

Designing quantum error correction codes for practical spin qudit systems

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The implementation of practical error correction protocols is essential for deployment of quantum information technologies. Ways of exploiting high-spin nuclei, which have multilevel quantum resources, have attracted interest in this context because they offer additional Hilbert space dimensions in a spatially compact and theoretically efficient structure. We present a quantitative analysis of the performance of a spin-qudit-based error-correctable quantum memory, with reference to the actual Hamiltonians of several potential candidate systems. First, the ideal code-word implemented on a spin-7/2 nucleus, which provides first-order Pauli- X , Y and Z error correction, has intrinsic infidelity due to mixed eigenstates under realistic conditions. We confirm that expansion to a spin-9/2 system with tailored code words can compensate for this infidelity. Second, we claim that electric field fluctuations, which are inevitable in real systems, should also be considered as a noise source, and we illustrate an encoding and decoding scheme for a multi-spin-qudit-based error correction code that can simultaneously compensate for both electric- and magnetic-field perturbations. Such strategies are important as we move beyond the current noisy-intermediate quantum era, and fidelities above two or three nines become crucial for the implementation of quantum technologies.

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I. INTRODUCTION

In the past decade, quantum information processing strategies deploying high-dimensional quantum objects have gained significant attention from researchers across a range of disciplines. This has well-known advantages in quantum error correction (QEC), for example, in the Gottesman-Kitaev-Preskill (GKP) code [1–3], which has been studied extensively on both theoretical [4–6] and experimental levels [7–10]. This approach can provide a spatially compact structure compared to qubit-based correction schemes [11–17]. In this context, high spin nuclei offer promising candidates for encoding fault-tolerant logical qubits [18,19], not only owing to their superior coherence times, but also because their potential multidimensional spin subspace yields a hardware-efficient structure. Moreover, several recent reports claim that spin-qudit-based information processing can outperform the same level of qubit-based counterparts in terms of resources [20] and gate [21] efficiency. Thus, various theoretical [22–27] and experimental [28,29] studies investigated ways of exploiting these systems as memory units for quantum technology. For instance, nuclei from $3d$ transition metals (such as Cu, V, Mn, hosting nuclear spins of $3/2$ or $5/2$) can provide correction of the Pauli- Z error; larger nuclei, such as Sb, Bi, La, which have spins of $7/2$ or $9/2$, can provide simultaneous correction of all Pauli- X , Y , and Z errors.

Practical implementation of these ideas, however, comes with several challenges arising from the actual physical interaction terms in the Hamiltonian, which are usually unavoidable in condensed matter systems. In the existing literature, the spin state has typically been modeled based on eigenstates of the S_z operator, with first- or second-order errors described as compositions of the ideal spin operators, S_x , S_y , and S_z . In all cases, a sufficiently large quantizing magnetic field B_0 will result in the states $|m_s\rangle$ and $|m_l\rangle$ being good eigenstates. However, but practical experiments are performed in finite field where this limit might not be reached, and therefore, there is demand for quantitative analysis regarding errors originating from this issue. We find that existing QEC protocols with ideal code-word designs are not suitable for an actual spin system with finite magnetic field, owing to perturbations such as hyperfine interaction, nuclear quadrupole interaction, crystal field splitting, and other terms, which lead to violations of the Knill-Laflamme (KL) criteria [12].

So far, since we are at the noisy intermediate-scale quantum (NISQ) era [30], control and readout fidelities of systems lie around two nines or lower [31–33], therefore, consideration for these weak perturbations has not been necessary. However, to go beyond the NISQ era and build a scalable quantum computer, we indeed need to tailor or reconstruct our code word to make our quantum error correction protocols valid under these various higher-order terms. In the case of qubit-based QEC protocols, such as the surface code [14,34–37], similar approaches (tailoring code words and decoders) have been reported [38–41] to compensate for biased error models. Moreover, optimization of a spin qudit code word [42] to counteract S_z dephasing has also been suggested, based on numerical simulation of the spin-bath interaction. We believe these approaches can be generalized; since we can

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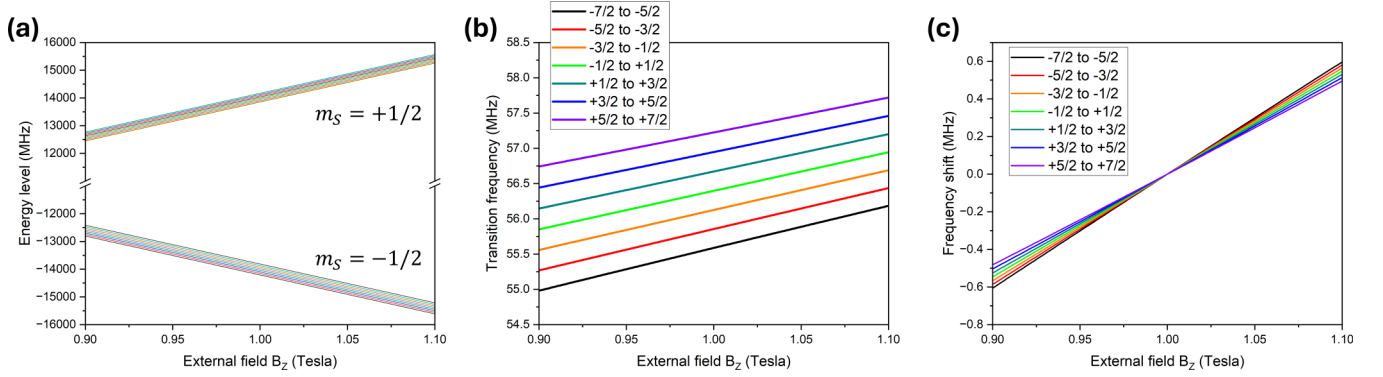


FIG. 1. (a) An energy level diagram for the spin states of the Si:Sb system. (b) The nuclear spin transition frequencies within the $m_s = -1/2$ manifold. (c) The shift of nuclear transition frequencies from their values at a field of 1 T as a function of magnetic field.

parameterize the full spin Hamiltonian of a single quantum object, the effects of a range of errors can be investigated at a fundamental level, and effective code-word designs can be found for each system.

In this article, we provide a quantitative analysis regarding these effects. First, the existence of any non-Ising-type interaction (e.g., the isotropic hyperfine interaction) leads to mixing of $|m_s, m_I\rangle$ eigenstates in the Z basis. We must take this mixing into account when tailoring code words. Interestingly, a spin-7/2 system, which has been suggested as the minimal spin dimension for correcting first-order S_x , S_y , and S_z errors, cannot host a code word meeting the KL criteria in the presence of such an interaction. As we show in Sec. II, the spin-9/2 system with tailored coefficients is required to deal with additional terms and conditions. Second, electron and nuclear spins with $m_s, m_I \geq 1$ often exhibit electric-field-sensitive Hamiltonian terms, as confirmed experimentally in donor spins in semiconductors [43,44] and nitrogen-vacancy centers in diamond [45,46]. Since, as is well known, magnons or phonons in condensed matter systems can cause magnetic or electric-field fluctuations, realistic first-order error protection requires code words robust against not only B_x , B_y , and B_z fluctuations, but also E_x , E_y , and E_z fluctuations. In Sec. III, we illustrate the construction of logical qubit code words providing this protection, encoded in multiple nuclear spin qudits, as suggested conceptually in Refs. [20,24], with a working example of three coupled nuclear spins 7/2. We focus on a model environment of group-V donors in Si with magnetic field on the scale of 1 T, but the approach we use here can be applied to a wide range of quantum high-spin systems in condensed matter.

II. TAILORED CODE WORD WITH SPIN HAMILTONIAN

One essential step in a quantum error correction protocol is, from a thermodynamic point of view, transferring extra entropy of the logical qubit into an ancillary qubit, and then transferring it into a thermal bath by initializing the ancillary qubit. Therefore, there must be coupling between the data qubit and the ancillary qubit, to perform, repeatedly, the entire cycle.

In practice, the coupling between spin systems (e.g., the hyperfine interaction between nuclear and electron spins) generates an asymmetry in the spin Hamiltonian. For example, the

Hamiltonian of the $S = 1/2, I = 7/2$ ^{123}Sb donor spin in Si is, to a good approximation

$$H = g\mu_B \mathbf{B} \cdot \mathbf{S} + g_I \mu_B \mathbf{B} \cdot \mathbf{I} + \mathbf{S} \cdot \mathbf{A} \cdot \mathbf{I}, \quad (1)$$

where $g\mu_B = 28.02$ GHz/T, $g_I \mu_B = 5.55$ MHz/T, and $A = 101.52$ MHz (obtained from Ref. [47]).

Figure 1(a) shows energy levels of this system, with an external magnetic field B_z around 1 T applied along Z . Focussing on the lower-energy group of eigenstates, those within the electron spin $m_s = -1/2$ manifold, Fig. 1(b) shows the seven $\Delta m_I = \pm 1$ transition frequencies between the nuclear spin states. To show more clearly the magnetic-field dependence of these frequencies, Fig. 1(c) shows the variation of the nuclear transition frequencies compared to their values at a magnetic field of 1 T. As we can see in Fig. 1(c), the gradients of these transition frequencies against B_z differ from each other, reflecting the fact that the states $|m_s, m_I\rangle$ are not the exact eigenstates of the system owing to the off-diagonal terms in the isotropic hyperfine interaction term. The consequence of this is that a fluctuation ΔB_z generates phase shifts on these levels differing from what is expected in the ideal cases described in, for example, Refs. [19,20]. We want also to note that $|m_I\rangle$ states mentioned hereafter are not perfect eigenstates of I_z , but those for which there is a dominant component in the superposition.

Under these circumstances, the ideal logical code words, $|0_L\rangle$ and $|1_L\rangle$, which are designed to protect against fluctuations of B_z , B_x , and B_y under the assumption that only the nuclear spin Zeeman term plays a role, and are given by [19,20]

$$\begin{aligned} |0_L\rangle &= \sqrt{\frac{3}{10}} \left| -\frac{7}{2} \right\rangle + \sqrt{\frac{7}{10}} \left| +\frac{3}{2} \right\rangle, \\ |1_L\rangle &= -\sqrt{\frac{3}{10}} \left| +\frac{7}{2} \right\rangle + \sqrt{\frac{7}{10}} \left| -\frac{3}{2} \right\rangle, \end{aligned} \quad (2)$$

do not satisfy the KL criteria under the influence of a fluctuation ΔB_z . This is demonstrated in Fig. 2, which shows the matrix elements $\langle 0_L | I_z | 0_L \rangle$ and $\langle 1_L | I_z | 1_L \rangle$ for magnetic fields around 1 T. Both are nonzero, and furthermore, they differ from each other, leading to a violation of the KL criteria [12]

$$\begin{aligned} \langle 0_L | A_i^\dagger A_j | 1_L \rangle &= 0, \\ \langle 0_L | A_i^\dagger A_j | 0_L \rangle - \langle 1_L | A_i^\dagger A_j | 1_L \rangle &= 0, \end{aligned} \quad (3)$$

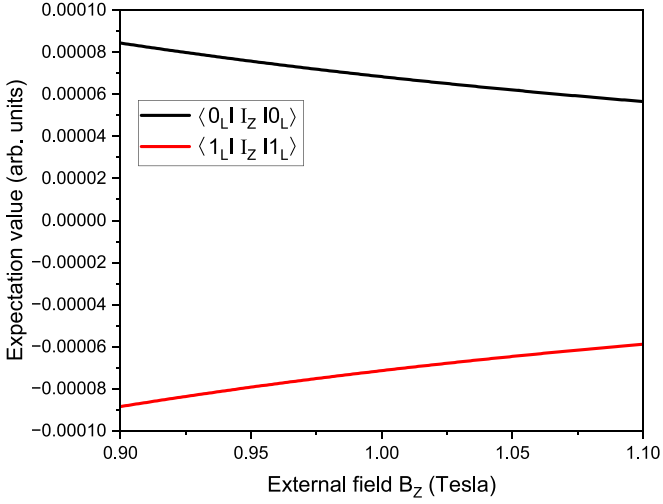


FIG. 2. The expectation value for $\langle 0_L | I_Z | 0_L \rangle$ and $\langle 1_L | I_Z | 1_L \rangle$ for original 7/2 code words, with practical environment of Si:Sb.

where $|0_L\rangle$ and $|1_L\rangle$ are the code words and A_i represent the errors we want to protect against, i.e., the identity, and fluctuations of B_X , B_Y , and B_Z .

This issue can be partially resolved if we also add an asymmetry in the code word. For example, we can encode to protect against ΔB_Z perturbations by introducing the small distortion (ϵ_1, ϵ_2) to the original code word, such that

$$\begin{aligned} |0_L\rangle &= \cos(\theta_0 + \epsilon_1) \left| -\frac{7}{2} \right\rangle + \sin(\theta_0 + \epsilon_1) \left| +\frac{3}{2} \right\rangle, \\ |1_L\rangle &= -\cos(\theta_0 + \epsilon_2) \left| +\frac{7}{2} \right\rangle + \sin(\theta_0 + \epsilon_2) \left| -\frac{3}{2} \right\rangle. \end{aligned} \quad (4)$$

Here, $\theta_0 = \cos^{-1}(\sqrt{\frac{3}{10}})$ and $\epsilon_1 = \epsilon_2 = 0$ corresponds to the ideal case. Through careful choice of the two free parameters, ϵ_1 and ϵ_2 , we can satisfy the condition $\langle 0_L | I_Z | 0_L \rangle = \langle 1_L | I_Z | 1_L \rangle$. The blue line in Fig. 3 shows a set of (ϵ_1, ϵ_2) that satisfying

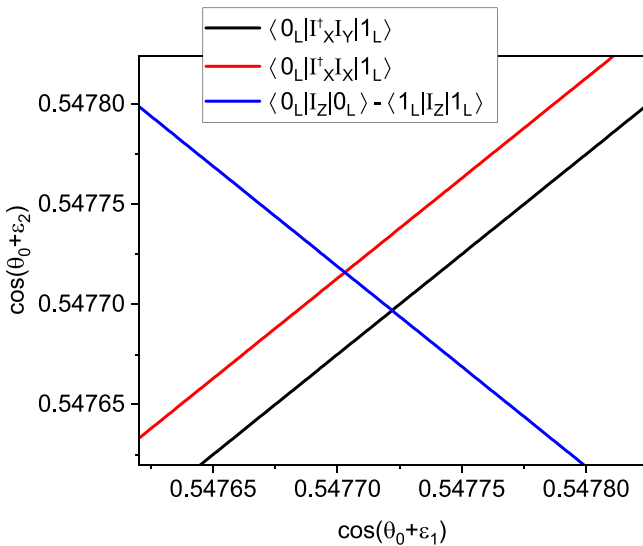


FIG. 3. The lines show set of ϵ_1 and ϵ_2 , which satisfying zero expectation value for $\langle 0_L | I_Z | 0_L \rangle - \langle 1_L | I_Z | 1_L \rangle$ (blue), $\langle 0_L | I_X^\dagger I_X | 1_L \rangle$ (red), and $\langle 0_L | I_Y^\dagger I_Y | 1_L \rangle$ (black).

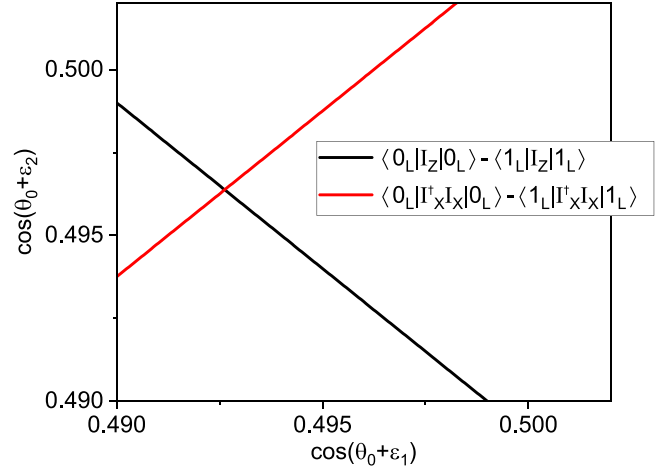


FIG. 4. The lines show a set of ϵ_1 and ϵ_2 , which satisfies zero expectation value for $\langle 0_L | I_Z | 0_L \rangle - \langle 1_L | I_Z | 1_L \rangle$ (black), $\langle 0_L | I_X^\dagger I_X | 0_L \rangle - \langle 1_L | I_X^\dagger I_X | 1_L \rangle$ (red), with ideal 9/2 code-word and additional ϵ_1 and ϵ_2 parameters.

$\langle 0_L | I_Z | 0_L \rangle - \langle 1_L | I_Z | 1_L \rangle = 0$, demonstrating that this quantity is zero along a line in (ϵ_1, ϵ_2) space.

However, in this spin-7/2 case, we cannot also satisfy the equivalent KL criteria for fluctuations ΔB_X and ΔB_Y through a choice of ϵ_1 and ϵ_2 : the red and black lines in Fig. 3 show the set of (ϵ_1, ϵ_2) , which have zero expectation value for $\langle 0_L | I_X^\dagger I_X | 1_L \rangle$ and $\langle 0_L | I_Y^\dagger I_Y | 1_L \rangle$, respectively. As is evident from Fig. 3, there is no choice of (ϵ_1, ϵ_2) for which all three quantities are zero, and therefore, there is no simple way to identify logical qubits resilient to all three perturbations.

However, extending to a spin-9/2 system, as offered, for example, by Bi dopants in Si, we acquire the resources required to deal with this asymmetry. As before, the isotropic hyperfine coupling of $I = 9/2$ and $S = 1/2$ generates mixing of $|m_S, m_I\rangle$ states, and an asymmetry in gradients of nuclear transition frequencies with magnetic field.

We model Bi dopants in Si using the same Hamiltonian as Sb dopants, Eq. (1), but with $S = 1/2$, $I = 9/2$, $g\mu_B = 28.02$ GHz/T, $g_I\mu_B = 6.841$ MHz/T, and $A = 1475.4$ MHz (also obtained from Ref. [47]). In this case, we can compensate for ΔB_X , ΔB_Y , and ΔB_Z fluctuations simultaneously, using the asymmetric code word

$$\begin{aligned} |0_L\rangle &= \cos(\theta_0 + \epsilon_1) \left| -\frac{9}{2} \right\rangle + \sin(\theta_0 + \epsilon_1) \left| +\frac{3}{2} \right\rangle, \\ |1_L\rangle &= \cos(\theta_0 + \epsilon_2) \left| +\frac{9}{2} \right\rangle + \sin(\theta_0 + \epsilon_2) \left| -\frac{3}{2} \right\rangle. \end{aligned} \quad (5)$$

The case with $\theta_0 = \cos^{-1}(\frac{1}{2})$ and $\epsilon_1 = \epsilon_2 = 0$ yields the original code word for the spin-9/2 QEC code [20]. To this, we can add the distortion (ϵ_1, ϵ_2) , to match the KL criteria to account for the presence of the hyperfine term in the Hamiltonian.

The black and red lines in Fig. 4 shows set of (ϵ_1, ϵ_2) , which have zero expectation value for $\langle 0_L | I_Z | 0_L \rangle - \langle 1_L | I_Z | 1_L \rangle$, $\langle 0_L | I_X^\dagger I_X | 0_L \rangle - \langle 1_L | I_X^\dagger I_X | 1_L \rangle$ for spin-9/2 logical code-word in Eq. (5), respectively. The set of (ϵ_1, ϵ_2) given by the black line also yields $\langle 0_L | I_X^\dagger I_Y | 0_L \rangle - \langle 1_L | I_X^\dagger I_Y | 1_L \rangle$ and the set represented by the red line yields $\langle 0_L | I_Y^\dagger I_Y | 0_L \rangle - \langle 1_L | I_Y^\dagger I_Y | 1_L \rangle$ and $\langle 0_L | I_Z^\dagger I_Z | 0_L \rangle - \langle 1_L | I_Z^\dagger I_Z | 1_L \rangle$ (as can be verified by considering the matrix elements or by numerical

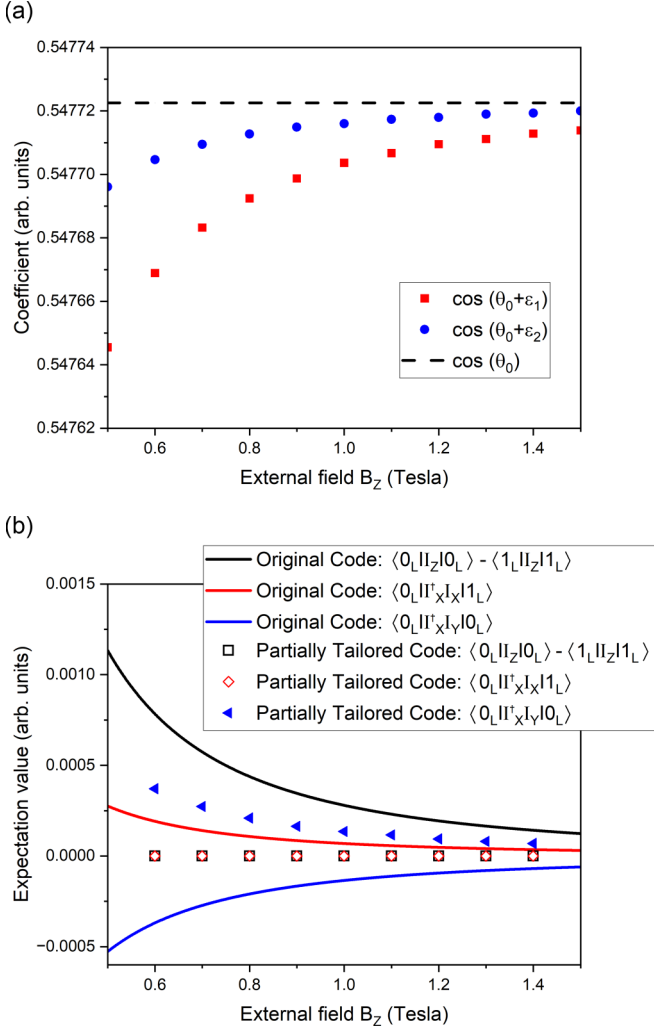


FIG. 5. (a) ϵ_1, ϵ_2 for the partially tailored 7/2 code words, which satisfies $\langle 0_L | I_Z | 0_L \rangle - \langle 1_L | I_Z | 1_L \rangle = 0$ and $\langle 0_L | I_X^\dagger I_X | 1_L \rangle = 0$ conditions. (b) The expectation values for KL criteria, for partially tailored code words and original 7/2 code words.

calculation). In this 9/2 case, owing to the additional m_I states between those making up the logical qubit, the other expectation values such as $\langle 0_L | I_X^\dagger I_X | 1_L \rangle$ or $\langle 0_L | I_Y^\dagger I_Y | 1_L \rangle$ are trivial zero. Thus, Fig. 4 shows the dependencies required to identify how to satisfy the KL criteria.

There exists a unique numerical solution, corresponding to $(\epsilon_1, \epsilon_2) \approx (0.0085, 0.0042)$ (in radians), satisfying the KL criteria

$$\begin{aligned} |0_L\rangle &= 0.492625 \left| -\frac{9}{2} \right\rangle + 0.870242 \left| +\frac{3}{2} \right\rangle, \\ |1_L\rangle &= 0.496375 \left| -\frac{3}{2} \right\rangle + 0.868208 \left| +\frac{9}{2} \right\rangle. \end{aligned} \quad (6)$$

While this choice of (ϵ_1, ϵ_2) , and therefore, the coefficients in Eq. (6), are the specific solution for an external field of 1 T, numerical solutions for other cases can be found easily by analogy.

In Fig. 5(a), we present the the amplitudes of $|m_I\rangle$ superposition components for the partially tailored code words for the spin-7/2 case, implemented for parameters for $^{123}\text{Sb}:\text{Si}$, as a function of magnetic field. These sets of (ϵ_1, ϵ_2) are

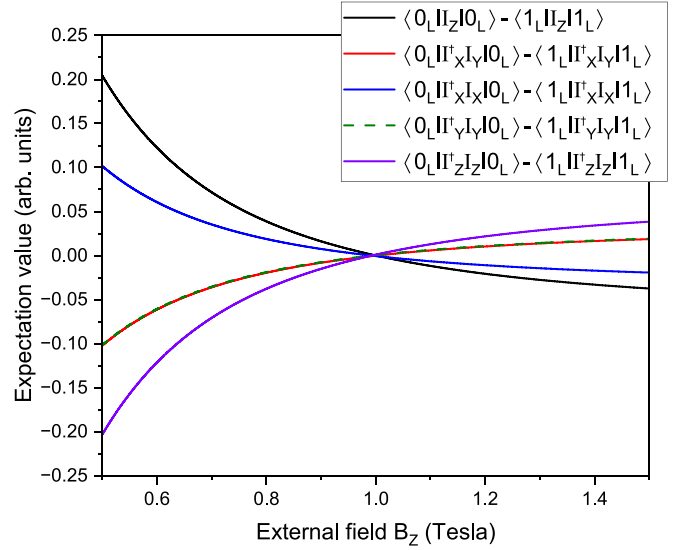


FIG. 6. The infidelity for fully tailored 9/2 code-word for 1 T coefficients case.

solutions for $\langle 0_L | I_Z | 0_L \rangle - \langle 1_L | I_Z | 1_L \rangle = 0$ and $\langle 0_L | I_X^\dagger I_X | 1_L \rangle = 0$, but do not give $\langle 0_L | I_Y^\dagger I_Y | 1_L \rangle = 0$ because, as shown in Fig. 3, there is no global solution for all three conditions. We plot the deviation from satisfaction of the KL criteria for these partially tailored and original (untailored) spin-7/2 code words in Fig. 5(b), optimized as a function of magnetic field. As expected, the original spin-7/2 code word exhibits a finite deviation across the field range. For these partially tailored cases, we can see that we can achieve zero expectation values for $\langle 0_L | I_Z | 0_L \rangle - \langle 1_L | I_Z | 1_L \rangle$ and $\langle 0_L | I_X^\dagger I_X | 1_L \rangle$ across the magnetic-field range, there is always a finite imperfection for the $\langle 0_L | I_Y^\dagger I_Y | 1_L \rangle$ case. An interesting feature is that the degree of imperfection decreases with increasing external field, and this is well matched with our intuition: a larger external field brings eigenstates closer to ideal S_Z eigenstates.

These deviations, on the scale of $\sim 10^{-3}$ to $\sim 10^{-4}$, indicate the experimental circumstances under which it is important to consider the error arising from the hyperfine interaction. Some state-of-the-art experiments demonstrate control fidelities for single spin qubits and qudits around the three nines level [32,33,43], so this source of error is already experimentally significant. The tailoring of the spin-7/2 code word reduces these infidelities by a factor of roughly 3, but still there remains a finite amount of error because S_X and S_Y errors cannot be compensated fully.

In Fig. 6, we illustrate the deviations of the KL criteria in Eq. (3) from 0 for the fully tailored spin-9/2 code word [Eq. (6)] implemented for parameters for Bi:Si at 1 T, as a function of the applied external field strength. In this field regime, the spin-9/2 cases exhibit a larger error compared to the spin-7/2 case shown in Fig. 5(b) because of the large hyperfine coupling of Bi dopants (~ 1400 MHz) compared to Sb dopants (~ 100 MHz). However, as described above, the infidelity can be suppressed to exactly zero by tailoring the code word for a particular magnetic field, 1 T in this case.

Thus we identify an approach to preparing robust quantum error-correctable memories using a nuclear spin-9/2, such as that offered by the Bi donor in Si, with a logical code word

tailored to account for the full spin Hamiltonian. Another practical way to avoid this problem is to use nuclear spin-7/2 without hyperfine coupling, as demonstrated in Ref. [44], storing the nucleus in the ionized state in which the donor electron is removed. Although the initial information encoding, detection (and correction) scheme requires coupling to an ancillary electron spin, the nuclear spin can be maintained in the ionized state during the memory storage time. This will minimize additional error due to asymmetry of the system Hamiltonian, depending on the ratio between memory storage time and the interaction time.

III. LOGICAL QUBIT CODE WORD PROTECTING AGAINST ELECTRIC- AND MAGNETIC-FIELD FLUCTUATIONS

Often in solid-state systems, the spin Hamiltonian can be coupled to an external electric field, particularly if the environment of the spin has broken inversion symmetry. Examples of Hamiltonian contributions through which electric field couplings can occur include, for nuclear spins, the nuclear quadrupole interaction $I \cdot Q \cdot I$ [43,44], and for electron spins, crystal field terms such as $S \cdot D \cdot S$ and higher-order terms [48–50]. (We note here that for other qudit candidates, such as superconducting circuits and trapped ions, practical implementations often rely on analogous higher-order Hamiltonian terms, to lift the degeneracies of transitions and allow spectral addressability.) From the engineering point of view, these spin-electric coupling terms are convenient, allowing coherent manipulation of spin states using electric control fields [43,44,51–53]. However, from the perspective of information integrity and fault-tolerant encoding, these properties also imply that environmental noise of both electric and magnetic fields can disrupt quantum states. Indeed, electric-field noise is known as one of the major limiting factors for spin coherence of nitrogen-vacancy centers [45,46]. By analogy with magnetic-field-fluctuation-related relaxation mechanisms such as thermal fluctuations of neighboring spins (including environmental nuclear spin bath, and other electron spins via the dipolar interaction), scattering of phonons or charge fluctuations can induce electric-field fluctuations at the site of spins [54].

This suggests that the first order operators that we should protect against are not simply S_X , S_Y , and S_Z , but instead those associated with both magnetic- and electric-field perturbations. Magnetic-field perturbations generate, to first-order, fluctuations of operators S_X , S_Y , and S_Z ; experiments on spin-electric couplings in condensed matter [43,44,51–53] suggest that we should also consider, to first order, operators of the form S_X^2 , S_Y^2 , S_Z^2 , $S_X S_Y$, $S_Y S_Z$, and $S_Z S_X$ arising from electric-field perturbations.

One way to make a logical code word robust to all of these error operators is by exploiting a single spin qudit dimension higher than 24 (i.e., $S \geq 23/2$). While second-order error correction of qubit-based code-word require minimum 11 qubits [55] (i.e., a Hilbert space dimension of 2048) this 24-dimensional spin qudit can provide efficient encoding, and can provide robustness under any perturbation, because all significant practical perturbations on a spin can be described as fluctuation of either electric field or magnetic field.

In terms of experimental candidates for implementation, careful engineering of a giant single molecule magnet has the potential to build a collective electron spin of $S = 23/2$ [56–58]. In this case, the spin Hamiltonian is described by a Zeeman term and a crystal field splitting term, and the logical codeword against first-order $\vec{E}$ and $\vec{B}$ fluctuations can be prepared by analogy with the procedure described in Sec. II. Similarly, the code word suggested in Ref. [20] will be able to correct also for second-order perturbations arising from fluctuations of $\vec{E}$ and $\vec{B}$:

$$\begin{aligned} |0_L\rangle &= +\sqrt{\frac{125}{1482}}\left|-\frac{23}{2}\right\rangle + \sqrt{\frac{874}{1482}}\left|-\frac{5}{2}\right\rangle + \sqrt{\frac{483}{1482}}\left|+\frac{15}{2}\right\rangle, \\ |1_L\rangle &= -\sqrt{\frac{125}{1482}}\left|+\frac{23}{2}\right\rangle + \sqrt{\frac{874}{1482}}\left|+\frac{5}{2}\right\rangle + \sqrt{\frac{483}{1482}}\left|-\frac{15}{2}\right\rangle. \end{aligned} \quad (7)$$

However, coherent and high-fidelity control of such a giant electron spin remains difficult. Additionally, electron spins usually have shorter relaxation times compared to nuclear spins, and there also might be relaxation processes related to the total angular momentum of the giant molecular spin.

Therefore, we propose an alternative scheme for implementing error correctable encoding, by using three coupled nuclear qudits of spin-7/2 implemented in solid-state environment. While trapped and laser-cooled nuclei can have additional metastable states with higher total angular momentum (such as Refs. [59–61]), a nucleus implemented in the solid state can be treated as having a constant total angular momentum I because we can neglect the motional degrees of freedom in this case. This means that the $|m_I\rangle$ (Z -axis projection) states form a complete basis set when a magnetic field is applied along the Z axis. These states also provide a good approximation to the eigenstates and represent a good quantum number. Therefore, code words that provide correction for only $|m_I\rangle$ would be sufficient for fault-tolerant encoding.

A logical code word satisfying the KL criteria for these error operators is

$$\begin{aligned} |0_L\rangle &= +\sqrt{\frac{2}{16}}\left|-\frac{7}{2}\right\rangle_{A,B,C} + \sqrt{\frac{7}{16}}\left|-\frac{3}{2}\right\rangle_{A,B,C} + \sqrt{\frac{7}{16}}\left|+\frac{5}{2}\right\rangle_{A,B,C} \\ |1_L\rangle &= +\sqrt{\frac{2}{16}}\left|+\frac{7}{2}\right\rangle_{A,B,C} + \sqrt{\frac{7}{16}}\left|+\frac{3}{2}\right\rangle_{A,B,C} - \sqrt{\frac{7}{16}}\left|-\frac{5}{2}\right\rangle_{A,B,C} \end{aligned} \quad (8)$$

where $|m_I\rangle_{A,B,C}$ corresponds to the tensor product of three nuclear spin states, $|m_I\rangle_A \otimes |m_I\rangle_B \otimes |m_I\rangle_C$. (A similar code-word given in Ref. [20], which consists of four spins-7/2 with the same coefficients, was proposed to protect against two individual errors. Here, three spins-7/2 are sufficient because we are considering a single error event, but accounting for second-order operators.) Suitable corresponding physical systems may be prepared, for example, by precise ion implantation in semiconductors incorporating control circuitry [47,62–64], or by chemical engineering of molecular magnets, such as $\text{Sc}_3\text{@C}_{82}$ [65,66]. While these three spin-7/2 qudits generate a large Hilbert space of dimension 512, we believe that this configuration is worth considering because it can host a code protecting against most practically relevant

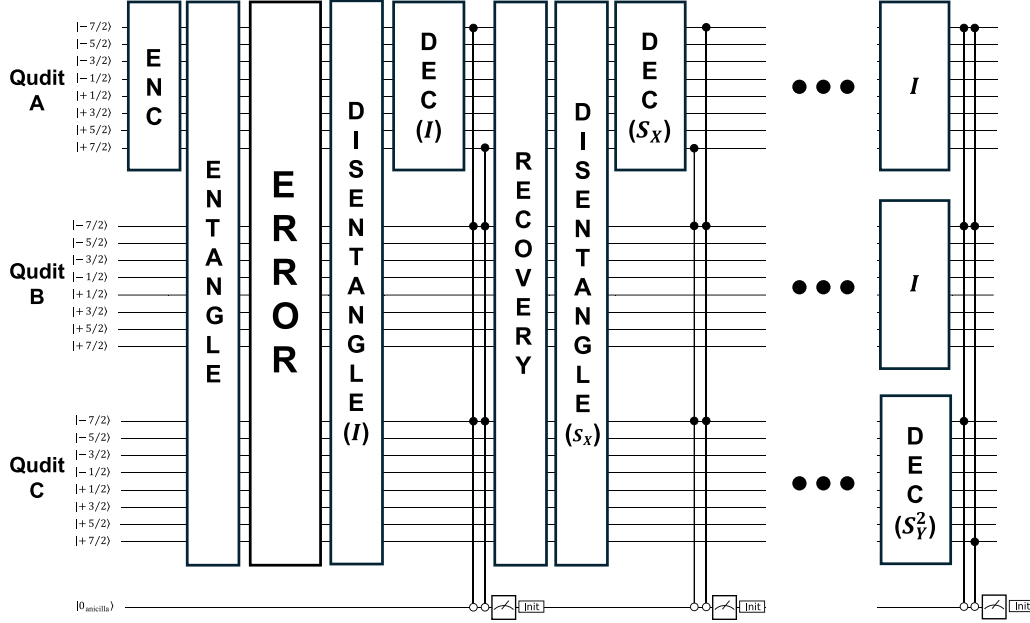


FIG. 7. The high-level encoding and decoding scheme for the three spins-7/2 code word.

electromagnetic perturbations; for individual memory elements, further complexity up is unlikely to be necessary for practical technologies.

We now consider the detail of the sequence for the encoding and decoding of this multiqudit error correction code, with the logical code word given in Eq. (8). For simplicity we assume that $|m_I\rangle_{A,B,C}$ are good eigenstates. A high-level schematic of the process is illustrated in Fig. 7, with more detail on pulse sequences given in Appendixes A and B. First, without loss of generality, we can start with the original qubit stored in a superposition of $|-7/2\rangle, |-5/2\rangle$ of the first nuclear spin (A), in the form $\alpha|-7/2\rangle_A + \beta|-5/2\rangle_A$, while the other two nuclear spins (B and C) are prepared in pure $|-7/2\rangle$ states. This starting state can be written as Eq. (9),

$$|\psi_0\rangle = (\alpha|-7/2\rangle_A + \beta|-5/2\rangle_A) \otimes |-7/2\rangle_B \otimes |-7/2\rangle_C. \quad (9)$$

The logical qubit state is first encoded in qudit A via the block of operations labeled “ENC.” After this block, the state of the system is

$$|\psi_1\rangle = \left[\alpha \left(\sqrt{\frac{2}{16}} \left| -\frac{7}{2} \right\rangle_A + \sqrt{\frac{7}{16}} \left| -\frac{3}{2} \right\rangle_A + \sqrt{\frac{7}{16}} \left| +\frac{5}{2} \right\rangle_A \right) + \beta \left(\sqrt{\frac{2}{16}} \left| +\frac{7}{2} \right\rangle_A + \sqrt{\frac{7}{16}} \left| +\frac{3}{2} \right\rangle_A - \sqrt{\frac{7}{16}} \left| -\frac{5}{2} \right\rangle_A \right) \right] \otimes \left| -\frac{7}{2} \right\rangle_B \otimes \left| -\frac{7}{2} \right\rangle_C. \quad (10)$$

The qudits B and C are then entangled by operation block “ENTANGLE” (as described in more detail in Appendix A), generating the state in Eq. (11), with $|0_L\rangle$ and $|1_L\rangle$ defined in Eq. (8),

$$|\psi_{\text{enc}}\rangle = \alpha|0_L\rangle + \beta|1_L\rangle. \quad (11)$$

Following encoding, perturbations affect the system during the main storage time (block labeled “ERROR”), after which the original information lies in multiple subspaces of the three qudits. Since all error subspaces satisfy the KL criteria, it is possible to decode and detect each error case sequentially, one by one, as indicated by blocks labeled “DISENTANGLE,” and “DEC” in Fig. 7.

Note that since this error correction code is not pure [67] (or nondegenerate), this decoding scheme differs from the first-order single spin qudit error correction code in Eq. (2), which corresponds to a “pure” code. The number of error operators ($I, S_X, S_X^2, S_Y, \dots, S_Y^2$) considered in this encoding is 12, bigger than dimension of the single spin-7/2 qudit space, 8. Therefore, there necessarily exists some mixing of logical zero and errored-logical zero (and also, logical one and errored-logical one). For example, because the S_X operator is Hermitian, the projection of the original $|0_L\rangle$ to S_X^2 errors can be rewritten as $\langle 0_L | S_X^2 | 0_L \rangle = \langle 0_L | S_X^\dagger S_X | 0_L \rangle = c_{\text{norm}}$, where c_{norm} stands for a finite nonzero normalization factor.

We now provide more details of the actual decoding sequence. After applying an error operator on the logical qubit, the state can be written as

$$\begin{aligned} E|\psi_{\text{enc}}\rangle = & \sqrt{1-\epsilon}(\alpha|0_L\rangle + \beta|1_L\rangle) \\ & + \sqrt{\epsilon_{X,A}}(\alpha S_{X,A}|0_L\rangle + \beta S_{X,A}|1_L\rangle) \\ & + \sqrt{\epsilon_{Y,A}}(\alpha S_{Y,A}|0_L\rangle + \beta S_{Y,A}|1_L\rangle) \\ & + \sqrt{\epsilon_{X^2,A}}(\alpha S_{X,A}^2|0_L\rangle + \beta S_{X,A}^2|1_L\rangle) \\ & + \dots, \end{aligned} \quad (12)$$

where $(\alpha|0_L\rangle + \beta|1_L\rangle)$ represents the original information and $\epsilon_{i,j}$ indicates the probability of an S_i error occurring on the qudit j . After this error, the “DISENTANGLE(I)” and “DEC(I)” gates can be expressed as the unitary operation U_{D-I} , which

satisfies

$$\begin{aligned} U_{D-I}|0_L\rangle &= |-7/2\rangle_A \otimes |-7/2\rangle_B \otimes |-7/2\rangle_C, \\ U_{D-I}|1_L\rangle &= |+7/2\rangle_A \otimes |-7/2\rangle_B \otimes |-7/2\rangle_C. \end{aligned} \quad (13)$$

As shown in Appendix B, this unitary transformation can be straightforwardly identified. The DISENTANGLE(I) block is simply the inverse of the ENTANGLE block, and DEC(I) is quite similar to the inverse of the ENC block. Then the conditional excitation from $|-7/2\rangle_A \otimes |-7/2\rangle_B \otimes |-7/2\rangle_C$, $|+7/2\rangle_A \otimes |-7/2\rangle_B \otimes |-7/2\rangle_C$ to the electron spin ancilla is applied to detect the “no-error” case.

At this stage, we must check the effect of U_{D-I} operator on other terms in the errored state, Eq. (12). The error terms such as $S_{X,A}|0_L\rangle$, $S_{X,A}|1_L\rangle$, $S_{Y,A}|0_L\rangle$, $S_{Y,A}|1_L\rangle$, $\dots$ all lie in a subspace orthogonal to both $|0_L\rangle$ and $|1_L\rangle$. Therefore, they contribute zero to the coefficients of $|-7/2\rangle_A \otimes |-7/2\rangle_B \otimes |-7/2\rangle_C$ and $|+7/2\rangle_A \otimes |-7/2\rangle_B \otimes |-7/2\rangle_C$ following the U_{D-I} operator, and are not affected by the conditional excitation to the ancilla.

In contrast, the error terms such as $S_{X,A}^2|0_L\rangle$, $S_{Y,A}^2|0_L\rangle$, $\dots$ have nonzero inner product with original $|0_L\rangle$, and the equivalent is true for $|1_L\rangle$. As a working example, consider the effect of the U_{D-I} operator on $S_{X,A}^2(\alpha|0_L\rangle + \beta|1_L\rangle)$; considering each logical qubit state, in turn,

$$\begin{aligned} U_{D-I}S_{X,A}^2|0_L\rangle &= U_{D-I}(c_{XX}|0_L\rangle + (1 - c_{XX})|P_{X^2,A,0}\rangle) \\ &= c_{XX}(|-7/2\rangle_A \otimes |-7/2\rangle_B \otimes |-7/2\rangle_C) \\ &\quad + (1 - c_{XX})U_{D-I}|P_{X^2,A,0}\rangle, \\ U_{D-I}S_{X,A}^2|1_L\rangle &= U_{D-I}(c_{XX}|1_L\rangle + (1 - c_{XX})|P_{X^2,A,1}\rangle) \\ &= c_{XX}(|+7/2\rangle_A \otimes |-7/2\rangle_B \otimes |-7/2\rangle_C) \\ &\quad + (1 - c_{XX})U_{D-I}|P_{X^2,A,1}\rangle. \end{aligned} \quad (14)$$

Here, we separate $S_{X,A}^2|0_L\rangle$ into two orthogonal components, $|0_L\rangle$ and $|P_{X^2,A,0}\rangle$ (and equivalent for $S_{X,A}^2|1_L\rangle$). Crucially, owing to the satisfaction of the KL criteria, $\langle 0_L|S_X^2|0_L\rangle = \langle 1_L|S_X^2|1_L\rangle$, the coefficient c_{XX} is the same in both.

Therefore, after conditional excitation of the ancilla, and allowing for the possibility of occurrence of any of the correctable errors, the state becomes

$$\begin{aligned} |\Psi\rangle &= (\text{remaining terms})|0_{\text{ancilla}}\rangle \\ &\quad + \sqrt{1 - \epsilon + c_{XX} + c_{XY} + \dots} \\ &\quad \times (\alpha|-7/2\rangle_A \otimes |-7/2\rangle_B \otimes |-7/2\rangle_C)|1_{\text{ancilla}}\rangle \\ &\quad + \sqrt{1 - \epsilon + c_{XX} + c_{XY} + \dots} \\ &\quad \times (\beta|+7/2\rangle_A \otimes |-7/2\rangle_B \otimes |-7/2\rangle_C)|1_{\text{ancilla}}\rangle, \end{aligned} \quad (15)$$

where $1 - \epsilon$ corresponds to the probability of no error occurring, c_{XX} corresponds to $\epsilon_{X^2}(0_L|S_X^2|0_L)$, c_{XY} corresponds to $\epsilon_{XY}(0_L|S_X S_Y|0_L)$, and so on.

If a measurement of the ancilla yields “1,” we conclude that an error did not occur and we can conclude the decoding process. Otherwise, we apply the “RECOVERY” block (which is the inverse of U_{D-I}). This restores the state to Eq. (12), but with the unerrored component projected out, and with a corresponding change to the normalization factor. Thus we can iterate the decoding process over each error case, as

shown in Fig. 7. Note that when we detect the error case $S_{X^2,A}$ (and other nonorthogonal cases), we need to decode for $|P_{X^2,A,0}\rangle$, $|P_{X^2,A,1}\rangle$ because the state has changed owing to the detection of the no-error case.

The entire decoding sequence is presented in Appendix B. One cycle of decoding requires roughly 1700 pulses. The typical relaxation times associated with S_Z and S_Z^2 are related to the phase coherence time T_2 , of the system, and other terms S_X , S_Y , S_X^2 , S_Y^2 , $\dots$, are related to the longitudinal relaxation time, T_1 . T_1 and T_2 are highly system dependent, but based on some assumptions we can simulate the efficiency this encoding. Given that each round of the error correction sequence requires around 1700 pulses, we can estimate the how the overall fidelity depends on the practical single-pulse fidelities.

In Appendix C, we provide an approximate estimate of the overall fidelity of the three-7/2-qudits error correction code. Under the assumption that the three qudits (qudits A , B , and C in our notation) and all first- and second-order errors have the same probability, achieving an advantage from this protocol requires a single-qudit gate duration (t_{gate}) shorter than 10^{-5} times T_{relax} , and a gate rotation angle error less than 10^{-4} . Note that these guidance values are relevant for the most general case where all perturbations are comparably important. Once a specific experimental environment is identified and characterized, it is possible to optimize protocol designs and pulse engineering techniques to correct only the dominant perturbations, thus enhancing performance and reducing circuit depth for the specific case. For example, in the usual solid-state spin systems, Z errors tend to occur frequently and are often the dominant source of decoherence. In that case, we can frequently check specific types of error (in this case, S_Z , $S_Z S_X$, $S_Z S_Y$, S_Z^2) on the decoding scheme, and effectively, reduce the required decoding scheme to about 500 pulses, thus improving the overall error correction fidelity for the same level of single-pulse fidelity.

IV. CONCLUSION AND FUTURE WORK

We demonstrated how to tailor fault-tolerant spin qudit code words which can protect quantum information under realistic Hamiltonians in practical environments. The ideal code word specified in Refs. [19,20] for the spin-7/2 case lies on somewhat unstable plateau originating from the strict spherical symmetry; even a single additional coupling term, which must exist because an ancillary qubit is essential for QEC, can destroy the symmetry, resulting in intrinsic infidelity. At the quantitative level, our numerical simulations provide baseline numbers for the maximum operational fidelity in example systems. For instance, a fault-tolerant code-word implementation on the nuclear spin-7/2 Si:Sb system with a typical X band or Q band electron paramagnetic resonance (EPR) setup will suffer unavoidable errors of around $\sim 10^{-3}$ to $\sim 10^{-4}$ owing to the hyperfine coupling with the electron ancilla. These effects will become important when the fidelity of operations improves to a comparable level, and they should be considered as we move beyond the current NISQ era and achieve fault-tolerant information processing. Thanks to the availability of nuclei having spin-9/2, we confirm that the extra levels available can be exploited to compensate for perturbations from coupling terms.

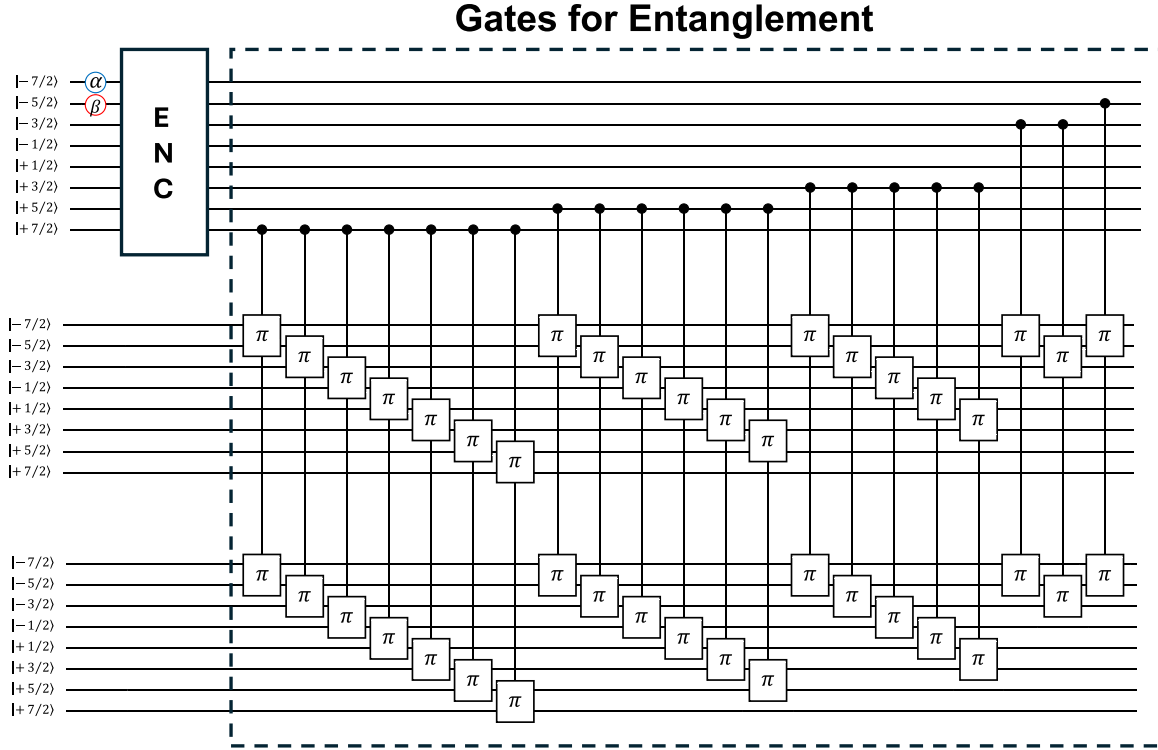


FIG. 8. The schematic diagram of the full encoding pulse sequence for the multiqutrit error correction code.

Another important characteristic of high-spin qutrits is their interaction with electric fields owing to the quadrupole interaction. While magnetic-field fluctuations generate first-order errors, electric-field fluctuations generate error operators that are second order in the spin operators. To simultaneously protect against both electric- and magnetic-field fluctuations, we present the simplest form of a multiqutrit fault-tolerant encoding and decoding process, which can, in principle, be expanded into more general cases.

Depending on the specific physical system, specialized code-word designs for specific errors (e.g., a code-word for correction of ΔB_z up to third-order, using a nuclear spin-7/2) can also be identified. In the case of electric-field-induced noise, we can benefit from the recent demonstration of electric-field control of nuclear spin qutrits [43,44] to develop models of how to mitigate associated errors; more accurate models can be provided as more experimental exploration is conducted in the field. This approach of multi-spin-qutrits error correction can also be adapted to other architectures, such as systems of trapped atoms or molecules [59–61], with some careful consideration of additional error pathways [4,6] may arise. We believe this work motivates future studies on strategies that exploit nuclear spin qutrits as basic building blocks of scalable quantum information processors.

ACKNOWLEDGMENTS

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DATA AVAILABILITY

The data that support the findings of this article are not publicly available. The data are available from the authors upon reasonable request.

APPENDIX A: SCHEMATIC DIAGRAM OF THE MULTIQUTRIT ERROR CORRECTION ENCODING SEQUENCE

Figure 8 shows full encoding sequence blocks for first-, second-, and third qutrit of multiqutrit error correction sequence. Without loss of generality, the starting state (original information on qutrit A, pure $|−7/2\rangle$ preparation on qutrits B and C) can be written in form of Eq. (A1)

$$|\psi_0\rangle = (\alpha|−\frac{7}{2}\rangle_A + \beta|−\frac{5}{2}\rangle_A) \otimes |−\frac{7}{2}\rangle_B \otimes |−\frac{7}{2}\rangle_C. \quad (\text{A1})$$

From this general starting states, gates of ENC code block [as shown in Fig. 9(a)] on qutrit A will change state into Eq. (A2),

$$|\psi_1\rangle = \left[\alpha \left(+\sqrt{\frac{2}{16}}|−\frac{7}{2}\rangle_A + \sqrt{\frac{7}{16}}|−\frac{3}{2}\rangle_A + \sqrt{\frac{7}{16}}|+\frac{5}{2}\rangle_A \right) + \beta \left(+\sqrt{\frac{2}{16}}|+\frac{7}{2}\rangle_A + \sqrt{\frac{7}{16}}|+\frac{3}{2}\rangle_A - \sqrt{\frac{7}{16}}|−\frac{5}{2}\rangle_A \right) \right] \otimes |−\frac{7}{2}\rangle_B \otimes |−\frac{7}{2}\rangle_C. \quad (\text{A2})$$

After the ENC code block, conditional π gates from qutrit A to qutrits B and C can generate entanglement of three qutrits, as shown in Fig. 8. (This is also labeled in ENTANGLE code

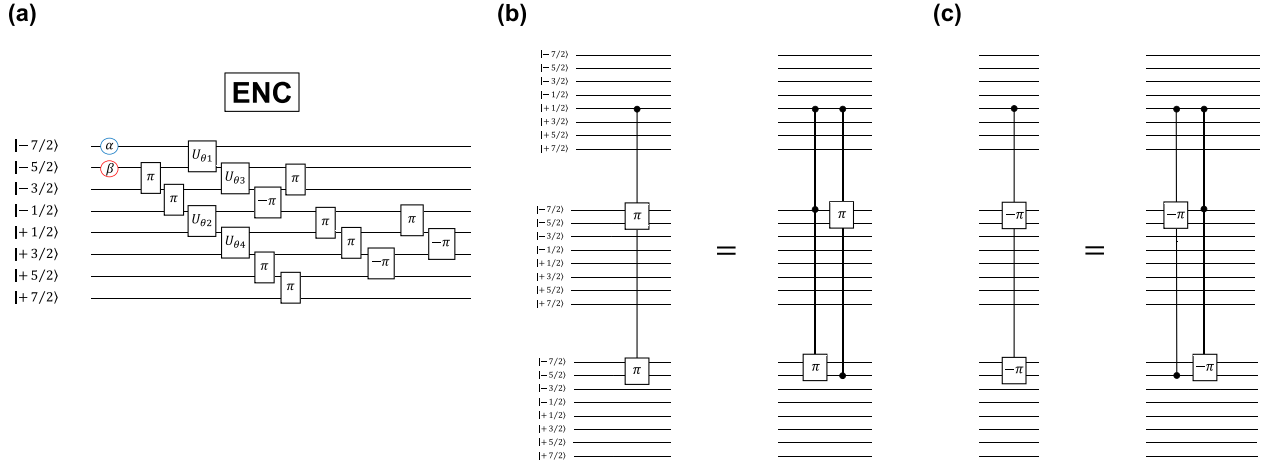


FIG. 9. (a) The schematic diagram for the ENC pulse block. Unitary rotations U_{θ_i} are around the y axis, with $\cos(\theta_i) = \sqrt{2/16}, \sqrt{7/16}, \sqrt{1/2}, \sqrt{7/9}$ for $i = 1, 2, 3, 4$, respectively. (b) The definition of controlled-double π . (c) The definition of the inverse of controlled-double π .

block, in Fig. 7.) In the diagram, controlled-double- π gates are designed to apply controlled π to the two other qudits, as defined in Fig. 9(b). Finally, the state will be encoded into Eq. (A3)

$$|\psi_{\text{enc}}\rangle = \alpha \left(\sqrt{\frac{2}{16}} \left| -\frac{7}{2} \right\rangle_{A,B,C} + \sqrt{\frac{7}{16}} \left| -\frac{3}{2} \right\rangle_{A,B,C} + \sqrt{\frac{7}{16}} \left| +\frac{5}{2} \right\rangle_{A,B,C} \right) + \beta \left(\sqrt{\frac{2}{16}} \left| +\frac{7}{2} \right\rangle_{A,B,C} + \sqrt{\frac{7}{16}} \left| +\frac{3}{2} \right\rangle_{A,B,C} - \sqrt{\frac{7}{16}} \left| -\frac{5}{2} \right\rangle_{A,B,C} \right). \quad (\text{A3})$$

APPENDIX B: SCHEMATIC DIAGRAM OF MULTIQUDIT ERROR CORRECTION DECODING SEQUENCE

In this Appendix, we briefly introduce the detailed decoding sequences for the I (“no-error”) case, the S_X error on qudit A case, and the S_X^2 error on qudit A case. This provides the general framework to detect and correct other error cases.

1. Decoding sequence for I case

After storage, our encoded state is represented by Eq. (12), and we will first apply the operation to detect the I case. The DISENTANGLE(I) gates, shown in Fig. 10 will transform the state into Eq. (A2), because this is simply the inverse

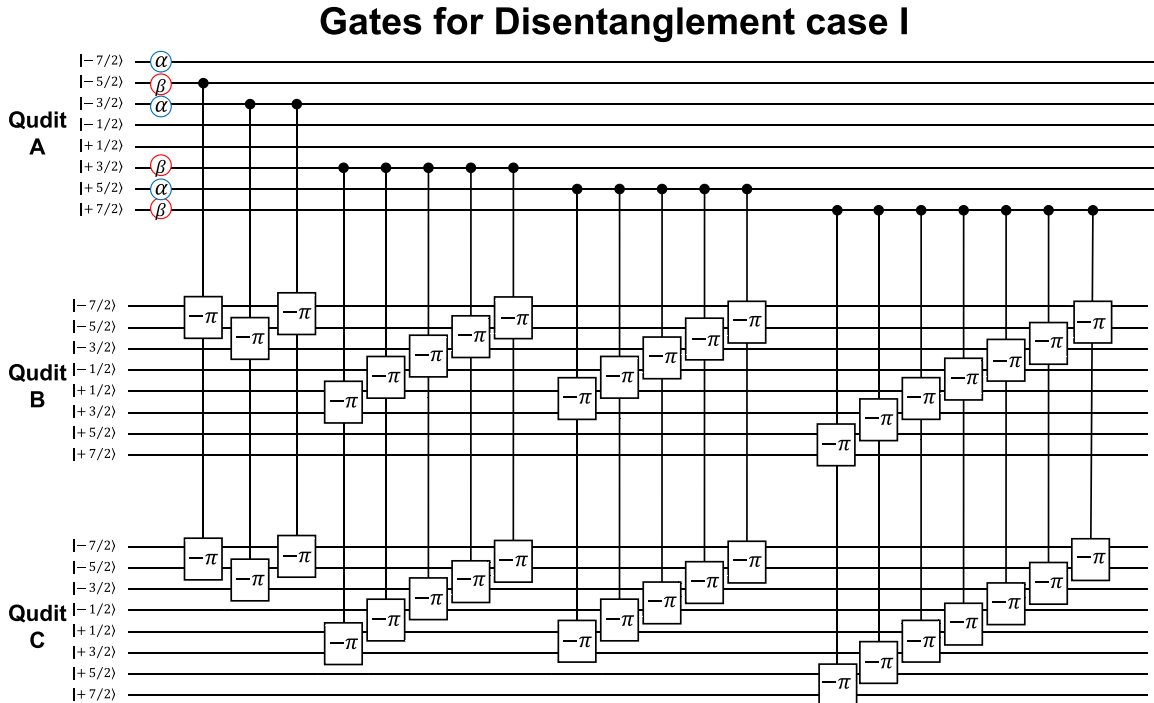


FIG. 10. The disentangle gates for I .

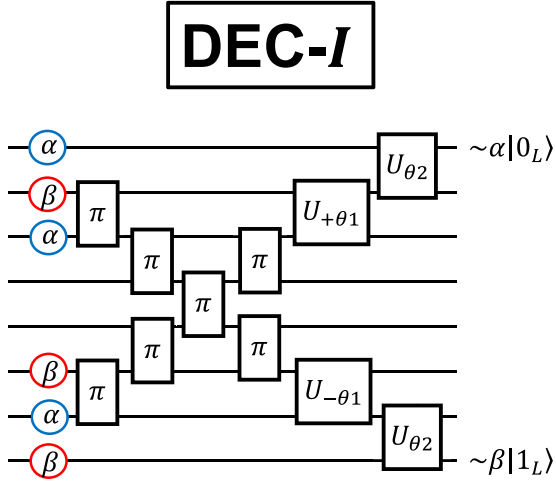


FIG. 11. The DEC gates for I. Unitary rotations U_{θ_i} are around y axis, with $\cos(\theta_i) = \sqrt{1/2}, \sqrt{2/16}$ for $i = 1, 2$, respectively.

of the ENTANGLE circuit we applied during the encoding sequence.

Then, we can apply the DEC- I operation in Fig. 11. This will send original $\sqrt{1-\epsilon}(\alpha|0_L\rangle + \beta|1_L\rangle)$ state into our target state

$$|\psi\rangle = (\sqrt{1-\epsilon} + C)(\alpha|-\frac{7}{2}\rangle_A + \beta|+\frac{7}{2}\rangle_A) \otimes |-\frac{7}{2}\rangle_B \otimes |-\frac{7}{2}\rangle_C + (\text{remaining terms}). \quad (\text{B1})$$

As discussed in the main text, these gates correspond to the U_{D-I} operation, and there can be additional coefficient arising from nonorthogonal terms. However, the target state still remains separable, and therefore, we can make a conditional excitation of the ancilla and finally measure (and detect) the “no-error” case. If the ancilla yields a “0” measurement, which means some kinds of error occurred, we move on to the next step. We first apply the RECOVERY operation (Fig. 7),

which is defined as the inverse of U_{D-I} , taking the state will back to a form similar to Eq. (12), but without no error term and with new normalization factor.

2. Decoding sequence for S_X error on qudit A case

Then, we can start the decoding of the $\sqrt{\epsilon_{X,A}}(\alpha S_{X,A}|0_L\rangle + \beta S_{X,A}|1_L\rangle)$ term. This state can be written as Eq. (B2)

$$\begin{aligned} |\psi_{S_{X,A}}\rangle = & \alpha \left(+\sqrt{\frac{2}{16}} \cdot \frac{\sqrt{7}}{2} \left| -\frac{5}{2} \right\rangle_A \left| -\frac{7}{2} \right\rangle_{B,C} \right. \\ & + \sqrt{\frac{7}{16}} \cdot \frac{\sqrt{12}}{2} \left| -\frac{5}{2} \right\rangle_A \left| -\frac{3}{2} \right\rangle_{B,C} \\ & + \sqrt{\frac{7}{16}} \cdot \frac{\sqrt{15}}{2} \left| -\frac{1}{2} \right\rangle_A \left| -\frac{3}{2} \right\rangle_{B,C} \\ & + \sqrt{\frac{7}{16}} \cdot \frac{\sqrt{12}}{2} \left| +\frac{3}{2} \right\rangle_A \left| +\frac{5}{2} \right\rangle_{B,C} \\ & + \sqrt{\frac{7}{16}} \cdot \frac{\sqrt{7}}{2} \left| +\frac{7}{2} \right\rangle_A \left| +\frac{5}{2} \right\rangle_{B,C} \Bigg) \\ & + \beta \left(+\sqrt{\frac{2}{16}} \cdot \frac{\sqrt{7}}{2} \left| +\frac{5}{2} \right\rangle_A \left| +\frac{7}{2} \right\rangle_{B,C} \right. \\ & + \sqrt{\frac{7}{16}} \cdot \frac{\sqrt{12}}{2} \left| +\frac{5}{2} \right\rangle_A \left| +\frac{3}{2} \right\rangle_{B,C} \\ & + \sqrt{\frac{7}{16}} \cdot \frac{\sqrt{15}}{2} \left| +\frac{1}{2} \right\rangle_A \left| +\frac{3}{2} \right\rangle_{B,C} \\ & - \sqrt{\frac{7}{16}} \cdot \frac{\sqrt{12}}{2} \left| -\frac{3}{2} \right\rangle_A \left| -\frac{5}{2} \right\rangle_{B,C} \\ & \left. - \sqrt{\frac{7}{16}} \cdot \frac{\sqrt{7}}{2} \left| -\frac{7}{2} \right\rangle_A \left| -\frac{5}{2} \right\rangle_{B,C} \right). \quad (\text{B2}) \end{aligned}$$

Gates for Disentanglement S_X on Qudit A

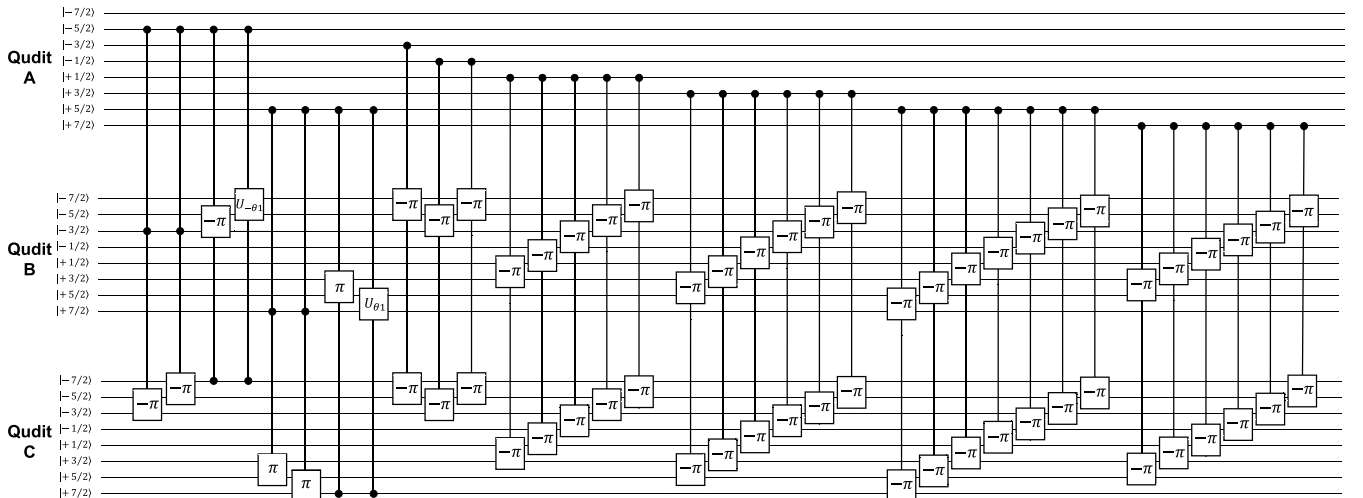


FIG. 12. The DISENTANGLE (S_X) operation on qudit A. The unitary rotation U_{θ_1} is around the y axis, with $\cos(\theta_1) = \sqrt{1/7}$.

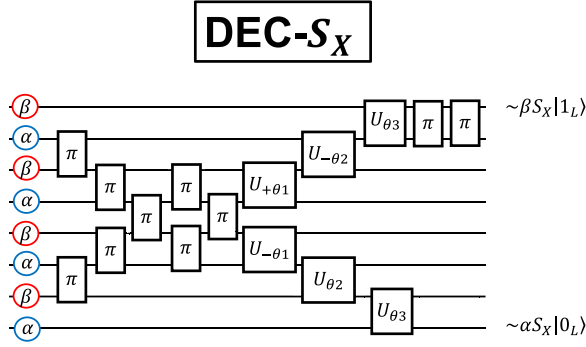


FIG. 13. The decoding gates for S_X on qudit A. Unitary rotations U_{θ_i} are around y axis, with $\cos(\theta_i) = \sqrt{14/29}, \sqrt{12/41}, \sqrt{7/48}$ for $i = 1, 2, 3$, respectively.

On this state, we will apply the $\text{DISENTANGLE}(S_X)$ operation, as shown in Fig. 12. This will change the state into

$$\begin{aligned}
 |\psi_{S_X,A}\rangle = & \left[\alpha \left(+\frac{7\sqrt{2}}{8} \left| -\frac{5}{2} \right\rangle_A + \frac{\sqrt{105}}{8} \left| -\frac{1}{2} \right\rangle_A \right. \right. \\
 & \left. \left. + \frac{\sqrt{84}}{8} \left| +\frac{3}{2} \right\rangle_A + \frac{7}{8} \left| +\frac{7}{2} \right\rangle_A \right) \right. \\
 & \left. + \beta \left(+\frac{7\sqrt{2}}{8} \left| +\frac{5}{2} \right\rangle_A + \frac{\sqrt{105}}{8} \left| +\frac{1}{2} \right\rangle_A - \frac{\sqrt{84}}{8} \left| -\frac{3}{2} \right\rangle_A \right. \right. \\
 & \left. \left. - \frac{7}{8} \left| -\frac{7}{2} \right\rangle_A \right) \right] \otimes \left| -\frac{7}{2} \right\rangle_B \otimes \left| -\frac{7}{2} \right\rangle_C. \quad (\text{B3})
 \end{aligned}$$

Then, we apply the $\text{DEC}(S_X)$ operation shown in Fig. 13. Then our $S_{X,A}$ error state $[\sqrt{\epsilon_{X,A}}(\alpha S_{X,A}|0_L\rangle + \beta S_{X,A}|1_L\rangle)]$ will sent to Eq. (B4), allowing detection of this error by condition excitation and projective measurement of the ancilla.

$$\begin{aligned}
 |\psi\rangle = & (\sqrt{\epsilon_{X,A}} + C) \left(\alpha \left| +\frac{7}{2} \right\rangle_A + \beta \left| -\frac{7}{2} \right\rangle_A \right) \otimes \left| -\frac{7}{2} \right\rangle_B \otimes \left| -\frac{7}{2} \right\rangle_C \\
 & + (\text{remaining terms}). \quad (\text{B4})
 \end{aligned}$$

3. Decoding sequence for S_X^2 error on qudit A case

Next, we can start the decoding of the $\sqrt{\epsilon_{X^2,A}}(\alpha S_{X,A}^2|0_L\rangle + \beta S_{X,A}^2|1_L\rangle)$ term. This state can be written as

$$\begin{aligned}
 |\psi_{S_X^2,A}\rangle = & c_0 \left[\alpha \left(+\sqrt{\frac{2}{16}} \frac{7}{4} \left| -\frac{7}{2} \right\rangle_A \left| -\frac{7}{2} \right\rangle_{B,C} \right. \right. \\
 & + \sqrt{\frac{2}{16}} \frac{2\sqrt{21}}{4} \left| -\frac{3}{2} \right\rangle_A \left| -\frac{7}{2} \right\rangle_{B,C} \\
 & + \sqrt{\frac{7}{16}} \frac{2\sqrt{21}}{4} \left| -\frac{7}{2} \right\rangle_A \left| -\frac{3}{2} \right\rangle_{B,C} \\
 & + \sqrt{\frac{7}{16}} \frac{27}{4} \left| -\frac{3}{2} \right\rangle_A \left| -\frac{3}{2} \right\rangle_{B,C} \\
 & \left. + \sqrt{\frac{7}{16}} \frac{4\sqrt{15}}{4} \left| +\frac{1}{2} \right\rangle_A \left| -\frac{3}{2} \right\rangle_{B,C} \right. \\
 & \left. + \sqrt{\frac{7}{16}} \frac{6\sqrt{5}}{4} \left| +\frac{1}{2} \right\rangle_A \left| +\frac{5}{2} \right\rangle_{B,C} \right. \\
 & \left. + \sqrt{\frac{7}{16}} \frac{19}{4} \left| +\frac{5}{2} \right\rangle_A \left| +\frac{5}{2} \right\rangle_{B,C} \right) \\
 & + \beta \left(+\sqrt{\frac{2}{16}} \frac{7}{4} \left| +\frac{7}{2} \right\rangle_A \left| +\frac{7}{2} \right\rangle_{B,C} \right. \\
 & + \sqrt{\frac{2}{16}} \frac{2\sqrt{21}}{4} \left| +\frac{3}{2} \right\rangle_A \left| +\frac{7}{2} \right\rangle_{B,C} \\
 & + \sqrt{\frac{7}{16}} \frac{2\sqrt{21}}{4} \left| +\frac{7}{2} \right\rangle_A \left| +\frac{3}{2} \right\rangle_{B,C} \\
 & + \sqrt{\frac{7}{16}} \frac{27}{4} \left| +\frac{3}{2} \right\rangle_A \left| +\frac{3}{2} \right\rangle_{B,C} \\
 & \left. + \sqrt{\frac{7}{16}} \frac{4\sqrt{15}}{4} \left| +\frac{1}{2} \right\rangle_A \left| +\frac{3}{2} \right\rangle_{B,C} \right. \\
 & \left. + \sqrt{\frac{7}{16}} \frac{6\sqrt{5}}{4} \left| +\frac{1}{2} \right\rangle_A \left| +\frac{5}{2} \right\rangle_{B,C} \right. \\
 & \left. + \sqrt{\frac{7}{16}} \frac{19}{4} \left| +\frac{5}{2} \right\rangle_A \left| +\frac{5}{2} \right\rangle_{B,C} \right) \right], \quad (\text{B5})
 \end{aligned}$$

where c_0 stands for a normalization factor, $\sqrt{\frac{8}{357}}$. Then, as discussed in main text, we can divide these into terms parallel and perpendicular to $|0_L\rangle$ (and $|1_L\rangle$).

$$\begin{aligned}
 |\psi_{S_X^2,A}\rangle = & c_{XX} [\alpha |0_L\rangle + \beta |1_L\rangle] \\
 & + (1 - c_{XX}) \left[\alpha \left(-\sqrt{\frac{2}{16}} \frac{14}{4} \left| -\frac{7}{2} \right\rangle_A \left| -\frac{7}{2} \right\rangle_{B,C} \right. \right. \\
 & + \sqrt{\frac{2}{16}} \frac{2\sqrt{21}}{4} \left| -\frac{3}{2} \right\rangle_A \left| -\frac{7}{2} \right\rangle_{B,C} \\
 & + \sqrt{\frac{7}{16}} \frac{2\sqrt{21}}{4} \left| -\frac{7}{2} \right\rangle_A \left| -\frac{3}{2} \right\rangle_{B,C} \\
 & + \sqrt{\frac{7}{16}} \frac{6}{4} \left| -\frac{3}{2} \right\rangle_A \left| -\frac{3}{2} \right\rangle_{B,C} \\
 & + \sqrt{\frac{7}{16}} \frac{4\sqrt{15}}{4} \left| +\frac{1}{2} \right\rangle_A \left| -\frac{3}{2} \right\rangle_{B,C} \\
 & + \sqrt{\frac{7}{16}} \frac{6\sqrt{5}}{4} \left| +\frac{1}{2} \right\rangle_A \left| +\frac{5}{2} \right\rangle_{B,C} \\
 & \left. - \sqrt{\frac{7}{16}} \frac{2}{4} \left| +\frac{5}{2} \right\rangle_A \left| +\frac{5}{2} \right\rangle_{B,C} \right) \\
 & + \beta \left(-\sqrt{\frac{2}{16}} \frac{14}{4} \left| +\frac{7}{2} \right\rangle_A \left| +\frac{7}{2} \right\rangle_{B,C} \right. \\
 & + \sqrt{\frac{2}{16}} \frac{2\sqrt{21}}{4} \left| +\frac{3}{2} \right\rangle_A \left| +\frac{7}{2} \right\rangle_{B,C} \\
 & + \sqrt{\frac{7}{16}} \frac{2\sqrt{21}}{4} \left| +\frac{7}{2} \right\rangle_A \left| +\frac{3}{2} \right\rangle_{B,C} \\
 & + \sqrt{\frac{7}{16}} \frac{27}{4} \left| +\frac{3}{2} \right\rangle_A \left| +\frac{3}{2} \right\rangle_{B,C} \\
 & \left. + \sqrt{\frac{7}{16}} \frac{4\sqrt{15}}{4} \left| +\frac{1}{2} \right\rangle_A \left| +\frac{3}{2} \right\rangle_{B,C} \right. \\
 & \left. + \sqrt{\frac{7}{16}} \frac{6\sqrt{5}}{4} \left| +\frac{1}{2} \right\rangle_A \left| +\frac{5}{2} \right\rangle_{B,C} \right. \\
 & \left. + \sqrt{\frac{7}{16}} \frac{19}{4} \left| +\frac{5}{2} \right\rangle_A \left| +\frac{5}{2} \right\rangle_{B,C} \right) \right], \quad (\text{B5})
 \end{aligned}$$

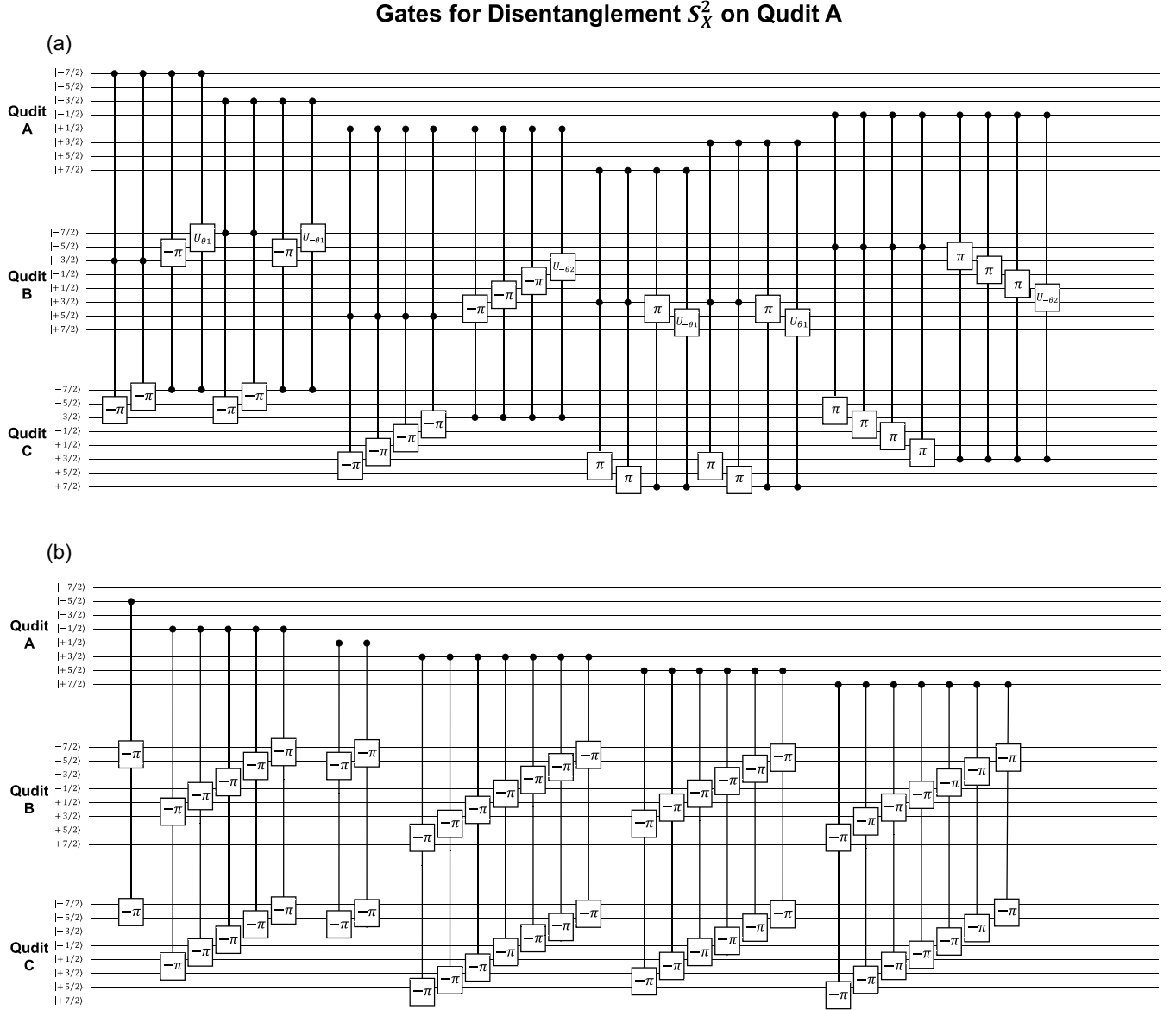


FIG. 14. The Disentangle gates for S_X^2 on qudit A. Unitary rotations U_{θ_i} are around y axis, with $\cos(\theta_i) = \sqrt{2/5}, \sqrt{4/7}$ for $i = 1, 2$, respectively. Pulses on diagram (a) applied first, and then pulses on (b) applied.

$$\begin{aligned}
 & + \sqrt{\frac{7}{16} \frac{2\sqrt{21}}{4}} \left| +\frac{7}{2} \right\rangle_A \left| +\frac{3}{2} \right\rangle_{B,C} \\
 & + \sqrt{\frac{7}{16} \frac{6}{4}} \left| +\frac{3}{2} \right\rangle_A \left| +\frac{3}{2} \right\rangle_{B,C} \\
 & + \sqrt{\frac{7}{16} \frac{4\sqrt{15}}{4}} \left| -\frac{1}{2} \right\rangle_A \left| +\frac{3}{2} \right\rangle_{B,C} \\
 & - \sqrt{\frac{7}{16} \frac{6\sqrt{5}}{4}} \left| -\frac{1}{2} \right\rangle_A \left| -\frac{5}{2} \right\rangle_{B,C} \\
 & + \sqrt{\frac{7}{16} \frac{2}{4}} \left| -\frac{5}{2} \right\rangle_A \left| -\frac{5}{2} \right\rangle_{B,C} \Bigg], \quad (B6)
 \end{aligned}$$

where c_{XX} stands for $\langle 0_L | S_X^2 | 0_L \rangle = \langle 1_L | S_X^2 | 1_L \rangle = \sqrt{\frac{8}{357}} \times \frac{21}{4}$. The second term with coefficient $(1 - c_{XX})$ corresponds to $(\alpha |P_{X^2,A,0}\rangle + \beta |P_{X^2,A,1}\rangle)$. Since the first term ($c_{XX}[\alpha |0_L\rangle + \beta |1_L\rangle]$) was already projected out during the detection of “no error,” detection of the second term only (combination of $|P_{X^2,A,0}\rangle, |P_{X^2,A,1}\rangle$) is sufficient at this step.

As above, we can first apply the $\text{DISENTANGLE}(S_X^2)$ block, as shown in Fig. 14.

Following this, the state can be written as

$$\begin{aligned}
 |\psi_{S_X^2,A}\rangle = & \left[\alpha \left(-\frac{14\sqrt{5}}{64} \left| -\frac{7}{2} \right\rangle_A + \frac{2\sqrt{105}}{64} \left| -\frac{3}{2} \right\rangle_A \right. \right. \\
 & \left. \left. + \frac{14\sqrt{15}}{64} \left| +\frac{1}{2} \right\rangle_A - \frac{2\sqrt{7}}{64} \left| +\frac{5}{2} \right\rangle_A \right) \right]
 \end{aligned}$$

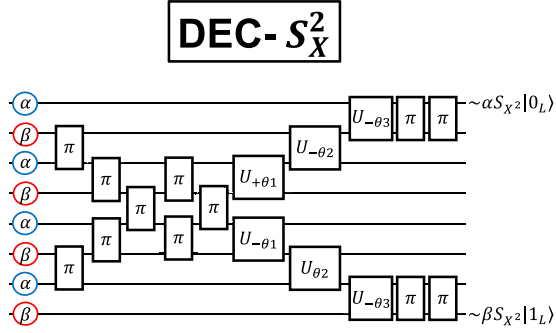


FIG. 15. The decoding gates for S_X^2 on qudit A. Unitary rotations U_{θ_i} are around y axis, with $\cos(\theta_i) = \sqrt{1/106}, \sqrt{15/121}, \sqrt{35/156}$ for $i = 1, 2, 3$, respectively.

$$\begin{aligned}
 & + \beta \left(-\frac{14\sqrt{5}}{64} \left| +\frac{7}{2} \right\rangle_A + \frac{2\sqrt{105}}{64} \left| +\frac{3}{2} \right\rangle_A \right. \\
 & \left. + \frac{14\sqrt{15}}{64} \left| -\frac{1}{2} \right\rangle_A + \frac{2\sqrt{7}}{64} \left| -\frac{5}{2} \right\rangle_A \right) \Big] \\
 & \otimes \left| -\frac{7}{2} \right\rangle_B \otimes \left| -\frac{7}{2} \right\rangle_C. \quad (B7)
 \end{aligned}$$

Finally, the $\text{DEC}(S_X^2)$ operation, shown in Fig. 15, is applied. This sends the $S_{X,A}^2$ error state $[\sqrt{\epsilon_{X^2,A}}(\alpha S_{X,A}^2 |0_L\rangle + \beta S_{X,A}^2 |1_L\rangle)]$ to the state

$$\begin{aligned}
 |\psi\rangle = & (\sqrt{\epsilon_{X^2,A}} + C)(\alpha \left| -\frac{7}{2} \right\rangle_A + \beta \left| +\frac{7}{2} \right\rangle_A) \otimes \left| -\frac{7}{2} \right\rangle_B \otimes \left| -\frac{7}{2} \right\rangle_C \\
 & + (\text{remaining terms}), \quad (B8)
 \end{aligned}$$

and by analogy with the above, conditional excitation and projective measurement of the ancilla detects this error case.

4. Decoding sequence for other errors

Figure 16 shows full the decoding sequence blocks for the first-, second- and third- qudits of multiqutrit error correction sequence. Each “DET” code block includes the sequence: Disentanglement – decoding – conditional ancilla measurement – recovery. Depending on the result, the ancilla qubit can be initialized and original information can be extracted. After full decoding and detection of the first qudit error case, as shown in Fig. 16, the error on the other two qudits can be decoded and detected by repeating the analogous procedure for each qudit in turn.

Here, we only provide detailed pulses composing the blocks ENC, DET-I, DET- S_X , DET- S_X^2 , but the sequences for the DET blocks can be obtained by calculating the unitary transformation between each error case and the state in Eq. (12). It is worth noting again that with each step of the detection scheme, we project out components of the error state in sequence.

APPENDIX C: OVERALL FIDELITY OF THE MULTIQUIT ERROR CORRECTION PROTOCOL

In this Appendix, we provide a quantitative analysis of the performance of the three-7/2-qudits error correction code. For the modeling, we assumed the following: (1) The three qudits (in our notation, qudits A, B, and C) have identical relaxation characteristics, environmental noise corresponding to the operators $S_X, S_Y, S_Z, S_X^2, S_Y^2, S_Z^2, S_X S_Y, S_Y S_Z, S_Z S_X$. (2) The relaxation rates of all noise channels are equal, i.e., $1/T_{\text{relax},S_X} = 1/T_{\text{relax},S_Y} = 1/T_{\text{relax},S_Z} = 1/T_{\text{relax},S_X^2} = 1/T_{\text{relax},S_Y^2} = 1/T_{\text{relax},S_Z^2} = 1/T_{\text{relax},S_X S_Y} = 1/T_{\text{relax},S_Y S_Z} = 1/T_{\text{relax},S_Z S_X}$. This modeling provides an approximate estimate of the performance of the error correction code. To check practical performance, a similar modeling approach can be applied once the target experimental system and its relaxation characteristics are

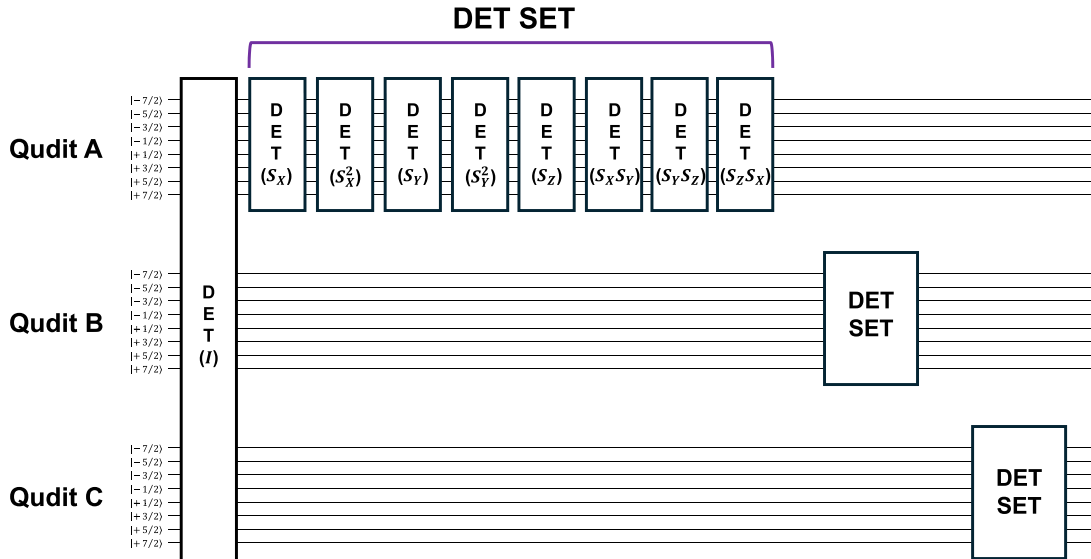


FIG. 16. The schematic diagram of full decoding pulse sequence for multi-qudit error correction code. Each DET code block including Disentanglement – decoding – conditional ancilla measure – recovery sequence for each error case.

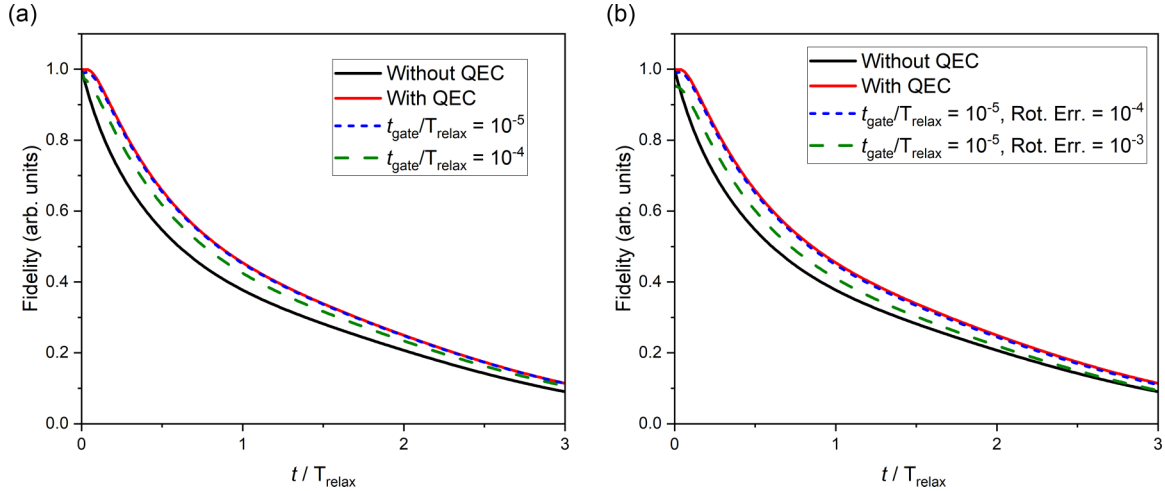


FIG. 17. (a) The qubit fidelity as a function of time without (black) and with (red) error correction. The effect of decoherence during the decoding sequence is simulated for condition of $t_{\text{gate}}/T_{\text{relax}} = 10^{-5}$ (blue), and $t_{\text{gate}}/T_{\text{relax}} = 10^{-4}$ (green). (b) The qubit fidelity simulation with decoherence during the decoding sequence ($t_{\text{gate}}/T_{\text{relax}} = 10^{-5}$), and imperfect gate rotation angle errors of 10^{-4} (blue), and 10^{-3} (green).

identified. In Fig. 17(a), we simulate the logical qubit fidelity with (red) and without (black) the error correction code, and we also include results that account for decoherence during decoding sequence (blue and green). Based on the simulation, we find that the single-qudit gate duration (t_{gate}) shorter than 10^{-5} times of relaxation time (T_{relax}) is required to obtain

a reasonable amount of fidelity gain from error correction protocol. Additionally, Fig. 17(b), presents the effects of imperfect gate rotation angles. This result implies that gate fidelities above three to four nines are required, which are expected to be achievable with state-of-art quantum devices and in near future implementations.

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- [1] D. Gottesman, A. Kitaev, and J. Preskill, Encoding a qubit in an oscillator, *Phys. Rev. A* **64**, 012310 (2001).
 - [2] S. Pirandola, S. Mancini, S. L. Braunstein, and D. Vitali, Minimal qudit code for a qubit in the phase-damping channel, *Phys. Rev. A* **77**, 032309 (2008).
 - [3] K. Noh, S. M. Girvin, and L. Jiang, Encoding an oscillator into many oscillators, *Phys. Rev. Lett.* **125**, 080503 (2020).
 - [4] V. V. Albert, J. P. Covey, and J. Preskill, Robust encoding of a qubit in a molecule, *Phys. Rev. X* **10**, 031050 (2020).
 - [5] B. Royer, S. Singh, and S. M. Girvin, Stabilization of finite-energy Gottesman-Kitaev-Preskill states, *Phys. Rev. Lett.* **125**, 260509 (2020).
 - [6] S. P. Jain, E. R. Hudson, W. C. Campbell, and V. V. Albert, Absorption-emission codes for atomic and molecular quantum information platforms, *Phys. Rev. Lett.* **133**, 260601 (2024).
 - [7] L. Hu, Y. Ma, W. Cai, X. Mu, Y. Xu, W. Wang, Y. Wu, H. Wang, Y. Song, C.-L. Zou *et al.*, Quantum error correction and universal gate set operation on a binomial bosonic logical qubit, *Nat. Phys.* **15**, 503 (2019).
 - [8] C. Flühmann and J. P. Home, Direct characteristic-function tomography of quantum states of the trapped-ion motional oscillator, *Phys. Rev. Lett.* **125**, 043602 (2020).
 - [9] N. Ofek, A. Petrenko, R. Heeres, P. Reinhold, Z. Leghtas, B. Vlastakis, Y. Liu, L. Frunzio, S. M. Girvin, L. Jiang *et al.*, Extending the lifetime of a quantum bit with error correction in superconducting circuits, *Nature (London)* **536**, 441 (2016).
 - [10] P. Campagne-Ibarcq, A. Eickbusch, S. Touzard, E. Zalys-Geller, N. E. Frattini, V. V. Sivak, P. Reinhold, S. Puri, S. Shankar, R. J. Schoelkopf *et al.*, Quantum error correction of a qubit encoded in grid states of an oscillator, *Nature (London)* **584**, 368 (2020).
 - [11] D. Gottesman, *Stabilizer Codes and Quantum Error Correction* (California Institute of Technology, Pasadena, CA, 1997).
 - [12] E. Knill and R. Laflamme, Theory of quantum error-correcting codes, *Phys. Rev. A* **55**, 900 (1997).
 - [13] E. Knill, R. Laflamme, and L. Viola, Theory of quantum error correction for general noise, *Phys. Rev. Lett.* **84**, 2525 (2000).
 - [14] Google Quantum AI, Suppressing quantum errors by scaling a surface code logical qubit, *Nature (London)* **614**, 676 (2023).
 - [15] B. M. Terhal, Quantum error correction for quantum memories, *Rev. Mod. Phys.* **87**, 307 (2015).
 - [16] D. P. DiVincenzo and P. W. Shor, Fault-tolerant error correction with efficient quantum codes, *Phys. Rev. Lett.* **77**, 3260 (1996).
 - [17] R. Laflamme, C. Miquel, J. P. Paz, and W. H. Zurek, Perfect quantum error correcting code, *Phys. Rev. Lett.* **77**, 198 (1996).
 - [18] A. Chiesa, E. Macaluso, F. Petiziol, S. Wimberger, P. Santini, and S. Carretta, Molecular nanomagnets as qubits with embedded quantum-error correction, *J. Phys. Chem. Lett.* **11**, 8610 (2020).
 - [19] J. A. Gross, Designing codes around interactions: The case of a spin, *Phys. Rev. Lett.* **127**, 010504 (2021).
 - [20] S. Lim, J. Liu, and A. Ardaavan, Fault-tolerant qubit encoding using a spin-7/2 qudit, *Phys. Rev. A* **108**, 062403 (2023).
 - [21] D. Janković, J.-G. Hartmann, M. Ruben, and P.-A. Hervieux, Noisy qudit vs multiple qubits: conditions on gate efficiency for enhancing fidelity, *npj Quantum Inf.* **10**, 59 (2024).

- [22] S. J. Lockyer, A. Chiesa, G. A. Timco, E. J. McInnes, T. S. Bennett, I. J. Vitorica-Yrezabal, S. Carretta, and R. E. Winpenny, Targeting molecular quantum memory with embedded error correction, *Chem. Sci.* **12**, 9104 (2021).
- [23] S. Carretta, D. Zueco, A. Chiesa, Á. Gómez-León, and F. Luis, A perspective on scaling up quantum computation with molecular spins, *Appl. Phys. Lett.* **118**, 240501 (2021).
- [24] S. Omanakuttan and J. A. Gross, Multispin Clifford codes for angular momentum errors in spin systems, *Phys. Rev. A* **108**, 022424 (2023).
- [25] S. Omanakuttan, V. Buchemavari, J. A. Gross, I. H. Deutsch, and M. Marvian, Fault-tolerant quantum computation using large spin-cat codes, *PRX Quantum* **5**, 020355 (2024).
- [26] S. Omanakuttan and T. J. Volkoff, Spin-squeezed Gottesman-Kitaev-Preskill codes for quantum error correction in atomic ensembles, *Phys. Rev. A* **108**, 022428 (2023).
- [27] S. Omanakuttan, A. Mitra, E. J. Meier, M. J. Martin, and I. H. Deutsch, Qudit entanglers using quantum optimal control, *PRX Quantum* **4**, 040333 (2023).
- [28] S. Lim, M. V. Vaganov, J. Liu, and A. Ardavan, Demonstrating experimentally the encoding and dynamics of an error-correctable logical qubit on a hyperfine-coupled nuclear spin qudit, *Phys. Rev. Lett.* **134**, 070603 (2025).
- [29] X. Yu, B. Wilhelm, D. Holmes, A. Vaartjes, D. Schwienbacher, M. Nurizzo, A. Kringhøj, M. R. V. Blankenstein, A. M. Jakob, P. Gupta *et al.*, Schrödinger cat states of a nuclear spin qudit in silicon, *Nat. Phys.* **21**, 362 (2025).
- [30] J. Preskill, Quantum computing in the NISQ era and beyond, *Quantum* **2**, 79 (2018).
- [31] A. Noiri, K. Takeda, T. Nakajima, T. Kobayashi, A. Sammak, G. Scappucci, and S. Tarucha, Fast universal quantum gate above the fault-tolerance threshold in silicon, *Nature (London)* **601**, 338 (2022).
- [32] X. Xue, M. Russ, N. Samkharadze, B. Undseth, A. Sammak, G. Scappucci, and L. M. Vandersypen, Quantum logic with spin qubits crossing the surface code threshold, *Nature (London)* **601**, 343 (2022).
- [33] J. Yoneda, K. Takeda, T. Otsuka, T. Nakajima, M. R. Delbecq, G. Allison, T. Honda, T. Kōdera, S. Oda, Y. Hoshi *et al.*, A quantum-dot spin qubit with coherence limited by charge noise and fidelity higher than 99.9%, *Nat. Nanotechnol.* **13**, 102 (2018).
- [34] F. Arute, K. Arya, R. Babbush, D. Bacon, J. C. Bardin, R. Barends, R. Biswas, S. Boixo, F. G. Brandao, D. A. Buell *et al.*, Quantum supremacy using a programmable superconducting processor, *Nature (London)* **574**, 505 (2019).
- [35] Y. Wu, W.-S. Bao, S. Cao, F. Chen, M.-C. Chen, X. Chen, T.-H. Chung, H. Deng, Y. Du, D. Fan *et al.*, Strong quantum computational advantage using a superconducting quantum processor, *Phys. Rev. Lett.* **127**, 180501 (2021).
- [36] Google Quantum AI, Exponential suppression of bit or phase errors with cyclic error correction, *Nature (London)* **595**, 383 (2021).
- [37] D. Bluvstein, S. J. Evered, A. A. Geim, S. H. Li, H. Zhou, T. Manovitz, S. Ebadi, M. Cain, M. Kalinowski, D. Hangleiter *et al.*, Logical quantum processor based on reconfigurable atom arrays, *Nature (London)* **626**, 58 (2024).
- [38] J. P. Bonilla Ataides, D. K. Tuckett, S. D. Bartlett, S. T. Flammia, and B. J. Brown, The XZZX surface code, *Nat. Commun.* **12**, 2172 (2021).
- [39] D. K. Tuckett, S. D. Bartlett, S. T. Flammia, and B. J. Brown, Fault-tolerant thresholds for the surface code in excess of 5% under biased noise, *Phys. Rev. Lett.* **124**, 130501 (2020).
- [40] D. K. Tuckett, A. S. Darmawan, C. T. Chubb, S. Bravyi, S. D. Bartlett, and S. T. Flammia, Tailoring surface codes for highly biased noise, *Phys. Rev. X* **9**, 041031 (2019).
- [41] Q. Xu, N. Mannucci, A. Seif, A. Kubica, S. T. Flammia, and L. Jiang, Tailored XZZX codes for biased noise, *Phys. Rev. Res.* **5**, 013035 (2023).
- [42] F. Petiziol, A. Chiesa, S. Wimberger, P. Santini, and S. Carretta, Counteracting dephasing in molecular nanomagnets by optimized qudit encodings, *npj Quantum Inf.* **7**, 133 (2021).
- [43] I. Fernández de Fuentes, T. Botzem, M. A. Johnson, A. Vaartjes, S. Asaad, V. Mourik, F. E. Hudson, K. M. Itoh, B. C. Johnson, A. M. Jakob, J. C. McCallum, D. N. Jamieson, A. S. Dzurak, and A. Morello, Navigating the 16-dimensional Hilbert space of a high-spin donor qudit with electric and magnetic fields, *Nat. Commun.* **15**, 1380 (2024).
- [44] S. Asaad, V. Mourik, B. Joecker, M. A. Johnson, A. D. Baczewski, H. R. Firdaus, M. T. Mądzik, V. Schmitt, J. J. Pla, F. E. Hudson *et al.*, Coherent electrical control of a single high-spin nucleus in silicon, *Nature (London)* **579**, 205 (2020).
- [45] M. Kim, H. J. Mamin, M. H. Sherwood, K. Ohno, D. D. Awschalom, and D. Rugar, Decoherence of near-surface nitrogen-vacancy centers due to electric field noise, *Phys. Rev. Lett.* **115**, 087602 (2015).
- [46] P. Jamonneau, M. Lesik, J. P. Tetienne, I. Alvizu, L. Mayer, A. Dréau, S. Kosen, J.-F. Roch, S. Pezzagna, J. Meijer, T. Teraji, Y. Kubo, P. Bertet, J. R. Maze, and V. Jacques, Competition between electric field and magnetic field noise in the decoherence of a single spin in diamond, *Phys. Rev. B* **93**, 024305 (2016).
- [47] A. Morello, J. J. Pla, P. Bertet, and D. N. Jamieson, Donor spins in silicon for quantum technologies, *Adv. Quantum Technol.* **3**, 2000005 (2020).
- [48] R. E. George, J. P. Edwards, and A. Ardavan, Coherent spin control by electrical manipulation of the magnetic anisotropy, *Phys. Rev. Lett.* **110**, 027601 (2013).
- [49] F. Dolde, M. W. Doherty, J. Michl, I. Jakobi, B. Naydenov, S. Pezzagna, J. Meijer, P. Neumann, F. Jelezko, N. B. Manson *et al.*, Nanoscale detection of a single fundamental charge in ambient conditions using the nv-center in diamond, *Phys. Rev. Lett.* **112**, 097603 (2014).
- [50] F. Dolde, H. Fedder, M. W. Doherty, T. Nöbauer, F. Rempp, G. Balasubramanian, T. Wolf, F. Reinhard, L. C. Hollenberg, F. Jelezko *et al.*, Electric-field sensing using single diamond spins, *Nat. Phys.* **7**, 459 (2011).
- [51] J. Liu, J. Mrozek, W. K. Myers, G. A. Timco, R. E. P. Winpenny, B. Kintzel, W. Plass, and A. Ardavan, Electric field control of spins in molecular magnets, *Phys. Rev. Lett.* **122**, 037202 (2019).
- [52] J. Liu, V. V. Laguta, K. Inzani, W. Huang, S. Das, R. Chatterjee, E. Sheridan, S. M. Griffin, A. Ardavan, and R. Ramesh, Coherent electric field manipulation of Fe³⁺ spins in PbTiO₃, *Sci. Adv.* **7**, eabf8103 (2021).
- [53] J. Liu, J. Mrozek, A. Ullah, Y. Duan, J. J. Baldoví, E. Coronado, A. Gaita-Arino, and A. Ardavan, Quantum coherent spin-electric control in a molecular nanomagnet at clock transitions, *Nat. Phys.* **17**, 1205 (2021).
- [54] S. Takahashi, I. Tupitsyn, J. Van Tol, C. Beedle, D. Hendrickson, and P. Stamp, Decoherence in crystals of

- quantum molecular magnets, *Nature (London)* **476**, 76 (2011).
- [55] D. Gottesman, Class of quantum error-correcting codes saturating the quantum hamming bound, *Phys. Rev. A* **54**, 1862 (1996).
- [56] A. Zabala-Lekuona, J. M. Seco, and E. Colacio, Single-molecule magnets: From Mn12-ac to dysprosium metallocenes, a travel in time, *Coord. Chem. Rev.* **441**, 213984 (2021).
- [57] M. Murugesu, S. Takahashi, A. Wilson, K. A. Abboud, W. Wernsdorfer, S. Hill, and G. Christou, Large mn25 single-molecule magnet with spin $S = 51/2$: magnetic and high-frequency electron paramagnetic resonance spectroscopic characterization of a giant spin state, *Inorg. Chem.* **47**, 9459 (2008).
- [58] S. Takahashi, J. van Tol, C. C. Beedle, D. N. Hendrickson, L.-C. Brunel, and M. S. Sherwin, Coherent manipulation and decoherence of $S = 10$ single-molecule magnets, *Phys. Rev. Lett.* **102**, 087603 (2009).
- [59] M. J. Boguslawski, Z. J. Wall, S. R. Vizvary, I. D. Moore, M. Bareian, D. T. C. Allcock, D. J. Wineland, E. R. Hudson, and W. C. Campbell, Raman scattering errors in stimulated-Raman-induced logic gates in $^{133}\text{Ba}^+$, *Phys. Rev. Lett.* **131**, 063001 (2023).
- [60] X. Yu, J. Mo, T. Lu, T. Y. Tan, and T. L. Nicholson, Magneto-optical trapping of a group-III atom, *Phys. Rev. A* **105**, L061101 (2022).
- [61] K. J. Arnold, R. Kaewuam, A. Roy, T. R. Tan, and M. D. Barrett, Blackbody radiation shift assessment for a lutetium ion clock, *Nat. Commun.* **9**, 1650 (2018).
- [62] G. Tosi, F. A. Mohiyaddin, V. Schmitt, S. Tenberg, R. Rahman, G. Klimeck, and A. Morello, Silicon quantum processor with robust long-distance qubit couplings, *Nat. Commun.* **8**, 450 (2017).
- [63] D. N. Jamieson, W. I. Lawrie, S. G. Robson, A. M. Jakob, B. C. Johnson, and J. C. McCallum, Deterministic doping, *Mater. Sci. Semicond. Proc.* **62**, 23 (2017).
- [64] K. Groot-Berning, T. Kornher, G. Jacob, F. Stopp, S. T. Dawkins, R. Kolesov, J. Wrachtrup, K. Singer, and F. Schmidt-Kaler, Deterministic single-ion implantation of rare-earth ions for nanometer-resolution color-center generation, *Phys. Rev. Lett.* **123**, 106802 (2019).
- [65] J. Rahmer, L. Dunsch, H. Dorn, J. Mende, and M. Mehring, W-and x-band epr investigation of $\text{Sc}_3@\text{C}_{82}$, *Magn. Reson. Chem.* **43**, S192 (2005).
- [66] Z. Liu, B.-W. Dong, H.-B. Meng, M.-X. Xu, T.-S. Wang, B.-W. Wang, C.-R. Wang, S.-D. Jiang, and S. Gao, Qubit crossover in the endohedral fullerene $\text{Sc}_3\text{C}_2@\text{C}_{80}$, *Chem. Sci.* **9**, 457 (2018).
- [67] A. R. Calderbank, E. M. Rains, P. M. Shor, and N. J. Sloane, Quantum error correction via codes over $\text{GF}(4)$, *IEEE Trans. Inf. Theory* **44**, 1369 (1998).