

Erratum: Two-body P -state energies at order α^6 [Phys. Rev. A **109**, 022819 (2024)]

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In calculation of the second-order contribution to energy of a two-body system with angular momentum $l = 1$,

$$\delta_2 E^{(6)} = \left\langle H^{(4)} \frac{1}{(E - H')} H^{(4)} \right\rangle, \quad (1)$$

we made a mistake that affects the final result for energy $E^{(6)}$. To calculate $\delta_2 E^{(6)}$ we used a general formula valid for an arbitrary state with angular momentum $l \geq 1$ from [1], and we carelessly set $l = 1$. The intermediate states in the resolvent have the angular momentum $L = l$ or $L = l \pm 2$. In order to use the result for $\delta_2 E^{(6)}$ from [1] for P states, one has to explicitly exclude intermediate states with $L = l - 2$. This contribution does not algebraically vanish due to the cancellation of $l - 1$ factors and was erroneously included in our calculation, affecting the final results. The following equations should therefore be corrected:

(1) Equation (66) should be

$$\mathcal{E}_{S12} = \frac{\mu^5}{m_1^2 m_2^2} \left[\left(\frac{5}{3n^5} - \frac{1}{n^4} - \frac{287}{90n^3} \right) \frac{g_1^2 g_2^2}{640} + \frac{1}{24} \left(\frac{1}{n^3} - \frac{1}{n^5} \right) \right]. \quad (66)$$

(2) Equation (68) should be

$$\begin{aligned} \mathcal{E}_{LS2} = & \frac{\mu^4}{m_1^2 m_2} \frac{g_2}{12} \left[\frac{1}{n^5} - \frac{1}{n^3} - g_1 \left(\frac{7}{20n^5} + \frac{1}{8n^4} - \frac{133}{720n^3} \right) + g_1^2 \left(-\frac{3}{20n^5} + \frac{1}{8n^4} + \frac{227}{720n^3} \right) \right] \\ & + \frac{\mu^5}{m_1^2 m_2^2} \frac{1}{12} \left[\frac{1}{n^3} - \frac{1}{n^5} + g_1 g_2 \left(\frac{7}{20n^5} + \frac{1}{8n^4} - \frac{133}{720n^3} \right) + g_1^2 g_2^2 \left(-\frac{1}{40n^5} + \frac{9}{320n^4} + \frac{187}{3200n^3} \right) \right]. \end{aligned} \quad (68)$$

(3) Equation (69) should be

$$\begin{aligned} \mathcal{E}_{SS} = & -\frac{\mu^3}{m_1 m_2} g_1 g_2 \left(\frac{1}{60n^5} + \frac{1}{18n^4} + \frac{47}{1620n^3} \right) + \frac{\mu^4}{m_1 m_2} \left(\frac{g_1}{m_2} + \frac{g_2}{m_1} \right) \left(\frac{1}{18n^5} + \frac{1}{18n^4} - \frac{5}{324n^3} \right) \\ & + \frac{\mu^5}{m_1^2 m_2^2} \left[\frac{g_1^2 g_2^2}{480} \left(\frac{5}{3n^5} - \frac{1}{n^4} - \frac{287}{90n^3} \right) + \frac{1}{30n^5} - \frac{1}{18n^4} - \frac{191}{1620n^3} \right] \\ & + \frac{2}{27} \left(\frac{1}{n^3} - \frac{1}{n^5} \right) \frac{\mu^5}{m_1 m_2} g_1 g_2 (r_{M1}^2 + r_{M2}^2). \end{aligned} \quad (69)$$

(4) Equation (70) should be

$$\begin{aligned} \mathcal{E}_{LL} = & \frac{\mu^3}{m_1 m_2} \frac{g_1 g_2}{4} \left(\frac{51}{50n^5} - \frac{7}{12n^4} - \frac{3697}{5400n^3} \right) + \frac{\mu^4}{m_1 m_2} \left[\left(\frac{g_1}{m_1} + \frac{g_2}{m_2} \right) g_1 g_2 \left(\frac{9}{200n^5} - \frac{3}{80n^4} - \frac{227}{2400n^3} \right) \right. \\ & + \left. \left(\frac{g_1}{m_2} + \frac{g_2}{m_1} \right) \left(-\frac{19}{150n^5} + \frac{1}{12n^4} + \frac{1171}{5400n^3} \right) \right] + \frac{\mu^5}{m_1^2 m_2^2} \left[\frac{g_1^2 g_2^2}{200} \left(\frac{7}{6n^5} - \frac{7}{8n^4} - \frac{1709}{720n^3} \right) \right. \\ & + \left. g_1 g_2 \left(-\frac{6}{25n^5} - \frac{3}{40n^4} + \frac{37}{1200n^3} \right) - \frac{g_1 + g_2}{10} \left(\frac{1}{n^3} - \frac{1}{n^5} \right) + \frac{2}{25n^5} - \frac{1}{12n^4} - \frac{1063}{5400n^3} \right] \\ & + \frac{\mu^5}{m_1 m_2} \frac{g_1 g_2}{9} \left(\frac{1}{n^3} - \frac{1}{n^5} \right) (r_{M1}^2 + r_{M2}^2). \end{aligned} \quad (70)$$

(5) Equation (83) should be

$$\begin{aligned} \mathcal{E}^{(6)} - \mathcal{E}_G^{(6)}|_{l=1} = & \frac{\mu^5}{9} \left(\frac{1}{n^3} - \frac{1}{n^5} \right) \left\{ \frac{\tilde{r}_{E1}^2 + \tilde{r}_{E2}^2}{m_1 m_2} + \frac{2}{3} r_{E1}^2 r_{E2}^2 + \frac{r_{E1}^2}{2 m_2^2} + \frac{r_{E2}^2}{2 m_1^2} + \frac{r_{EE1}^4 + r_{EE2}^4}{5} + \frac{3}{8 m_1^2 m_2^2} \left(1 - \frac{g_1^2 g_2^2}{16} \right) \right. \\ & + \tilde{L} \cdot \tilde{s}_1 \left[-\frac{g_1 (\tilde{r}_{M1}^2 + \tilde{r}_{E2}^2)}{m_1 m_2} + \frac{r_{E1}^2 + (1 - g_1) r_{E2}^2 - g_1 \tilde{r}_{M1}^2}{m_1^2} + \frac{3}{4 m_1^2 m_2^2} \left(2 - g_1 + \frac{g_1^2 g_2^2 - 16}{16} \right) \right] \\ & + \tilde{L} \cdot \tilde{s}_2 \left[-\frac{g_2 (\tilde{r}_{M2}^2 + \tilde{r}_{E1}^2)}{m_1 m_2} + \frac{(1 - g_2) r_{E1}^2 + r_{E2}^2 - g_2 \tilde{r}_{M2}^2}{m_2^2} + \frac{3}{4 m_1^2 m_2^2} \left(2 - g_2 + \frac{g_1^2 g_2^2 - 16}{16} \right) \right] \\ & + \tilde{s}_1 \cdot \tilde{s}_2 \left[\frac{2}{3} \frac{g_1 g_2}{m_1 m_2} (\tilde{r}_{M1}^2 + \tilde{r}_{M2}^2) - \frac{1}{4 m_1 m_2} \left(g_2 \frac{g_1 - 2}{m_1^2} + g_1 \frac{g_2 - 2}{m_2^2} \right) \right. \\ & \quad \left. - \frac{(g_1 - 2)^2 (g_2 - 2)^2 + 4 (g_1 - 2)^2 + 4 (g_2 - 2)^2 + 4 (g_1 - 2)(g_2 - 2)(g_1 + g_2 - 4)}{32 m_1^2 m_2^2} \right] \\ & + (L^i L^j)^{(2)} s_1^i s_2^j \left[\frac{g_1 g_2}{m_1 m_2} (\tilde{r}_{M1}^2 + \tilde{r}_{M2}^2) + \frac{3}{10 m_1 m_2} \left(g_2 \frac{g_1 - 2}{m_1^2} + g_1 \frac{g_2 - 2}{m_2^2} \right) \right. \\ & \quad \left. - \frac{3 [(g_1 - 2)^2 (g_2 - 2)^2 + 4 (g_1 - 2)^2 + 4 (g_2 - 2)^2 + 4 (g_1 - 2)(g_2 - 2)(g_1 + g_2 - 4)]}{16 m_1^2 m_2^2} \right] \Big\}. \end{aligned} \quad (83)$$

(6) Equation (84) should be

$$\delta E_{\text{pos}} = \mathcal{E}^{(6)} - \mathcal{E}_G^{(6)}|_{l=1, m_1=m_2} = 0. \quad (84)$$

(7) Equation (86) should be

$$\delta E_{\text{pos}}(s = 1, j = 0) = 0. \quad (86)$$

(8) Equation (94) should be

$$\begin{aligned} E_{\text{fs}}^{(6)} = & \frac{3}{2} E_{L\mu}(n = 2, g_\mu = 2, s_N = 1/2) \\ = & \mu \frac{(Z\alpha)^6}{64} \left[\frac{5}{4} + \frac{1}{4} \frac{\mu}{m_N} + \left(-\frac{19}{18} + \frac{851}{900} g_N^2 \right) \left(\frac{\mu}{m_N} \right)^2 + \left(-\frac{3}{4} + \frac{5}{72} g_N - \frac{2179}{1800} g_N^2 \right) \left(\frac{\mu}{m_N} \right)^3 \right. \\ & \left. + \left(\frac{11}{36} - \frac{5}{72} g_N + \frac{53}{200} g_N^2 \right) \left(\frac{\mu}{m_N} \right)^4 - \mu^2 \left(r_E^2 + \frac{3}{4 m_N^2} \right) \left(1 - \frac{\mu^2}{m_N^2} \right) \right]. \end{aligned} \quad (94)$$

(9) Equation (B2) should be

$$E_{\text{pos}}^{(6)}(n^3 P_0) = m \alpha^6 \left(-\frac{69}{512 n^6} + \frac{119}{240 n^5} - \frac{1}{3 n^4} - \frac{833}{4320 n^3} - \frac{a_1^2 + 6 a_2}{24 \pi^2 n^3} \right), \quad (B2)$$

(10) The sentence after Eq. (B7) should be: The presented formulas agree with [13, Eq. (204)] for all states.

(11) Value $E_{\text{fs}}^{(6)}$ in Table I for $\mu^3\text{He}^+$ is changed to 0.004 19.

In summary, the general formula for arbitrary $l > 1$ calculated in Ref. [1] was shown to be valid also for the positronium $l = 1$ states. This statement is a result of a surprising cancellation of two effects. First, the direct application of the $l > 1$ formula from Ref. [1] to the positronium $l = 1$ case leads to a spurious contribution coming from intermediate states with $L = l - 2$ in the second-order correction, which should vanish for P states. Second, the $l > 1$ formula from Ref. [1] omits local-interaction terms in the first-order operators, which vanish for $l > 1$ but contribute for $l = 1$. Surprisingly, these two effects cancel each other identically, so that the formula derived for the $l > 1$ states stays valid also for the positronium $l = 1$ state. This means that our corrected result agrees with the one obtained in Refs. [2] and [3] for all positronium levels. Note that this conclusion is valid only for systems with pointlike particles with $g = 2$.

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Data availability. No data were created or analyzed in this study.

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- [1] J. Zatorski, V. Patkóš, and K. Pachucki, Quantum electrodynamics of two-body systems with arbitrary masses up to α^6 order, *Phys. Rev. A* **106**, 042804 (2022).
 [2] A. Czarnecki, K. Melnikov, and A. Yelkhovsky, Positronium S -state spectrum: Analytic results at $O(m\alpha^6)$, *Phys. Rev. A* **59**, 4316 (1999).
 [3] J. Zatorski, $O(m\alpha^6)$ corrections to energy levels of positronium with nonvanishing orbital angular momentum, *Phys. Rev. A* **78**, 032103 (2008).