not depend on the shape of the neighborhood V_{ϵ} but only on its volume. Additionally, this expression is not valid for a neighborhood centered at the origin ($r_{\text{ts}} = 0$) when $h \rightarrow 0$ because it is not possible to evaluate this limit with V_{ϵ} entirely inside or entirely outside the sphere of radius h . To obtain a finite value in this limit, we can analytically calculate expression (B1) for a V_{ϵ} as a ball of radius ϵ centered at the origin, with $h < \epsilon$, where V_{in} is defined by $0 \leq R \leq h$ and V_{out} by $h < R \leq \epsilon$. This results in

$$V(\Omega_{\epsilon}(0)) = \frac{\epsilon^2(3-2\epsilon) - h^2}{1-h^2}, \quad h < \epsilon, \quad (\text{B3})$$

which, in the limit as $h \rightarrow 0$, results in $\epsilon^2(3-2\epsilon)$.

For separable states, the neighborhood V_{ϵ} can only be outside the sphere of radius h . By directly integrating the measure (A7), we obtain

$$V(\Omega_{\epsilon}^{\otimes}) = \frac{1}{2\pi(1-h^2)} \int_{V_{\epsilon}} \frac{dV}{R}. \quad (\text{B4})$$

Again, taking the approximation of V_{ϵ} as infinitesimal at a distance r_{ts} from the origin, we find that

$$V(\Omega_{\epsilon}^{\otimes}(\mathbf{r}_{\text{ts}})) = \frac{V_{\epsilon}}{2\pi(1-h^2)r_{\text{ts}}}, \quad h < r_{\text{ts}} < 1. \quad (\text{B5})$$

APPENDIX C: RMT METHODS FOR N QUBITS

This Appendix aims to present the generalization of the methods implemented in the previous appendix for application in systems composed of N qubits. To begin, we introduce the notation that allows us to efficiently express the N -partite state and the diagonal elements of the target state. Subsequently, we represent the joint probability density function of the diagonal elements of the target state in integral form and solve these integrals in the Laplace space. We apply the Vandermonde operator of partial derivatives and invert the Laplace transform using the residue theorem. As a final result, we obtain the probability density function of the target state derived from states composed of N qubits, as a function of the Bloch radius of the target state and the set of probabilities that define the CG map. We once again note that the target state considered is always monopartite, hence $D = 2$.

We start with Eq. (27), which states a relationship between $f_{\rho}(\vec{\lambda})$, the joint PDF of the eigenvalues of ρ , and $\Psi_{\rho}(\vec{\rho})|_{\vec{\rho}=\vec{\lambda}}$, the joint PDF of its diagonal elements (evaluated in the eigenvalues of ρ).

We consider a pure N -partite state of the form

$$|\psi\rangle = \sum_l c_l |l\rangle, \quad l = (l_1, \dots, l_N), \quad (\text{C1})$$

where $l_i = 0, 1$ represents the state of the i th qubit, and $|l\rangle$ is an element of the computational basis.

Density matrices of N qubits are distributed according to the FS measure [28], their diagonal elements $x_l = |c_l|^2$ so that $\sum_l x_l = 1$ and $x_l \geq 0$ have a uniform distribution over the $(2^N - 1)$ -simplex.

We may then exclude the last element $l = (1, \dots, 1)$ and implement the condition $0 \leq 1 - \sum_{l \neq (1, \dots, 1)} x_l$ so we can write the distribution of the diagonal elements of ρ as

$$dv = (2^N - 1)! \prod_{l \neq (1, \dots, 1)} dx_l, \quad (\text{C2})$$

where the constant factor corresponds to the volume of the standard $(2^N - 1)$ -simplex.

Applying the CG map on the state (C1), we obtain a target state whose diagonal elements are of the form

$$\rho_{00} = \sum_l \bar{l} \cdot p x_l, \quad \rho_{11} = \sum_l l \cdot p x_l, \quad (\text{C3})$$

where $\bar{l} = (\bar{l}_1, \dots, \bar{l}_N)$ with $\bar{l}_i = 1 - l_i$ and $p = (p_1, \dots, p_N)$. As in the case when $N = 2$, it must hold that $\sum_i p_i = 1$, which implies that $\rho_{00} + \rho_{11} = 1$, defining thus a mapping from the $(2^N - 1)$ -simplex to the one-simplex. The joint probability density function of the diagonal elements (ρ_{00}, ρ_{11}) of ρ , is now given only in terms of ρ_{00} by the integral

$$\begin{aligned} \Psi(\rho_{00}) &= (2^N - 1)! \int_H \prod_{l \neq (1, \dots, 1)} dx_l \\ &= (2^N - 1)! \int_{\mathbb{R}_+^S} \prod_{\ell} dx_{\ell} \Theta(\rho_{00} - \tilde{\rho}_{00}) \Theta(\rho_{11} - \tilde{\rho}_{11}) \\ &\equiv (2^N - 1)! A(\rho_{00}, \rho_{11}), \end{aligned} \quad (\text{C4})$$

where $S = 2^N - 2$, ℓ index the elements of the computational basis excluding the first $(0, \dots, 0)$ and last $(1, \dots, 1)$

elements. H represents the preimage of ρ_{00} , defined as the region where the inequalities $0 \leq x_{\ell}$, $0 \leq \rho_{00} - \tilde{\rho}_{00} = x_{0\dots 0}$, and $0 \leq \rho_{11} - \tilde{\rho}_{11} = x_{1\dots 1}$ hold, with

$$\tilde{\rho}_{00} = \sum_{\ell} \bar{\ell} \cdot p x_{\ell}, \quad (\text{C5})$$

$$\tilde{\rho}_{11} = \sum_{\ell} \ell \cdot p x_{\ell}. \quad (\text{C6})$$

Now, in order to calculate the operator $\Delta(\vec{\rho})\Psi_{\rho}(\rho_{00}, \rho_{11})$, with $\vec{\rho} = (\rho_{00}, \rho_{11})$, we calculate the Laplace transform of the area function $A(\rho_{00}, \rho_{11})$. Let us proceed:

$$\begin{aligned} \tilde{A}(s_0, s_1) &= \mathcal{L}_{s_0, s_1}[A(\rho_{00}, \rho_{11})] \\ &= \int_{\mathbb{R}_+^S} \prod_{\ell} dx_{\ell} \mathcal{L}_{s_0, s_1}[\Theta(\rho_{00} - \tilde{\rho}_{00})\Theta(\rho_{11} - \tilde{\rho}_{11})] \\ &= \frac{1}{s_0 s_1} \int_{\mathbb{R}_+^S} \prod_{\ell} dx_{\ell} e^{-s_0 \tilde{\rho}_{00}} e^{-s_1 \tilde{\rho}_{11}} \\ &= \frac{1}{s_0 s_1} \prod_{\ell} \int_{\mathbb{R}_+^S} dx_{\ell} e^{-(\bar{\ell} \cdot p s_0 + \ell \cdot p s_1) x_{\ell}} \\ &= \prod_l (\bar{l} \cdot p s_0 + l \cdot p s_1)^{-1}. \end{aligned} \quad (\text{C7})$$

The change in the index ℓ to l in the last step of the previous calculation comes from the result that $\sum_i p_i = 1$, thus, when $l = (1, \dots, 1)$ we obtain $l \cdot p s_i = s_i$ and $\bar{l} \cdot p s_i = 0$, and analogously for $l = (1, \dots, 1)$.

Then, applying the operator $\Delta(-\partial_{(\rho_{00}, \rho_{11})})$ in the Laplace space and subsequently inverting the transform, we get

$$\begin{aligned} \Delta(-\partial_{\vec{\rho}})A(\rho_{00}, \rho_{11}) &= \mathcal{L}_{\rho_{00}, \rho_{11}}^{-1}[(s_0 - s_1)\tilde{A}(s_0, s_1)] = \frac{1}{(2\pi i)^2} \oint_{\gamma} \oint_{\gamma} ds_0 ds_1 (s_0 - s_1) \prod_l (\bar{l} \cdot p s_0 + l \cdot p s_1)^{-1} e^{s_0 \rho_{00}} e^{s_1 \rho_{11}} \\ &= (2^N - 1)! \sum \text{Res} \left[(s_0 - s_1) \prod_l (\bar{l} \cdot p s_0 + l \cdot p s_1)^{-1} e^{s_0 \rho_{00}} e^{s_1 \rho_{11}}; s_0, s_1 \right], \end{aligned} \quad (\text{C8})$$

where γ is a simple closed curve sufficiently large so that it contains all the poles of the function.

We begin by calculating the residues with respect to s_0 , which arise when $s_0 = -\frac{l' \cdot p}{\bar{l}' \cdot p} s_1$, excluding the case $\bar{l}' = (0, \dots, 0)$. When $l \cdot p$ yields a distinct value for each l , there are $2^N - 1$ simple poles. Consequently, for a given l' , the residue is

$$\begin{aligned} &\lim_{s_0 \rightarrow -\frac{l' \cdot p}{\bar{l}' \cdot p} s_1} \left[\left(s_0 + \frac{l' \cdot p}{\bar{l}' \cdot p} s_1 \right) (s_0 - s_1) \prod_l (\bar{l} \cdot p s_0 + l \cdot p s_1)^{-1} e^{s_0 \rho_{00}} e^{s_1 \rho_{11}} \right] \\ &= - \prod_{l \neq l'} \left\{ [(l \cdot p)(\bar{l}' \cdot p) - (\bar{l} \cdot p)(l' \cdot p)]^{-1} (\bar{l}' \cdot p)^{2^N - 3} s_1^{-(2^N - 2)} e^{s_1(a_1 - \frac{l' \cdot p}{\bar{l}' \cdot p} \rho_{00})} \Theta\left(a_1 - \frac{l' \cdot p}{\bar{l}' \cdot p} \rho_{00}\right) \right\}. \end{aligned} \quad (\text{C9})$$

The expression (C9) has only one pole with respect to s_1 , which is a pole at zero of order $2^N - 2$. We use the Laurent expansion

$$\frac{e^{s_1(\rho_{11} - \frac{l' \cdot p}{\bar{l}' \cdot p} \rho_{00})}}{s_1^{(2^N - 2)}} = \sum_{n=0}^{\infty} \frac{(\rho_{11} - \frac{l' \cdot p}{\bar{l}' \cdot p} \rho_{00})^n}{n!} s_1^{n - (2^N - 2)} \quad (\text{C10})$$

to find that the residue takes the form

$$\frac{1}{(2^N - 3)!} \left(\rho_{11} - \frac{l' \cdot p}{\bar{l}' \cdot p} \rho_{00} \right)^{2^N - 3}. \quad (\text{C11})$$

Finally, we can compile the previous results by summing all the residues to determine that

$$\Delta(-\partial_{\vec{p}^{(l)}})\Psi(\rho_{00}, \rho_{11}) = -\frac{(2^N - 1)!}{(2^N - 3)!} \sum_{l' \neq (0, \dots, 0)} (\vec{l}' \cdot p \rho_{11} - l' \cdot p \rho_{00})^{2^N - 3} \Theta(\vec{l}' \cdot p \rho_{11} - l' \cdot p \rho_{00}) \\ \times \prod_{l \neq l'} [(l \cdot p)(\vec{l}' \cdot p) - (l' \cdot p)(\vec{l} \cdot p)]^{-1}. \quad (\text{C12})$$

We can finally gather the result in a compact expression. We define a transformed vector, that depends on both the set of probabilities p and the multi-index l , with components $\tilde{p}_i^{(l)} = (2l_i - 1)p_i$. $\tilde{p}^{(l)}$ is the probability vector with signs flipped according to l . We end up with

$$P_N(p, r_{\text{ts}}) = c_N r_{\text{ts}} \sum_{l \neq (0, \dots, 0)} \frac{(\sum_i \tilde{p}_i^{(l)} - r_{\text{ts}})^{2^N - 3} \Theta(\sum_i \tilde{p}_i^{(l)} - r_{\text{ts}})}{\prod_{l' \neq (0, \dots, 0)} l' \cdot \tilde{p}^{(l)}}, \quad (\text{C13})$$

with the normalization constant $c_N = (2^N - 1)(2^N - 2)/2^{2^N - 2}$. Notice that, due to the unitary invariance of the problem, r_{ts} can be taken as the radius of the Bloch vector independently of the direction of the polarization of the state. The above formula constitutes the main result of the appendix.

Note that the analytical formula in Eq. (C13) does not admit cases where some pair of probabilities p_k are equal. Such cases can be dealt with before inverting the Laplace transforms. For example, the case with all probabilities being equal, $p_k = \frac{1}{N}$, leads to the formula

$$\tilde{\Psi}(s_0, s_1) = N^{2^N} (2^N - 1)! \prod_{n=0}^N [s_0(N - n) + s_1 n]^{-\binom{N}{n}}. \quad (\text{C14})$$

In this case, the poles of the function to be inverted are of binomial order as a function of N , which makes it difficult to find an analytical expression for arbitrary N . However, once the size of the system is fixed, it is possible to apply Laplace transform techniques to invert the corresponding expressions.

APPENDIX D: AVERAGE-STATE INTEGRALS

We define the average state in the preimage of a neighborhood V_ϵ as

$$\varrho^{\text{as}} = \frac{1}{V(\Omega_\epsilon)} \int_{\Omega_\epsilon} d\mu \varrho, \quad (\text{D1})$$

where ϱ is a state in the space of bipartite pure states. Since the trace is linear, it is straightforward to see that $\text{Tr} \varrho^{\text{as}} = 1$. Additionally, since the measure $d\mu$ is non-negative, the integral divided by the volume $V(\Omega_\epsilon)$ represents a convex sum of pure states ϱ , which generally results in a mixed state, not a pure state.

From Eq. (16), it follows that a density matrix in the space of pure states takes the form

$$\varrho = \cos^2 \frac{\eta}{2} |\mathbf{n}_1\rangle\langle\mathbf{n}_1| \otimes |\mathbf{n}_2\rangle\langle\mathbf{n}_2| \\ + \sin^2 \frac{\eta}{2} |-\mathbf{n}_1\rangle\langle-\mathbf{n}_1| \otimes |-\mathbf{n}_2\rangle\langle-\mathbf{n}_2| \\ + \frac{1}{2} \sin \eta (e^{i\gamma} |-\mathbf{n}_1\rangle\langle\mathbf{n}_1| \otimes |-\mathbf{n}_2\rangle\langle\mathbf{n}_2| + \text{H.c.}), \quad (\text{D2})$$

where the third term is cyclic in γ , so it does not contribute to the integral with the measure in Eq. (A6). Taking the density matrices $|\mathbf{n}_i\rangle\langle\mathbf{n}_i| = \frac{1}{2}(I + \mathbf{n}_i \cdot \boldsymbol{\sigma})$ and expanding the tensor products, the integral in (D1) becomes

$$\int d\mu \varrho = \frac{1}{4} \int d\mu \left[I \otimes I + (\mathbf{r}_1 \cdot \boldsymbol{\sigma}) \otimes I \right. \\ \left. + I \otimes (\mathbf{r}_2 \cdot \boldsymbol{\sigma}) + \sum_{ij} n_{1i} n_{2j} \sigma_i \otimes \sigma_j \right], \quad (\text{D3})$$

where we recall that $\mathbf{r}_i = r \mathbf{n}_i$ with $r = \cos \eta$. In this way, the problem reduces to integrating the Bloch vectors $\mathbf{r}_i$ and the components of the tensor product $\mathbf{n}_1 \otimes \mathbf{n}_2$ on Ω_ϵ . We use the parametrization in Eq. (A1) of the Bloch vectors, together with Eq. (A5) to calculate the integrals first over the preimage Ω_R in (κ, v, γ) coordinates and then over all R in the neighborhood V_ϵ .

The integrals of the Bloch vectors can be expressed as

$$\int_{\Omega_\epsilon} d\mu \mathbf{r}_i = \int_{V_\epsilon} dV R \mathcal{R}_{\Theta\Phi} \int_{\Omega_R} dV_\Omega \mathbf{a}_i, \quad (\text{D4})$$

where, by direct integration, it follows that

$$\int_{\Omega_R} dV_\Omega \mathbf{a}_1 = \frac{(\kappa_f - 1)[3 - h(\kappa_f^2 + \kappa_f + 1)]}{2\pi(1 - h)^2(1 + h)} \hat{\mathbf{k}}, \quad (\text{D5})$$

and that the integral of $\mathbf{a}_2$ is obtained by exchanging h for $-h$. By performing the integration over V_ϵ , we obtain

$$\int_{\Omega_\epsilon} d\mu \mathbf{r}_1 = -\frac{1 - h}{2\pi h^2(1 + h)} A + \frac{1}{2\pi(1 - h^2)(1 + h)} B, \\ \int_{\Omega_\epsilon} d\mu \mathbf{r}_2 = \frac{1 + 4h + h^2}{2\pi h^2(1 + h)^2} A + \frac{1}{2\pi(1 - h^2)(1 + h)} C, \quad (\text{D6})$$

with V_{in} and V_{out} defined as in Eq. (B1), and

$$A = \int_{V_{\text{in}}} dV R, \\ B = \int_{V_{\text{out}}} dV (R^{-1} - 1)[3 - h(R^{-2} + R^{-1} + 1)]R, \\ C = \int_{V_{\text{out}}} dV (R^{-1} - 1)[3 + h(R^{-2} + R^{-1} + 1)]R.$$

We compute the integral of $n_{1i}n_{2j}$ as the components of the integral of the tensor product as follows

$$\begin{aligned} \int_{\Omega_\epsilon} d\mu \mathbf{n}_1 \otimes \mathbf{n}_2 &= \int_{\Omega_\epsilon} d\mu (\kappa R)^{-2} \mathbf{r}_1 \otimes \mathbf{r}_2 \\ &= \int_{V_\epsilon} dV (\mathcal{R}_{\Theta\Phi} \otimes \mathcal{R}_{\Theta\Phi}) \end{aligned} \quad (\text{D7})$$

$$\times \int_{\Omega_R} dV_\Omega \kappa^{-2} \mathbf{a}_1 \otimes \mathbf{a}_2. \quad (\text{D8})$$

A direct integration yields

$$\begin{aligned} \int_{\Omega_R} dV_\Omega \kappa^{-2} \mathbf{a}_1 \otimes \mathbf{a}_2 &= \frac{(\kappa_f - 1)^2 [h^2 \kappa_f (\kappa_f + 2) - 3]}{4\pi (1 - h^2)^2 \kappa_f} \mathbb{I} \\ &\quad - \frac{3(\kappa_f^2 - 1)(h^2 \kappa_f^2 - 1)}{4\pi (1 - h^2)^2 \kappa_f} \hat{\mathbf{k}} \otimes \hat{\mathbf{k}}, \end{aligned} \quad (\text{D9})$$

where $\mathbb{I} = \hat{\mathbf{i}} \otimes \hat{\mathbf{i}} + \hat{\mathbf{j}} \otimes \hat{\mathbf{j}} + \hat{\mathbf{k}} \otimes \hat{\mathbf{k}}$. Using that $\mathbb{I}$ is invariant under the product of rotations $(\mathcal{R}_{\Theta\Phi} \otimes \mathcal{R}_{\Theta\Phi})$, we have

$$\begin{aligned} \int_{\Omega_\epsilon} d\mu \mathbf{n}_1 \otimes \mathbf{n}_2 &= \left(-\frac{(1-h)A'}{2\pi h(1+h)^2} + \frac{B'}{4\pi(1-h^2)^2} \right) \mathbb{I} \\ &\quad - \frac{3C'}{4\pi(1-h^2)^2}, \end{aligned} \quad (\text{D10})$$

where

$$A' = \int_{V_{\text{in}}} dV,$$

$$B' = \int_{V_{\text{out}}} dV R(R^{-1} - 1)^2 [h^2 R^{-1}(R^{-1} + 2) - 3],$$

$$C' = \int_{V_{\text{out}}} dV R^{-1}(R^{-2} - 1)(h^2 R^{-2} - 1) \mathbf{R} \otimes \mathbf{R}.$$

Now, we take V_ϵ as an infinitesimal neighborhood at a distance r_{ts} from the origin. The integrals (D6) and (D10) separate into two expressions: when the neighborhood is entirely within the sphere of radius h , and another when it is entirely outside. By substituting these expressions into Eq. (D1), along with the volume $V(\Omega_\epsilon(r_{\text{ts}}))$ from Eq. (B2), we finally obtain

$$\begin{aligned} \varrho^{\text{as}} &= \frac{1}{4} I \otimes I + c_1 (\hat{\mathbf{r}}_{\text{ts}} \cdot \boldsymbol{\sigma}) \otimes I + c_2 I \otimes (\hat{\mathbf{r}}_{\text{ts}} \cdot \boldsymbol{\sigma}) \\ &\quad + c_3 (\hat{\mathbf{r}}_{\text{ts}} \cdot \boldsymbol{\sigma}) \otimes (\hat{\mathbf{r}}_{\text{ts}} \cdot \boldsymbol{\sigma}) + c_4 \sum_i \sigma_i \otimes \sigma_i, \end{aligned} \quad (\text{D11})$$

with the following c_i coefficients:

$$\begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix} = \begin{cases} \frac{1}{12h(1+h)} \begin{pmatrix} -(1-h^2)r_{\text{ts}} \\ (1+4h+h^2)r_{\text{ts}} \\ 0 \\ -h(1-h) \end{pmatrix} & \text{if } 0 \leq r_{\text{ts}} < h \\ \frac{1}{24(1-h^2)r_{\text{ts}}^2} \begin{pmatrix} 2r_{\text{ts}}(1+h)(3r_{\text{ts}}^2 - h(r_{\text{ts}}^2 + r_{\text{ts}} + 1)) \\ 2r_{\text{ts}}(1-h)(3r_{\text{ts}}^2 + h(r_{\text{ts}}^2 + r_{\text{ts}} + 1)) \\ 3(1+r_{\text{ts}})(r_{\text{ts}}^2 - h^2) \\ -(1-r_{\text{ts}})(3r_{\text{ts}}^2 - h^2(2r_{\text{ts}} + 1)) \end{pmatrix} & \text{if } h < r_{\text{ts}} < 1. \end{cases} \quad (\text{D12})$$

In the space of separable bipartite states, we define the average state of the preimage of a neighborhood V_ϵ analogously as

$$\varrho^{\text{as}} = \frac{1}{V(\Omega_\epsilon^\otimes)} \int_{\Omega_\epsilon^\otimes} d\mu^\otimes \varrho, \quad (\text{D13})$$

where all density matrices of separable states take the form

$$\begin{aligned} \varrho &= |\mathbf{n}_1\rangle\langle\mathbf{n}_1| \otimes |\mathbf{n}_2\rangle\langle\mathbf{n}_2| \\ &= \frac{1}{4} \left(I \otimes I + (\mathbf{n}_1 \cdot \boldsymbol{\sigma}) \otimes I \right. \\ &\quad \left. + I \otimes (\mathbf{n}_2 \cdot \boldsymbol{\sigma}) + \sum_{ij} n_{1i} n_{2j} \sigma_i \otimes \sigma_j \right), \end{aligned} \quad (\text{D14})$$

reducing the problem to integrating the vectors $\mathbf{n}_i$ and the tensor product $\mathbf{n}_1 \otimes \mathbf{n}_2$ similarly to the general case. We use the parametrization given in Eq. (A1), and, together with Eq. (A4) and $r = 1$, we express u in terms of R to perform the integration with the measure $d\mu^\otimes$ in Eq. (A7). All integrals in

the variable v are straightforward, yielding

$$\int_{\Omega_\epsilon^\otimes} d\mu^\otimes \mathbf{n}_1 = \frac{1}{2\pi(1-h^2)(1-h)} \int_{V_\epsilon} dV \frac{R^2 - h}{R^3} \mathbf{R}. \quad (\text{D15})$$

The integral of $\mathbf{n}_2$ is obtained from the previous one by exchanging h with $-h$, and

$$\int_{\Omega_\epsilon^\otimes} d\mu^\otimes \mathbf{n}_1 \otimes \mathbf{n}_2 = \frac{1}{4\pi(1-h^2)^2} \int_{V_\epsilon} dV (A'' - B'' \mathbb{I}), \quad (\text{D16})$$

where

$$A'' = \int_{V_\epsilon} dV R^{-5} [R^4 + R^2 + h^2(R^2 - 3)] \mathbf{R} \otimes \mathbf{R},$$

$$B'' = \int_{V_\epsilon} dV R^{-3} (1 - R^2)(R^2 - h^2).$$

Recall that the preimage of a target state contains separable states only if it is outside the sphere of radius h in the Bloch ball; for this reason, there is a unique integration region, unlike the general case. Taking V_ϵ as an infinitesimal neighborhood at a distance $r_{\text{ts}} > h$ from the origin, and using the volume

computed in Eq. (B5), we find that the average state has the same form as in Eq. (D11) but with the following coefficients:

$$\begin{aligned} c_1 &= \frac{r_{ts}^2 - h}{4(1-h)r_{ts}}, & c_2 &= \frac{r_{ts}^2 + h}{4(1+h)r_{ts}}, \\ c_3 &= \frac{r_{ts}^4 + r_{ts}^2 + h^2(r_{ts}^2 - 3)}{8(1-h^2)r_{ts}^2}, \\ c_4 &= -\frac{(1-r_{ts}^2)(r_{ts}^2 - h^2)}{8(1-h^2)r_{ts}^2}. \end{aligned} \quad (D17)$$

The general expression in Eq. (D11) for the average state explicitly reveals that the functions c_i are independent of the polarization of ρ . Furthermore, by reorienting $\mathbf{r}_{ts}$, these functions can be determined as averages of expectation values. If we set $\hat{\mathbf{r}}_{ts} = \hat{\mathbf{k}}$, the average state takes the form

$$\varrho^{\text{as}} = \frac{1}{4}\sigma_{0,0} + c_1\sigma_{3,0} + c_2\sigma_{0,3} + c_3\sigma_{3,3} + c_4 \sum_{i=1}^3 \sigma_{i,i}, \quad (D18)$$

where σ_{μ_1, μ_2} is defined as in Eq. (42). It is then straightforward to observe that the expectation values averaged in Eq. (43) are expressed in terms of the functions c_i as follows:

$$\begin{aligned} \varrho_{0,0}^{\text{as}} &= 1, \\ \varrho_{3,0}^{\text{as}} &= 4c_1, \\ \varrho_{0,3}^{\text{as}} &= 4c_2, \\ \varrho_{3,3}^{\text{as}} &= 4(c_3 + c_4), \\ \varrho_{1,1}^{\text{as}} &= \varrho_{2,2}^{\text{as}} = 4c_4, \end{aligned} \quad (D19)$$

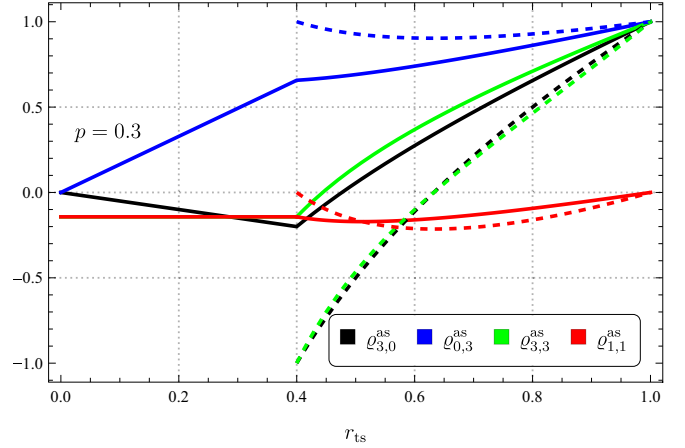


FIG. 14. Components of the average state ϱ^{as} as a function of the r_{ts} . The components are in the tensor product of the Pauli basis. The graphs with solid lines show the case where the set of pure bipartite states over which it was averaged includes both entangled and nonentangled states. The graphs with dotted lines represent the case in which the averaging was performed over a set of pure bipartite states that are nonentangled, i.e., are product states.

with $\varrho_{\mu_1, \mu_2}^{\text{as}} = 0$ otherwise. That is, in the basis of Hermitian matrices $\sigma_{\mu_1 \mu_2}$, the average state ϱ for a target state polarized along the z axis has only six nonzero entries, four of which are independent. These can be measured as averages of expectation values and are directly comparable to the functions c_i derived here. Figure 14 displays the coefficients $\varrho_{\mu_1 \mu_2}^{\text{as}}$ for both the general case and the separable case with $N = 2$.

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