

Qudit-based quantum error-correcting codes from irreducible representations of $SU(d)$

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Qudits naturally correspond to multilevel quantum systems, which have the potential to enable more efficient quantum information processing, but their reliability is contingent upon quantum error-correction capabilities. In this paper, we present a general procedure for constructing error-correcting qudit codes through the irreducible representations of $SU(d)$ for any odd integer $d \geq 3$. Using the Weyl character formula and inner product of characters, we deduce the relevant branching rules, through which we identify the physical Hilbert spaces that contain valid code spaces. We then discuss how two forms of permutation invariance and the Heisenberg-Weyl symmetry of $\mathfrak{su}(d)$ can be exploited to simplify the construction of error-correcting codes. Finally, we use our procedure to construct an infinite class of error-correcting codes encoding a logical qudit into $(d-1)^2$ physical qudits.

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I. INTRODUCTION

Despite our best efforts to suppress quantum noise, it continues to stymie reliable large-scale quantum computation [1,2]. Quantum states are inextricably coupled to their environment and this inevitably leads to the corruption of fragile quantum information [3,4]. Good quantum error-correcting codes are thus vital to the realization of the full potential of quantum computers [5,6].

Quantum error correction is typically performed by encoding quantum information across a sufficiently large number of physical states [7]. Although most quantum error-correcting codes encode logical qubits into multiple physical qubits, qudit-based computation offers a myriad of advantages. One theoretical benefit is that the number of computational units needed to obtain a logical Hilbert space of the same dimension is reduced by a factor of $\log_2 d$ [8], where d is the dimension of one qudit. Moreover, developing qudit codes offers practical advantages, which has been demonstrated experimentally [9]. Many physical systems like superconducting circuits naturally have more than two energy levels [10] and isolating a two-dimensional subspace leads to leakage errors. Using a physical system as a qudit would involve the whole Hilbert space in the encoding and error correction, thereby avoiding leakage [11].

While error-correcting codes for qudits have been constructed before [12–16], representation theory may offer new insight into this. In [17], Gross explored how to construct

error-correcting codes through the irreducible representations of $SU(2)$. These are suitable for encoding qubits into single-spin systems since they possess $\mathfrak{su}(2)$ symmetry. However, the concept behind his procedure could be generalized to other physical systems with a Lie algebra of Hamiltonians [17], including $\mathfrak{su}(d)$ -symmetric qudits. The idea is to choose our code space within a vector space on which some irreducible representation (irrep) of a Lie group G , restricted to a finite subgroup $K \leq G$, is defined. By construction, we can implement any operation $O \in K$ on the logical qudit and the code space is endowed with a Lie algebra $\mathfrak{g}$ of physically relevant error operators.

Recently, Herbert *et al.* worked on the generalization to $SU(3)$, yielding a method for constructing error-correcting codes for qutrits [18]. The natural next step would be to consider the general case of $SU(d)$ and, through this, formulate a recipe for constructing error-correcting codes that encode a logical qudit of higher dimension into multiple physical qudits (Fig. 1). Notably, these codes are protected against $\mathfrak{su}(d)$ errors, i.e., phase and dit flips. This extension is by no means a simple task, not least because it requires us to deal with arbitrarily large code spaces and arbitrarily large sets of error-correction conditions. However, as we will soon show, symmetry can considerably simplify the problem.

In this paper, we present a general procedure for constructing distance-3 quantum error-correcting codes for qudits of odd dimension $d \geq 3$ through the irreducible representations of $SU(d)$. To allow error correction, we restrict the $SU(d)$ irreps to $HW(d)$, the Heisenberg-Weyl group, which was chosen since its image under certain irreps contains the qudit generalizations of the qubit Pauli operators [19–21]. We then determine the branching rules for $SU(d) \downarrow HW(d)$, from which we can deduce which irreps are defined on symmetric vector spaces that contain valid code spaces. This is followed by the use of the Heisenberg-Weyl symmetry of $\mathfrak{su}(d)$ to show that some error-correction conditions are equivalent to each other and thereby obtain a reduced set of sufficient

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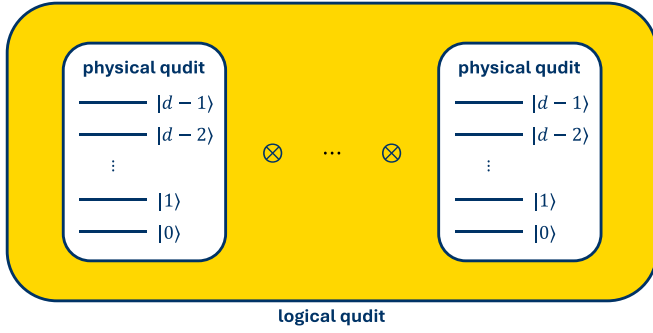


FIG. 1. Logical qudit encoded into multiple physical qudits, each of dimension d . This is a direct analog of encoding a logical qubit into many physical qubits.

conditions for a code to be error correcting. We then show that it is convenient to consider so-called *sparse* and *doubly permutation-invariant* codes and use these simplifications to construct a class of error-correcting codes encoding a logical qudit, for any odd integer $d \geq 5$, into $(d-1)^2$ physical qudits.

II. IRREPS AND CODE SPACES

A *code* $\mathcal{C}$ is defined as an embedding of a logical Hilbert space $\mathcal{H}_L$ into a physical Hilbert space $\mathcal{H}_P$ [22–26]. The image of $\mathcal{C}$ is called the *code space*, whose elements are referred to as *code words* [22,23]. Since we are dealing with qudits, the logical Hilbert space is $\mathcal{H}_L = \mathbb{C}^d$, for which we use the orthonormal basis $\{|k\rangle_L \mid k \in \mathbb{Z}_d\}$. We then denote by $|\bar{k}\rangle$ the image of $|k\rangle_L$ under $\mathcal{C}$, i.e., the code word corresponding to the logical qudit state $|k\rangle_L$.

A. Irreps of $SU(d)$ on symmetric spaces

We choose $\mathcal{H}_P$ to be a vector space on which an irrep of $SU(d)$ is defined, thereby ensuring that the code space is preserved by any $SU(d)$ operation (i.e., any logical single-qudit gate, up to a global phase factor). Physically, a natural choice for $\mathcal{H}_P$ would be a symmetric vector space. By the symmetrization postulate for indistinguishable particles, the states of a system of bosons are necessarily permutation invariant (or totally symmetric) [27–30]. Moreover, the ground subspace of any Heisenberg ferromagnet’s Hilbert space contains permutation-invariant states [31,32] and this model of quantum magnetism is naturally realized in numerous physical systems [33]. Separately from this physical motivation, there are also mathematical reasons for choosing symmetric vector spaces: constructing a basis for such spaces is simple and the action of the error operators in $\mathfrak{su}(d)$ obeys a straightforward and easily generalizable pattern. We therefore aim to construct codes that map to a symmetric physical Hilbert space.

In line with this, we give a brief overview of some relevant results about the irreps of $SU(d)$, particularly those defined on symmetric vector spaces. For further information, readers may consult Refs. [34–36].

We begin by noting that there is a one-to-one correspondence between the irreps of the Lie group $SU(d)$, the associated Lie algebra $\mathfrak{su}(d)$, and its complexification $\mathfrak{su}(d)_{\mathbb{C}} \cong \mathfrak{sl}(d, \mathbb{C})$ [34]. Thus, in a not-so-egregious abuse of

notation, we will denote these maps by the same symbol. More importantly, this means that we can study the irreps of $SU(d)$ by applying the general theory of semisimple Lie algebras to $\mathfrak{sl}(d, \mathbb{C})$.

By the theorem of the highest weight [34], each irrep of $\mathfrak{sl}(d, \mathbb{C})$ is uniquely determined by its highest weight ω , which in turn can be identified with a multiplet of non-negative integers $(\omega_0, \dots, \omega_{d-2}) \in \mathbb{N}_0^{d-1}$. Moreover, it can be shown that the irrep with highest weight $\omega = (N, 0, \dots, 0)$, for any positive integer $N \in \mathbb{N}$, is on a symmetric vector space $\text{Sym}^N(\mathbb{C}^d)$ [36]. We shall denote this irrep by $\pi_{d,N}$. For the rest of this paper, we will only be considering irreps of this type.

Having chosen a set of candidates for $\mathcal{H}_P$, let us now construct a basis for these vector spaces. Fix $N \in \mathbb{N}$. For each $\mathbf{u} \in \mathbb{N}_0^d$ with $\sum_{\xi \in \mathbb{Z}_d} u_{\xi} = N$, we define the multiset

$$\mathcal{M}_{\mathbf{u}} = \left\{ \underbrace{0, \dots, 0}_{u_0}, \dots, \underbrace{d-1, \dots, d-1}_{u_{d-1}} \right\} \quad (1)$$

and thereafter the *permutation-invariant vector*

$$|S_{\mathbf{u}}\rangle = \sum_{\substack{i_1, \dots, i_N \text{ such that} \\ \{i_1, \dots, i_N\} = \mathcal{M}_{\mathbf{u}}}} |i_1\rangle \otimes \dots \otimes |i_N\rangle. \quad (2)$$

From this, we can construct the set

$$\left\{ |S_{\mathbf{u}}\rangle \mid \mathbf{u} \in \mathbb{N}_0^d, \sum_{\xi \in \mathbb{Z}_d} u_{\xi} = N \right\}, \quad (3)$$

which is a basis for $\text{Sym}^N(\mathbb{C}^d)$. To see why, note that it has the correct size and that, for $\mathbf{u} \neq \mathbf{v}$, $|S_{\mathbf{u}}\rangle$ and $|S_{\mathbf{v}}\rangle$ are orthogonal and thus linearly independent.

This is, however, not the end of the story. Thanks to the way tensor product representations are defined, codes whose range is $\text{Sym}^N(\mathbb{C}^d)$ can transversally implement any logical operation $\mathbf{O} \in SU(d)$. Unfortunately, the Eastin-Knill theorem [37–39] precludes such codes from being error correcting. In order to obtain a proper subspace of $\text{Sym}^N(\mathbb{C}^d)$ that remains invariant under some set of logical operations, we must restrict the irreps to a finite subgroup of logical operations that we want to implement—the Heisenberg-Weyl group.

B. Heisenberg-Weyl group

Let $d \geq 3$ be an odd integer, $\mathbb{Z}_d$ be the ring of integers modulo d , and $\mathbb{Z}_d^{\times}$ be the multiplicative group of integers modulo d . For any vector space V , let $\text{GL}(V)$ be the group of invertible linear maps $V \rightarrow V$.

We define the *Heisenberg-Weyl group*, otherwise known as the qudit Pauli group, to be the set

$$\text{HW}(d) = \left\{ \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} \mid a, b, c \in \mathbb{Z}_d \right\} \quad (4)$$

with the operation matrix multiplication and identity element $\text{diag}(1, 1, 1)$. Since we want to be able to implement any operation $\mathbf{O} \in \text{HW}(d)$ on the logical qudit, we are interested in the faithful d -dimensional irreps of $\text{HW}(d)$, which we now construct. For each $z \in \mathbb{Z}_d^{\times}$, we define $\rho_z : \text{HW}(d) \rightarrow \text{GL}(\mathbb{C}^d)$

as the group homomorphism that maps

$$\begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} \mapsto \zeta_d^{cz} X_d^a Z_d^{bz}, \quad (5)$$

where $\zeta_d = \exp(2\pi i/d)$ and

$$X_d = \sum_{j \in \mathbf{Z}_d} |j \oplus 1\rangle \langle j|, \quad (6)$$

$$Z_d = \sum_{k \in \mathbf{Z}_d} \zeta_d^k |k\rangle \langle k|. \quad (7)$$

These are all of the faithful d -dimensional irreps of $\text{HW}(d)$ [40,41]. Above, the symbol $\oplus$ is used to denote addition modulo d . It will be useful to know that the character of ρ_z is the map $\chi_{\rho_z} : \text{HW}(d) \rightarrow \mathbf{C}$ given by

$$\begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} \mapsto d \zeta_d^{cz} \delta_{a0} \delta_{b0}. \quad (8)$$

For more information about character theory, we refer readers to Refs. [34–36].

Before proceeding, we note that it is necessary to restrict ourselves to odd d because, for any $z \in \mathbf{Z}_d^\times$, the group $\rho_z(\text{HW}(d))$ is not a subgroup of $\text{SU}(d)$ if d is even. A quick check shows that, whenever d is even, the determinant of Z_d , which is an element of $\rho_z(\text{HW}(d))$ for any $z \in \mathbf{Z}_d^\times$, is -1 .

C. Branching rules and valid code spaces

Now that we have identified a set of candidate physical Hilbert spaces from $\text{SU}(d)$ irreps as well as a finite subgroup of logical operations, the next step is to determine which of these candidate physical Hilbert spaces contain valid code spaces.

In our analysis, we consider the embedding $\text{HW}(d) \hookrightarrow \text{SU}(d)$ given by the map ρ_1 . A *valid code space* is thus defined as a direct sum of subspaces $W \subset \text{Sym}^N(\mathbf{C}^d)$ on which ρ_1 may be defined. But why is this the appropriate definition? First things first, we define our code $\mathcal{C}$ in such a way that, for all $k \in \mathbf{Z}_d$,

$$\bar{Z}_d |\bar{0}\rangle = |\bar{0}\rangle, \quad (9)$$

$$\bar{X}_d |\bar{k}\rangle = |\bar{k} \oplus 1\rangle, \quad (10)$$

where the bars over operators indicate that we are taking the image under the relevant $\text{SU}(d)$ irrep. Now, since ρ_1 is the fundamental irrep of $\text{HW}(d)$, it follows that the action of $\bar{X}_d$ and $\bar{Z}_d$ on valid code spaces, under our definition, is given by

$$\bar{X}_d : |\bar{k}\rangle \mapsto |\bar{k} \oplus 1\rangle, \quad (11)$$

$$\bar{Z}_d : |\bar{k}\rangle \mapsto \zeta_d^k |\bar{k}\rangle, \quad (12)$$

for all $k \in \mathbf{Z}_d$. Hence our definition ensures that the corresponding operations X_L and Z_L on a logical qudit can be implemented in the way they are typically defined.

A remark is in order. It is worth noting that, although our work primarily adheres to this convention, other embeddings and definitions are technically just as valid. After all, there is nothing fundamental about the phase factor that Z_L applies to

each $|k\rangle_L \in \mathcal{H}_L$. We could, for instance, have defined its action on $\mathcal{H}_L$ to be $|k\rangle_L \mapsto \zeta_d^{-k} |k\rangle_L$ or even $|k\rangle_L \mapsto \zeta_d^{2k} |k\rangle_L$. In such cases, we then define a valid code space as a direct sum of subspaces on which ρ_{d-1} and ρ_2 may be defined. We discuss this in greater detail towards the end; for now, we will anchor our analysis in standard definitions.

We can determine the values of N for which the vector space $\text{Sym}^N(\mathbf{C}^d)$ contains a valid code space by studying how $\pi_{d:N}|_{\text{HW}(d)}$ [which is in general a reducible representation of $\text{HW}(d)$] decomposes into a direct sum of $\text{HW}(d)$ irreps. The rules for such a decomposition of an irrep when restricted to a subgroup are called *branching rules*; we denote by $\text{SU}(d) \downarrow \text{HW}(d)$ all the branching rules for all irreps of $\text{SU}(d)$ when restricted to $\text{HW}(d)$. One way to deduce a branching rule is to use the inner product between characters. In particular, the multiplicity of the irrep ρ_η (for any $\eta \in \mathbf{Z}_d^\times$) in the decomposition of $\pi_{d:N}|_{\text{HW}(d)}$ into a direct sum of irreps is given by [36,42]

$$\left\langle \chi_{\rho_\eta}, \chi_{\pi_{d:N}|_{\text{HW}(d)}} \right\rangle = \frac{1}{|\text{HW}(d)|} \sum_{g \in \text{HW}(d)} \chi_{\rho_\eta}^*(g) \chi_{\pi_{d:N}|_{\text{HW}(d)}}(g). \quad (13)$$

We have already calculated χ_{ρ_η} previously in Eq. (8), so it remains to evaluate $\chi_{\pi_{d:N}|_{\text{HW}(d)}}$. In fact, it suffices to determine the values that $\chi_{\pi_{d:N}|_{\text{HW}(d)}}$ take in the center $Z(\text{HW}(d))$ because χ_{ρ_η} vanishes for all elements $g \in \text{HW}(d) \setminus Z(\text{HW}(d))$. We do this in the following lemma.

Lemma 1. For any $N \in \mathbf{N}$, the character of the irrep $\pi_{d:N}|_{\text{HW}(d)}$ takes on the following values in the center $Z(\text{HW}(d))$:

$$\chi_{\pi_{d:N}}(\zeta_d^\ell \mathbf{I}) = \binom{N+d-1}{N} \zeta_d^{\ell N} \quad (14)$$

for all $\ell \in \mathbf{Z}_d$.

Proof. We use the Weyl character formula expressed in terms of complete symmetric polynomials [36]. In particular, for the special case of $\pi_{d:N}$, we have for all $\mathbf{M} \in \text{SU}(d)$,

$$\chi_{\pi_{d:N}}(\mathbf{M}) = \sum_{1 \leq i_1 \leq \dots \leq i_N \leq d} \lambda_{i_1} \lambda_{i_2} \dots \lambda_{i_N}, \quad (15)$$

where $\lambda_1, \dots, \lambda_d$ are the eigenvalues of $\mathbf{M}$.

Recall that, for any $\ell \in \mathbf{Z}_d$, the eigenvalues of $\zeta_d^\ell \mathbf{I}$ are all ζ_d^ℓ . By a simple counting argument, it is easy to see that the number of increasing functions $\{1, \dots, N\} \rightarrow \{1, \dots, d\}$ is $\binom{N+d-1}{N}$. The result follows immediately from these observations. ■

With the above lemma established, we are now ready to state and prove the branching rules.

Theorem 1. For any $N \in \mathbf{N}$ with $\gcd(N, d) = 1$, the branching rules for $\text{SU}(d) \downarrow \text{HW}(d)$ are

$$\pi_{d:N}|_{\text{HW}(d)} = \rho_\eta^{\oplus \frac{1}{d} \dim(\text{Sym}^N(\mathbf{C}^d))}, \quad (16)$$

where $\eta \in \mathbf{Z}_d^\times$ is such that $\eta \equiv N \pmod{d}$. (In this particular case, $\oplus$ is used to denote a direct sum.)

Proof. First note that

$$\dim(\text{Sym}^N(\mathbf{C}^d)) = \binom{N+d-1}{N}. \quad (17)$$

We then use Eq. (13) to obtain the multiplicity of the $\text{HW}(d)$ irreps in $\pi_{d:N}|\text{HW}(d)\rangle$. We fix $\eta \in \mathbf{Z}_d^\times$ and observe that, for any $m \in \mathbf{N}$ and any $N \equiv \eta \pmod{d}$,

$$\begin{aligned} & \left\langle \chi_{\rho_m}, \chi_{\pi_{d:N}|\text{HW}(d)} \right\rangle \\ &= \frac{1}{|\text{HW}(d)|} \sum_{\ell \in \mathbf{Z}_d} \chi_{\rho_m}^*(\zeta_d^\ell \mathbf{l}) \cdot \chi_{\pi_{d:N}|\text{HW}(d)}(\zeta_d^\ell \mathbf{l}) \\ &= \frac{1}{d^3} \sum_{\ell \in \mathbf{Z}_d} d \zeta_d^{-\ell m} \cdot \dim(\text{Sym}^N(\mathbf{C}^d)) \zeta_d^{\ell \eta} \\ &= \frac{1}{d^2} \dim(\text{Sym}^N(\mathbf{C}^d)) \underbrace{\sum_{\ell \in \mathbf{Z}_d} \zeta_d^{\ell(\eta-m)}}_{=d \delta_{m\eta}} \\ &= \frac{1}{d} \dim(\text{Sym}^N(\mathbf{C}^d)) \delta_{m\eta}, \end{aligned} \quad (18)$$

where we have used the fact that $\chi_{\rho_m}(g) = 0$ for all $g \in \text{HW}(d) \setminus Z(\text{HW}(d))$. The result then follows directly. ■

As a direct consequence of Theorem 1, we have the following corollary.

Corollary 1. If $N \equiv 1 \pmod{d}$, then the vector space $\text{Sym}^N(\mathbf{C}^d)$ contains a valid code space.

Proof. By Theorem 1, ρ_1 is in the decomposition of $\pi_{d:N}|\text{HW}(d)\rangle$ if $N \equiv 1 \pmod{d}$. ■

In fact, more can be said beyond Corollary 1. If $N \equiv 1 \pmod{d}$, $\text{Sym}^N(\mathbf{C}^d)$ does not only contain a valid code space—it is a valid code space. This follows from the fact that, if $N \equiv 1 \pmod{d}$, the only $\text{HW}(d)$ irrep in the decomposition of the representation $\pi_{d:N}|\text{HW}(d)\rangle$ is ρ_1 . Of course, as noted earlier, we still cannot have $\mathcal{C}(\mathbf{C}^d) = \text{Sym}^N(\mathbf{C}^d)$ if we want $\mathcal{C}$ to be error correcting. But, as Theorem 1 shows, $\text{Sym}^N(\mathbf{C}^d)$ has proper subsets that work as valid code spaces, which is what we aimed to achieve. In fact, we have considerable choice over these subsets since $\text{Sym}^N(\mathbf{C}^d)$ as a whole is a valid code space.

D. Code structure

To reiterate, the logical Hilbert space in our case is $\mathcal{H}_L = \mathbf{C}^d$. The physical Hilbert space, on the other hand, is taken to be $\mathcal{H}_P = \text{Sym}^N(\mathbf{C}^d)$ for some $N \equiv 1 \pmod{d}$. For the remainder of this article, it is assumed that $N \equiv 1 \pmod{d}$, unless otherwise stated.

In addition, we note that a code $\mathcal{C}$ is uniquely determined by its image on $\{|0\rangle_L\}$ since, by Eq. (10), specifying $|\bar{0}\rangle$ fixes all the other code words. Thus we will often be defining our codes by specifying the code word $|\bar{0}\rangle$.

III. REDUCED KNILL-LAFLAMME CONDITIONS

A code $\mathcal{C} : \mathcal{H}_L \rightarrow \mathcal{H}_P$ is called a *quantum error-correcting code* (QECC) that corrects a set $\mathcal{E}$ of errors if and

only if for all $|\bar{i}\rangle, |\bar{j}\rangle \in \mathcal{C}(\mathcal{H}_L)$ and all $\mathbf{E}_a, \mathbf{E}_b \in \mathcal{E}$,

$$\langle \bar{i} | \mathbf{E}_a^\dagger \mathbf{E}_b | \bar{j} \rangle = c_{a,b} \langle \bar{i} | \bar{j} \rangle \quad (19)$$

for some set $\{c_{a,b}\}$ of constants that only depend on $\mathbf{E}_a, \mathbf{E}_b$ [23,43]. This set of equations is also known as the *Knill-Laflamme conditions*. Notably, it suffices to show that these conditions are satisfied on a basis of error operators [23,43].

With $\mathcal{H}_P$ being a vector space on which a representation of $\text{SU}(d)$ is defined, it would be mathematically elegant to consider the Lie algebra $\mathfrak{su}(d)$ as our error operators [44]—more precisely, the image of $\mathfrak{su}(d)$ under the relevant representation. Remarkably, this mathematical assumption coincides with what is physically relevant. The elements of $\mathfrak{su}(d)$ correspond to dit flips, phase flips, or combinations of both. Of course, there is also the case of no error occurring, i.e., the error operator is $\mathbf{l}$.

In constructing a basis for $\mathcal{E}$, it would be useful to define the following matrices for all $(j, k) \in \mathbf{Z}_d^2$, $\ell \in \mathbf{Z}_d$:

$$\mathbf{S}_d^{(j,k)} = |j\rangle \langle k| + |k\rangle \langle j|, \quad (20)$$

$$\mathbf{A}_d^{(j,k)} = -i|j\rangle \langle k| + i|k\rangle \langle j|, \quad (21)$$

$$\mathbf{D}_d^{(\ell)} = |\ell\rangle \langle \ell| - |\ell \oplus 1\rangle \langle \ell \oplus 1|, \quad (22)$$

which are essentially the qudit generalizations of the Pauli error operators on qubits. Physically, $\mathbf{S}_d^{(j,k)}$ are dit flips, $\mathbf{D}_d^{(\ell)}$ are phase flips, and $\mathbf{A}_d^{(j,k)}$ are combinations of dit and phase flips. Having defined these matrices, let us now explicitly construct a basis for $\mathcal{E}$ [45]:

$$\begin{aligned} \mathcal{B}_{\mathcal{E}} = \{ & \mathbf{l}, \tilde{\mathbf{S}}_d^{(j,k)}, \tilde{\mathbf{A}}_d^{(j,k)}, \tilde{\mathbf{D}}_d^{(\ell)} \mid 0 \leq j < k \leq d-1, \\ & 0 \leq \ell \leq d-2 \}, \end{aligned} \quad (23)$$

where the tilde indicates that it is the image under the relevant irrep, i.e., $\tilde{\mathbf{G}} = \pi_{d:N}(\mathbf{G})$ for every $\mathbf{G} \in \mathfrak{su}(d)$.

As was the case in Refs. [17,18,46], we can leverage the symmetry of the Lie algebra with respect to the finite subgroup to reduce the number of error-correction conditions we need to check. The following lemma establishes the Heisenberg-Weyl symmetry of $\mathfrak{su}(d)$ elements.

Lemma 2. Observe that

$$\bar{\mathbf{X}}_d^\dagger \tilde{\mathbf{S}}_d^{(j,k)} \bar{\mathbf{X}}_d = \tilde{\mathbf{S}}_d^{(j \oplus 1, k \oplus 1)}, \quad (24)$$

$$\bar{\mathbf{X}}_d^\dagger \tilde{\mathbf{A}}_d^{(j,k)} \bar{\mathbf{X}}_d = \tilde{\mathbf{A}}_d^{(j \oplus 1, k \oplus 1)}, \quad (25)$$

$$\bar{\mathbf{X}}_d^\dagger \tilde{\mathbf{D}}_d^{(\ell)} \bar{\mathbf{X}}_d = \tilde{\mathbf{D}}_d^{(\ell \oplus 1)}, \quad (26)$$

$$\bar{\mathbf{Z}}_d^\dagger \tilde{\mathbf{S}}_d^{(j,k)} \bar{\mathbf{Z}}_d = \text{Re}(\zeta_d^{k-j}) \tilde{\mathbf{S}}_d^{(j,k)} - \text{Im}(\zeta_d^{k-j}) \tilde{\mathbf{A}}_d^{(j,k)}, \quad (27)$$

for all $0 \leq j < k \leq d-1$, $0 \leq \ell \leq d-2$. Above, $\ominus$ denotes subtraction modulo d .

Proof. These follow directly from definitions; explicit calculations can be found in Appendix B. ■

Using the Heisenberg-Weyl symmetry of $\mathfrak{su}(d)$, we obtain a reduced set of conditions for a code to be error correcting.

Theorem 2. A code $\mathcal{C} : \mathbf{C}^d \rightarrow \text{Sym}^N(\mathbf{C}^d)$ is a QECC if it satisfies the following conditions:

$$(C1) \quad \langle \bar{i} | \tilde{\mathbf{S}}_d^{(0,n)} | \bar{j} \rangle = 0 \text{ for all } i, j \in \mathbf{Z}_d \text{ with } i \neq j \text{ and } n \in \{1, \dots, \frac{d-1}{2}\},$$

$$\begin{aligned}
 \text{(C2)} \quad & \langle \bar{i} | \tilde{\Gamma}_d^{(p,q)} \tilde{\Upsilon}_d^{(r,s)} | \bar{j} \rangle = c_{(\tilde{\Gamma}_d^{(p,q)}, \tilde{\Upsilon}_d^{(r,s)})} \delta_{ij} \text{ for all } i, j \in \mathbf{Z}_d, \\
 & \Gamma, \Upsilon \in \{\mathbf{S}, \mathbf{A}\}, 0 \leq p < q \leq d-1, \text{ and } 0 \leq r < s \leq d-1, \\
 \text{(C3)} \quad & \langle \bar{k} | \tilde{\mathbf{D}}_d^{(d-2)} | \bar{k} \rangle = c_{(1, \tilde{\mathbf{D}}_d^{(d-2)})} \text{ for all } k \in \mathbf{Z}_d, \\
 \text{(C4)} \quad & \langle \bar{k} | \tilde{\mathbf{D}}_d^{(\ell)} \tilde{\mathbf{D}}_d^{(d-2)} | \bar{k} \rangle = c_{(\tilde{\mathbf{D}}_d^{(\ell)}, \tilde{\mathbf{D}}_d^{(d-2)})} \text{ for all } k \in \mathbf{Z}_d \text{ and } \\
 & \ell \in \{0, \dots, d-2\},
 \end{aligned}$$

for some set of constants $\{c_{(\mathbf{E}, \mathbf{F})} \mid \mathbf{E}, \mathbf{F} \in \mathcal{E}\}$.

Proof. See Appendix C. ■

In practice, we will be considering codes with permutation invariance, i.e., codes defined by the codeword

$$|\bar{0}\rangle = \sum_{\mathbf{u} \in \mathbf{N}_0^d} \beta_{\mathbf{u}} |S_{\mathbf{u}}\rangle, \quad (28)$$

for some constants $\{\beta_{\mathbf{u}} \mid \mathbf{u} \in \mathbf{N}_0^d\}$. Plugging this into the inner products in (C1) to (C4) above, we see that each inner product gives rise to a quadratic form, whose solutions (zero sets) are precisely the desired combinations of coefficients $\{\beta_{\mathbf{u}}\}$. Explicit examples and calculations for $d = 3$ can be found in Ref. [18].

IV. SPARSE AND DOUBLY PERMUTATION-INVARIANT CODES

Owing to the Heisenberg-Weyl symmetry of $\mathfrak{su}(d)$, the Knill-Laflamme conditions have been somewhat reduced—from d^6 quadratic forms to $d^6 - 2d^5 + d^4 + d^3/2 + d/2$, as can be seen from the delineation of the reduced conditions in Appendix C. However, we evidently still have a lot of conditions to satisfy, so devising a procedure for constructing error-correcting codes for a qudit with large d , let alone a general odd-dimensional qudit, remains a Herculean task. It therefore behooves us to make further simplifications. Abstracting the ideas in Ref. [18] to a general form that works for any odd-dimensional qudit, we consider two types of codes and show how these further reduce the Knill-Laflamme conditions to just three quadratic forms.

A. Sparse codes

First, we consider sparse codes, which naturally protect against most errors involving dit flips. Informally, these codes map logical qudit states to orthogonal subspaces of $\mathcal{H}_P$ that are sufficiently “far” apart such that any sequence of at most two distinct dit flips will not send a code word to a subspace containing a code word. To get an intuition for what would be a good metric and minimum distance to use in our definition, see Lemma D1 in Appendix D.

For each d , we define the following sets for all $N \equiv 1 \pmod d$ and all $v \in \mathbf{Z}_d$:

$$\mathcal{U}_{d,N,v} = \left\{ \mathbf{u} \in \mathbf{N}_0^d \mid \sum_{\xi \in \mathbf{Z}_d} u_{\xi} = N, \bigoplus_{\xi \in \mathbf{Z}_d} \xi u_{\xi} = v \right\}. \quad (29)$$

Suppose that $\mathcal{C} : \mathbf{C}^d \rightarrow \text{Sym}^N(\mathbf{C}^d)$ is a code with

$$|\bar{0}\rangle = \sum_{\mathbf{u} \in \mathcal{Z}_{d,N,0}} \alpha_{\mathbf{u}} |S_{\mathbf{u}}\rangle \quad (30)$$

for some set of constants $\{\alpha_{\mathbf{u}} \mid \mathbf{u} \in \mathcal{Z}_{d,N,0}\}$. We say that $\mathcal{C}$ is *sparse* if

$$\sum_{\xi \in \mathbf{Z}_d} |u_{\xi} - u'_{\xi+\delta}| > 4 \quad (31)$$

for all $\mathbf{u}, \mathbf{u}' \in \mathcal{Z}_{d,N,0}$ with $\alpha_{\mathbf{u}}, \alpha_{\mathbf{u}'} \neq 0$ and all $\delta \in \mathbf{Z}_d$.

Having formally defined sparse codes, we can now be more precise about what we mean by the natural protection they offer against most dit flip errors.

Theorem 3. If a code $\mathcal{C} : \mathbf{C}^d \rightarrow \text{Sym}^N(\mathbf{C}^d)$ is sparse, then $\mathcal{C}$ satisfies Condition (C1) and Condition (C2) with $(p, q) \neq (r, s)$.

Proof. See Appendix D. ■

We note that a sparse code does not necessarily satisfy Condition (C2) with $(p, q) = (r, s)$. To see why, observe that the image of $|S_{\mathbf{u}}\rangle \in \text{Sym}^N(\mathbf{C}^d)$ under two identical dit flips could have a nonzero component along $|S_{\mathbf{u}}\rangle$. Thus, given $0 \leq p < q \leq d-1$, the inner products $\langle \bar{k} | \tilde{\Gamma}_d^{(p,q)} \tilde{\Upsilon}_d^{(p,q)} | \bar{k} \rangle$ might not vanish for all $k \in \mathbf{Z}_d$, in which case we really have to make sure that they are all equal to each other. However, the sparsity of a code does not guarantee this.

B. Doubly permutation-invariant codes

Next, we consider doubly permutation-invariant codes, which are meant to deal with the phase flip errors. To get a sense of such codes, consider the action of a phase flip on $|S_{\mathbf{u}}\rangle \in \text{Sym}^N(\mathbf{C}^d)$:

$$\tilde{\mathbf{D}}_d^{(\ell)} |S_{\mathbf{u}}\rangle = (u_{\ell} - u_{\ell+1}) |S_{\mathbf{u}}\rangle, \quad (32)$$

which shows that $|S_{\mathbf{u}}\rangle$ is an eigenvector of $\tilde{\mathbf{D}}_d^{(\ell)}$ with an eigenvalue that is antisymmetric with respect to u_{ℓ} and $u_{\ell+1}$. Roughly speaking, it is therefore reasonable to conjecture that the invariance of a code word under permutations of $u_1, \dots, u_{d-1}$ would lead to the vanishing of a lot of the inner products in Conditions (C3) and (C4).

Let us now be formal about our description of doubly permutation-invariant codes. To do this, we first define the following sets:

$$\begin{aligned}
 \mathcal{V}_{d,N} &= \left\{ \mathbf{v} \in \mathbf{N}_0^d \mid v_1 \equiv \dots \equiv v_{d-1} \pmod d, \right. \\
 &\quad \left. \sum_{\xi \in \mathbf{Z}_d} v_{\xi} = N \right\}, \quad (33)
 \end{aligned}$$

$$\mathcal{W}_{d,N} = \mathcal{V}_{d,N} \cap \left\{ \mathbf{w} \in \mathbf{N}_0^d \mid \bigoplus_{\xi \in \mathbf{Z}_d} \xi w_{\xi} = 0 \right\}. \quad (34)$$

For each d , define the relation $\mathcal{R}$ on the set $\mathcal{V}_{d,N}$ by $(v_0, v_1, \dots, v_{d-1}) \mathcal{R} (v'_0, v'_1, \dots, v'_{d-1})$ if and only if $(v'_0, v'_1, \dots, v'_{d-1}) = (v_0, v_{\sigma(1)}, \dots, v_{\sigma(d-1)})$ for some $\sigma \in S_{d-1}$. A quick inspection shows that $\mathcal{R}$ is an equivalence relation. Note that restricting to $\mathcal{W}_{d,N} \subset \mathcal{V}_{d,N}$ gives us an equivalence relation $\tilde{\mathcal{R}}$ on $\mathcal{W}_{d,N}$.

For each $\mathbf{v} \in \mathcal{V}_{d,N}$, we can define a *doubly permutation-invariant vector* by

$$||S_{\mathbf{v}}\rangle = \sum_{\mathbf{v}' \in [\mathbf{v}]_{\mathcal{R}}} |S_{\mathbf{v}'}\rangle. \quad (35)$$

We then define a *doubly symmetric vector space* as

$$\text{DSym}^N(\mathbf{C}^d) = \text{span}_{\mathbf{C}}\{||S_{\mathbf{v}}\rangle\rangle \mid \mathbf{v} \in \mathcal{V}_{d,N}\}. \quad (36)$$

Now, we say that a code is a *doubly permutation-invariant code* if it maps to a doubly symmetric physical Hilbert space, i.e.,

$$\mathcal{H}_P = \text{DSym}^N(\mathbf{C}^d). \quad (37)$$

Equivalently, a code is doubly permutation invariant if the code word for $|0\rangle_L$ is of the form

$$|\bar{0}\rangle = \sum_{\mathbf{w} \in f(\mathcal{W}_{d,N}/\hat{\mathcal{R}})} \alpha_{\mathbf{w}} ||S_{\mathbf{w}}\rangle\rangle \quad (38)$$

for some set of constants $\{\alpha_{\mathbf{w}} \mid \mathbf{w} \in f(\mathcal{W}_{d,N}/\hat{\mathcal{R}})\}$ and some injection $f : \mathcal{W}_{d,N}/\hat{\mathcal{R}} \rightarrow \mathcal{W}_{d,N}$ that maps each equivalence class to a chosen representative.

We can use the double permutation invariance of a code to simplify Conditions (C3) and (C4).

Theorem 4. Let $\mathcal{C} : \mathbf{C}^d \rightarrow \text{DSym}^N(\mathbf{C}^d)$ be a doubly permutation-invariant code. If $\mathcal{C}$ satisfies

$$(\text{QF1}) \quad \langle \bar{d}-1 | \tilde{\mathbf{D}}_d^{(d-2)} | \bar{d}-1 \rangle = 0,$$

$$(\text{QF2}) \quad \langle \bar{0} | \tilde{\mathbf{D}}_d^{(d-2)} \tilde{\mathbf{D}}_d^{(d-2)} | \bar{0} \rangle = \langle \bar{d}-1 | \tilde{\mathbf{D}}_d^{(d-2)} \tilde{\mathbf{D}}_d^{(d-2)} | \bar{d}-1 \rangle,$$

then it satisfies Condition (C3) and Condition (C4).

Proof. See Appendix E. ■

C. SDPI codes

Now, we combine both types and introduce the notion of a *sparse and doubly permutation-invariant (SDPI) code*. By itself, the sparsity of a code does not guarantee that the code would satisfy Condition (C2) with $(p, q) = (r, s)$. However, when this is combined with double permutation invariance, the condition simplifies into just one quadratic equation.

Theorem 5. Let $\mathcal{C} : \mathbf{C}^d \rightarrow \text{DSym}^N(\mathbf{C}^d)$ be an SDPI code. If $\mathcal{C}$ satisfies

$$(\text{QF3}) \quad \langle \bar{0} | \tilde{\mathbf{S}}_d^{(0,1)} \tilde{\mathbf{S}}_d^{(0,1)} | \bar{0} \rangle = \langle \bar{d}-1 | \tilde{\mathbf{S}}_d^{(0,1)} \tilde{\mathbf{S}}_d^{(0,1)} | \bar{d}-1 \rangle,$$

then it satisfies Condition (C2) with $(p, q) = (r, s)$.

Proof. See Appendix F. ■

We now arrive at one of the main results of this paper. Corollary 2 shows that, for SDPI codes, the d^6 quadratic forms that make up the Knill-Laflamme conditions reduce to just three.

Corollary 2. If an SDPI code $\mathcal{C} : \mathbf{C}^d \rightarrow \text{DSym}^N(\mathbf{C}^d)$ satisfies the quadratic forms QF1 to QF3, then $\mathcal{C}$ is a QECC.

Proof. This follows trivially from Theorems 3 to 5. ■

D. Procedure

Now that we have established Corollary 2, we can apply this knowledge to devise a procedure for constructing error-correcting qudit codes that work for any odd integer $d \geq 3$. The procedure is as follows.

(1) Choose a dimension d for the logical qudit and the number of physical qudits N , noting that it must satisfy $N \equiv 1 \pmod{d}$.

(2) Pick a subset $\mathcal{S} \subset f(\mathcal{W}_{d,N}/\hat{\mathcal{R}})$ such that $\mathcal{C}$ is sparse if

$$\alpha_{\mathbf{s}} \neq 0, \quad \alpha_{\mathbf{t}} = 0 \quad (39)$$

for all $\mathbf{s} \in \mathcal{S}$ and $\mathbf{t} \in f(\mathcal{W}_{d,N}/\hat{\mathcal{R}}) \setminus \mathcal{S}$.

(3) Define the code $\mathcal{C} : \mathbf{C}^d \rightarrow \text{DSym}^N(\mathbf{C}^d)$ by

$$|\bar{0}\rangle = \sum_{\mathbf{s} \in \mathcal{S}} \alpha_{\mathbf{s}} ||S_{\mathbf{s}}\rangle\rangle, \quad (40)$$

with $\alpha_{\mathbf{s}}$ arbitrary. Recall from Sec. IID that this suffices in defining $\mathcal{C}$ since the other code words are given by $|\bar{k}\rangle = \bar{X}_d^k |\bar{0}\rangle$, for any $k \in \mathbf{Z}_d$.

(4) Evaluate the inner products in QF1 to QF3 for the code defined above and solve the system of quadratic forms for nontrivial solution(s) $\{\alpha_{\mathbf{s}} \mid \mathbf{s} \in \mathcal{S}\}$.

(5) If a nontrivial solution does not exist, return to Step (2) and choose another subset $\mathcal{S}$ or return to Step (1) and choose a larger N .

When picking a subset $\mathcal{S}$ in Step (2), note that, for a code to be error correcting, there necessarily exist $\mathbf{s}', \mathbf{s}'' \in \mathcal{S}$ such that $s'_i > s'_0$ for some $i \in \{1, \dots, d-1\}$ and $s''_j < s''_0$ for some $j \in \{1, \dots, d-1\}$. To see why, observe that QF1 gives us

$$\sum_{\mathbf{s} \in \mathcal{S}} \beta_{\mathbf{s}} \langle S_{\mathbf{s}} | S_{\mathbf{s}} \rangle ||\alpha_{\mathbf{s}}||^2 = 0 \quad (41)$$

for some set of coefficients $\{\beta_{\mathbf{s}} \mid \mathbf{s} \in \mathcal{S}\}$. If the aforementioned necessary condition is not satisfied, then $\beta_{\mathbf{s}} > 0$ or $\beta_{\mathbf{s}} < 0$ for all $\mathbf{s} \in \mathcal{S}$. This then implies that there is no nontrivial solution to Eq. (41).

V. SOME ERROR-CORRECTING SDPI CODES

With the procedure established, we can now employ it to construct concrete examples of error-correcting SDPI codes.

A. Codes for qutrits

An example of an error-correcting SDPI code for qutrits is the map $\mathcal{C}_1 : \mathbf{C}^3 \rightarrow \text{DSym}^{13}(\mathbf{C}^3)$ defined by

$$|\bar{0}\rangle = \frac{1}{9} \sqrt{\frac{41}{5}} ||S_{(13,0,0)}\rangle\rangle + \frac{1}{9\sqrt{55}} ||S_{(4,0,9)}\rangle\rangle + \frac{1}{18\sqrt{385}} ||S_{(3,5,5)}\rangle\rangle. \quad (42)$$

We now show how we arrive at this through our procedure. For $N = 13$, observe that we can pick the subset $\mathcal{S} = \{(13, 0, 0), (4, 0, 9), (3, 5, 5)\}$. We then obtain the following system of linear equations in ξ_1, ξ_2, ξ_3 (or, equivalently, quadratic equations in $||\alpha_{(13,0,0)}||, ||\alpha_{(4,0,9)}||, ||\alpha_{(3,5,5)}||$):

$$13\xi_1 - \xi_2 - 2\xi_3 = 0, \quad (43)$$

$$169\xi_1 - 121\xi_2 + 4\xi_3 = 0, \quad (44)$$

$$13\xi_1 + 71\xi_2 - 22\xi_3 = 0, \quad (45)$$

where ξ_1, ξ_2, ξ_3 are defined as

$$\xi_1 = \langle S_{(13,0,0)} | S_{(13,0,0)} \rangle ||\alpha_{(13,0,0)}||^2, \quad (46)$$

$$\xi_2 = \langle S_{(4,0,9)} | S_{(4,0,9)} \rangle ||\alpha_{(4,0,9)}||^2, \quad (47)$$

$$\xi_3 = \langle S_{(3,5,5)} | S_{(3,5,5)} \rangle ||\alpha_{(3,5,5)}||^2. \quad (48)$$

Solving the linear system using MATHEMATICA or elementary linear-algebraic methods yields the code in Eq. (42).

B. Codes for five-level qudits and beyond

For $d = 5$, an example is the code $\mathcal{C}_2 : \mathbb{C}^5 \rightarrow \text{DSym}^{16}(\mathbb{C}^5)$ defined by

$$|\bar{0}\rangle = \frac{1}{5\sqrt{5}} \|S_{(16,0,0,0,0)}\rangle + \frac{1}{5}\sqrt{\frac{2}{5005}} \|S_{(6,10,0,0,0)}\rangle + \frac{1}{175\sqrt{4290}} \|S_{(0,4,4,4,4)}\rangle. \quad (49)$$

Similarly, for $d = 7$, we have the code $\mathcal{C}_3 : \mathbb{C}^7 \rightarrow \text{DSym}^{36}(\mathbb{C}^7)$ defined by

$$|\bar{0}\rangle = \frac{1}{7}\sqrt{\frac{13}{7}} \|S_{(36,0,0,0,0,0,0)}\rangle + \frac{1}{14}\sqrt{\frac{3}{5883955}} \|S_{(8,28,0,0,0,0,0)}\rangle + \frac{1}{300179880\sqrt{60500902}} \|S_{(0,6,6,6,6,6,6)}\rangle. \quad (50)$$

In fact, the form of the code words given for $d = 5$ and $d = 7$ can be generalized. Theorem 6 below shows that we can encode a logical qudit of any odd dimension $d \geq 5$ into $(d-1)^2$ physical qudits.

Theorem 6. Let $\mathbf{a} = ((d-1)^2, \underbrace{0, \dots, 0}_{d-1})$, $\mathbf{b} = (d+1, d(d-3), \underbrace{0, \dots, 0}_{d-2})$, $\mathbf{c} = (0, \underbrace{d-1, \dots, d-1}_{d-1})$. For each odd integer $d \geq 5$, there exists an error-correcting SDPI code with the logical code word

$$|\bar{0}\rangle = \alpha_{\mathbf{a}} \|S_{\mathbf{a}}\rangle + \alpha_{\mathbf{b}} \|S_{\mathbf{b}}\rangle + \alpha_{\mathbf{c}} \|S_{\mathbf{c}}\rangle \quad (51)$$

for some coefficients $\alpha_{\mathbf{a}}$, $\alpha_{\mathbf{b}}$, $\alpha_{\mathbf{c}}$. In particular, these coefficients are given by

$$\alpha_{\mathbf{a}} = \sqrt{\frac{d^3 - 5d^2 + d - 1}{2d^4 - 6d^3}}, \quad (52)$$

$$\alpha_{\mathbf{b}} = \sqrt{\left(\frac{(d-1)^2}{\mathbf{b}}\right)^{-1} \frac{(d-1)(1 - d\alpha_{\mathbf{a}}^2)}{d^2 + d}}, \quad (53)$$

$$\alpha_{\mathbf{c}} = \sqrt{\left(\frac{(d-1)^2}{\mathbf{c}}\right)^{-1} \left(1 - \alpha_{\mathbf{a}}^2 + \frac{1-d}{d^2 + d} \alpha_{\mathbf{b}}^2\right)}, \quad (54)$$

where we have used multinomial coefficients in the expressions for $\alpha_{\mathbf{b}}$ and $\alpha_{\mathbf{c}}$.

Proof. See Appendix G. ■

C. Alternative definitions

As mentioned in Sec. II C, we could also consider alternative definitions of the action of $\mathbf{Z}_L$ on $\mathcal{H}_L$. In general, we could define it, for any $\eta \in \mathbb{Z}_d^\times$, as

$$\mathbf{Z}_L |k\rangle_L = \zeta_d^{k\eta} |k\rangle_L \quad (55)$$

for all $k \in \mathbb{Z}_d$. Ultimately, we are still able to implement all qudit Pauli logical operations and that is what really matters anyway.

Once we have chosen $\eta \in \mathbb{Z}_d^\times$, we must then restrict ourselves to SDPI codes that map to $\mathcal{H}_P = \text{DSym}^N(\mathbb{C}^d)$ with $N \equiv \eta \pmod{d}$.

Now, let us consider a concrete example of such codes. Suppose we fix $\eta = d-1$. In this case, we can construct an error-correcting SDPI code for $d = 7$ mapping to a physical Hilbert space of smaller dimension than the previous example. In particular, this is the map $\mathcal{C}_4 : \mathbb{C}^7 \rightarrow \text{DSym}^{20}(\mathbb{C}^7)$ defined by

$$|\bar{0}\rangle = \frac{1}{7}\sqrt{\frac{3}{7}} \|S_{(20,0,0,0,0,0,0)}\rangle + \frac{1}{14\sqrt{6783}} \|S_{(6,14,0,0,0,0,0)}\rangle + \frac{1}{11760\sqrt{230945}} \|S_{(2,3,3,3,3,3,3)}\rangle. \quad (56)$$

VI. CONCLUSION

In conclusion, we formulated a general procedure for constructing quantum error-correcting codes for qudits of any odd dimension $d \geq 3$. To do so, we determined the branching rules for $\text{SU}(d) \downarrow \text{HW}(d)$, which to our knowledge have not yet been reported in the literature. Moreover, by exploiting various forms of symmetry, we managed to reduce the Knill-Laflamme conditions from a Brodbingnagian set of d^6 quadratic forms to just 3. This allowed us to devise a general procedure, which we then used to explicitly construct an infinite class of error-correcting codes that encode a logical qudit into $(d-1)^2$ physical qudits.

Our work opens a broad spectrum of promising directions for further research. On the theoretical side, it would be interesting to extend our framework to a universal set of logical qudit gates. Currently, our procedure yields error-correcting codes that only allow us to perform qudit Pauli operations, which do not form a universal gate set. It has been shown that the set $\mathbb{C}(d) \cup \{\text{CSUM}, \text{T}\}$ is a universal set of qudit gates [47–51]. Above, $\mathbb{C}(d)$ is the single-qudit Clifford group, CSUM is the generalization of the CNOT gate for qubits, and T is a single-qudit non-Clifford gate. Similar to the qubit case [17], one possible avenue for future work would then be to consider the single-qudit Clifford group as the finite subgroup of logical operations and subsequently determine how to implement CSUM and T.

Furthermore, it would be worthwhile to investigate, for each d , the minimum value of N for which there exists an error-correcting SDPI code that maps to $\text{DSym}^N(\mathbb{C}^d)$. This would give insight into how resource-efficient our error-correcting SDPI codes are, i.e., how many physical qudits are required to encode a single logical qudit. In principle, this work can be done computationally, but we expect that this would become very computationally intensive as d grows. To obtain general inequalities that work for the set of all odd integers $d \geq 3$ or at least some subset thereof, a proper mathematical analysis of the problem would probably be required.

Finally, going beyond our abstract representation-theoretic foray into quantum error correction on qudits, another natural next step would be to find efficient encoding and decoding schemes and thereafter experimentally realize our error-correcting codes. Since the physical Hilbert space of our codes was chosen to be a vector space on which $\text{SU}(d)$ irreps are

defined, a natural physical implementation would be with ultracold atoms that realize $SU(d)$ spin models [52–54], arrays of high-spin nuclei in silicon [55], or nuclear spin ensembles in GaAs quantum dots [56].

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APPENDIX A: TENSOR PRODUCT REPRESENTATIONS

Many of the subsequent proofs assume knowledge about tensor product representations, so we have included some useful background information.

Suppose $\Phi_1 : G \rightarrow GL(V_1)$ and $\Phi_2 : G \rightarrow GL(V_2)$ are representations of a Lie group G . Then the tensor product representation $\Phi_1 \otimes \Phi_2$ is given by

$$(\Phi_1 \otimes \Phi_2)(g) = \Phi_1(g) \otimes \Phi_2(g) \quad (A1)$$

for all $g \in G$ [34].

Now, let $\hat{\Phi}_1 : \mathfrak{g} \rightarrow \text{End}(V_1)$ and $\hat{\Phi}_2 : \mathfrak{g} \rightarrow \text{End}(V_2)$ be the corresponding representations of the associated Lie algebra $\mathfrak{g}$. The tensor product representation $\hat{\Phi}_1 \otimes \hat{\Phi}_2$ is given by

$$(\hat{\Phi}_1 \otimes \hat{\Phi}_2)(G) = \hat{\Phi}_1(G) \otimes I + I \otimes \hat{\Phi}_2(G) \quad (A2)$$

for all $G \in \mathfrak{g}$ [34].

APPENDIX B: PROOF OF LEMMA 2

Proof of Lemma 2. We first show that these equations are true for the fundamental irrep $\pi_{d;1}$. We observe that

$$X_d^\dagger S_d^{(j,k)} X_d = \left(\sum_{p \in \mathbb{Z}_d} |p\rangle \langle p \oplus 1| \right) (|j\rangle \langle k| + |k\rangle \langle j|) \left(\sum_{p \in \mathbb{Z}_d} |p \oplus 1\rangle \langle p| \right) \quad (B1)$$

$$= |j \ominus 1\rangle \langle j|j\rangle \langle k|k\rangle \langle k \ominus 1| + |k \ominus 1\rangle \langle k|k\rangle \langle j|j\rangle \langle j \ominus 1| \quad (B2)$$

$$= |j \ominus 1\rangle \langle k \ominus 1| + |k \ominus 1\rangle \langle j \ominus 1| \quad (B3)$$

$$= S_d^{(j \ominus 1, k \ominus 1)}, \quad (B4)$$

$$X_d^\dagger A_d^{(j,k)} X_d = \left(\sum_{p \in \mathbb{Z}_d} |p\rangle \langle p \oplus 1| \right) (-i|j\rangle \langle k| + i|k\rangle \langle j|) \left(\sum_{p \in \mathbb{Z}_d} |p \oplus 1\rangle \langle p| \right) \quad (B5)$$

$$= -i|j \ominus 1\rangle \langle k \ominus 1| + i|k \ominus 1\rangle \langle j \ominus 1| \quad (B6)$$

$$= A_d^{(j \ominus 1, k \ominus 1)}, \quad (B7)$$

$$X_d^\dagger D_d^{(\ell)} X_d = \left(\sum_{p \in \mathbb{Z}_d} |p\rangle \langle p \oplus 1| \right) (|\ell\rangle \langle \ell| - |\ell \oplus 1\rangle \langle \ell \oplus 1|) \left(\sum_{p \in \mathbb{Z}_d} |p \oplus 1\rangle \langle p| \right) \quad (B8)$$

$$= |\ell \ominus 1\rangle \langle \ell \ominus 1| - |\ell\rangle \langle \ell| \quad (B9)$$

$$= D_d^{(\ell \ominus 1)}, \quad (B10)$$

$$Z_d^\dagger S_d^{(j,k)} Z_d = \left(\sum_{p \in \mathbb{Z}_d} \zeta_d^{-p} |p\rangle \langle p| \right) (|j\rangle \langle k| + |k\rangle \langle j|) \left(\sum_{p \in \mathbb{Z}_d} \zeta_d^p |p\rangle \langle p| \right) \quad (B11)$$

$$= \zeta_d^{k-j} |j\rangle \langle k| + \zeta_d^{j-k} |k\rangle \langle j| \quad (B12)$$

$$= \text{Re}(\zeta_d^{k-j}) (|j\rangle \langle k| + |k\rangle \langle j|) - \text{Im}(\zeta_d^{k-j}) (-i|j\rangle \langle k| + i|k\rangle \langle j|) \quad (B13)$$

$$= \text{Re}(\zeta_d^{k-j}) \tilde{S}_d^{(j,k)} - \text{Im}(\zeta_d^{k-j}) \tilde{A}_d^{(j,k)}, \quad (B14)$$

as claimed.

We then proceed by applying Eqs. (A1) and (A2). Notice that, since X_d and Z_d are unitary, we have

$$\bar{X}_d^\dagger \tilde{S}_d^{(j,k)} \bar{X}_d = (X_d^\dagger)^{\otimes N} \left(\sum_{i=0}^{N-1} I^{\otimes i} \otimes S^{(j,k)} \otimes I^{\otimes (N-1-i)} \right) X_d^{\otimes N} \quad (B15)$$

$$= \sum_{i=0}^{N-1} I^{\otimes i} \otimes (X_d^\dagger S^{(j,k)} X_d) \otimes I^{\otimes(N-1-i)} \quad (\text{B16})$$

$$= \sum_{i=0}^{N-1} I^{\otimes i} \otimes S_d^{(j \oplus 1, k \oplus 1)} \otimes I^{\otimes(N-1-i)} \quad (\text{B17})$$

$$= \tilde{S}_d^{(j \oplus 1, k \oplus 1)}, \quad (\text{B18})$$

$$\bar{X}_d^\dagger \tilde{A}_d^{(j,k)} \bar{X}_d = (X_d^\dagger)^{\otimes N} \left(\sum_{i=0}^{N-1} I^{\otimes i} \otimes A^{(j,k)} \otimes I^{\otimes(N-1-i)} \right) X_d^{\otimes N} \quad (\text{B19})$$

$$= \sum_{i=0}^{N-1} I^{\otimes i} \otimes (X_d^\dagger A^{(j,k)} X_d) \otimes I^{\otimes(N-1-i)} \quad (\text{B20})$$

$$= \sum_{i=0}^{N-1} I^{\otimes i} \otimes A_d^{(j \oplus 1, k \oplus 1)} \otimes I^{\otimes(N-1-i)} \quad (\text{B21})$$

$$= \tilde{A}_d^{(j \oplus 1, k \oplus 1)}, \quad (\text{B22})$$

$$\bar{X}_d^\dagger \tilde{D}_d^{(\ell)} \bar{X}_d = (X_d^\dagger)^{\otimes N} \left(\sum_{i=0}^{N-1} I^{\otimes i} \otimes D_d^{(\ell)} \otimes I^{\otimes(N-1-i)} \right) X_d^{\otimes N} \quad (\text{B23})$$

$$= \sum_{i=0}^{N-1} I^{\otimes i} \otimes (X_d^\dagger D_d^{(\ell)} X_d) \otimes I^{\otimes(N-1-i)} \quad (\text{B24})$$

$$= \sum_{i=0}^{N-1} I^{\otimes i} \otimes D_d^{(\ell \oplus 1)} \otimes I^{\otimes(N-1-i)} \quad (\text{B25})$$

$$= \tilde{D}_d^{(\ell \oplus 1)}, \quad (\text{B26})$$

$$\bar{Z}_d^\dagger \tilde{S}_d^{(j,k)} \bar{Z}_d = (Z_d^\dagger)^{\otimes N} \left(\sum_{i=0}^{N-1} I^{\otimes i} \otimes S_d^{(j,k)} \otimes I^{\otimes(N-1-i)} \right) Z_d^{\otimes N} \quad (\text{B27})$$

$$= \sum_{i=0}^{N-1} I^{\otimes i} \otimes (Z_d^\dagger S_d^{(j,k)} Z_d) \otimes I^{\otimes(N-1-i)} \quad (\text{B28})$$

$$= \sum_{i=0}^{N-1} I^{\otimes i} \otimes [\text{Re}(\zeta_d^{k-j}) S_d^{(j,k)} - \text{Im}(\zeta_d^{k-j}) A_d^{(j,k)}] \otimes I^{\otimes(N-1-i)} \quad (\text{B29})$$

$$= \text{Re}(\zeta_d^{k-j}) \tilde{S}_d^{(j,k)} - \text{Im}(\zeta_d^{k-j}) \tilde{A}_d^{(j,k)}, \quad (\text{B30})$$

as claimed. ■

APPENDIX C: PROOF OF THEOREM 2

In addition to Lemma 2 in the main text, the proof of Theorem 2 will require extensive use of the following lemmas.

Lemma C1. Suppose $\mathbf{u}, \mathbf{v} \in \text{Sym}^N(\mathbb{C}^d)$ satisfy $\bar{Z}_d \mathbf{u} = \zeta_d^{w_u} \mathbf{u}$ and $\bar{Z}_d \mathbf{v} = \zeta_d^{w_v} \mathbf{v}$, with $\zeta_d^{w_u} \neq \zeta_d^{w_v}$. Then

$$\mathbf{u}^\dagger \mathbf{v} = 0. \quad (\text{C1})$$

Proof. Observe that

$$\bar{Z}_d \mathbf{u} = \zeta_d^{w_u} \mathbf{u} \implies \mathbf{u}^\dagger \bar{Z}_d \mathbf{v} = \zeta_d^{w_u} \mathbf{u}^\dagger \mathbf{v}, \quad (\text{C2})$$

$$\bar{Z}_d \mathbf{v} = \zeta_d^{w_v} \mathbf{v} \implies \mathbf{u}^\dagger \bar{Z}_d \mathbf{v} = \zeta_d^{w_v} \mathbf{u}^\dagger \mathbf{v}. \quad (\text{C3})$$

Thus $(\zeta_d^{w_u} - \zeta_d^{w_v}) \mathbf{u}^\dagger \mathbf{v} = 0$, which implies that $\mathbf{u}^\dagger \mathbf{v} = 0$ since $\zeta_d^{w_u} \neq \zeta_d^{w_v}$. ■

Lemma C2. Let $|S_u\rangle \in \text{Sym}^N(\mathbb{C}^d)$. Observe that for all $0 \leq p < q \leq d-1$ and all $0 \leq \ell \leq d-2$

$$\tilde{S}_d^{(p,q)} |S_u\rangle = (u_p + 1) |S_v\rangle + (u_q + 1) |S_w\rangle, \quad (\text{C4})$$

$$\tilde{D}_d^{(\ell)} |S_u\rangle = (u_\ell - u_{\ell+1}) |S_u\rangle, \quad (\text{C5})$$

where $\mathbf{v} = (u_0, \dots, u_p + 1, \dots, u_q - 1, \dots, u_{d-1})$ and $\mathbf{w} = (u_0, \dots, u_p - 1, \dots, u_q + 1, \dots, u_{d-1})$.

(We adopt the convention that $|S_t\rangle = \mathbf{0}$ if there exists k such that $t_k < 0$ or $t_k > N$.)

Proof. Note that for each $i \in \mathbf{Z}_N$

$$(\mathbb{I}^{\otimes i} \otimes \mathbf{S}_d^{(p,q)} \otimes \mathbb{I}^{\otimes(N-1-i)}) |S_{\mathbf{u}}\rangle = \sum_{\mathbf{u}' \in [\mathbf{u}]_{\mathcal{R}}} (\mathbb{I}^{\otimes i} \otimes \mathbf{S}_d^{(p,q)} \otimes \mathbb{I}^{\otimes(N-1-i)}) (|u'_0\rangle \otimes \cdots \otimes |u'_{d-1}\rangle) \quad (\text{C6})$$

$$\begin{aligned} &= \sum_{\mathbf{u}' \in [\mathbf{u}]_{\mathcal{R}}, u'_i = p} (\mathbb{I}^{\otimes i} \otimes \mathbf{S}_d^{(p,q)} \otimes \mathbb{I}^{\otimes(N-1-i)}) (|u'_0\rangle \otimes \cdots \otimes |p\rangle \otimes \cdots \otimes |u'_{d-1}\rangle) \\ &\quad + \sum_{\mathbf{u}' \in [\mathbf{u}]_{\mathcal{R}}, u'_i = q} (\mathbb{I}^{\otimes i} \otimes \mathbf{S}_d^{(p,q)} \otimes \mathbb{I}^{\otimes(N-1-i)}) (|u'_0\rangle \otimes \cdots \otimes |q\rangle \otimes \cdots \otimes |u'_{d-1}\rangle) \end{aligned} \quad (\text{C7})$$

$$\begin{aligned} &= \sum_{\mathbf{u}' \in [\mathbf{u}]_{\mathcal{R}}, u'_i = p} |u'_0\rangle \otimes \cdots \otimes |q\rangle \otimes \cdots \otimes |u'_{d-1}\rangle \\ &\quad + \sum_{\mathbf{u}' \in [\mathbf{u}]_{\mathcal{R}}, u'_i = q} |u'_0\rangle \otimes \cdots \otimes |p\rangle \otimes \cdots \otimes |u'_{d-1}\rangle \end{aligned} \quad (\text{C8})$$

$$\begin{aligned} &= \sum_{\mathbf{w}' \in [\mathbf{w}]_{\mathcal{R}}, w'_i = q} |w'_0\rangle \otimes \cdots \otimes |q\rangle \otimes \cdots \otimes |w'_{d-1}\rangle \\ &\quad + \sum_{\mathbf{v}' \in [\mathbf{v}]_{\mathcal{R}}, v'_i = p} |v'_0\rangle \otimes \cdots \otimes |p\rangle \otimes \cdots \otimes |v'_{d-1}\rangle. \end{aligned} \quad (\text{C9})$$

Thus we have

$$\tilde{\mathbf{S}}_d^{(p,q)} |S_{\mathbf{u}}\rangle = \sum_{i \in \mathbf{Z}_N} \left(\sum_{\mathbf{w}' \in [\mathbf{w}]_{\mathcal{R}}, w'_i = q} |w'_0\rangle \otimes \cdots \otimes |q\rangle \otimes \cdots \otimes |w'_{d-1}\rangle + \sum_{\mathbf{v}' \in [\mathbf{v}]_{\mathcal{R}}, v'_i = p} |v'_0\rangle \otimes \cdots \otimes |p\rangle \otimes \cdots \otimes |v'_{d-1}\rangle \right) \quad (\text{C10})$$

$$= v_p \sum_{\mathbf{v}' \in [\mathbf{v}]_{\mathcal{R}}} |v'_0\rangle \otimes \cdots \otimes |v'_{d-1}\rangle + w_q \sum_{\mathbf{w}' \in [\mathbf{w}]_{\mathcal{R}}} |w'_0\rangle \otimes \cdots \otimes |w'_{d-1}\rangle \quad (\text{C11})$$

$$= (u_p + 1) |S_{\mathbf{v}}\rangle + (u_q + 1) |S_{\mathbf{w}}\rangle, \quad (\text{C12})$$

where we have used the fact that, for each $\mathbf{v}' \in [\mathbf{v}]_{\mathcal{R}}$, the tensor product $|v'_0\rangle \otimes \cdots \otimes |v'_{d-1}\rangle$ was counted v_p times in the sum

$$\sum_{i \in \mathbf{Z}_N} \sum_{\mathbf{v}' \in [\mathbf{v}]_{\mathcal{R}}, v'_i = p} |v'_0\rangle \otimes \cdots \otimes |p\rangle \otimes \cdots \otimes |v'_{d-1}\rangle. \quad (\text{C13})$$

Similarly, we have

$$\tilde{\mathbf{D}}_d^{(\ell)} |S_{\mathbf{u}}\rangle = \left(\sum_{i \in \mathbf{Z}_N} \mathbb{I}^{\otimes i} \otimes \mathbf{D}_d^{(\ell)} \otimes \mathbb{I}^{\otimes(N-1-i)} \right) \left(\sum_{\mathbf{u}' \in [\mathbf{u}]_{\mathcal{R}}} |u'_0\rangle \otimes \cdots \otimes |u'_{d-1}\rangle \right) \quad (\text{C14})$$

$$= \sum_{i \in \mathbf{Z}_N} \left(\sum_{\mathbf{u}' \in [\mathbf{u}]_{\mathcal{R}}, u'_i = \ell} |u'_0\rangle \otimes \cdots \otimes |\ell\rangle \otimes \cdots \otimes |u'_{d-1}\rangle - \sum_{\mathbf{u}' \in [\mathbf{u}]_{\mathcal{R}}, u'_i = \ell+1} |u'_0\rangle \otimes \cdots \otimes |\ell+1\rangle \otimes \cdots \otimes |u'_{d-1}\rangle \right) \quad (\text{C15})$$

$$= u_{\ell} \sum_{\mathbf{u}' \in [\mathbf{u}]_{\mathcal{R}}} |u'_0\rangle \otimes \cdots \otimes |u'_{d-1}\rangle - u_{\ell+1} \sum_{\mathbf{u}' \in [\mathbf{u}]_{\mathcal{R}}} |u'_0\rangle \otimes \cdots \otimes |u'_{d-1}\rangle \quad (\text{C16})$$

$$= (u_{\ell} - u_{\ell+1}) |S_{\mathbf{u}}\rangle, \quad (\text{C17})$$

as claimed. ■

It is also helpful to delineate the Knill-Laflamme conditions in our particular case:

(KL1) $\langle \tilde{i} | \tilde{\mathbf{S}}_d^{(p,q)} | \tilde{j} \rangle = c_{(\tilde{i}, \tilde{\mathbf{S}}_d^{(p,q)})} \delta_{ij}$ for all $i, j \in \mathbf{Z}_d$ and $0 \leq p < q \leq d-1$;

(KL2) $\langle \tilde{i} | \tilde{\mathbf{A}}_d^{(p,q)} | \tilde{j} \rangle = c_{(\tilde{i}, \tilde{\mathbf{A}}_d^{(p,q)})} \delta_{ij}$ for all $i, j \in \mathbf{Z}_d$ and $0 \leq p < q \leq d-1$;

(KL3) $\langle \tilde{i} | \tilde{\mathbf{D}}_d^{(\ell)} | \tilde{j} \rangle = c_{(\tilde{i}, \tilde{\mathbf{D}}_d^{(\ell)})} \delta_{ij}$ for all $i, j \in \mathbf{Z}_d$ and $0 \leq \ell \leq d-2$;

(KL4) $\langle \tilde{i} | \tilde{\Gamma}_d^{(p,q)} \tilde{\Upsilon}_d^{(r,s)} | \tilde{j} \rangle = c_{(\tilde{i}, \tilde{\Gamma}_d^{(p,q)}, \tilde{\Upsilon}_d^{(r,s)})} \delta_{ij}$ for all $i, j \in \mathbf{Z}_d$, $\Gamma, \Upsilon \in \{\mathbf{S}, \mathbf{A}\}$, $0 \leq p < q \leq d-1$, and $0 \leq r < s \leq d-1$;

(KL5) $\langle \tilde{i} | \tilde{\Lambda} \tilde{\mathbf{D}}_d^{(\ell)} | \tilde{j} \rangle = c_{(\tilde{i}, \tilde{\Lambda}, \tilde{\mathbf{D}}_d^{(\ell)})} \delta_{ij}$ for all $i, j \in \mathbf{Z}_d$, $\tilde{\Lambda} \in \{\tilde{\mathbf{S}}_d^{(p,q)}, \tilde{\mathbf{A}}_d^{(p,q)} \mid 0 \leq p < q \leq d-1\}$ and $\ell \in \{0, \dots, d-2\}$;

(KL6) $\langle \tilde{i} | \tilde{\mathbf{D}}_d^{(\ell)} \tilde{\Lambda} | \tilde{j} \rangle = c_{(\tilde{i}, \tilde{\mathbf{D}}_d^{(\ell)}, \tilde{\Lambda})} \delta_{ij}$ for all $i, j \in \mathbf{Z}_d$, $\tilde{\Lambda} \in \{\tilde{\mathbf{S}}_d^{(p,q)}, \tilde{\mathbf{A}}_d^{(p,q)} \mid 0 \leq p < q \leq d-1\}$ and $\ell \in \{0, \dots, d-2\}$;

(KL7) $\langle \bar{i} | \tilde{D}_d^{(p)} \tilde{D}_d^{(q)} | \bar{j} \rangle = c_{(\tilde{D}_d^{(p)}, \tilde{D}_d^{(q)})} \delta_{ij}$ for all $i, j \in \mathbf{Z}_d$ and $p, q \in \{0, \dots, d-2\}$.

Above, we have already used the fact that the elements of $\mathcal{E}$ are Hermitian. (Here, we adopt physicists' definition of a Lie algebra, which differs from that of mathematicians by a factor of i .) There is, therefore, no need to separately consider redundant conditions like $\langle \bar{i} | \tilde{S}_d^{(p,q)} | \bar{j} \rangle = 0$.

Now, let us proceed with proving Theorem 2.

Proof of Theorem 2. We systematically prove each KL condition one at a time as follows.

Condition KL1. We first show that $\langle \bar{k} | \tilde{S}_d^{(0,n)} | \bar{k} \rangle = 0$ for all $k \in \mathbf{Z}_d$ and $n \in \{1, \dots, \frac{d-1}{2}\}$. We fix $k' \in \mathbf{Z}_d$ and $n' \in \{1, \dots, \frac{d-1}{2}\}$. Observe that, for any $\mathbf{u} \in \mathcal{U}_{d,N,k'}$,

$$\bar{Z}_d |S_{(u_0+1, \dots, u_{n'}-1, \dots, u_{d-1})}\rangle = \zeta_d^{k'-n'} |S_{(u_0+1, \dots, u_{n'}-1, \dots, u_{d-1})}\rangle, \quad (C18)$$

$$\bar{Z}_d |S_{(u_0-1, \dots, u_{n'}+1, \dots, u_{d-1})}\rangle = \zeta_d^{k'+n'} |S_{(u_0-1, \dots, u_{n'}+1, \dots, u_{d-1})}\rangle. \quad (C19)$$

Since $d \nmid n'$, it follows that $\zeta_d^{k'} \notin \{\zeta_d^{k'-n'}, \zeta_d^{k'+n'}\}$. By Lemma C1 and Lemma C2, we have $\langle \bar{k}' | \tilde{S}_d^{(0,n')} | S_{\mathbf{u}} \rangle = 0$ for all $\mathbf{u} \in \mathcal{U}_{d,N,k'}$. Since $|\bar{k}'\rangle \in \text{span}_{\mathbb{C}}\{|S_{\mathbf{u}}\rangle \mid \mathbf{u} \in \mathcal{U}_{d,N,k'}\}$, this implies that $\langle \bar{k}' | \tilde{S}_d^{(0,n')} | \bar{k}' \rangle = 0$, as claimed.

Combining the above result with Hypothesis (C1), we have

$$\langle \bar{i} | \tilde{S}_d^{(0,n)} | \bar{j} \rangle = 0 \quad (C20)$$

for all $i, j \in \mathbf{Z}_d, n \in \{1, \dots, \frac{d-1}{2}\}$.

To get from this to Condition KL1, we use the Heisenberg-Weyl symmetry of $\mathfrak{su}(d)$. We fix $i', j' \in \mathbf{Z}_d$ and $0 \leq p' < q' \leq d-1$. By Lemma 2, we have

$$\begin{aligned} \tilde{S}_d^{(p',q')} &= \bar{X}_d^{p'} \tilde{S}_d^{(0,q'-p')} (\bar{X}_d^\dagger)^{p'} = \bar{X}_d^{q'} \tilde{S}_d^{(d-q'+p',0)} (\bar{X}_d^\dagger)^{q'} \\ &= \bar{X}_d^{q'} \tilde{S}_d^{(0,d-q'+p')} (\bar{X}_d^\dagger)^{q'}, \end{aligned} \quad (C21)$$

where we have used the symmetry of $\tilde{S}_d^{(d-q'+p',0)}$ in the last equality. Thus

$$\begin{aligned} \langle \bar{i}' | \tilde{S}_d^{(p',q')} | \bar{j}' \rangle &= \langle \bar{i}' \ominus p' | \tilde{S}_d^{(0,q'-p')} | \bar{j}' \ominus p' \rangle \\ &= \langle \bar{i}' \ominus q' | \tilde{S}_d^{(0,d-q'+p')} | \bar{j}' \ominus q' \rangle. \end{aligned} \quad (C22)$$

Note that either $q' - p' \in \{1, \dots, \frac{d-1}{2}\}$ or $d - q' + p' \in \{1, \dots, \frac{d-1}{2}\}$. By Eq. (C20), we are thus guaranteed that $\langle \bar{i}' | \tilde{S}_d^{(p',q')} | \bar{j}' \rangle = 0$.

Condition KL2. We fix $i', j' \in \mathbf{Z}_d$ and $0 \leq p' < q' \leq d-1$. By Condition KL1, we have $\langle \bar{i}' | \tilde{S}_d^{(p',q')} | \bar{j}' \rangle = 0$. Observe that by Lemma 2,

$$\begin{aligned} \langle \bar{i}' | \tilde{A}_d^{(p',q')} | \bar{j}' \rangle &= \frac{1}{\text{Im}(\zeta_d^{q'-p'})} [\text{Re}(\zeta_d^{q'-p'}) \langle \bar{i}' | \tilde{S}_d^{(p',q')} | \bar{j}' \rangle \\ &\quad - \langle \bar{i}' | \bar{Z}_d^\dagger \tilde{S}_d^{(p',q')} \bar{Z}_d | \bar{j}' \rangle] \end{aligned} \quad (C23)$$

$$\begin{aligned} &= \frac{1}{\text{Im}(\zeta_d^{q'-p'})} [\text{Re}(\zeta_d^{q'-p'}) \langle \bar{i}' | \tilde{S}_d^{(p',q')} | \bar{j}' \rangle \\ &\quad - \zeta_d^{j'-i'} \langle \bar{i}' | \tilde{S}_d^{(p',q')} | \bar{j}' \rangle] \end{aligned} \quad (C24)$$

$$= 0. \quad (C25)$$

Note that $\text{Im}(\zeta_d^{q'-p'}) \neq 0$ since d is odd and $p' \neq q'$.

Condition KL3. We first show that $\langle \bar{i} | \tilde{D}_d^{(\ell)} | \bar{j} \rangle = 0$ for all $i, j \in \mathbf{Z}_d, i \neq j$ and $\ell \in \{0, \dots, d-2\}$. We fix $i', j' \in \mathbf{Z}_d, i' \neq j'$ and $\ell' \in \{0, \dots, d-2\}$. Note that $\bar{Z}_d \tilde{D}_d^{(\ell')} | \bar{j}' \rangle = \zeta_d^{j'} \tilde{D}_d^{(\ell')} | \bar{j}' \rangle$. Since $\zeta_d^{i'} \neq \zeta_d^{j'}$, we can use Lemma C1 to deduce that $\langle \bar{i}' | \tilde{D}_d^{(\ell')} | \bar{j}' \rangle = 0$.

Now, it remains to prove that Hypothesis (C3) implies that $\langle \bar{k} | \tilde{D}_d^{(\ell)} | \bar{k} \rangle = c_{(1, \tilde{D}_d^{(\ell)})}$ for all $k \in \mathbf{Z}_d$ and $\ell \in \{0, \dots, d-2\}$. We fix $k' \in \mathbf{Z}_d$ and $\ell' \in \{0, \dots, d-2\}$. Using Lemma 2, we obtain

$$\begin{aligned} \langle \bar{k}' | \tilde{D}_d^{(\ell')} | \bar{k}' \rangle &= \langle \bar{k}' | (\bar{X}_d^\dagger)^{d-2-\ell'} \tilde{D}_d^{(d-2)} \bar{X}_d^{d-2-\ell'} | \bar{k}' \rangle \\ &= \langle (d-2-\ell') \oplus k' | \tilde{D}_d^{(d-2)} | (d-2-\ell') \oplus k' \rangle \\ &= c_{(1, \tilde{D}_d^{(d-2)})}. \end{aligned} \quad (C26)$$

Condition KL4. This is identical to Hypothesis (C2).

Condition KL5. We fix $i', j' \in \mathbf{Z}_d, \tilde{\Lambda}' \in \{\tilde{S}_d^{(p,q)}, \tilde{A}_d^{(p,q)} \mid 0 \leq p < q \leq d-1\}$, $\ell' \in \{0, \dots, d-2\}$. Observe that $\bar{Z}_d \tilde{D}_d^{(\ell')} | \bar{j}' \rangle = \zeta_d^{j'} \tilde{D}_d^{(\ell')} | \bar{j}' \rangle$, from which we can deduce by Lemma C1 that

$$\langle \bar{k} | \tilde{D}_d^{(\ell')} | \bar{j}' \rangle = 0 \iff k = j'. \quad (C27)$$

We proceed by using the completeness relation:

$$\begin{aligned} \langle \bar{i}' | \tilde{\Lambda}' \tilde{D}_d^{(\ell')} | \bar{j}' \rangle &= \langle \bar{i}' | \tilde{\Lambda}' \left(\sum_{k \in \mathbf{Z}_d} |\bar{k}\rangle \langle \bar{k}| \right) \tilde{D}_d^{(\ell')} | \bar{j}' \rangle \\ &= \langle \bar{i}' | \tilde{\Lambda}' | \bar{j}' \rangle \langle \bar{j}' | \tilde{D}_d^{(\ell')} | \bar{j}' \rangle = 0, \end{aligned} \quad (C28)$$

where the last line follows from the form of Conditions KL1 and KL2 proven above, i.e., $\langle \bar{i} | \tilde{\Lambda} | \bar{j} \rangle = 0$ for all $i, j \in \mathbf{Z}_d$ and $\tilde{\Lambda} \in \{\tilde{S}_d^{(p,q)}, \tilde{A}_d^{(p,q)} \mid 0 \leq p < q \leq d-1\}$.

Condition KL6. The proof for this condition is very similar to that for Condition KL5. Notice that, even after swapping the order of $\tilde{\Lambda}'$ and $\tilde{D}_d^{(\ell')}$ in Eq. (C28), we still get zero.

Condition KL7. First, we show that $\langle \bar{i} | \tilde{D}_d^{(p)} \tilde{D}_d^{(q)} | \bar{j} \rangle = 0$ for all $i, j \in \mathbf{Z}_d$ and $p, q \in \{0, \dots, d-2\}$. We fix $i', j' \in \mathbf{Z}_d, i \neq j$ and $p', q' \in \{0, \dots, d-2\}$. Note that $\bar{Z}_d \tilde{D}_d^{(p')} | \bar{i}' \rangle = \zeta_d^{i'} \tilde{D}_d^{(p')} | \bar{i}' \rangle$ and $\bar{Z}_d \tilde{D}_d^{(q')} | \bar{j}' \rangle = \zeta_d^{j'} \tilde{D}_d^{(q')} | \bar{j}' \rangle$. Since $i' \neq j'$, it follows from Lemma C1 that

$$\langle \bar{i}' | \tilde{D}_d^{(p')} \tilde{D}_d^{(q')} | \bar{j}' \rangle = 0. \quad (C29)$$

Next, we prove that $\langle \bar{k} | \tilde{D}_d^{(p)} \tilde{D}_d^{(q)} | \bar{k} \rangle = 0$ for all $k \in \mathbf{Z}_d, 0 \leq p \leq q \leq d-2$. We fix $k' \in \mathbf{Z}_d, 0 \leq p' \leq q' \leq d-2$. By Lemma 2, we have

$$\begin{aligned} \tilde{D}_d^{(p')} \tilde{D}_d^{(q')} &= (\bar{X}_d^\dagger)^{d-2-q'} \tilde{D}_d^{(d-2-q'+p')} \bar{X}_d^{d-2-q'} (\bar{X}_d^\dagger)^{d-2-q'} \\ &\quad \times \tilde{D}_d^{(d-2)} \bar{X}_d^{d-2-q'} \end{aligned} \quad (C30)$$

$$= (\bar{X}_d^\dagger)^{d-2-q'} \tilde{D}_d^{(d-2-q'+p')} \tilde{D}_d^{(d-2)} \bar{X}_d^{d-2-q'}. \quad (C31)$$

We then calculate

$$\begin{aligned} \langle \bar{k}' | \tilde{D}_d^{(p')} \tilde{D}_d^{(q')} | \bar{k}' \rangle &= \langle \bar{k}' \oplus (d-2-q') | \tilde{D}_d^{(d-2-q'+p')} \tilde{D}_d^{(d-2)} \\ &\quad \times |\bar{k}' \oplus (d-2-q')\rangle = 0, \end{aligned} \quad (C32)$$

where the last equality follows from Hypothesis (C4). Note that $d - 2 - q' + p' \geq 0$.

Finally, observe that $\tilde{D}_d^{(p)} \tilde{D}_d^{(q)} = \tilde{D}_d^{(q)} \tilde{D}_d^{(p)}$ for all $0 \leq p \leq q \leq d - 2$. This implies that if $\langle \bar{k} | \tilde{D}_d^{(p)} \tilde{D}_d^{(q)} | \bar{k} \rangle = 0$ for all $k \in \mathbf{Z}_d$, $0 \leq p \leq q \leq d - 2$; then

$$\langle \bar{k} | \tilde{D}_d^{(p)} \tilde{D}_d^{(q)} | \bar{k} \rangle = 0 \quad (\text{C33})$$

for all $k \in \mathbf{Z}_d$, $p, q \in \{0, \dots, d - 2\}$. ■

APPENDIX D: PROOF OF THEOREM 3

We first prove the following lemma.

Lemma D1. Let $0 \leq p < q \leq d - 1$ and $0 \leq r < s \leq d - 1$. Let $\Gamma, \Upsilon \in \{\mathbf{S}, \mathbf{A}\}$. We fix $v \in \mathbf{Z}_d$ and let $\mathbf{u} \in \mathcal{U}_{d,N,v}$.

(1) If $p \neq q$ and $r \neq s$, then

$$\tilde{\Gamma}_d^{(p,q)} \tilde{\Upsilon}_d^{(r,s)} |S_{\mathbf{u}}\rangle \in \text{span}_{\mathbf{C} \setminus \{0\}}\{|S_{\mathbf{a}}\rangle, |S_{\mathbf{b}}\rangle, |S_{\mathbf{c}}\rangle, |S_{\mathbf{d}}\rangle\}, \quad (\text{D1})$$

with $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}$ defined as follows:

- $a_p = u_p + 1$, $a_q = u_q - 1$, $a_r = u_r + 1$, $a_s = u_s - 1$, and $a_i = u_i$ for all $i \in \mathbf{Z}_d \setminus \{p, q, r, s\}$,
- $b_p = u_p - 1$, $b_q = u_q + 1$, $b_r = u_r + 1$, $b_s = u_s - 1$, and $b_i = u_i$ for all $i \in \mathbf{Z}_d \setminus \{p, q, r, s\}$,
- $c_p = u_p + 1$, $c_q = u_q - 1$, $c_r = u_r - 1$, $c_s = u_s + 1$, and $c_i = u_i$ for all $i \in \mathbf{Z}_d \setminus \{p, q, r, s\}$,
- $d_p = u_p - 1$, $d_q = u_q + 1$, $d_r = u_r - 1$, $d_s = u_s + 1$, and $d_i = u_i$ for all $i \in \mathbf{Z}_d \setminus \{p, q, r, s\}$.

(2) If $p \neq r$ and $q = s$, then

$$\tilde{\Gamma}_d^{(p,q)} \tilde{\Upsilon}_d^{(r,s)} |S_{\mathbf{u}}\rangle \in \text{span}_{\mathbf{C} \setminus \{0\}}\{|S_{\mathbf{a}}\rangle, |S_{\mathbf{b}}\rangle, |S_{\mathbf{c}}\rangle, |S_{\mathbf{d}}\rangle\}, \quad (\text{D2})$$

with $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}$ defined as follows:

- $a_p = u_p + 1$, $a_q = u_q - 2$, $a_r = u_r + 1$, $a_i = u_i$ for all $i \in \mathbf{Z}_d \setminus \{p, q, r\}$,
- $b_p = u_p + 1$, $b_q = u_q$, $b_r = u_r - 1$, $b_i = u_i$ for all $i \in \mathbf{Z}_d \setminus \{p, q, r\}$,
- $c_p = u_p - 1$, $c_q = u_q$, $c_r = u_r + 1$, $c_i = u_i$ for all $i \in \mathbf{Z}_d \setminus \{p, q, r\}$,
- $d_p = u_p - 1$, $d_q = u_q + 2$, $d_r = u_r - 1$, $d_i = u_i$ for all $i \in \mathbf{Z}_d \setminus \{p, q, r\}$.

(3) If $p = r$ and $q \neq s$, then

$$\tilde{\Gamma}_d^{(p,q)} \tilde{\Upsilon}_d^{(r,s)} |S_{\mathbf{u}}\rangle \in \text{span}_{\mathbf{C} \setminus \{0\}}\{|S_{\mathbf{a}}\rangle, |S_{\mathbf{b}}\rangle, |S_{\mathbf{c}}\rangle, |S_{\mathbf{d}}\rangle\}, \quad (\text{D3})$$

with $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}$ defined as follows:

- $a_p = u_p + 1$, $a_q = u_q - 2$, $a_s = u_s + 1$, $a_i = u_i$ for all $i \in \mathbf{Z}_d \setminus \{p, q, s\}$,
- $b_p = u_p + 1$, $b_q = u_q$, $b_s = u_s - 1$, $b_i = u_i$ for all $i \in \mathbf{Z}_d \setminus \{p, q, s\}$,
- $c_p = u_p - 1$, $c_q = u_q$, $c_s = u_s + 1$, $c_i = u_i$ for all $i \in \mathbf{Z}_d \setminus \{p, q, s\}$,
- $d_p = u_p - 1$, $d_q = u_q + 2$, $d_s = u_s - 1$, $d_i = u_i$ for all $i \in \mathbf{Z}_d \setminus \{p, q, s\}$.

Proof. This follows from two applications of Lemma C2. ■

We then proceed with proving Theorem 3.

Proof of Theorem 3. We prove the contrapositive. There are two cases to consider. Condition (C1) is not satisfied or Condition (C2) with $(p, q) \neq (r, s)$ is not satisfied.

Condition (C1). Suppose Condition (C1) is not satisfied, i.e., for some $i', j' \in \mathbf{Z}_d$ and some $n' \in \{1, \dots, \frac{d-1}{2}\}$,

$$\langle \bar{i}' | \tilde{S}_d^{(0,n')} | \bar{j}' \rangle \neq 0. \quad (\text{D4})$$

This implies that there exists $\mathbf{u}, \mathbf{u}' \in \mathcal{U}_{d,N,0}$ such that $\alpha_{\mathbf{u}}, \alpha_{\mathbf{u}'} \neq 0$ and

$$\langle S_{u_{i'}, \dots, u_{d-1}, u_0, \dots, u_{i'-1}} | \tilde{S}_d^{(0,n')} | S_{u_{j'}, \dots, u_{d-1}, u_0, \dots, u_{j'-1}} \rangle \neq 0. \quad (\text{D5})$$

Hence it follows by Lemma C2 that one of the following is true:

- $\langle S_{u_{i'}, \dots, u_{d-1}, u_0, \dots, u_{i'-1}} | S_{u_{j'+1}, \dots, u_{j'+n'-1}, \dots, u_{j'-1}} \rangle \neq 0$,
- $\langle S_{u_{i'}, \dots, u_{d-1}, u_0, \dots, u_{i'-1}} | S_{u_{j'-1}, \dots, u_{j'+n'+1}, \dots, u_{j'-1}} \rangle \neq 0$.

Either way, we can set $\delta' = j' - i'$ and note that

$$\sum_{\xi \in \mathbf{Z}_d} |u_{\xi} - u'_{\xi+\delta'}| = 2 \leq 4. \quad (\text{D6})$$

Thus $\mathcal{C}$ is not sparse.

Condition (C2) with $(p, q) \neq (r, s)$. Suppose that, for some $i', j' \in \mathbf{Z}_d$ and some $0 \leq p' < q' \leq d - 1$, $0 \leq r' < s' \leq d - 1$ with $(p', q') \neq (r', s')$,

$$\langle \bar{i}' | \tilde{\Gamma}_d^{(p',q')} \tilde{\Upsilon}_d^{(r',s')} | \bar{j}' \rangle \neq 0. \quad (\text{D7})$$

This implies that there exists $\mathbf{u}, \mathbf{u}' \in \mathcal{Z}_{d,N,0}$ such that $\alpha_{\mathbf{u}}, \alpha_{\mathbf{u}'} \neq 0$ and

$$\langle S_{u_{i'}, \dots, u_{d-1}, u_0, \dots, u_{i'-1}} | \tilde{\Gamma}_d^{(p',q')} \tilde{\Upsilon}_d^{(r',s')} | S_{u_{j'}, \dots, u_{d-1}, u_0, \dots, u_{j'-1}} \rangle \neq 0. \quad (\text{D8})$$

By Lemma D1, we see that, for $\delta' = j' - i'$,

$$\sum_{\xi \in \mathbf{Z}_d} |u_{\xi} - u'_{\xi+\delta'}| \in \{2, 4\}. \quad (\text{D9})$$

Hence we conclude that $\mathcal{C}$ is not sparse. ■

APPENDIX E: PROOF OF THEOREM 4

Proof of Theorem 4. We prove each condition systematically.

Condition (C3). We first show that $\langle \bar{0} | \tilde{D}_d^{(d-2)} | \bar{0} \rangle = 0$. We fix an injection $f : \mathcal{W}_{d,N}/\tilde{\mathcal{R}} \rightarrow \mathcal{W}_{d,N}$ that maps each equivalence class to a chosen representative. Then

$$\langle \bar{0} | \tilde{D}_d^{(d-2)} | \bar{0} \rangle = \sum_{\mathbf{w} \in f(\mathcal{W}_{d,N}/\tilde{\mathcal{R}})} |\alpha_{\mathbf{w}}|^2 \sum_{\mathbf{w}' \in [\mathbf{w}]_{\tilde{\mathcal{R}}}} \langle S_{\mathbf{w}'} | \tilde{D}_d^{(d-2)} | S_{\mathbf{w}'} \rangle \quad (\text{E1})$$

$$= \sum_{\mathbf{w} \in f(\mathcal{W}_{d,N}/\tilde{\mathcal{R}})} |\alpha_{\mathbf{w}}|^2 \sum_{\mathbf{w}' \in [\mathbf{w}]_{\tilde{\mathcal{R}}}} (w'_{d-2} - w'_{d-1}) \quad (\text{E2})$$

$$= 0, \quad (\text{E3})$$

where going from Eq. (E2) to Eq. (E3) requires the observation that, for each $\mathbf{w}' \in [\mathbf{w}]_{\tilde{\mathcal{R}}}$, there exists a unique element $\mathbf{w}'' \in [\mathbf{w}]_{\tilde{\mathcal{R}}}$ such that $w'_{d-2} = w''_{d-1}$, $w'_{d-1} = w''_{d-2}$, and $w'_i = w''_i$ for all $i \in \{0, \dots, d - 3\}$. (It is in arguments like this that we see the second form of permutation invariance in action.)

Next, we combine the above argument with Heisenberg-Weyl symmetry to show that $\langle \bar{k} | \tilde{D}_d^{(d-2)} | \bar{k} \rangle = 0$ for all $k \in \{1, \dots, d - 3\}$. We fix $k' \in \{1, \dots, d - 3\}$ and an injection $f : \mathcal{W}_{d,N}/\tilde{\mathcal{R}} \rightarrow \mathcal{W}_{d,N}$ that maps each equivalence class to a chosen representative. Notice that

$$\langle \bar{k}' | \tilde{D}_d^{(d-2)} | \bar{k}' \rangle = \langle \bar{0} | (\bar{X}_d^{\dagger})^{k'} \tilde{D}_d^{(d-2)} \bar{X}_d^{k'} | \bar{0} \rangle \quad (\text{E4})$$

$$= \langle \bar{0} | \tilde{D}_d^{(d-2-k')} | \bar{0} \rangle \quad (E5)$$

$$= \sum_{\mathbf{w} \in f(\mathcal{W}_{d,N}/\hat{\mathcal{R}})} |\alpha_{\mathbf{w}}|^2 \times \sum_{\mathbf{w}' \in [\mathbf{w}]_{\hat{\mathcal{R}}}} \langle S_{\mathbf{w}'} | \tilde{D}_d^{(d-2-k')} | S_{\mathbf{w}'} \rangle \quad (E6)$$

$$= \sum_{\mathbf{w} \in f(\mathcal{W}_{d,N}/\hat{\mathcal{R}})} |\alpha_{\mathbf{w}}|^2 \times \sum_{\mathbf{w}' \in [\mathbf{w}]_{\hat{\mathcal{R}}}} (w'_{d-2-k'} - w'_{d-1-k'}) \quad (E7)$$

$$= 0. \quad (E8)$$

Finally, it remains to show that $\langle \overline{d-2} | \tilde{D}_d^{(d-2)} | \overline{d-2} \rangle = -\langle \overline{d-1} | \tilde{D}_d^{(d-2)} | \overline{d-1} \rangle$. We fix an injection $f: \mathcal{W}_{d,N}/\hat{\mathcal{R}} \rightarrow \mathcal{W}_{d,N}$ that maps each equivalence class to a chosen representative. Observe that

$$\langle \overline{d-2} | \tilde{D}_d^{(d-2)} | \overline{d-2} \rangle = \sum_{\mathbf{w} \in f(\mathcal{W}_{d,N}/\hat{\mathcal{R}})} |\alpha_{\mathbf{w}}|^2 \times \sum_{\mathbf{w}' \in [\mathbf{w}]_{\hat{\mathcal{R}}}} (w'_0 - w'_1) \quad (E9)$$

$$= - \sum_{\mathbf{w} \in f(\mathcal{W}_{d,N}/\hat{\mathcal{R}})} |\alpha_{\mathbf{w}}|^2 \times \sum_{\mathbf{w}' \in [\mathbf{w}]_{\hat{\mathcal{R}}}} (w''_{d-1} - w''_0) \quad (E10)$$

$$= - \langle \overline{d-1} | \tilde{D}_d^{(d-2)} | \overline{d-1} \rangle, \quad (E11)$$

where we have used the fact that, for each $\mathbf{w}' \in [\mathbf{w}]_{\hat{\mathcal{R}}}$, there exists a unique $\mathbf{w}'' \in [\mathbf{w}]_{\hat{\mathcal{R}}}$ such that $w'_1 = w''_{d-1}$, $w'_{d-1} = w''_1$, and $w'_i = w''_i$ for all $i \in \{0, 2, 3, \dots, d-2\}$.

Condition (C4). First, we fix $k' \in \{0, \dots, d-3\}$ and observe that

$$\langle \bar{k}' | \tilde{D}_d^{(d-2)} \tilde{D}_d^{(d-2)} | \bar{k}' \rangle = \langle \bar{0} | \tilde{D}_d^{(d-2-k')} \tilde{D}_d^{(d-2-k')} | \bar{0} \rangle \quad (E12)$$

$$= \sum_{\mathbf{w} \in f(\mathcal{W}_{d,N}/\hat{\mathcal{R}})} |\alpha_{\mathbf{w}}|^2 \sum_{\mathbf{w}' \in [\mathbf{w}]_{\hat{\mathcal{R}}}} \langle S_{\mathbf{w}'} | \tilde{D}_d^{(d-2-k')} \tilde{D}_d^{(d-2-k')} | S_{\mathbf{w}'} \rangle \quad (E13)$$

$$= \sum_{\mathbf{w} \in f(\mathcal{W}_{d,N}/\hat{\mathcal{R}})} |\alpha_{\mathbf{w}}|^2 \sum_{\mathbf{w}' \in [\mathbf{w}]_{\hat{\mathcal{R}}}} (w'_{d-2-k'} - w'_{d-1-k'})^2 \quad (E14)$$

$$= \sum_{\mathbf{w} \in f(\mathcal{W}_{d,N}/\hat{\mathcal{R}})} |\alpha_{\mathbf{w}}|^2 \sum_{\mathbf{w}'' \in [\mathbf{w}]_{\hat{\mathcal{R}}}} (w''_{d-2} - w''_{d-1})^2 \quad (E15)$$

$$= \langle \bar{0} | \tilde{D}_d^{(d-2)} \tilde{D}_d^{(d-2)} | \bar{0} \rangle, \quad (E16)$$

$$\langle \overline{d-2} | \tilde{D}_d^{(d-2)} \tilde{D}_d^{(d-2)} | \overline{d-2} \rangle = \langle \bar{0} | \tilde{D}_d^{(0)} \tilde{D}_d^{(0)} | \bar{0} \rangle \quad (E17)$$

$$= \sum_{\mathbf{w} \in f(\mathcal{W}_{d,N}/\hat{\mathcal{R}})} |\alpha_{\mathbf{w}}|^2 \sum_{\mathbf{w}' \in [\mathbf{w}]_{\hat{\mathcal{R}}}} (w'_0 - w'_1)^2 \quad (E18)$$

$$= \sum_{\mathbf{w} \in f(\mathcal{W}_{d,N}/\hat{\mathcal{R}})} |\alpha_{\mathbf{w}}|^2 \sum_{\mathbf{w}'' \in [\mathbf{w}]_{\hat{\mathcal{R}}}} (w''_{d-1} - w''_0)^2 \quad (E19)$$

$$= \langle \overline{d-1} | \tilde{D}_d^{(d-2)} \tilde{D}_d^{(d-2)} | \overline{d-1} \rangle. \quad (E20)$$

Together with Hypothesis QF2, these imply that, for all $k \in \mathbf{Z}_d$, $\langle \bar{k} | \tilde{D}_d^{(d-2)} \tilde{D}_d^{(d-2)} | \bar{k} \rangle = c_{(\tilde{D}_d^{(d-2)}, \tilde{D}_d^{(d-2)})}$ for some $c_{(\tilde{D}_d^{(d-2)}, \tilde{D}_d^{(d-2)})}$.

Now, we fix $\ell' \in \{0, \dots, d-4\}$ and $m' \in \mathbf{Z}_d$. Observe that

$$\langle \bar{m}' | \tilde{D}_d^{(\ell')} \tilde{D}_d^{(d-2)} | \bar{m}' \rangle = \langle \bar{0} | \tilde{D}_d^{(\ell' \ominus m')} \tilde{D}_d^{[(d-2) \ominus m']} | \bar{0} \rangle \quad (E21)$$

$$= \sum_{\mathbf{w} \in f(\mathcal{W}_{d,N}/\hat{\mathcal{R}})} |\alpha_{\mathbf{w}}|^2 \sum_{\mathbf{w}' \in [\mathbf{w}]_{\hat{\mathcal{R}}}} (w'_{\ell' \ominus m'} - w'_{(\ell'+1) \ominus m'}) \times (w'_{(d-2) \ominus m'} - w'_{(d-1) \ominus m'}) \quad (E22)$$

$$= 0, \quad (E23)$$

since, for each $\mathbf{w}' \in [\mathbf{w}]_{\hat{\mathcal{R}}}$, there is either of the following:

- a unique $\mathbf{w}'' \in [\mathbf{w}]_{\hat{\mathcal{R}}}$ such that $w'_{\ell' \ominus m'} = w''_{(\ell'+1) \ominus m'}$, $w'_{(\ell'+1) \ominus m'} = w''_{\ell' \ominus m'}$, and $w'_i = w''_i$ for all $i \in \mathbf{Z}_d \setminus \{\ell' \ominus m', (\ell'+1) \ominus m'\}$ or
- a unique $\mathbf{w}''' \in [\mathbf{w}]_{\hat{\mathcal{R}}}$ such that $w'_{(d-2) \ominus m'} = w'''_{(d-1) \ominus m'}$, $w'_{(d-1) \ominus m'} = w'''_{(d-2) \ominus m'}$, and $w'_i = w'''_i$ for all $i \in \mathbf{Z}_d \setminus \{(d-2) \ominus m', (d-1) \ominus m'\}$.

Finally, we fix $n' \in \mathbf{Z}_d$ and observe that

$$\langle \bar{n}' | \tilde{D}_d^{(d-3)} \tilde{D}_d^{(d-2)} | \bar{n}' \rangle = \langle \bar{0} | \tilde{D}_d^{[(d-3) \ominus n']} \tilde{D}_d^{[(d-2) \ominus n']} | \bar{0} \rangle \quad (E24)$$

$$= \sum_{\mathbf{w} \in f(\mathcal{W}_{d,N}/\hat{\mathcal{R}})} |\alpha_{\mathbf{w}}|^2 \sum_{\mathbf{w}' \in [\mathbf{w}]_{\hat{\mathcal{R}}}} (w'_{(d-3) \ominus n'} - w'_{(d-2) \ominus n'}) \times (w'_{(d-2) \ominus n'} - w'_{(d-1) \ominus n'}) \quad (E25)$$

$$= 0, \quad (E26)$$

since, for each $\mathbf{w}' \in [\mathbf{w}]_{\hat{\mathcal{R}}}$, there is one of the following:

- a unique $\mathbf{w}'' \in [\mathbf{w}]_{\hat{\mathcal{R}}}$ such that $w'_{(d-3) \ominus n'} = w''_{(d-1) \ominus n'}$, $w'_{(d-1) \ominus n'} = w''_{(d-3) \ominus n'}$, and $w'_i = w''_i$ for all $i \in \mathbf{Z}_d \setminus \{(d-3) \ominus n', (d-1) \ominus n'\}$,
- a unique $\mathbf{w}''' \in [\mathbf{w}]_{\hat{\mathcal{R}}}$ such that $w'_{(d-3) \ominus n'} = w'''_{(d-2) \ominus n'}$, $w'_{(d-2) \ominus n'} = w'''_{(d-3) \ominus n'}$, and $w'_i = w'''_i$ for all $i \in \mathbf{Z}_d \setminus \{(d-3) \ominus n', (d-2) \ominus n'\}$, or
- a unique $\mathbf{w}'''' \in [\mathbf{w}]_{\hat{\mathcal{R}}}$ such that $w'_{(d-2) \ominus n'} = w''''_{(d-1) \ominus n'}$, $w'_{(d-1) \ominus n'} = w''''_{(d-2) \ominus n'}$, and $w'_i = w''''_i$ for all $i \in \mathbf{Z}_d \setminus \{(d-2) \ominus n', (d-1) \ominus n'\}$. ■

APPENDIX F: PROOF OF THEOREM 5

Proof of Theorem 5. We first show that $\langle \bar{0} | \tilde{S}_d^{(0,1)} \tilde{S}_d^{(0,1)} | \bar{0} \rangle = \langle \bar{1} | \tilde{S}_d^{(0,1)} \tilde{S}_d^{(0,1)} | \bar{1} \rangle$. Using Heisenberg-Weyl symmetry, we have

$$\langle \bar{1} | \tilde{S}_d^{(0,1)} \tilde{S}_d^{(0,1)} | \bar{1} \rangle = \langle \bar{0} | \tilde{X}_d^\dagger \tilde{S}_d^{(0,1)} \tilde{X}_d \tilde{S}_d^{(0,1)} \tilde{X}_d | \bar{0} \rangle = \langle \bar{0} | \tilde{S}_d^{(0,d-1)} \tilde{S}_d^{(0,d-1)} | \bar{0} \rangle. \quad (F1)$$

Moreover, observe that, for any given injection $f: \mathcal{W}_{d,N}/\hat{\mathcal{R}} \rightarrow \mathcal{W}_{d,N}$ that maps each equivalence class to a

chosen representative, we have

$$\tilde{S}_d^{(0,d-1)} |\bar{0}\rangle = \sum_{\mathbf{w} \in f(\mathcal{W}_{d,N}/\hat{\mathcal{R}})} |\alpha_{\mathbf{w}}|^2 \sum_{\mathbf{w}' \in [\mathbf{w}]_{\hat{\mathcal{R}}}} [(w'_0 + 1)w'_{d-1} + w'_0(w'_{d-1} + 1)] |S_{\mathbf{w}'}\rangle \quad (\text{F2})$$

$$= \sum_{\mathbf{w} \in f(\mathcal{W}_{d,N}/\hat{\mathcal{R}})} |\alpha_{\mathbf{w}}|^2 \sum_{\mathbf{w}'' \in [\mathbf{w}]_{\hat{\mathcal{R}}}} [(w''_0 + 1)w''_1 + w''_0(w''_1 + 1)] |S_{\mathbf{w}''}\rangle \quad (\text{F3})$$

$$= \tilde{S}_d^{(0,1)} |\bar{0}\rangle, \quad (\text{F4})$$

from which the conclusion follows directly.

Next, we show that $\langle \bar{k} | \tilde{S}_d^{(0,1)} \tilde{S}_d^{(0,1)} | \bar{k} \rangle = \langle \overline{d-1} | \tilde{S}_d^{(0,1)} \tilde{S}_d^{(0,1)} | \overline{d-1} \rangle$ for all $k \in \{2, \dots, d-1\}$. We fix $k' \in \{2, \dots, d-2\}$. Note that

$$\langle \bar{k}' | \tilde{S}_d^{(0,1)} \tilde{S}_d^{(0,1)} | \bar{k}' \rangle = \langle \bar{0} | \tilde{S}_d^{(d-k', d-k'+1)} \tilde{S}_d^{(d-k', d-k'+1)} | \bar{0} \rangle, \quad (\text{F5})$$

$$\langle \overline{d-1} | \tilde{S}_d^{(0,1)} \tilde{S}_d^{(0,1)} | \overline{d-1} \rangle = \langle \bar{0} | \tilde{S}_d^{(1,2)} \tilde{S}_d^{(1,2)} | \bar{0} \rangle. \quad (\text{F6})$$

Again, for any injection $f : \mathcal{W}_{d,N}/\hat{\mathcal{R}} \rightarrow \mathcal{W}_{d,N}$ that maps each equivalence class to a chosen representative, we have

$$\tilde{S}_d^{(1,2)} |\bar{0}\rangle = \sum_{\mathbf{w} \in f(\mathcal{W}_{d,N}/\hat{\mathcal{R}})} |\alpha_{\mathbf{w}}|^2 \sum_{\mathbf{w}' \in [\mathbf{w}]_{\hat{\mathcal{R}}}} [(w'_1 + 1)w'_2 + w'_1(w'_2 + 1)] |S_{\mathbf{w}'}\rangle \quad (\text{F7})$$

$$= \sum_{\mathbf{w} \in f(\mathcal{W}_{d,N}/\hat{\mathcal{R}})} |\alpha_{\mathbf{w}}|^2 \sum_{\mathbf{w}'' \in [\mathbf{w}]_{\hat{\mathcal{R}}}} [(w''_{d-k'} + 1)w''_{d-k'+1} + w''_{d-k'}(w''_{d-k'+1} + 1)] |S_{\mathbf{w}''}\rangle \quad (\text{F8})$$

$$= \tilde{S}_d^{(d-k', d-k'+1)} |\bar{0}\rangle, \quad (\text{F9})$$

as required.

Combining the two results above with Hypothesis QF3, we have thus far proven that, for all $k \in \mathbf{Z}_d$, we have $\langle \bar{k} | \tilde{S}_d^{(0,1)} \tilde{S}_d^{(0,1)} | \bar{k} \rangle = c_{(\tilde{S}_d^{(0,1)}, \tilde{S}_d^{(0,1)})}$ for some $c_{(\tilde{S}_d^{(0,1)}, \tilde{S}_d^{(0,1)})}$. As we show below, this implies that, for all $k \in \mathbf{Z}_d$ and $0 \leq p < q \leq d-1$, we have $\langle \bar{k} | \tilde{\Gamma}_d^{(p,q)} \tilde{\Upsilon}_d^{(p,q)} | \bar{k} \rangle = c_{(\tilde{\Gamma}_d^{(p,q)}, \tilde{\Upsilon}_d^{(p,q)})}$ for some $c_{(\tilde{\Gamma}_d^{(p,q)}, \tilde{\Upsilon}_d^{(p,q)})}$. We fix $0 \leq p' < q' \leq d-1$. Observe that $\langle \bar{p}' | \tilde{S}_d^{(p',q')} \tilde{S}_d^{(p',q')} | \bar{p}' \rangle = \langle \bar{q}' | \tilde{S}_d^{(p',q')} \tilde{S}_d^{(p',q')} | \bar{q}' \rangle = \langle \bar{0} | \tilde{S}_d^{(0,1)} \tilde{S}_d^{(0,1)} | \bar{0} \rangle$:

$$\langle \bar{q}' | \tilde{S}_d^{(p',q')} \tilde{S}_d^{(p',q')} | \bar{q}' \rangle = \langle \bar{q}' | \tilde{X}_d^{q'} \tilde{S}_d^{(0,d+p'-q')} (\tilde{X}_d^\dagger)^{q'} \tilde{X}_d^{q'} \tilde{S}_d^{(0,d+p'-q')} (\tilde{X}_d^\dagger)^{q'} | \bar{q}' \rangle \quad (\text{F10})$$

$$= \langle \bar{0} | \tilde{S}_d^{(0,d+p'-q')} \tilde{S}_d^{(0,d+p'-q')} | \bar{0} \rangle \quad (\text{F11})$$

$$= \sum_{\mathbf{w} \in f(\mathcal{W}_{d,N}/\hat{\mathcal{R}})} |\alpha_{\mathbf{w}}|^2 \sum_{\mathbf{w}' \in [\mathbf{w}]_{\hat{\mathcal{R}}}} [(w'_0 + 1)w'_{d+p'-q'} + w'_0(w'_{d+p'-q'} + 1)]^2 \quad (\text{F12})$$

$$= \sum_{\mathbf{w} \in f(\mathcal{W}_{d,N}/\hat{\mathcal{R}})} |\alpha_{\mathbf{w}}|^2 \sum_{\mathbf{w}'' \in [\mathbf{w}]_{\hat{\mathcal{R}}}} [(w''_0 + 1)w''_1 + w''_0(w''_1 + 1)]^2 \quad (\text{F13})$$

$$= \langle \bar{0} | \tilde{S}_d^{(0,1)} \tilde{S}_d^{(0,1)} | \bar{0} \rangle, \quad (\text{F14})$$

$$\langle \bar{p}' | \tilde{S}_d^{(p',q')} \tilde{S}_d^{(p',q')} | \bar{p}' \rangle = \langle \bar{p}' | \tilde{X}_d^{p'} \tilde{S}_d^{(0,q'-p')} (\tilde{X}_d^\dagger)^{p'} \tilde{X}_d^{p'} \tilde{S}_d^{(0,q'-p')} (\tilde{X}_d^\dagger)^{p'} | \bar{p}' \rangle \quad (\text{F15})$$

$$= \langle \bar{0} | \tilde{S}_d^{(0,q'-p')} \tilde{S}_d^{(0,q'-p')} | \bar{0} \rangle \quad (\text{F16})$$

$$= \sum_{\mathbf{w} \in f(\mathcal{W}_{d,N}/\hat{\mathcal{R}})} |\alpha_{\mathbf{w}}|^2 \sum_{\mathbf{w}' \in [\mathbf{w}]_{\hat{\mathcal{R}}}} [(w'_0 + 1)w'_{q'-p'} + w'_0(w'_{q'-p'} + 1)]^2 \quad (\text{F17})$$

$$= \sum_{\mathbf{w} \in f(\mathcal{W}_{d,N}/\hat{\mathcal{R}})} |\alpha_{\mathbf{w}}|^2 \sum_{\mathbf{w}'' \in [\mathbf{w}]_{\hat{\mathcal{R}}}} [(w''_0 + 1)w''_1 + w''_0(w''_1 + 1)]^2 \quad (\text{F18})$$

$$= \langle \bar{0} | \tilde{S}_d^{(0,1)} \tilde{S}_d^{(0,1)} | \bar{0} \rangle. \quad (\text{F19})$$

Moreover, we notice that, for any $k \in \mathbf{Z}_d \setminus \{p', q'\}$,

$$\langle \bar{k} | \tilde{S}_d^{(p',q')} \tilde{S}_d^{(p',q')} | \bar{k} \rangle = \langle \bar{0} | \tilde{S}_d^{(p' \ominus k, q' \ominus k)} \tilde{S}_d^{(p' \ominus k, q' \ominus k)} | \bar{0} \rangle \quad (\text{F20})$$

$$= \sum_{\mathbf{w} \in f(\mathcal{W}_{d,N}/\hat{\mathcal{R}})} |\alpha_{\mathbf{w}}|^2 \sum_{\mathbf{w}' \in [\mathbf{w}]_{\hat{\mathcal{R}}}} [(w'_{p \ominus k} + 1)w'_{q' \ominus k} + w'_{p \ominus k}(w'_{q' \ominus k} + 1)]^2 \quad (\text{F21})$$

$$= \sum_{\mathbf{w} \in f(\mathcal{W}_{d,N}/\hat{\mathcal{R}})} |\alpha_{\mathbf{w}}|^2 \sum_{\mathbf{w}'' \in [\mathbf{w}]_{\hat{\mathcal{R}}}} [(w''_1 + 1)w''_2 + w''_1(w''_2 + 1)]^2 \quad (\text{F22})$$

$$= \langle \bar{0} | \tilde{S}_d^{(1,2)} \tilde{S}_d^{(1,2)} | \bar{0} \rangle \quad (\text{F23})$$

$$= \langle \overline{d-1} | \tilde{S}_d^{(0,1)} \tilde{S}_d^{(0,1)} | \overline{d-1} \rangle. \quad (\text{F24})$$

This shows that $\langle \bar{k} | \tilde{S}_d^{(p',q')} \tilde{S}_d^{(p',q')} | \bar{k} \rangle = c_{(\tilde{S}_d^{(p',q')}, \tilde{S}_d^{(p',q')})}$ for all $k \in \mathbf{Z}_d$. By a similar argument, we can show that $\langle \bar{k} | \tilde{A}_d^{(p',q')} \tilde{A}_d^{(p',q')} | \bar{k} \rangle = c_{(\tilde{A}_d^{(p',q')}, \tilde{A}_d^{(p',q')})}$, $\langle \bar{k} | \tilde{S}_d^{(p',q')} \tilde{A}_d^{(p',q')} | \bar{k} \rangle = c_{(\tilde{S}_d^{(p',q')}, \tilde{A}_d^{(p',q')})}$ and $\langle \bar{k} | \tilde{A}_d^{(p',q')} \tilde{S}_d^{(p',q')} | \bar{k} \rangle = c_{(\tilde{A}_d^{(p',q')}, \tilde{S}_d^{(p',q')})}$ for all $k \in \mathbf{Z}_d$. [The specific action of $\tilde{A}_d^{(p',q')} \tilde{A}_d^{(p',q')}$, $\tilde{S}_d^{(p',q')} \tilde{A}_d^{(p',q')}$, and $\tilde{A}_d^{(p',q')} \tilde{S}_d^{(p',q')}$ on any given basis vector differs from that of $\tilde{S}_d^{(p',q')} \tilde{S}_d^{(p',q')}$, but the same reasoning (using double permutation invariance) still works.]

It remains to show that $\langle \bar{i} | \tilde{\Gamma}_d^{(p,q)} \tilde{\Gamma}_d^{(p,q)} | \bar{j} \rangle = 0$ for all $i, j \in \mathbf{Z}_d$ with $i \neq j$ and $0 \leq p < q \leq d-1$. Note that it suffices to do this for $\Gamma = \Upsilon = \mathbf{S}$; a similar argument works for combinations with $\mathbf{A}$. We fix $i', j' \in \mathbf{Z}_d$ with $i \neq j$ and $0 \leq p' < q' \leq d-1$. Observe that

$$\begin{aligned} \tilde{S}_d^{(p' \ominus j', q' \ominus j')} \tilde{S}_d^{(p' \ominus j', q' \ominus j')} | \bar{0} \rangle &= \sum_{\mathbf{w} \in f(\mathcal{W}_{d,N}/\hat{\mathcal{R}})} |\alpha_{\mathbf{w}}|^2 \sum_{\mathbf{w}' \in [\mathbf{w}]_{\hat{\mathcal{R}}}} \{ (w_{p' \ominus j'} + 2)(w_{p' \ominus j'} + 1) |S_{\mathbf{u}'}\rangle \\ &\quad + [w_{p' \ominus j'}(w_{q' \ominus j'} + 1) + (w_{p' \ominus j'} + 1)w_{q' \ominus j'}] |S_{\mathbf{w}'}\rangle + (w_{q' \ominus j'} + 2)(w_{q' \ominus j'} + 1) |S_{\mathbf{v}'}\rangle \}, \end{aligned} \quad (\text{F25})$$

where $\mathbf{u}'$ and $\mathbf{v}'$ are defined by

- $u'_{p' \ominus j'} = w'_{p' \ominus j'} + 2$, $u'_{q' \ominus j'} = w'_{q' \ominus j'} - 2$, and $u'_i = w'_i$ for all $i \in \mathbf{Z}_d \setminus \{p' \ominus j', q' \ominus j'\}$,
- $v'_{p' \ominus j'} = w'_{p' \ominus j'} - 2$, $v'_{q' \ominus j'} = w'_{q' \ominus j'} + 2$, and $v'_i = w'_i$ for all $i \in \mathbf{Z}_d \setminus \{p' \ominus j', q' \ominus j'\}$.

We then have

$$\langle \bar{i}' | \tilde{S}_d^{(p',q')} \tilde{S}_d^{(p',q')} | \bar{j}' \rangle = \langle \bar{i}' \ominus j' | \tilde{S}_d^{(p' \ominus j', q' \ominus j')} \tilde{S}_d^{(p' \ominus j', q' \ominus j')} | \bar{0} \rangle = 0, \quad (\text{F26})$$

where the last equality follows from Eq. (F25) and the hypothesis that $\mathcal{C}$ is sparse. ■

APPENDIX G: PROOF OF THEOREM 6

Proof of Theorem 6. With the code word in Eq. (51), we get the following system of quadratic forms:

$$(d-1)^2 \|\alpha_{\mathbf{a}}\|^2 + (3d-1) \|\alpha_{\mathbf{b}}\|^2 - (d-1) \|\alpha_{\mathbf{c}}\|^2 = 0, \quad (\text{G1})$$

$$\begin{aligned} (d-1)^4 \|\alpha_{\mathbf{a}}\|^2 - (d^4 - 5d^3 + 4d^2 - 5d + 1) \|\alpha_{\mathbf{b}}\|^2 \\ + (d-1) \|\alpha_{\mathbf{c}}\|^2 = 0, \end{aligned} \quad (\text{G2})$$

$$\begin{aligned} (d-1)^2 \|\alpha_{\mathbf{a}}\|^2 + (2d^3 - 4d^2 - 3d - 1) \|\alpha_{\mathbf{b}}\|^2 \\ - (2d^2 - 3d + 1) \|\alpha_{\mathbf{c}}\|^2 = 0. \end{aligned} \quad (\text{G3})$$

Using MATHEMATICA, we see that a nontrivial solution exists. The coefficients given in Eqs. (G1) to (G3) form a (normalized) solution to the system of quadratic forms. ■

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