

Generalized Schawlow-Townes limit

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We study a class of feedback oscillators realized by a phase-insensitive amplifier in positive feedback, where either the amplifier or the feedback element may determine the oscillator's linewidth. The spectral purity of the output of such a device originates from basic demands of quantum mechanics and causality. The resulting expression generalizes the Schawlow-Townes limit, which is itself one component of a standard quantum limit for feedback oscillators. Recently realized bad-cavity oscillators, such as superradiant lasers and solid-state masers, can saturate this generalized Schawlow-Townes limit. This limit can be surpassed through appropriate quantum engineering, for example, by atomic spin squeezing in a superradiant laser.

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I. INTRODUCTION

The spectral purity of a laser, given by the (modified) Schawlow-Townes formula [1–8], can be understood in a simple and general picture of an oscillator realized by an amplifier in positive feedback [9]. In this picture, depicted in Figs. 1(a) and 1(c), the mean oscillation frequency is dictated by the frequency selectivity of the feedback element and its mean amplitude by the nonlinearity of the amplifier. Quantum fluctuations in the oscillator's frequency and amplitude are determined by the unavoidable quantum noises from the amplifier and the out-coupler, necessary elements in any feedback oscillator. Linearized quantum noise analysis then produces the modified Schawlow-Townes formula for the linewidth [9]

$$\Gamma_{\text{ST}} = \frac{\hbar\Omega_0\kappa_F^2}{2P} \quad (1)$$

of the emitted signal at frequency Ω_0 , with power P , and κ_F is the linewidth of the frequency selective feedback element (such as a cavity in a laser). The Schawlow-Townes limit can be understood as one facet in an equal trade-off between quantum fluctuations in the phase and amplitude of the emitted signal, i.e., a standard quantum limit (SQL) for feedback oscillators. Knowing the precise origin of this basic result from such a minimal system-agnostic model opens the door to the possibility of oscillators that operate beyond the Schawlow-Townes limit and to identifying the resources necessary to realize them.

Lasers, and more generally feedback oscillators, need not necessarily operate in the regime where the feedback element is more frequency selective than the amplifier [i.e., the

so-called “good-cavity” regime depicted in Fig. 1(c)]. Bad-cavity oscillators [10–20] [see Fig. 1(d)] can operate beyond the Schawlow-Townes limit [21], by employing an amplifier whose linewidth is smaller than that of the cavity enclosing it. Moreover, the output of such oscillators can be immune to a variety of technical noises originating in the cavity [17,19,22–24] since the amplifying gain medium is well isolated from its environment.

We derive fundamental quantum limits on the noise performance of all feedback oscillators (i.e., those operating in the good as well as bad-cavity regimes) using a generalized, minimal, and system-agnostic model of such devices. The resulting “generalized Schawlow-Townes limit” applies to oscillators operating in both the good- and bad-cavity regimes, and can be identified as one facet of a SQL. The generalized Schawlow-Townes limit we derive has been previously derived from specific models of lasers [4,25,26]. By adopting a minimal, system-agnostic model, we show that this limit is a natural bound on the frequency stability of any feedback oscillator due to the implications of causality and Heisenberg uncertainty alone, rather than the microphysics of spontaneous emission. This unifies previous models of specific devices such as good-cavity lasers [1], hydrogen masers [27], superradiant lasers [23], subluminal lasers [17], and other devices. Furthermore, we find that this quantum limit can be surpassed, a point which has evaded previous analyses. This can be done, for example, by spin squeezing the atomic gain medium in a superradiant realization of a bad-cavity laser.

II. QUANTUM-LIMITED FEEDBACK OSCILLATOR

Consider the positive feedback amplifier configuration shown in Fig. 1(a). The output of a phase-insensitive amplifier with (frequency-dependent) response $G[\Omega]$ is coupled back into its input after attenuation by a factor η and a feedback delay of τ_F ; the remaining fraction of the signal is coupled out of the loop to derive the out-of-loop field $\hat{a}_{\text{out}}$. The equations of motion for the system are obtained by going around the loop in Fig. 1(a) employing the linearity of each of the elements

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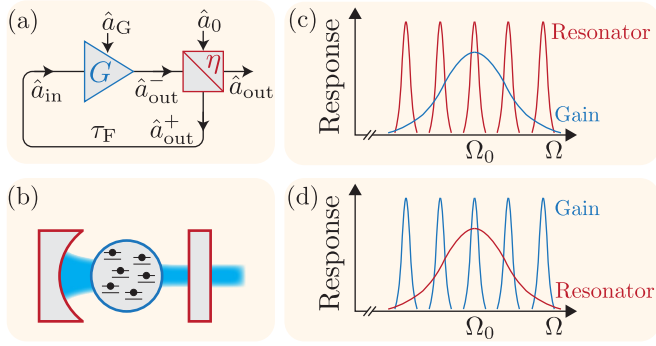


FIG. 1. (a) Good- and bad-cavity feedback oscillators can be abstractly modeled as a quantum electronic or optical circuit consisting of an amplifier, beam splitter, and time delay. (b) An example of a bad-cavity feedback oscillator is a hydrogen maser or a superradiant laser where the atomic linewidth is narrower than the cavity's. (c) The gain medium and resonator response as a function of frequency for oscillators in the “good-cavity” regime where the resonator linewidth is narrow compared to the gain medium's. (d) The gain medium and resonator profiles in the “bad-cavity” regime where the resonator linewidth is large compared with the gain medium's.

and of the feedback path. The Heisenberg-picture operators in the frequency domain are then related by [9]

$$\begin{aligned}\hat{a}_{\text{out}}^-[\Omega] &= G[\Omega]\hat{a}_{\text{in}}[\Omega] + \sqrt{G[\Omega]^2 - 1}\hat{a}_G^+[\Omega], \\ \hat{a}_{\text{out}}^+[\Omega] &= -i\sqrt{\eta}\hat{a}_{\text{out}}^-[\Omega] + \sqrt{1-\eta}\hat{a}_0[\Omega], \\ \hat{a}_{\text{out}}[\Omega] &= \sqrt{1-\eta}\hat{a}_{\text{out}}^-[\Omega] + i\sqrt{\eta}\hat{a}_0[\Omega], \\ \hat{a}_{\text{in}}[\Omega] &= e^{i\Omega\tau_F}\hat{a}_{\text{out}}^+[\Omega].\end{aligned}\quad (2)$$

Here, $\hat{a}_G$ ($\hat{a}_0$) represents the unavoidable noise added at the amplifier [28,29] (out-coupler). Note that the last equation is the Fourier transform of $\hat{a}_{\text{in}}(t) = \hat{a}_{\text{out}}^+(t - \tau_F)$, which expresses the delay in the feedback path.

Mean-field dynamics govern the saturation of the oscillator's gain medium [9]. The gain will saturate to a point where “gain = loss.” Since the loss is given by the out-coupler's reflectivity η , the gain at the resonance frequency Ω_0 will be $|G[\Omega_0]| = 1/\sqrt{\eta}$.

Applying the principle of causality to the amplifier fixes the frequency dependence of $G[\Omega]$. Causality determines the *minimum* phase $\phi[\Omega] \equiv \arg G[\Omega]$ that is stipulated by the gain $|G[\Omega]|$ (see Appendix B and [30–32])

$$\phi[\Omega] = -\frac{1}{\pi}P \int_{-\infty}^{\infty} \frac{\ln |G[\Omega']|}{\Omega' - \Omega} d\Omega'. \quad (3)$$

If the amplifier has its peak gain at Ω_0^G , then the first derivative of its magnitude response is zero at this frequency and in the vicinity of this frequency it has the form

$$|G[\Omega_0^G + \omega]| = 1/\sqrt{\eta} - a\omega^2 + O(\omega^3), \quad (4)$$

where $\omega \equiv \Omega - \Omega_0^G$ and $a > 0$. Since its magnitude response is symmetric near Ω_0^G , Eq. (3) requires $\phi[\Omega_0^G] \approx 0$ such that the amplifier adds no phase on resonance. We write this relation with an approximation sign as the amplifier's magnitude response is not guaranteed to be symmetric far from resonance

and this may cause the integral in Eq. (3) to differ slightly from zero.

With respect to the feedback element's resonance frequency $\Omega_0^F = \Omega_0^G - \delta\omega$, the amplifier's phase response is then

$$\phi[\Omega_0^F + \omega] = \tau_G(\omega - \delta\omega) + O(\omega^2), \quad (5)$$

and causality requires that $\tau_G > 0$, which can be seen by taking the derivative of Eq. (3) evaluated at $\Omega = \Omega_0^G$. Any small amplifier phase offset on resonance $\phi[\Omega_0^G]$ can also be absorbed into the $\tau_G\delta\omega$ term.

The input-output relation of the oscillator, obtained by solving Eq. (2), can be put in the form

$$\hat{a}_{\text{out}}[\Omega] = H_0[\Omega]\hat{a}_0[\Omega] + H_G[\Omega]\hat{a}_G^+[\Omega], \quad (6)$$

where H_0, H_G are the transfer functions from the two essential input noises $\hat{a}_0$ and $\hat{a}_G$, respectively. When the magnitude and phase of $G[\Omega]$ are given by Eqs. (4) and (5), for frequency shifts ω small around the resonance, i.e., $|\omega|(\tau_G + \tau_F) \ll 1$, and for small resonance frequency offsets between the amplifier and feedback elements, these transfer functions are given explicitly by

$$H_0[\Omega_0 + \omega] \approx H_G[\Omega_0 + \omega] \approx \frac{\sqrt{\eta} - 1/\sqrt{\eta}}{i\omega(\tau_G + \tau_F)}, \quad (7)$$

where the frequency pulling from the amplifier modifies the system's resonance frequency from Ω_0^F to $\Omega_0 = \Omega_0^F + \delta\omega/(1 + \tau_F/\tau_G)$.

The linewidth of the emitted output is given in terms of the noise spectrum of its phase quadrature by [9], $\Gamma = (\omega^2/2|\alpha|^2)\bar{S}_{pp}^{\text{out}}[\Omega_0 + \omega]$, where $|\alpha|^2$ is the mean photon flux out of the oscillator and $\bar{S}_{pp}^{\text{out}}[\Omega] = \int_{-\infty}^{\infty} \frac{1}{2} \langle \{\hat{p}_{\text{out}}[\Omega], \hat{p}_{\text{out}}[\Omega']\} \rangle \frac{d\Omega'}{2\pi}$ is the noise spectrum of the output phase quadrature $\hat{p}_{\text{out}} \equiv i(\hat{a}^\dagger - \hat{a})/\sqrt{2}$, which is related to the spectrum of the input noises as

$$\begin{aligned}\bar{S}_{pp}^{\text{out}}[\Omega] &= |H_0[\Omega]|^2 \bar{S}_{pp}^0[\Omega] + |H_G[\Omega]|^2 \bar{S}_{pp}^G[\Omega] \\ &\quad - 2\text{Re}[H_0[\Omega]^* H_G[\Omega] \bar{S}_{pp}^{0,G}[\Omega]].\end{aligned}\quad (8)$$

Assuming the noise modes are in thermal states, i.e., $\bar{S}_{pp}^0[\Omega] = \frac{1}{2} + \bar{n}_0$, $\bar{S}_{pp}^G[\Omega] = \frac{1}{2} + \bar{n}_G$, and $\bar{S}_{pp}^{0,G}[\Omega] = 0$, we have

$$\bar{S}_{pp}^{\text{out}}[\Omega_0 + \omega] = \frac{(\sqrt{\eta} - 1/\sqrt{\eta})^2}{\omega^2(\tau_G + \tau_F)^2} (1 + 2\bar{n}_{\text{th}}), \quad (9)$$

where $\bar{n}_{\text{th}} = (\bar{n}_0 + \bar{n}_G)/2$ is the mean thermal occupation of the two noise modes. Using this expression for the phase noise spectrum, the linewidth takes the simple form

$$\Gamma = \frac{\hbar\Omega_0}{2P} \frac{1 + 2\bar{n}_{\text{th}}}{(\kappa_F^{-1} + \kappa_G^{-1})^2}, \quad (10)$$

and the oscillation frequency is

$$\Omega_0 = \Omega_0^F + \frac{\delta\omega}{1 + \kappa_F/\kappa_G}. \quad (11)$$

Here, $P = \hbar\Omega_0|\alpha|^2$ is the mean power of the output; the linewidths of the feedback loop and that of the gain medium are $\kappa_{F,G} \equiv (1 - \eta)/\tau_{F,G}$ (we assume that $1 - \eta \ll 1$ and the feedback loop has a high quality factor).

The above expressions for the linewidth and oscillation frequency apply to both good- and bad-cavity oscillators. Indeed, from Eq. (10) we see that the linewidths of the amplifier and feedback loop “add in parallel,” so that the resulting oscillator’s output linewidth is set by the narrower of the gain medium and the feedback loop. For example, when the gain is broadband, i.e., $\kappa_G \gg \kappa_F$, the linewidth expression reduces to the usual Schawlow-Townes formula $\Gamma_{ST} \approx (\hbar\Omega_0\kappa_F^2/P)(\frac{1}{2} + \bar{n}_{th})$ that applies to good-cavity lasers. For this reason, the expression in Eq. (10) will henceforth be referred to as the generalized Schawlow-Townes linewidth and denoted Γ_{GST} .

Our result agrees with previous calculations of laser linewidths [4,25,26,33,34]. Importantly, our expression applies even more broadly. Previous results relied on models of the detailed physics of atoms in optical cavities. In contrast, we made no assumptions about the detailed construction of our oscillator. Instead, we make the minimal assumption that the oscillator is formed by an amplifier placed in positive feedback, obeying the constraints of causality and quantum mechanics. As a result, we show that Eqs. (10) and (11) apply to lasers and masers, as was previously known, but also to the much broader class of feedback oscillators including devices such as phase-shift oscillators and crystal oscillators.

Similarly, Eq. (11) characterizes the influence of the amplifier and feedback element on the device’s oscillation frequency, depending on the ratio of their linewidths. In the good (bad) cavity regime with $\kappa_F \ll \kappa_G$ ($\kappa_F \gg \kappa_G$), the oscillation frequency is determined primarily by the feedback element (amplifier). In the intermediate regime, the amplifier exerts a degree of frequency pulling on the oscillator’s frequency.

III. STANDARD QUANTUM LIMIT FOR FEEDBACK OSCILLATORS

The generalized Schawlow-Townes linewidth for feedback oscillators is one facet of a SQL. To wit, when no quantum enhancement is employed, the uncertainty principle for the output field, quantified in terms of the product of the noise spectra of the quadratures of the output can be bounded as [[35], Sec. IV.C] $\bar{S}_{qq}^{out}[\Omega]\bar{S}_{pp}^{out}[\Omega] \geq \frac{1}{4}(|H_0[\Omega]|^2 + |H_G[\Omega]|^2)^2$. Explicitly,

$$\bar{S}_{qq}^{out}[\Omega]\bar{S}_{pp}^{out}[\Omega] \gtrsim [\omega(\kappa_F^{-1} + \kappa_G^{-1})]^{-4}; \quad (12)$$

this inequality can be interpreted as a SQL for generalized feedback oscillators. (Here the approximation is valid for $1 - \eta \ll 1$.) Comparing Eq. (12) to Eq. (9), we see that the Schawlow-Townes limit arises from the SQL in the case that the minimal uncertainty is distributed equally between the two quadratures.

The SQL can be evaded by engineering an asymmetry between the two quadratures, or by modifying the oscillator’s dynamics and hence its transfer functions. For instance, if the in-coupled and amplifier’s noise modes are phase-squeezed with squeezing parameters r_0 and r_G , respectively, the oscillator has a linewidth given by $\Gamma = \Gamma_{GST}(e^{-2r_0} + e^{-2r_G})/2$.

IV. APPLICATION TO SUPERRADIANT LASERS

In what follows, we show that superradiant lasers can saturate the “bad-cavity” Schawlow-Townes linewidth limit and

explore how the linewidth of such a laser can be narrowed beyond this limit by squeezing or entangling its atoms and/or incident electromagnetic field. It was shown previously that superradiant lasers can saturate the “bad-cavity” Schawlow-Townes linewidth limit [13,23,36], but to our knowledge it has not previously been understood how their linewidths may be further suppressed through quantum engineering.

A superradiant laser can be modeled as a collection of N spin-1/2 systems, almost entirely in their excited state, interacting with an optical-cavity mode such that the atomic transition is much narrower than that of the cavity. The effective spin- $N/2$ system, described by the collective angular momentum operators $\{\hat{S}_x, \hat{S}_y, \hat{S}_z\}$, resonantly interacts with a single-cavity mode via the Jaynes-Cummings Hamiltonian [23,37] $\hat{H}_{int} = i\hbar g(\hat{a}\hat{S}_+ - \hat{a}^\dagger\hat{S}_-)$. Here, $\hat{S}_\pm = \hat{S}_x \pm i\hat{S}_y$ are the spin raising and lowering operators and $\hat{a}^\dagger, \hat{a}$ are those of the cavity mode. In a superradiant laser, the atoms are almost entirely in the excited state. It is then useful to use the Holstein-Primakoff mapping to relate the atomic spin operators to an effective bosonic mode $\hat{b}$ by [38,39], $\hat{S}_+ = \sqrt{2N}\sqrt{1 - \hat{b}^\dagger\hat{b}/(2N)}\hat{b} = \hat{S}_+^-$. Since the atoms are almost all in the excited state, by assumption, $\hat{b}^\dagger\hat{b}/(2N) \ll 1$, in which case the interaction Hamiltonian $\hat{H}_{int}$ describes a nondegenerate parametric amplifier.

To describe fluctuations in the atomic and cavity subsystems, we linearize the interaction by taking $\hat{a} = \alpha + \delta\hat{a}$ and $\hat{b} = \beta + \delta\hat{b}$ where $\delta\hat{a}$ and $\delta\hat{b}$ represent small quantum fluctuations on top of the large mean-field values α and β . Working to leading order in $\delta\hat{a}$ and $\delta\hat{b}$, we have $\hat{S}_+ \approx \sqrt{2N - |\beta|^2}\hat{b}$ and so on. A consistent accounting of the fluctuations requires a description of the spontaneous emission of the atoms and fluctuations in the field entering through the laser’s partially reflective mirror. We do so using input-output theory [40] and find that the superradiant laser evolves according to

$$\begin{aligned} \dot{\hat{a}} &= -\frac{\kappa_F}{2}\hat{a} - \sqrt{2N - |\beta|^2}g\hat{b}^\dagger - \sqrt{\kappa_F}\hat{a}_{in}, \\ \dot{\hat{b}}^\dagger &= -\frac{\kappa_G}{2}\hat{b}^\dagger - \sqrt{2N - |\beta|^2}g\hat{a} - \sqrt{\kappa_G}\hat{b}_{in}^\dagger, \\ \dot{\hat{a}}_{out} &= \sqrt{\kappa_F}\hat{a} + \hat{a}_{in}. \end{aligned} \quad (13)$$

For simplicity, we assume the atomic and cavity frequencies coincide, but this assumption can be lifted to investigate frequency pulling effects at the expense of somewhat more complicated algebraic expressions.

The classical steady state of the laser follows from demanding that the expectation values α, β do not evolve in time. This gives

$$\begin{aligned} 0 &= \beta\left(2N - \frac{\kappa_F\kappa_G}{4g^2} - |\beta|^2\right), \\ \alpha &= -\frac{2g}{\kappa_F}\sqrt{2N - |\beta|^2}\beta^*, \end{aligned} \quad (14)$$

so the system has a lasing threshold at $C \equiv 8g^2N/(\kappa_F\kappa_G) = 1$; here C is the collective cooperativity. When $C < 1$, the steady-state amplitude is given by $\beta = 0$ and the system has no macroscopic output. When $C > 1$, the steady-state amplitudes

are given by

$$|\beta|^2 = 2N(1 - C^{-1}), \quad |\alpha| = \sqrt{\kappa_G/\kappa_F} |\beta|, \quad (15)$$

implying that the laser emits a mean photon flux $|\alpha_{\text{out}}|^2 = 2N\kappa_G(1 - C^{-1})$.

Fluctuations on top of the steady-state flux can be computed from Eq. (13) by linearizing about the steady state in Eq. (15). Fourier transforming and solving for the output field-mode fluctuations $\delta\hat{a}_{\text{out}}$ gives

$$\delta\hat{a}_{\text{out}}[\omega] \approx -i \frac{\delta\hat{a}_{\text{in}}[\omega] + \delta\hat{b}_{\text{in}}^\dagger[\omega]}{(\kappa_F^{-1} + \kappa_G^{-1})\omega}. \quad (16)$$

This expression is approximated for frequency detunings small about resonance, which is the regime relevant for determining the linewidth of the output. The spectra of the oscillator's output amplitude and phase quadrature fluctuations then follow

$$\bar{S}_{xx}^{\text{out}} = \frac{\bar{S}_{xx}^a \pm 2\bar{S}_{xx}^{ab} + \bar{S}_{xx}^b}{[(\kappa_F^{-1} + \kappa_G^{-1})\omega]^2}, \quad (17)$$

where all spectra should be evaluated at the frequency $\Omega_0 + \omega$ and $x \in q, p$ is the amplitude or phase quadrature, related to the lowering operator by $\hat{q} \equiv (\hat{a}^\dagger + \hat{a})/\sqrt{2}$ and $\hat{p} \equiv i(\hat{a}^\dagger - \hat{a})/\sqrt{2}$; here, the $\pm$ signs correspond to q, p . In particular, the spectrum of phase fluctuations gives the linewidth; assuming the electromagnetic and atomic incident modes are in thermal states, the oscillator's linewidth is

$$\Gamma = \frac{\hbar\Omega_0}{2P} \frac{1 + 2\bar{n}_{\text{th}}}{(\kappa_F^{-1} + \kappa_G^{-1})^2}, \quad (18)$$

where $\bar{n}_{\text{th}}$ is the average thermal population of the atomic and field subsystems and $P = \hbar\Omega|\alpha_{\text{out}}|^2$ is the mean output power.

Indeed, the superradiant laser is a specific instance of the abstracted feedback oscillator analyzed above. Here, κ_F and κ_G characterize uncoupled linewidths of the superradiant laser's optical cavity and atoms. These are specific instances of the feedback element and amplifier linewidths discussed in previous sections such that this oscillator's generalized Schawlow-Townes linewidth is given by Eq. (10). Comparing Eq. (18) with Eq. (10), we see that the superradiant laser saturates the generalized Schawlow-Townes linewidth limit. For the superradiant laser, $\hat{a}_{\text{in}}$ and $\hat{b}_{\text{in}}$ are the quantum noise modes coupled into the system by the out-coupler and atoms, respectively, and are specific instances of out-coupler and amplifier modes $\hat{a}_0$ and $\hat{a}_G$. The superradiant laser saturates the generalized Schawlow-Townes limit in the “good-cavity,” “bad-cavity,” and intermediate regimes, all of which can thus be probed experimentally using these devices.

V. QUANTUM-ENHANCED SUPERRADIANT LASER

A superradiant laser without quantum-engineered inputs and outputs has a linewidth limited by Eq. (18). From Eq. (17), we see that the linewidth of a superradiant laser can be suppressed beyond the Schawlow-Townes limit by either squeezing the atomic and/or cavity field, or correlating the in-coupled noise modes $\hat{a}_{\text{in}}$ and $\hat{b}_{\text{in}}$. While the optical mode $\hat{a}_{\text{in}}$ represents a single spatial mode, defined by the cavity, and

can be readily squeezed, the mode $\hat{b}_{\text{in}}$ represents the collection of all spatial modes the atoms emit photons into besides the cavity mode. This huge collection of modes cannot be readily squeezed with existing technology.

Instead, we consider directly squeezing the collective atomic excitation [41–48]. This can be modeled by adding a spin-squeezing term to the interaction Hamiltonian, which can either take the form of a one-axis twisting interaction $\hbar\chi_1(\hat{S}_x - \hat{S}_y)^2$ [41,42,49,50], or a two-axis twisting interaction $(i\hbar\chi_2/2)(\hat{S}_-^2 - \hat{S}_+^2)$ [48,50–52]. In either case, after making the Holstein-Primakoff approximation the interaction Hamiltonian in the rotating frame is

$$\hat{H}_{\text{int}} = i\hbar\sqrt{2N}g(\hat{a}\hat{b} - \hat{a}^\dagger\hat{b}^\dagger) + i\hbar N\chi(\hat{b}^2 - \hat{b}^{\dagger 2}). \quad (19)$$

Here χ is either $\chi_{1,2}$ depending on whether the interaction is one or two-axis twisting.

Proceeding as in the case of a conventional superradiant laser, we find that the system has a lasing threshold when $C = (1 + s)^{-1}$, where $s \equiv \kappa_F\chi/(2g^2)$ is a dimensionless factor characterizing the strength of the spin-squeezing interaction [53]. As before, we linearize about the mean-field values to find the spectra of the oscillators' quantum fluctuations. The crucial upshot of the result is that, in contrast to the case with no spin squeezing, we now have an asymmetry between the amplitude and phase quadratures. In the simple case of Fourier frequencies close to resonance and when spin squeezing is weak (see Appendix C for exact expressions), the output amplitude and phase quadratures are given by

$$\delta\hat{q}_a^{\text{out}}[\omega] = -i \frac{\delta\hat{q}_a^{\text{in}}[\omega] - \delta\hat{q}_b^{\text{in}}[\omega]}{(\kappa_F^{-1} + \kappa_G^{-1})\omega}, \quad (20)$$

$$\delta\hat{p}_a^{\text{out}}[\omega] = \frac{\delta\hat{p}_a^{\text{in}}[\omega] + \delta\hat{p}_b^{\text{in}}[\omega]}{s(C - 2) + i(\kappa_F^{-1} + \kappa_G^{-1})\omega}. \quad (21)$$

Notably, the phase quadrature transfer function no longer has a pole at $\omega = 0$, allowing the phase-quadrature fluctuations to be substantially reduced below the generalized Schawlow-Townes limit. From Eq. (21), the linewidth of the spin-squeezed superradiant laser's output is given by (see Appendix D for details)

$$\Gamma \approx \Gamma_{\text{GST}} \left[1 - \left(\frac{\Gamma_{\text{GST}}}{\gamma s^2} \right)^{-1} \tan^{-1} \left(\frac{\Gamma_{\text{GST}}}{\gamma s^2} \right) \right]^{1/2}, \quad (22)$$

where $\gamma = [4 \ln(2)/\pi^2](2 - C)^2/(\kappa_F^{-1} + \kappa_G^{-1})$ characterizes the linewidth reduction due to spin squeezing. The linewidth narrows as the level of spin squeezing increases, as shown in the inset of Fig. 2.

Figure 2 shows the spectra of several types of quantum-engineered superradiant lasers. Without quantum enhancement, these lasers' phase quadrature fluctuations are limited by the generalized Schawlow-Townes limit. This limit can be evaded by injecting squeezed light into the device's input-output coupler, however, such an improvement is limited to, at most, a halving of the frequency noise. Further improvement requires modifying the system's underlying dynamics, for example, by spin squeezing the atomic excitation that produces gain.

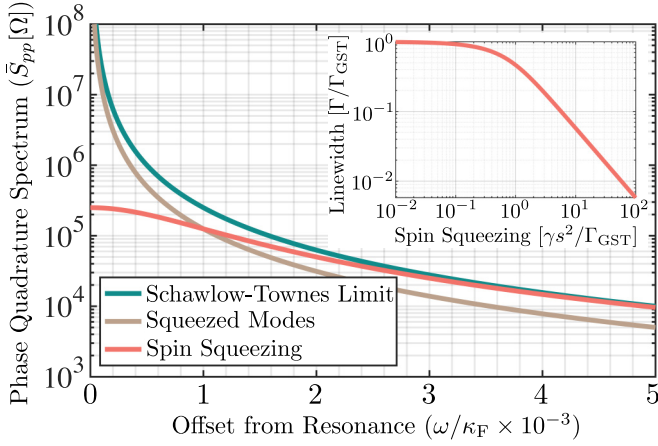


FIG. 2. The phase quadrature spectra for a laser in the intermediate, i.e., not good or bad cavity, regime where $\kappa_F = \kappa_G$. This spectrum is symmetric with respect to $\omega = 0$. The “Schawlow-Townes” curve plots the generalized Schawlow-Townes limit for such an oscillator, and the “squeezed modes” curve plots the bound for an oscillator with highly squeezed vacuum light injected into its input-output coupler. The “spin-squeezing” curve plots the spectra for a spin-squeezed superradiant laser with the spin-squeezing factor $s|C - 2| = 0.002$. The inset shows the linewidth decrease as spin-squeezing levels increase, as quantified by Eq. (22).

VI. CONCLUSION

We derived a generalized Schawlow-Townes limit, applicable to “good” or “bad” cavity feedback oscillators, including, but not limited to, lasers and masers. This limit is a consequence of necessary consistency conditions (arising from causality and quantum mechanics) that any feedback oscillator must obey. Having identified the quantum origins of this bound, we find that it is not a fundamental limit, but a SQL. A superradiant laser can, in principle, saturate this SQL, and operate at the generalized Schawlow-Townes limit. Appropriate quantum engineering can evade the SQL; in the case of a superradiant laser, it can be evaded by atomic spin squeezing. Given the promise of “bad” cavity lasers, and in particular, superradiant lasers, as an avenue for realizing local oscillators that are not limited by technical noises, our results establish fundamental benchmarks to gauge their performance against, and a route for their continued improvement beyond the SQL.

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DATA AVAILABILITY

The data that support the findings of this article are openly available [54].

APPENDIX A: FOURIER TRANSFORM NOTATIONS AND CONVENTIONS

Given a Heisenberg picture operator $\hat{X}(t)$, which may or may not be Hermitian, we define its Fourier transform $\hat{X}[\Omega]$ by

$$\hat{X}[\Omega] \equiv \int_{-\infty}^{\infty} \hat{X}(t) e^{i\Omega t} dt. \quad (A1)$$

The inverse Fourier transform is then given by

$$\hat{X}(t) \equiv \int_{-\infty}^{\infty} \hat{X}[\Omega] e^{-i\Omega t} \frac{d\Omega}{2\pi}. \quad (A2)$$

We define the symmetrized double-sided cross-correlation spectrum between operators $\hat{X}$ and $\hat{Y}$ by [9,55]

$$\bar{S}_{XY}[\Omega] \equiv \int_{-\infty}^{\infty} \frac{1}{2} \langle \{\hat{X}^\dagger[\Omega], \hat{Y}[\Omega']\} \rangle \frac{d\Omega'}{2\pi}, \quad (A3)$$

where $\{\hat{A}, \hat{B}\} = \hat{A}\hat{B} + \hat{B}\hat{A}$ denotes an anticommutator between the operators $\hat{A}$ and $\hat{B}$.

APPENDIX B: IMPLICATIONS OF CAUSALITY FOR THE DISPERSION RELATIONS OF LINEAR SYSTEMS

Let $G(t)$ be the time-domain response of a linear device such as an amplifier. Any physical device must be causal, so its linear response satisfies

$$G(t) = G(t)\Theta(t) = \begin{cases} G(t), & \text{if } t \geq 0, \\ 0, & \text{if } t < 0, \end{cases} \quad (B1)$$

where $\Theta(t)$ is the Heaviside step function. The response of a physical amplifier should also be real-valued such that $G(t) \in \mathbb{R} \forall t$.

In order for $G(t)$ to have well-defined Fourier and Laplace transforms, we assume it is Lebesgue integrable such that

$$\int_0^{\infty} |G(t)| dt < \infty. \quad (B2)$$

We define the Laplace transform $\tilde{F}(s)$ of a function $F(t)$ by

$$\tilde{F}(s) := \int_0^{\infty} F(t) e^{-st} dt. \quad (B3)$$

For a causal function, $F(t) = F(t)\Theta(t)$ such that

$$\tilde{F}(s) = \int_{-\infty}^{\infty} F(t) e^{-st} dt, \quad (B4)$$

and we define the Fourier transform $F[\Omega]$ of $F(t)$ by

$$F[\Omega] := \tilde{F}[-i\Omega] = \int_{-\infty}^{\infty} F(t) e^{i\Omega t} dt. \quad (B5)$$

Similarly, we define the Fourier transform of an operator $\hat{X}(t)$ by

$$\hat{X}[\Omega] := \int_{-\infty}^{\infty} \hat{X}(t) e^{i\Omega t} dt, \quad (B6)$$

which is valid whether or not $\hat{X}(t)$ is Hermitian.

We now want to relate the device's magnitude and phase response. To do so, we can express $G[\Omega]$ as $G[\Omega] = |G[\Omega]| \exp(i\phi[\Omega])$. Assuming the linear response of an amplifier is causal, real-valued, Lebesgue-integrable, and its Laplace transform has no zeros in the right-half-plane and converges at high frequencies such that

$$\lim_{\text{Re}[s] \rightarrow \infty} \frac{\ln(\tilde{G}(s))}{s} = 0, \quad (\text{B7})$$

the magnitude and phase response of a noninverting amplifier are related by [32]

$$\phi[\Omega] = -\frac{2\Omega}{\pi} P \int_0^\infty \frac{\ln |G[\Omega']|}{\Omega'^2 - \Omega^2} d\Omega', \quad (\text{B8})$$

where P denotes the Cauchy principal value. We are primarily interested in the magnitude and phase response near the frequency corresponding to the amplifier's peak amplitude response Ω_0^G . To leading order in ω/Ω_0^G , the magnitude and phase are related as

$$\phi[\Omega_0^G + \omega] \approx -\frac{1}{\pi} P \int_{-\infty}^\infty \frac{\ln |G[\Omega_0^G + \omega']|}{\omega' - \omega} d\omega'. \quad (\text{B9})$$

We use the assumption that $\omega \ll \Omega_0^G$ to extend the lower bound of the integral from $-\Omega_0^G$ to $-\infty$.

We now show that causality of the amplifier fixes the essential behavior of $G[\Omega]$. If the amplifier has its peak gain at Ω_0^G , then the first derivative of its magnitude response is zero at this frequency. Then, in the vicinity of this frequency, the magnitude response has the form

$$|G[\Omega_0^G + \omega]| = 1/\sqrt{\eta} - a\omega^2 + O(\omega^3), \quad (\text{B10})$$

where $\omega \equiv \Omega - \Omega_0^G$ and $a > 0$. Since its magnitude response is symmetric near Ω_0^G Eq. (B9) requires $\phi[\Omega_0^G] \approx 0$ such that the amplifier adds no phase on resonance. We write this relation with an approximation sign as the amplifier's magnitude response is not guaranteed to be symmetric far from resonance, and this may cause the integral in Eq. (B9) to differ slightly from zero. Then, to leading order, the (minimum) phase response is

$$\phi[\Omega_0^G + \omega] = \tau_G \omega + O(\omega^2). \quad (\text{B11})$$

Here, τ_G is some characteristic constant with the dimension of time. This constant has to be positive: indeed, Eq. (B9) implies that

$$\frac{d\phi[\Omega_0^G + \omega]}{d\omega} = \frac{1}{\pi} P \int \frac{\ln \left| \frac{G[\Omega_0^G + \omega]}{G[\Omega_0^G + \omega']} \right|}{(\omega' - \omega)^2} d\omega'; \quad (\text{B12})$$

for an amplifier $\ln |G[\Omega_0^G]/G[\Omega_0^G + \omega]| \geq 0$ (since $|G[\Omega]|$ has a local maximum at $\Omega = \Omega_0^G$), and so $d\phi[\Omega_0^G + \omega]/d\omega > 0$. The last inequality is strict since an amplifier must have $|G[\Omega_0^G]/G[\Omega_0^G + \omega']| > 1$ for some ω' . As a result the coefficient of the first-order term in the Taylor expansion of $\phi[\Omega_0^G + \omega]$ will always be positive for an amplifier, i.e., $\tau_G > 0$.

We find it convenient to work with respect to the resonant frequency of the feedback element Ω_0^F . Using $\delta\omega = \Omega_0^G - \Omega_0^F$, we can rewrite Eq. (B11) relative to this frequency as

$$\phi[\Omega_0^F + \omega] = \tau_G(\omega - \delta\omega) + O(\omega^2). \quad (\text{B13})$$

APPENDIX C: DETAILS OF A SUPERRADIANT LASER WITH SPIN SQUEEZING

From the Hamiltonian

$$\hat{H}_{\text{int}} = i\hbar\sqrt{2N} g(\hat{a}\hat{b} - \hat{a}^\dagger\hat{b}^\dagger) + i\hbar N \chi(\hat{b}^2 - \hat{b}^{\dagger 2}), \quad (\text{C1})$$

we find that the equations of motion for a spin-squeezed superradiant laser are given by

$$\partial_t \hat{a} = -\frac{\kappa_F}{2} \hat{a} - \sqrt{2N - |\beta|^2} g \hat{b}^\dagger - \sqrt{\kappa_F} \hat{a}_{\text{in}},$$

$$\begin{aligned} \partial_t \hat{b}^\dagger &= -\frac{\kappa_G}{2} \hat{b}^\dagger - \sqrt{2N - |\beta|^2} g \hat{a} \\ &\quad + 2\chi(N - |\beta|^2) \hat{b} - \sqrt{\kappa_G} \hat{b}_{\text{in}}, \end{aligned}$$

$$\hat{a}_{\text{out}} = \sqrt{\kappa_F} \hat{a} + \hat{a}_{\text{in}}. \quad (\text{C2})$$

Assuming $\beta \in \mathbb{R}$ such that the macroscopic field is aligned with the oscillator's amplitude quadrature [56], and making the Holstein-Primakoff approximation that $|\beta|^2 \ll N$, we find that the steady-state mean-field amplitudes are the solutions to the equations

$$\begin{aligned} \alpha &= -\frac{2g}{\kappa_F} \sqrt{2N - |\beta|^2} \beta, \\ 0 &= \beta[2N(1 + s - C^{-1}) - (1 + 2s)|\beta|^2]. \end{aligned} \quad (\text{C3})$$

We note that these equations reduce to Eq. (14) in the main text in the case that $\chi = 0$ and there is no two-axis twisting interaction. We see that as before, the system has a lasing transition below which $\beta = 0$ and the system has no mean output and above which the system has a macroscopic output given by

$$\begin{aligned} |\beta|^2 &= 2N \frac{1 + s - C^{-1}}{1 + 2s}, \\ |\alpha| &= \sqrt{\frac{\kappa_G(1 + sC)}{\kappa_F(1 + 2s)}} |\beta|. \end{aligned} \quad (\text{C4})$$

As expected, these equations reduce to Eq. (15) in the main text in the absence of a two-axis twisting interaction.

Using Eq. (C4), Eq. (C2) tells us that the system's quantum fluctuations are governed by

$$\begin{aligned} \partial_t \delta \hat{a} &= -\frac{\kappa_F}{2} \delta \hat{a} - \sqrt{2N} g \sqrt{\frac{s + C^{-1}}{1 + 2s}} \delta \hat{b}^\dagger - \sqrt{\kappa_F} \delta \hat{a}_{\text{in}}, \\ \partial_t \delta \hat{b}^\dagger &= -\frac{\kappa_G}{2} \delta \hat{b}^\dagger - \sqrt{2N} g \sqrt{\frac{s + C^{-1}}{1 + 2s}} \delta \hat{a} \\ &\quad + \frac{\kappa_G s(2 - C)}{2(1 + 2s)} \delta \hat{b} - \sqrt{\kappa_G} \delta \hat{b}_{\text{in}}, \\ \delta \hat{a}_{\text{out}} &= \sqrt{\kappa_F} \delta \hat{a} + \delta \hat{a}_{\text{in}}. \end{aligned} \quad (\text{C5})$$

We emphasize that we use the Holstein-Primakoff approximation here to neglect fluctuation terms inside the quantity $\sqrt{2N - \hat{b}^\dagger \hat{b}}$ and replace this with $\sqrt{2N - |\beta|^2}$. As a result, these equations are only valid for $|\beta|^2 \ll 2N$. Defining the

amplitude and phase quadratures for mode i as $\hat{q}_i \equiv (\hat{a}_i^\dagger + \hat{a}_i)/\sqrt{2}$ and $\hat{p}_i \equiv i(\hat{a}_i^\dagger - \hat{a}_i)/\sqrt{2}$, Fourier transforming, we

find that the oscillator's output amplitude quadrature is given by

$$\delta\hat{q}_a^{\text{out}}[\omega] = \frac{-i\kappa_F\kappa_G(1+Cs)}{[\kappa_F(1+2s) + \kappa_G(1+Cs)]\omega} \delta\hat{q}_a^{\text{in}}[\omega] + \frac{i\kappa_F\kappa_G(1+2s)\sqrt{(1+2s)(1+Cs)}}{[\kappa_F(1+2s) + \kappa_G(1+Cs)]\omega} \delta\hat{q}_b^{\text{in}}[\omega]. \quad (\text{C6})$$

As in the main text, we neglect nonleading terms in ω since we are primarily interested in the oscillator's noise near resonance. Similarly, the oscillator's output phase quadrature is given by

$$\begin{aligned} \delta\hat{p}_a^{\text{out}}[\omega] &= \frac{(1+2s)\kappa_F\kappa_G + i\omega([1 + (4-C)s]\kappa_G - [1+2s]\kappa_F)}{i\omega(\kappa_F + \kappa_G) + s[(C-2)\kappa_F\kappa_G + 2i\kappa_F\omega - i(C-4)\kappa_G\omega]} \delta\hat{p}_a^{\text{in}}[\omega] \\ &+ \frac{(1+2s)\sqrt{\frac{1+Cs}{1+2s}}\kappa_F\kappa_G}{i\omega(\kappa_F + \kappa_G) + s[(C-2)\kappa_F\kappa_G + 2i\kappa_F\omega - i(C-4)\kappa_G\omega]} \delta\hat{p}_b^{\text{in}}[\omega], \end{aligned} \quad (\text{C7})$$

where we neglect quadratic terms in ω .

Assuming that χ and ω are both small parameters, we can simplify the preceding equations to

$$\begin{aligned} \delta\hat{q}_a^{\text{out}}[\omega] &= \frac{-i\kappa_F\kappa_G}{(\kappa_F + \kappa_G)\omega} (\delta\hat{q}_a^{\text{in}}[\omega] - \delta\hat{q}_b^{\text{in}}[\omega]), \\ \delta\hat{p}_a^{\text{out}}[\omega] &= \frac{\kappa_F\kappa_G(\delta\hat{p}_a^{\text{in}}[\omega] + \delta\hat{p}_b^{\text{in}}[\omega])}{s(C-2)\kappa_F\kappa_G + i\omega(\kappa_F + \kappa_G)}. \end{aligned} \quad (\text{C8})$$

APPENDIX D: LINEWIDTH OF A SUPERRADIANT LASER WITH SPIN SQUEEZING

For an oscillator with a flat frequency noise spectrum, the linewidth is given in terms of the frequency noise spectrum by $\Gamma = \bar{S}_{\dot{\phi}\dot{\phi}}/(2\pi)$ [57]. For an oscillator with a frequency noise spectrum that varies with the frequency offset from resonance, the relation between the frequency spectrum and linewidth is generally more complicated, but is given approximately by [57]

$$\Gamma = \sqrt{8 \ln(2)A}, \quad (\text{D1})$$

where

$$A := \int_0^{\Omega_{\text{cut}}} \frac{\omega^2}{4\pi^2|\alpha_{\text{out}}|^2} \bar{S}_{pp}^{\text{out}}[\omega] \frac{d\omega}{2\pi}, \quad (\text{D2})$$

and Ω_{cut} is determined by

$$\frac{16 \ln(2)|\alpha_{\text{out}}|^2}{\pi \Omega_{\text{cut}}} = \bar{S}_{pp}^{\text{out}}[\Omega_{\text{cut}}]. \quad (\text{D3})$$

To arrive at a relatively simple expression for the linewidth, we assume that the amount of spin-squeezing is sufficiently small that we can approximate $\bar{S}_{pp}^{\text{out}}[\Omega_{\text{cut}}]$ by $\bar{S}_{pp}^{\text{out}}[\Omega_{\text{cut}}]_{s=0}$ in Eq. (D3). Making this approximation and evaluating the integral in Eq. (D2) using $\bar{S}_{pp}^{\text{out}}[\omega]$ from Eq. (C8), we find

$$\Gamma \approx \sqrt{\Gamma_{\text{GST}}^2 - \gamma \Gamma_{\text{GST}} s^2 \tan^{-1} \left(\frac{\Gamma_{\text{GST}}}{\gamma s^2} \right)}, \quad (\text{D4})$$

where

$$\gamma := \frac{4 \ln(2)(C-2)^2 \kappa_F \kappa_G}{\pi^2 (\kappa_F + \kappa_G)} \quad (\text{D5})$$

characterizes the linewidth reduction due to the spin-squeezing interaction. As expected, the linewidth reduces to the Schawlow-Townes linewidth in the absence of a spin-squeezing interaction, i.e., for $s = 0$.

APPENDIX E: LINEWIDTH OF A LASER BASED ON A NONDEGENERATE SQUEEZER

We can also construct a laser with stability at the generalized Schawlow-Townes limit above by using a nondegenerate squeezer, in the form of a crystal with a $\chi^{(2)}$ nonlinearity and a strong pump field, as the amplifier. Such an amplifier has two input fields, $\hat{a}_{\text{in}}$ and $\hat{b}_{\text{in}}$, and two output fields, $\hat{a}_{\text{out}}$ and $\hat{b}_{\text{out}}$, related by

$$\begin{aligned} \hat{a}_{\text{out}}^-[\Omega] &= \cosh(r)\hat{a}_{\text{in}}[\Omega] + \sinh(r)\hat{b}_{\text{in}}^\dagger[\Omega], \\ \hat{b}_{\text{out}}^-[\Omega] &= \cosh(r)\hat{b}_{\text{in}}[\Omega] + \sinh(r)\hat{a}_{\text{in}}^\dagger[\Omega], \end{aligned} \quad (\text{E1})$$

from which we see that we can identify the gain for input mode $\hat{a}_{\text{in}}$ with $G[\Omega] = \cosh(r)$ and the ancillary mode $\hat{a}_G^\dagger$ with $\hat{b}_{\text{in}}^\dagger$. Importantly, the phase-matching bandwidth for $\chi^{(2)}$ devices can be extremely large, on the order of THz [58], so the amplification of the gain medium is roughly independent of frequency. In this case, the equations for the oscillator [Eq. (2)] become

$$\begin{aligned} \hat{a}_{\text{out}}^-[\Omega] &= G\hat{a}_{\text{in}}[\Omega] + \sqrt{G^2 - 1}\hat{b}_{\text{in}}^\dagger[\Omega], \\ \hat{a}_{\text{out}}^+[\Omega] &= -i\sqrt{\eta}\hat{a}_{\text{out}}^-[\Omega] + \sqrt{1-\eta}\hat{a}_0[\Omega], \\ \hat{a}_{\text{out}}[\Omega] &= \sqrt{1-\eta}\hat{a}_{\text{out}}^-[\Omega] + i\sqrt{\eta}\hat{a}_0[\Omega], \\ \hat{a}_{\text{in}}[\Omega] &= e^{i\Omega\tau_F}\hat{a}_{\text{out}}^+[\Omega]. \end{aligned} \quad (\text{E2})$$

Assuming the noise modes $\hat{a}_0$ and $\hat{b}_{\text{in}}^\dagger$ are uncorrelated, we find that the oscillator's output phase quadrature spectrum is given by

$$\bar{S}_{pp}^{\text{out}}[\Omega_0 + \omega] = \frac{(\sqrt{\eta} - 1/\sqrt{\eta})^2}{(\omega\tau_F)^2} (\bar{S}_{pp}^{b,\text{in}}[\Omega] + \bar{S}_{pp}^0[\Omega]). \quad (\text{E3})$$

Assuming $\bar{S}_{pp}^{b,\text{in}}$ and $\bar{S}_{pp}^0$ are independent of frequency and that $1 - \eta \ll 1$, we find that the $\chi^{(2)}$ oscillator's linewidth, $\Gamma_{\chi^{(2)}}$, is

$$\Gamma_{\chi^{(2)}} = \frac{\hbar\Omega_0\kappa_F^2}{2P}(\bar{S}_{pp}^{b,\text{in}} + \bar{S}_{pp}^0), \quad (\text{E4})$$

which saturates the generalized Schawlow-Townes limit when the input noise modes are in vacuum states and $\bar{S}_{pp}^{b,\text{in}}[\Omega] = \bar{S}_{pp}^0[\Omega] = 1/2$. If the noise modes $\hat{a}_0$ and $\hat{b}_{\text{in}}^\dagger$ are instead in squeezed states with squeezing parameters r_0 and r_b , respectively, the oscillator linewidth is narrowed to $\Gamma_{\chi^{(2)}}(e^{-2r_0} + e^{-2r_b})/2$.

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