

Extracting fine topological features of high-dimensional fields in structured light

An Aloysius Wang^{✉,*}, Yunqi Zhang, Yuxi Cai, and Chao He^{✉,†}

Department of Engineering Science, *University of Oxford*, Parks Road, Oxford OX1 3PJ, United Kingdom



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Topological invariants play a central role in structured light, providing robust, discrete descriptors of complex optical fields. In an effort to systematically expand such invariants, we detail and generalize two complementary mechanisms for extracting lower-dimensional components of a higher-dimensional field: one associated with phase vortices, in which a gauge structure is relevant, and one based on polarization, for which it is not. Together, these mechanisms form part of a broader framework for defining topological invariants in high-dimensional fields, enabling high-density and robust information encoding through topological field engineering.

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I. INTRODUCTION

Engineered spatially varying high-dimensional fields provide a rich setting in which topological invariants arise naturally, and recent efforts have begun to exploit these invariants for robust information manipulation [1–3]. At a high level, the design process for such topological encoding schemes typically involves two key steps. The first is an engineering step, whereby constraints on the field are designed and realized, for example, through the manipulation of initial [4] and boundary conditions [1,5], material properties [6], and system geometry [7]. These design considerations are deliberately chosen to enforce a restricted class of admissible fields that separates into distinct, inequivalent families, leading to the second, mathematical step in which topological tools are used to assign an index distinguishing qualitatively different fields [2].

A particularly suitable setting for exploring such topological encoding schemes is provided by optics and photonics, which naturally involve a wide range of high-dimensional fields [8], including the optical field itself, which is vectorial in nature, as well as structured materials whose internal parameters and optical response are commonly described by spatially varying manifold-valued fields [9]. Moreover, modern developments provide a mature set of techniques for the flexible synthesis, modulation, and measurement of such fields [9–13]. These advantages have enabled recent work on optical Stokes skyrmions [14], whereby S^2 -valued polarization fields are engineered to satisfy appropriate boundary conditions and classified by a topological index associated with a map in second cohomology [2], and on axis-geometry-based skyrmions, in which the optical response of a designed

matter field plays an analogous role to polarization [6], among many other related developments [2,3,14,15].

Another classic example of topological encoding in optics and photonics is provided by optical vortices [16–18], which are fields containing phase singularities and whose topology is characterized by a winding number. Alternative schemes based on winding numbers also include subsequent work exploring other S^1 -valued quantities derived from the Stokes parameters as candidates for encoding and transmission [19–21].

More generally, the examples in optics and photonics discussed above illustrate that a foundational technique in topological field engineering is the extraction of a lower-dimensional, manifold-valued field from an underlying higher-dimensional physical field, with analogous constructions appearing in disciplines such as acoustics [22], fluid dynamics [23], cosmology [24], soft-matter physics [25] and condensed-matter physics [26–28]. These varied demonstrations across physical domains motivate abstracting the strategies used to extract topologically nontrivial lower-dimensional fields, thereby providing a unifying framework for topological encoding across disciplines and suggesting new routes toward increasing both the amount and diversity of information that can be robustly encoded in high-dimensional fields.

To this end, two main approaches can be identified. The central idea is to reduce the effective parameter space of the field through a projection onto a lower-dimensional manifold. This process naturally yields two complementary components: the projected field itself, and the information required to reconstruct the original field from its projection, corresponding, for example, to polarization and phase in optical fields. These two components then form the basis of two distinct approaches to topological encoding (see Fig. 1).

In this work we present a general framework for the two contrasting constructions introduced above, and demonstrate concrete examples of each through traditional optical vortices and alternative phases derived from Stokes parameters. In the former, we show that a gauge-independent topological invariant—defined by the homotopy class of a bundle isomorphism canonically induced by the field—exists and can

*Contact author: alloysius.wang@gmail.com

†Contact author: chao.he@eng.ox.ac.uk

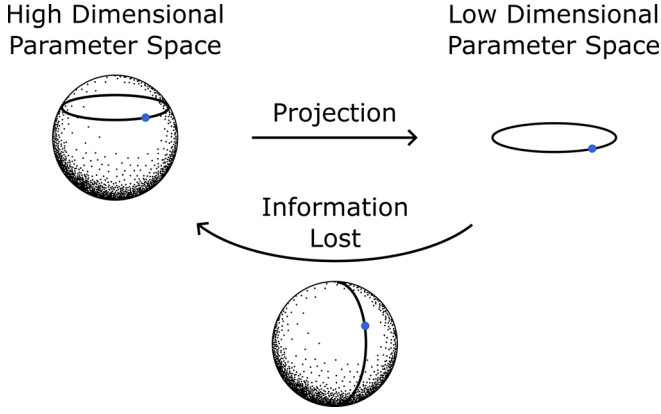


FIG. 1. Concept. Schematic of the framework applied to the projection map $S^2 \rightarrow S^1$ defined by the azimuthal angle on the sphere. Points in the higher-dimensional parameter space are projected onto a lower-dimensional manifold, thereby giving rise to a corresponding lower-dimensional field; the blue dot in the figure indicates one representative point. The information lost under the projection, which is required to reconstruct the original point from its projected value, gives rise to a second complementary field taking values in the fibers of the projection map.

be accessed experimentally via the induced map on first cohomology associated with a gauge-dependent $U(1)$ -valued field (see the Appendix 1). This provides a rigorous formalization of the topology of optical vortices. In the latter, we demonstrate how the resulting one-dimensional topological invariant relates to higher-dimensional invariants. In particular, when the reduced field originates from a skyrmion, the induced phase vortices behave as effective particles, for which creation, annihilation, and braiding operations can be performed. These results illustrate the conceptual richness of topological field engineering, and demonstrate how techniques for extracting lower-dimensional subcomponents can be used to resolve fine topological features of a field. Our work thereby paves the way for robust, high-density topological information encoding, analysis, and manipulation in high-dimensional physical systems.

II. RESULTS

To keep the presentation accessible, we use optical phase and polarization as guiding examples throughout the paper, with technical details provided in the Appendix 1. Fundamental to our discussion is the $\mathbb{C}^2$ -valued electric field, which represents the two transverse complex degrees of freedom of a monochromatic optical field and serves as the higher-dimensional physical field from which lower-dimensional components, including optical phase and polarization, are derived.

The phenomenological origin of phase and polarization lies in the fact that the only directly accessible quantity of the electric field is its intensity, $I = \langle E_x E_x^* \rangle + \langle E_y E_y^* \rangle$, which is invariant under the change

$$(E_x, E_y) \mapsto (e^{i\theta} E_x, e^{i\theta} E_y).$$

Therefore, in the absence of interference with additional fields, electric field values that differ only by a common factor

$g \in U(1)$ are experimentally indistinguishable and can be regarded as representing the same physical state. This motivates the identification of all such field values as a single physical quantity.

More precisely, this experimental indistinguishability is captured by the orbits of the right $U(1)$ action on $\mathbb{C}^2$ defined by

$$(E_x, E_y)g = (gE_x, gE_y), \quad g \in U(1). \quad (1)$$

Under this action, the point of zero intensity is distinguished, since it is a fixed point, whereas any nonzero field can be decomposed into its intensity I , which is invariant under the action, and its associated Jones vector $J \in S^3$, which is not. When each Jones vector is further identified up to physically indistinguishable values, the result defines the polarization state. Mathematically, this reduction corresponds to taking the quotient of S^3 by the $U(1)$ action above, yielding the Hopf projection

$$\mathcal{P}_{\text{Hopf}} : S^3 \longrightarrow S^2,$$

which maps a Jones vector to its associated Stokes vector [29].

Before proceeding to a discussion of phase, it is instructive to recast the constructions of intensity and the Jones vector presented above within our framework of projections. To this end, define

$$\mathcal{P}_I : \mathbb{C}^2 \longrightarrow [0, \infty), \quad \mathcal{P}_I(E_x, E_y) = |E_x|^2 + |E_y|^2,$$

so that for a spatially varying electric field $\mathbf{E} : \Omega \rightarrow \mathbb{C}^2$, the associated intensity distribution is given by $I = \mathcal{P}_I \circ \mathbf{E}$. In passing from the electric field to intensity, information about the field is lost, since many different values $(E_x, E_y) \in \mathbb{C}^2$ correspond to the same intensity. A natural question is then: Given the intensity distribution I , what additional information is required to recover the full information contained in the original field $\mathbf{E}$?

To answer this, we can begin by examining the constraints that I imposes on $\mathbf{E}$. For example, if in fact $I(u) = 0$ for some $u \in \Omega$, then we know for certain that $\mathbf{E}(u) = 0$. On the other hand, if $I(u) \neq 0$, then $\mathbf{E}(u)$ lies on the 3-sphere defined by $\|\mathbf{E}(u)\| = \sqrt{I(u)}$. Thus, we can form a space by attaching to each $u \in \Omega$ a single point when $I(u) = 0$, and a copy of S^3 when $I(u) \neq 0$. The additional information required to recover $\mathbf{E}$ is then precisely a lift of Ω into this space. More formally, this qualitative description motivates the pullback construction in the category **Top** [30], as detailed in the Appendix 1.

For projections with a structure as simple as that of $\mathcal{P}_I$, we can further simplify by noting that if $I(u) = 0$, no additional information is required to reconstruct $\mathbf{E}$, and such points can simply be excluded from the domain. In this case, the lift

$$J : \mathbf{E}^{-1}(\mathbb{C}^2 \setminus \{0\}) \longrightarrow S^3$$

is precisely the field of Jones vectors. Therefore, one can view I and J as a complementary pair: I describes the information retained by the projection $\mathcal{P}_I$, while J encodes the information “lost” under $\mathcal{P}_I$.

In the same vein, polarization and phase play the respective roles of I and J for the map $\mathcal{P}_{\text{Hopf}}$, with the former representing the information retained by the projection and the latter encoding the additional information required to recover

the Jones vector from a given polarization state. However, the construction here is more subtle, as $\mathcal{P}_{\text{Hopf}}$ is itself more intricate. Indeed, the projection $\mathcal{P}_I$ is topologically trivial in that, when restricted to $\mathbb{C}^2 \setminus \{0\}$, it induces the global product decomposition

$$\mathbb{C}^2 \setminus \{0\} \cong S^3 \times (0, \infty),$$

so that the S^3 -valued Jones vector is obtained simply by normalising $\mathbf{E}$. By contrast, $\mathcal{P}_{\text{Hopf}}$ admits no such global splitting, since although each fiber is homeomorphic to S^1 , the total space does not decompose as $S^3 \cong S^2 \times S^1$. Consequently, there is no globally consistent way to choose an S^1 -valued complement to polarization that reconstructs the Jones vector. More concretely, this obstruction arises because $\mathcal{P}_{\text{Hopf}}$ endows S^3 , through the right group action defined in Eq. (1), with the structure of a nontrivial principal $U(1)$ bundle over S^2 , which therefore admits no global section [29].

Nonetheless, the principal bundle structure ensures that it remains meaningful to speak of phase differences, namely, that $J \cdot g$ differs from J by the phase $g \in U(1)$. Using this observation, one can now address the question: given the Stokes field $\mathcal{S} = \mathcal{P}_{\text{Hopf}} \circ J$, what additional data are required to reconstruct J ? As above, we can approach this by first considering the constraints that $\mathcal{S}$ imposes on J . In particular, if J is defined on a domain Ω , then to each $u \in \Omega$ we can associate the fiber $\mathcal{P}_{\text{Hopf}}^{-1}(\mathcal{S}(u))$, together with its inherited right $U(1)$ action, representing the possible values of J at u . These fibers assemble into a principal $U(1)$ bundle over Ω , namely, the pullback bundle $\mathcal{S}^*S^3$, so that, in direct analogy with the intensity case, recovering J amounts to specifying a lift of Ω into $\mathcal{S}^*S^3$.

The difference now lies in how this lift can be reduced to a $U(1)$ -valued field, to which the usual topological considerations apply. The key idea here is that, while there is no canonical way to assign a phase consistently across all Jones vectors at once, given a lift s , it is nevertheless possible to compare it with another lift t by asking, “how far ahead is t of s ” at each point in Ω ? More specifically, this comparison is captured by the unique map $g : \Omega \rightarrow U(1)$ satisfying

$$t(u) = s(u)g(u).$$

Conversely, given a map $g : \Omega \rightarrow U(1)$, the lift t defined by the equation above corresponds to the field that is “ g ahead of s ” at each point of Ω . Thus, fixing a reference lift s , we obtain a bijection

$$\phi_s : \{\text{lifts of } \Omega \text{ into } \mathcal{S}^*S^3\} \rightarrow \{\text{maps } \Omega \rightarrow U(1)\}$$

given by $t \mapsto g$, which we term the phase associated with s . For any lift t , the $U(1)$ -valued function $\phi_s(t)$ is naturally interpreted as the phase of t relative to s . Assigning to each lift its associated phase then gives rise to a map

$$\begin{aligned} & \{\text{lifts of } \Omega \text{ into } \mathcal{S}^*S^3\} \\ & \rightarrow \text{Bij}(\{\text{lifts of } \Omega \text{ into } \mathcal{S}^*S^3\}, \{\text{maps } \Omega \rightarrow U(1)\}) \end{aligned}$$

given by $s \mapsto \phi_s$.

Physically, this interpretation of phase reflects the non-triviality of the bundle $U(1) \hookrightarrow S^3 \rightarrow S^2$, which in general prevents a canonical reference from being chosen. Without

such a reference, there is no gauge-independent way to interpret an arbitrary lift s as a $U(1)$ -valued field; as such, the only object canonically associated to s is the correspondence that uses s as a reference to assign to every other field with the same polarization structure a “phase relative to s .” In other words, ϕ_s can be regarded as a gauge-independent object that captures all the phase information of the underlying field.

The topological content of this definition follows from the fact that the assignment $s \mapsto \phi_s$ descends to homotopy classes (see the Appendix 1), with homotopies of lifts taken through sections of the pullback bundle and homotopies of maps $\Omega \rightarrow U(1)$ taken in the usual sense. More specifically, one has

$$\begin{aligned} & [\text{lifts of } \Omega \text{ into } \mathcal{S}^*S^3] \\ & \rightarrow \text{Bijections}([\text{lifts of } \Omega \text{ into } \mathcal{S}^*S^3], [\Omega, U(1)]), \end{aligned}$$

given by

$$[s] \mapsto \phi_{[s]} := ([t] \mapsto [\phi_s(t)]),$$

where $[\cdot]$ denotes homotopy classes. Thus, the object $\phi_{[s]}$ can be regarded as a gauge-independent quantity that encodes all relevant topological information of the underlying phase structure and provides the most general definition of phase vortices. Moreover, since both $[\text{lifts of } \Omega \text{ into } \mathcal{S}^*S^3]$ and $[\Omega, U(1)]$ are discrete, $\phi_{[s]}$ itself lies in a discrete space and therefore defines a topological index associated with the phase distribution. Importantly, the index $\phi_{[s]}$ is well defined for all electric field distributions, not only for the commonly studied orbital angular momentum modes.

Lastly, to relate this construction to more conventional approaches, note that if a reference s_0 is specified, a gauge-dependent correspondence

$$[\text{lifts of } \Omega \text{ into } \mathcal{S}^*S^3] \rightarrow H^1(\Omega; \mathbb{Z}),$$

given by

$$[s] \mapsto \phi_{[s_0]}([s])^* \eta,$$

where $\eta \in H^1(S^1; \mathbb{Z})$ denotes the canonical generator, can also be defined. This correspondence recovers the usual interpretation of optical vortices, in which the class $\phi_{[s_0]}([s])^* \eta$ is given by winding numbers of the phase of s relative to s_0 . However, while this class can be accessed experimentally, it is in fact gauge-dependent and therefore less fundamental than the gauge-independent $\phi_{[s]}$. A different perspective on the construction above, formulated in terms of homotopy classes of bundle isomorphisms, is given in the Appendix 1.

As a concrete example, we consider topological phase structures in paraxial optical skyrmions formed from a superposition of Laguerre-Gaussian (LG) modes. Letting $|0\rangle$ and $|1\rangle$ denote right- and left-circular polarizations, respectively, it is a well-known result [31] that the superposition

$$\text{LG}_0^{\ell_1} |0\rangle + \text{LG}_0^{\ell_2} |1\rangle, \quad \ell_1 \neq \ell_2,$$

gives rise to standard Néel-type skyrmions of degree $\pm(\ell_1 - \ell_2)$, depending on the asymptotic behavior of the individual modes (see the Appendix 3 for conventions). From this, it is clear that different superpositions of modes can give rise to similar polarization structures, differing only by a trivial coordinate transformation and otherwise indistinguishable

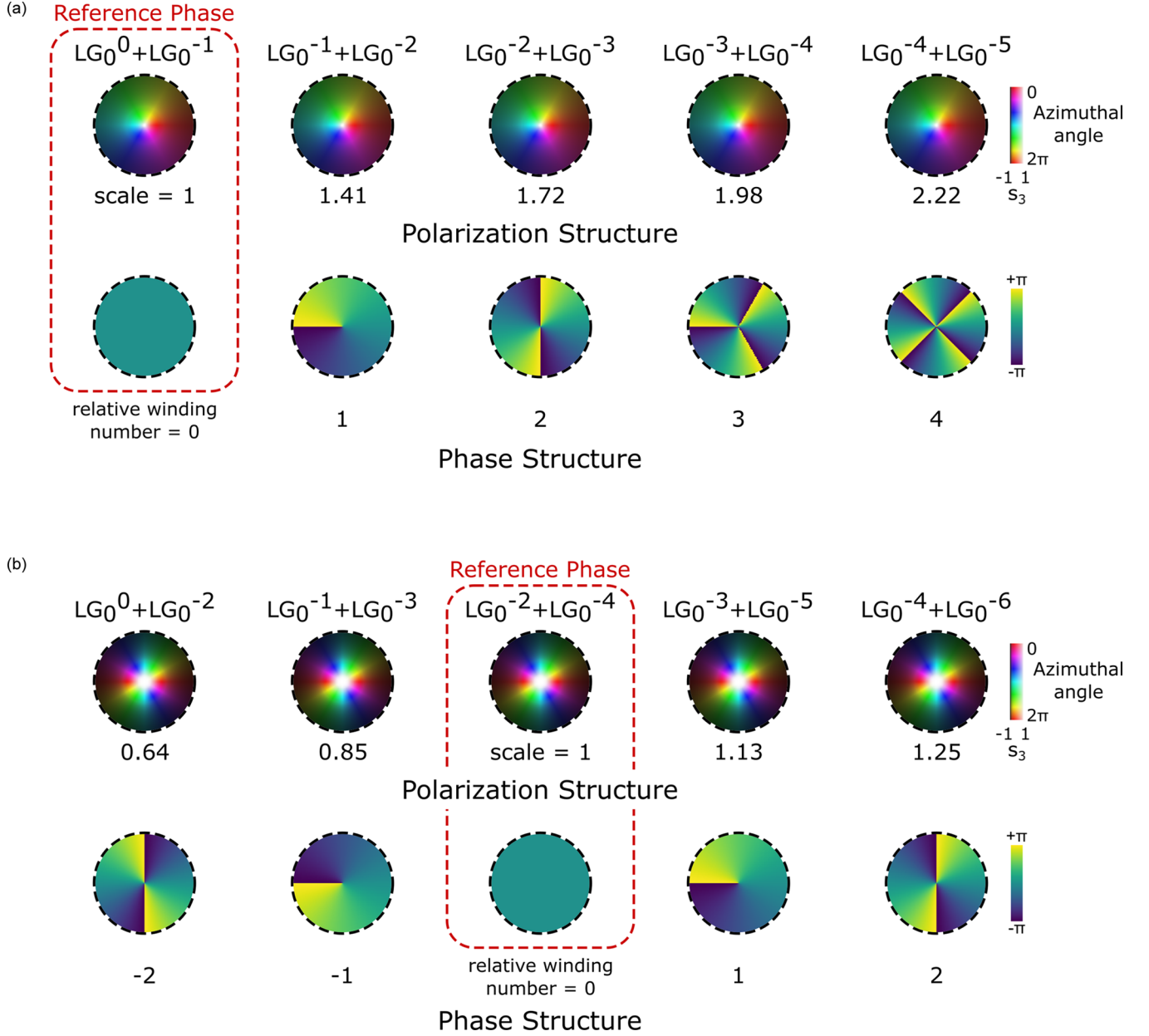


FIG. 2. Distinguishing optical skyrmions by phase. (a) Stokes fields of degree-1 Néel-type skyrmion formed from superpositions of different combinations of LG modes. Different scaling factors are used to highlight the indistinguishability of the underlying polarization structure. The relative phase structure with respect to the mode $LG_0^0 |0\rangle + LG_0^{-1} |1\rangle$ is also shown, from which the relative winding numbers can be computed. For brevity, $|0\rangle$ and $|1\rangle$ are omitted from the figure. (b) Stokes fields of degree-2 Néel-type skyrmion formed from superpositions of different combinations of LG modes. The reference phase is taken to be $LG_0^{-2} |0\rangle + LG_0^{-4} |1\rangle$, highlighting the gauge dependence of the computed relative winding numbers.

from the polarization structure alone. For example, Fig. 2 shows various superpositions of modes that produce standard Néel-type skyrmions of degrees 1 and 2, which, when appropriately scaled, have virtually identical polarization distributions.

However, these states are in fact distinguishable, as they possess topologically distinct phase structures. Figure 2(a), for example, uses the mode $s_0 = LG_0^0 |0\rangle + LG_0^{-1} |1\rangle$ as a reference to compute the relative phase structure of the remaining modes. This is done by first computing a

diffeomorphism ψ such that the transformed polarization field $\mathcal{P}_{\text{Hopf}} \circ s \circ \psi^{-1}$ agrees with $\mathcal{P}_{\text{Hopf}} \circ s_0$, and then computing the relative phase $\phi_{s_0}(s \circ \psi^{-1})$. Figure 2(a) gives approximate scaling factors that characterize the mean scaling of the diffeomorphism ψ in each case.

From the computed phase structure, we observe that different winding numbers arise, indicating that the resulting superpositions of modes are topologically distinct. Additionally, Fig. 2(b) shows the case of degree-2 Néel-type skyrmions, where $LG_0^{-2} |0\rangle + LG_0^{-4} |1\rangle$ is taken as the

reference. This highlights the gauge dependence of the computed relative winding numbers, particularly in complex structured fields such as optical skyrmions, where a canonical reference is not available. As such, the relative winding numbers computed in Fig. 2 are not intrinsic to the field; rather, it is the ability to use any field as a reference to compute a relative winding number that is intrinsic.

To summarize our theoretical framework, the pairs (intensity, Jones vector) and (optical phase, polarization) illustrate two distinct mechanisms for extracting lower-dimensional fields from a higher-dimensional field via projection. Specifically, (1) the projected component, encoding the information retained under projection; and (2) a lift of an associated pullback construction, encoding the information lost through projection. The most general topological content of each component is given, in case (1), by the homotopy class of the resulting map through admissible fields, and in case (2), by the homotopy class of the corresponding lift through sections of admissible fields. Moreover, we demonstrate that in case (2), where the pullback is a G -principal bundle, cohomological tools provide a direct means of accessing the associated topological content.

To round out the discussion, a third pair, arising from the projection map that defines phase via the Stokes parameters, is presented in the Appendix 2. In this case, the resulting topologically nontrivial field falls under case (1) and carries a topological index taking values in $H^1(\Omega; \mathbb{Z})$. Moreover, we show that when the underlying field has integer skyrmion number, the corresponding phase singularities behave as quasiparticles.

Beyond the examples discussed here, many other projection maps arise naturally in optics and photonics, and the framework developed in this work applies equally to their analysis. These include direct generalizations of the Hopf projection to fully vectorial $\mathbb{C}^3$ -valued electric fields, together with corresponding constructions for $SU(2)$ -valued matter fields describing optical retarders. More generally, one can also consider projections defined on the space of positive semidefinite Hermitian matrices arising from the coherency matrix in incoherent or partially coherent light, as well as projections on the space of Mueller matrices describing the corresponding depolarising media. These directions are left for future work.

In conclusion, we have developed a general framework for extracting and characterising topologically nontrivial lower-dimensional fields from higher-dimensional physical fields. Through the various examples developed in the paper, we have shown how this perspective unifies seemingly different constructions and makes explicit the mechanisms by which topological information arises and is classified. This framework can thus provide useful principles for the future engineering of robust topological structures in physical fields, with potential applications in topologically robust information storage, transfer, and manipulation.

DATA AVAILABILITY

There are no publicly available research data or software supporting this paper. Requests for further information or data should be sent to the authors.

APPENDIX: MATHEMATICAL FRAMEWORK

1. Extracting topologically nontrivial lower-dimensional subcomponents

In this Appendix we present the theoretical framework underlying our approach. Throughout this work, we consider continuous fields defined over a domain Ω and taking values in a space Y , which represent the physically engineered high-dimensional fields from which lower-dimensional subcomponents are extracted. Let $F : \Omega \rightarrow Y$ be such a field, and let $\mathcal{P} : U \rightarrow Z$ be a continuous map defined on a subset $U \subseteq Y$, where Z is the target manifold. One possible lower-dimensional field is obtained by restricting F to the preimage $F^{-1}(U) \subseteq \Omega$ and composing with $\mathcal{P}$, yielding a map

$$f : F^{-1}(U) \rightarrow Z, \quad f = \mathcal{P} \circ F|_{F^{-1}(U)}.$$

By contrast, the information lost under the map $\mathcal{P}$ can be understood in terms of the pullback construction. Specifically, one can form the fiber product of f and $\mathcal{P}$ given by

$$F^{-1}(U) \times_Z U = \{(x, u) \in F^{-1}(U) \times U \mid f(x) = \mathcal{P}(u)\},$$

and the associated lift $s_F : F^{-1}(U) \rightarrow F^{-1}(U) \times_Z U$, given by

$$s_F(x) = (x, F(x)).$$

Accordingly, our analysis can be viewed from two complementary perspectives: the reduced field f , which captures the information retained under $\mathcal{P}$, and the lift s_F , which encodes the redundancy of the original field description F with respect to $\mathcal{P}$. Characterizing these maps through appropriate topological indices then provides a systematic means of quantifying and classifying the information encoded in the field, and will be the focus of the remainder of this section. Before proceeding to a more detailed analysis, we note that conceptually, the constructions above can be summarized by the diagram below, consisting of two pullback squares.

$$\begin{array}{ccccc} \Omega & \xrightarrow{F} & Y & & \\ \uparrow & & \uparrow & & \\ F^{-1}(U) = \Omega \times_Y U & \xrightarrow{\quad} & U & \xrightarrow{\mathcal{P}} & Z \\ \downarrow s_F & \nearrow f & \downarrow & & \\ F^{-1}(U) \times_Z U & & & & \end{array}$$

The most general topological information carried by the pair $(F, \mathcal{P})$ is given by the homotopy classes of the induced maps. Specifically, the reduced field f and its associated lift s_F define topological indices

$$[f] \in [F^{-1}(U), Z], \quad [s_F] \in [F^{-1}(U), F^{-1}(U) \times_Z U],$$

where homotopies of both f and s_F are taken through admissible fields, as defined by physical constraints or engineered design choices, and, in the case of s_F , through sections of the pullback (f is fixed). However, determining these homotopy classes directly is typically intractable in practice, motivating the use of more computable, though potentially coarser, topological invariants such as those derived from homology or cohomology. For example, in the case of optical skyrmions,

one can restrict to compactifiable fields, and take

$$Y = \mathbb{C}^2, \quad U = Y \setminus \{(0, 0)\}, \\ Z = S^2, \quad \mathcal{P}(z) = \mathcal{P}_{\text{Hopf}}(z/\|z\|),$$

where $\mathcal{P}_{\text{Hopf}}$ denotes the usual Hopf projection mapping a Jones vector to the corresponding Stokes parameters. In this case, the skyrmion number is the associated topological index that can be computed directly via an integral formula, and corresponds to the induced map on second cohomology associated with the compactified polarization field.

As alluded to in the main text, this strategy can also be used to define S^1 -valued subcomponents of an optical field by further projection from the Poincaré sphere, for example, via maps of the form

$$s \mapsto \frac{\langle u, s \rangle + \langle v, s \rangle i}{|\langle u, s \rangle + \langle v, s \rangle i|}, \quad (\text{A1})$$

for fixed vectors $u, v \in S^2$. Keeping with the presentation, and using $\tilde{\cdot}$ to denote the maps and spaces arising from this further projection, we can take

$$\tilde{Y} = S^2, \quad \tilde{U} = S^2 \setminus \{\pm u \times v\}, \quad \tilde{Z} = S^1.$$

Phases defined in this way can then be associated with a topological index given by the induced map

$$\tilde{f}^* \in \text{Hom}(H^1(S^1; \mathbb{Z}), H^1(\tilde{F}^{-1}(\tilde{U}); \mathbb{Z})),$$

which reduces to a computation of winding numbers once a generator of $H^1(S^1; \mathbb{Z})$ is fixed.

Having established the construction for f , we now turn to the situation for s_F , where the definition of topological indices is more subtle due to the restriction that homotopies must be taken through sections of the pullback. To simplify, we restrict attention to the setting in which $U \rightarrow Z$ is a G -principal bundle, so that the pullback $F^{-1}(U) \times_Z U$ is itself a G -principal bundle, denoted f^*U . In this setting the pullback bundle is in fact trivial as a G -principal bundle since s_F is a global section. More specifically, s_F defines a bundle isomorphism $\Phi_F : f^*U \rightarrow F^{-1}(U) \times G$ given by

$$\Phi_F(x, F(x)g) = (x, g),$$

where $\cdot$ denotes the corresponding right G -action on U , and hence determines a gauge-independent homotopy class

$$[\Phi_F] \in [f^*U, F^{-1}(U) \times G],$$

where homotopies are taken through bundle isomorphisms. Note that this perspective, while appearing different from the approach based on sections and lifts presented in the main text, is equivalent, owing to the correspondence between bundle trivializations and global lifts. This correspondence is also used implicitly in the proofs below.

We show now that there exists a correspondence

$$[f^*U, F^{-1}(U) \times G] \cong [F^{-1}(U), G].$$

Let $\Phi : f^*U \rightarrow F^{-1}(U) \times G$ be any choice of bundle isomorphism, corresponding to a choice of gauge. We can associate with any other bundle isomorphism $\Psi : f^*U \rightarrow F^{-1}(U) \times G$ a map $F^{-1}(U) \rightarrow G$, defined by composing the trivial section $s : F^{-1}(U) \rightarrow F^{-1}(U) \times G$ with $\Phi \circ \Psi^{-1}$

and then projecting onto G . Setting $\pi_G : F^{-1}(U) \times G \rightarrow G$ to be the projection onto G , to see that this assignment

$$[\Psi] \mapsto [\pi_G \circ (\Psi \circ \Phi^{-1})^*s]$$

establishes the desired correspondence, note that if the bundle isomorphisms Ψ and Υ define the same homotopy class, and if H_t is a homotopy through bundle isomorphisms with $H_0 = \Psi$ and $H_1 = \Upsilon$, then $\pi_G \circ (H_t \circ \Phi^{-1})^*s$ defines a homotopy between the maps $\pi_G \circ (\Psi \circ \Phi^{-1})^*s$ and $\pi_G \circ (\Upsilon \circ \Phi^{-1})^*s$. Conversely, if $s_0, s_1 : F^{-1}(U) \rightarrow G$ are two homotopic maps, connected by a homotopy h_t , then h_t induces a homotopy through bundle isomorphisms given by

$$\Phi(\Psi_t^{-1}(x, g)) = (x, h_t(x)g)$$

such that $(\Psi_0 \circ \Phi^{-1})^*s = (\text{id}, s_0)$ and $(\Psi_1 \circ \Phi^{-1})^*s = (\text{id}, s_1)$.

In the case where $G = U(1)$, we further have the standard identification

$$[F^{-1}(U), U(1)] \cong H^1(F^{-1}(U); \mathbb{Z}),$$

which completes the correspondence

$$[f^*U, F^{-1}(U) \times G] \cong H^1(F^{-1}(U); \mathbb{Z}),$$

$$[\Psi] \mapsto (\pi_G \circ (\Psi \circ \Phi^{-1})^*s)^*\eta,$$

where $\eta \in H^1(S^1; \mathbb{Z})$ is the canonical generator. This once again reduces to a computation of winding numbers.

2. Tangle and induced phase singularities in optical skyrmions

In this section we discuss induced phases of the form given in Eq. (A1) and their relation to optical skyrmions. Here we examine maps where u and v are orthogonal, with particular focus on the three maps $\mathcal{P}_i : S^2 \setminus \{\pm e_i\} \rightarrow S^1$, $i = 1, 2, 3$, given by

$$\mathcal{P}_1(s) = \frac{\langle e_2, s \rangle + \langle e_3, s \rangle i}{|\langle e_2, s \rangle + \langle e_3, s \rangle i|}, \\ \mathcal{P}_2(s) = \frac{\langle e_3, s \rangle + \langle e_1, s \rangle i}{|\langle e_3, s \rangle + \langle e_1, s \rangle i|}, \\ \mathcal{P}_3(s) = \frac{\langle e_1, s \rangle + \langle e_2, s \rangle i}{|\langle e_1, s \rangle + \langle e_2, s \rangle i|},$$

where e_1, e_2, e_3 denote the standard basis for $\mathbb{R}^3$.

We begin by introducing notation and discussing how the framework introduced in the Appendix 1 applies to the projection maps $\mathcal{P}_i$. In what follows, denote by $\mathcal{S} : \Omega \rightarrow S^2$ the polarization field defined on the closed unit disk $\Omega = \overline{B_1(0)}$. We now set the s_i -induced phase to be

$$\phi_i := (\mathcal{P}_i \circ \mathcal{S})|_{\Omega_i},$$

where

$$\Omega_i := S^{-1}(S^2 \setminus \{\pm e_i\})$$

is the domain obtained by removing the singular points for which $\mathcal{S}(z) = \pm e_i$. Then, as derived above, each ϕ_i has a topological index given by the induced map on first cohomology

$$\phi_i^* \in \text{Hom}(H^1(S^1; \mathbb{Z}), H^1(\Omega_i; \mathbb{Z})).$$

Setting $\eta \in H^1(S^1; \mathbb{Z})$ to be the canonical generator, we have the identification

$$\text{Hom}(H^1(S^1; \mathbb{Z}), H^1(\Omega_i; \mathbb{Z})) \cong H^1(\Omega_i; \mathbb{Z}) \quad \phi_i^* \mapsto \phi_i^* \eta,$$

where the latter can be computed as a collection of winding numbers associated with the singularities $\mathcal{S}(z) = \pm e_i$. More specifically, for any curve γ representing a class $[\gamma] \in H_1(\Omega_i; \mathbb{Z})$, the pairing

$$\phi_i^* \eta([\gamma])$$

is equal to the winding number of ϕ_i along γ .

Before a more in-depth discussion of the cohomology class $\phi_i^* \eta$, note that the complementary component associated with $\mathcal{P}_i$ is trivial, since the projection induces a product decomposition

$$S^2 \setminus \{\pm e_i\} \cong S^1 \times (-1, 1).$$

As in the case of Jones vectors discussed in the main text, this decomposition implies that the pullback bundle is trivial, admitting an isomorphism

$$\Omega_i \times_{S^1} S^2 \cong \Omega_i \times (-1, 1).$$

Accordingly, sections can be identified with maps $\Omega_i \rightarrow (-1, 1)$. Since $(-1, 1)$ is contractible, it follows that all sections are homotopic, and hence the only nontrivial topological content associated with $\mathcal{P}_i$ is captured by the cohomology class $\phi_i^* \eta$.

We now turn to the central observation of this section, namely that when the field $\mathcal{S}$ is a skyrmion, the singularities $\mathcal{S}(z) = \pm e_i$ and the fields ϕ_i exhibit particle-like behavior. In the derivation below, we focus primarily on ϕ_3 , noting that an analogous argument applies to ϕ_1 and ϕ_2 .

Our starting point is the notion of the skyrmion potential, introduced independently in [1,31], which we develop further in this section. Recall that for an S^2 -compactifiable field $\mathcal{S}$, the skyrmion number of the field is the integer-valued quantity defined by

$$\deg \mathcal{S} = \int_{\Omega} \mathcal{S}^* \omega_0,$$

where $\omega_0 \in \Lambda^2(S^2)$ is the standard normalized volume form on S^2 . Expressed in spherical coordinates,

$$\omega_0 = \frac{1}{4\pi} \sin \theta \, d\theta \wedge d\phi,$$

and one easily verifies that on $S^2 \setminus \{\pm e_3\}$, the following 1-form $\nu = -\frac{1}{4\pi} \cos \theta \, d\phi$ satisfies $\omega_0 = d\nu$.

Therefore,

$$\int_{\Omega_3} \mathcal{S}^* \omega_0 = \int_{\Omega_3} \mathcal{S}^* d\nu = \int_{\Omega_3} d\mathcal{S}^* \nu.$$

In particular, by Stokes' theorem, for any smooth manifold M with boundary embedded in Ω_3 , one has

$$\int_M \mathcal{S}^* \omega_0 = \int_{\partial M} \mathcal{S}^* \nu.$$

As is typical in physics derivations, we now compute $\deg \mathcal{S}$ via a limiting process, by considering loops of progressively smaller radius around each singularity. More formally, let

$\{M_n\}$ be a sequence of manifolds with boundary contained in Ω_3 and converging to Ω . Then

$$\deg \mathcal{S} = \lim_{n \rightarrow \infty} \int_{M_n} \mathcal{S}^* \omega_0 = \lim_{n \rightarrow \infty} \int_{\partial M_n} \mathcal{S}^* \nu.$$

To incorporate ϕ_3 , we note that a direct computation yields

$$\mathcal{S}^* \nu = -\frac{1}{4\pi} s_3 \, d(\arg \phi_3).$$

Now, let $\{z_1^+, \dots, z_k^+\}$ and $\{z_1^-, \dots, z_l^-\}$ denote the points within Ω at which $\mathcal{S}(z) = +e_3$ and $\mathcal{S}(z) = -e_3$, respectively, and let $\{\gamma_1^+, \dots, \gamma_k^+\}$ and $\{\gamma_1^-, \dots, \gamma_l^-\}$ denote positively oriented loops around each of these points. Lastly, let $\Gamma = \partial\Omega$. One then has

$$\begin{aligned} \deg \mathcal{S} = & -(\mathbf{1}_{\{1\}}(s_3|_{\Gamma}) - \mathbf{1}_{\{-1\}}(s_3|_{\Gamma})) \frac{1}{2} \phi_3^* \eta([\Gamma]) \\ & + \sum_{i=1}^k \frac{1}{2} \phi_3^* \eta([\gamma_i^+]) - \sum_{i=1}^l \frac{1}{2} \phi_3^* \eta([\gamma_i^-]). \end{aligned}$$

Therefore, this gives the following particle-like interpretation of singularities:

(1) If the boundary is right-circularly polarized, it carries charge $-\phi_3^* \eta([\Gamma])/2$, whereas if it is left-circularly polarized, it carries charge $+\phi_3^* \eta([\Gamma])/2$.

(2) Singularities with $\mathcal{S}(z) = e_3$ carry charge $\phi_3^* \eta([\gamma^+])/2$, where γ^+ is any positively oriented loop enclosing only that singularity. Equivalently, each such singularity carries charge equal to one-half the winding number of ϕ_3 about that point.

(3) Singularities with $\mathcal{S}(z) = -e_3$ carry charge $-\phi_3^* \eta([\gamma^-])/2$, where γ^- is any positively oriented loop enclosing only that singularity. Equivalently, each such singularity carries charge equal to minus one-half the winding number of ϕ_3 about that point.

To further develop the analogy between singularities and quasiparticles, we consider their evolution under a homotopy of skyrmions. In this case, singularities with nonzero charge cannot spontaneously disappear, as this would amount to contracting the homotopically nontrivial map $\phi_3|_{\gamma}$ to a point, where γ is a curve encircling the singularity. On the other hand, if the singularity is removable (that is, if ϕ_3 extends continuously to the singular point), then it necessarily has zero charge.

Note that while the total particle number can vary, the total charge is conserved. Indeed, Fig. 3 shows the evolution of a standard order-2 Néel-type skyrmion passing through a horizontal linear retarder with retardance 2π , in which s_2 - and s_3 -particle division and recombination can be observed. Before delving into the figure in greater detail, we note that these observations imply that the worldlines of such phase singularities must originate and terminate at creation and recombination events, and therefore define an embedded graph in $\Omega \times [0, 1]$. Since the homotopy is taken through skyrmions, we can further identify $\partial\Omega$ so that the embedded graph lies in $S^2 \times [0, 1]$, and hence defines an element of a variant of the category **Tangle** on S^2 .

There are, however, a few technical differences which make the worldlines of phase singularities more general than **Tangle**. The first is that three or more particle interactions can

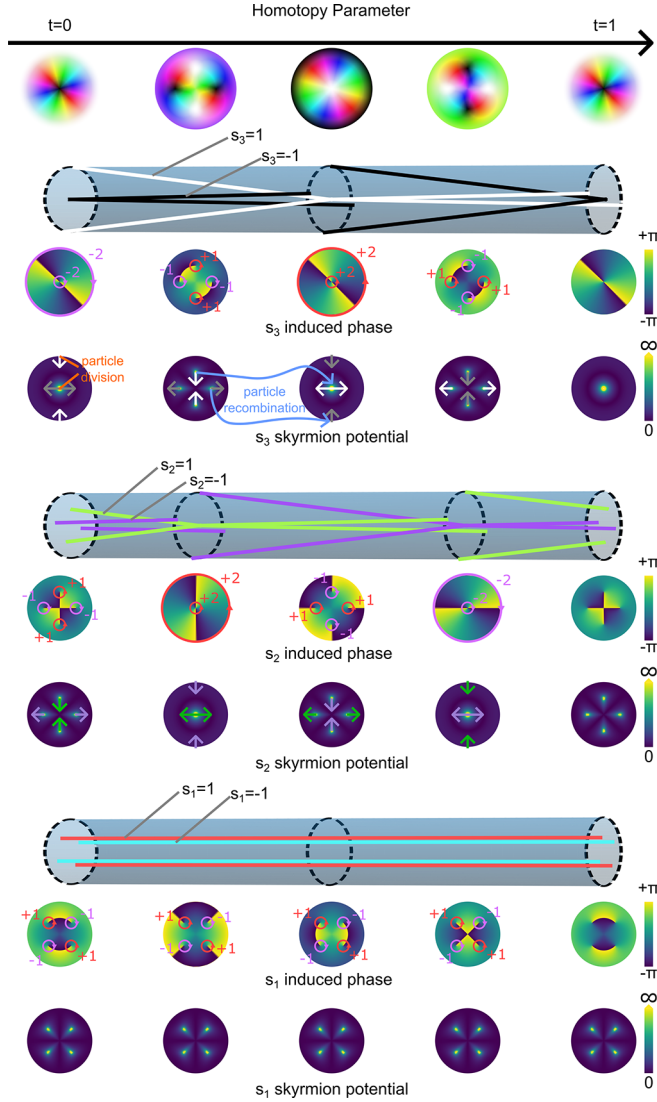


FIG. 3. Singularities of induced phases. The fields ϕ_1 , ϕ_2 , and ϕ_3 , together with the corresponding skyrmion potentials, along a homotopy (with parameter t) of a standard Néel-type skyrmion passing through a horizontal linear retarder with retardance 2π . The winding numbers associated with each singularity are shown alongside the plots of the induced phase, while arrows are overlaid on the skyrmion potential to indicate the direction of motion of each particle. Particle division and recombination events are also shown. Lastly, the world-lines of the different phase singularities and the elements of **Tangle?** defined by each pair of polarization states $\{\pm e_1\}$, $\{\pm e_2\}$, and $\{\pm e_3\}$ are also shown.

occur, and the second is that the expansion of a singularity into a finite area, although nongeneric for smooth functions, is nevertheless possible, so that one should really consider subspaces that homotopically retract onto a path, together with appropriate isotopies of such spaces. Nonetheless,

without going into technical details, we regard the world-lines of such phase singularities as elements of a “category” **Tangle?** and by repeating the argument above for rotations of the skyrmion potential, a homotopy of skyrmions defines a map from any finite collection of polarization states $\{s_1, \dots, s_p\}$ into **Tangle?** obtained by considering the world-lines of the corresponding phase singularities. Given the importance of **Tangle?** in modern topological computing, this perspective suggests a natural connection between the dynamics of phase singularities and diagrammatic models of computation [32,33].

As a concrete illustration of the particle-like nature of the phase singularities obtained through projection, Fig. 3 shows the evolution of the s_1 -, s_2 -, and s_3 -induced phases of a standard second-order Néel-type skyrmion passing through a horizontal linear retarder with retardance 2π . The winding numbers associated with each singularity are shown in the diagram, together with the trajectories of the corresponding particles, including division and recombination events. The skyrmion potential in each case is also shown, demonstrating that phase singularities correspond to divergences of the skyrmion potential. Lastly, the element of **Tangle?** defined by each pair of polarization states $\{\pm e_1\}$, $\{\pm e_2\}$, and $\{\pm e_3\}$ is also shown, from which it is clear that nontrivial braiding of the s_2 - and s_3 -singularities occurs.

3. LG beam conventions

In this section we adopt the following convention for the electric field of Laguerre-Gaussian modes. The electric field for a beam with initial waist radius w_0 and wavelength λ is written as

$$LG_p^l(r, \phi, z) = \sqrt{\frac{2p!}{\pi(p+|l|)!}} \frac{1}{w(z)} \left(\frac{r\sqrt{2}}{w(z)} \right)^{|l|} \times e^{-\frac{r^2}{w(z)^2}} L_p^{|l|} \left(\frac{2r^2}{w(z)^2} \right) e^{-ik\frac{z^2}{2R(z)} - il\phi + i\psi(z)},$$

where $z_R = \pi w_0^2/\lambda$ and

$$w(z) = w_0 \sqrt{1 + \left(\frac{z}{z_R} \right)^2},$$

$$R(z) = z \left(1 + \left(\frac{z_R}{z} \right)^2 \right),$$

$$\psi(z) = (2p + |l| + 1) \arctan \left(\frac{z}{z_R} \right),$$

and $L_p^{|l|}$ is the associated Laguerre polynomial of degree p and order $|l|$.

In all simulations, the normalized values $w_0 = 1.0$ m and $\lambda = 1.0$ m are adopted, and the beam is evaluated at $z = 0$ m.

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