

Interacting Bose gases in twisted-bilayer optical lattices

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Recent experiments have realized ultracold gases in twisted-bilayer optical lattices. We show that interacting bosons in these lattices present a highly nontrivial ground-state physics resulting from the interplay between inter- and intralayer hopping and interactions. This physics is crucially determined by site clusterization, which we properly take into account by developing a specifically tailored cluster Gutzwiller approach. Clusterization results in a large variety of different Mott-like phases characterized by typically different occupations of the clusters, and in the appearance of pockets of sites in between which particles can freely move, but which remain for any practical purposes disconnected from the rest of the sites. This peculiar phase, which resembles the well-known Bose glass phase, may occur even for commensurate twist angles and is further enhanced when the twisting is incommensurate. Moreover, in the incommensurate case, the formation of mobility islands may occur even without interlayer hopping solely due to interlayer interactions.

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I. INTRODUCTION

Recent years have witnessed major breakthroughs in solid-state studies of twisted-bilayer materials and moiré physics [1–3], including the observation of anomalous superconductivity [4–6] and insulating behavior [7–10]. Whereas at magic commensurate twisting angles a periodic superlattice (moiré) modulation develops [1], at incommensurate twistings the system presents quasidisorder, resembling quasicrystals [11].

Recently, ultracold atoms in optical potentials have been revealed as a suitable platform for the quantum simulation of twisted-bilayer physics [12–18]. Twisted-bilayer square lattices using two coupled internal states as effective bilayers in a synthetic dimension [12] have been recently realized by employing atomic Bose-Einstein condensates loaded into spin-dependent optical lattices [19]. Optical lattice experiments, which are characterized by an exquisite experimental control of all relevant parameters (most relevantly the interparticle interactions, hopping amplitudes, and the lattice geometry) open exciting perspectives for the study of the properties of interacting particles in twisted potentials.

Short-range interacting bosons in regular periodic lattices present as a function of the chemical potential and hopping rate—either a superfluid or a Mott insulator phase in the ground state [20]. The presence of disorder or quasidisorder results in the appearance of the Bose glass (BG) [21–23], an insulator which, in contrast to the Mott phase, presents a finite compressibility. Recently, Bose glasses have been discussed

in the context of quasicrystalline structures [24–26], including single-layer twisted potentials, where weak superfluidity was reported for commensurate angles [27]. Very recently, it was shown that bosons in twisted-bilayer optical lattices may present in the absence of interlayer hopping a BG-like phase for commensurate angles, due to the presence of interlayer interactions [28].

In this paper, we investigate the ground-state properties of interacting ultracold bosons in a twisted-bilayer optical lattice. Due to interlayer tunneling, the lattice splits into clusters of two sites, each one belonging to a different layer, which are connected by a significant interlayer hopping. We develop a specific Gutzwiller Ansatz that properly takes into account clusterization. Equipped with this method, we show that the interplay between interactions and the twist-induced lattice clusterization results in a rich ground-state physics, for both commensurate and incommensurate twistings. Mott-like phases are characterized by a nontrivial dependence of the filling of the different clusters with the chemical potential, typically resulting in a density wave modulation. Bose-glass-like phases may occur even for commensurate angles due to the formation of mobility islands. These islands, which are similar to the “quantum wheels” recently predicted in quasicrystals [25], are formed by sites among which particles may efficiently delocalize, but which are very poorly connected to the rest of the lattice. For all practical purposes, optical lattice experiments in this regime will then be characterized by localization of particles within these regions, whether the system is actually in a BG phase or in a strongly inhomogeneous superfluid phase. Our Gutzwiller mean-field type approach is not capable of distinguishing these two cases, and hence employs the term *BG-like*, but for the purposes of finite timescale experiments the distinction is not of significant consequence. On the other hand, for sufficiently large hopping amplitudes,

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the eventual percolation of these islands leads to an unambiguous superfluid phase [24,29], in which particles can efficiently travel macroscopic distances.

The paper is organized as follows. In Sec. II we introduce the model, the cluster Gutzwiller approach employed throughout the paper, as well as the percolation analysis. Section III is devoted to the case of commensurate twistings and the formation of mobility islands. Section IV deals with incommensurate twisting angles, discussing separately the case with and without interlayer hopping. Finally, in Sec. V we summarize our conclusions.

II. MODEL AND METHODS

A. Hamiltonian

In the following, we consider contact-interacting bosons in two layers of square optical lattices, where one layer is twisted by an angle θ with respect to the other. We focus in particular on the case in which this arrangement is engineered in the way proposed in Ref. [12], and recently realized experimentally in Ref. [19], in which two internal levels of the atoms act as two effective layers in a synthetic dimension, whereas a state-dependent potential results in the twisted-bilayer geometry. For square lattices, a periodic moiré pattern is obtained for twist angles $\theta = \theta(m, n) \equiv \arccos(\frac{2mn}{m^2+n^2})$ ($n, m \in \mathbb{Z}$). The Hamiltonian of the system acquires the form

$$\begin{aligned}
 H = & -t \sum_{\alpha} \sum_{\langle j, j' \rangle} a_{\alpha, j}^{\dagger} a_{\alpha, j'} - \sum_{\alpha \neq \bar{\alpha}} \sum_{j, j'} t_{\perp}(j, j') a_{\alpha, j}^{\dagger} a_{\bar{\alpha}, j'} \\
 & + \frac{U}{2} \sum_{\alpha} \sum_j \hat{n}_{\alpha, j} (\hat{n}_{\alpha, j} - 1) - \mu \sum_{\alpha} \sum_j \hat{n}_{\alpha, j} \\
 & + \sum_{\alpha} \sum_{j, j'} V_{\perp}(j, j') \hat{n}_{\alpha, j} \hat{n}_{\bar{\alpha}, j'}, \quad (1)
 \end{aligned}$$

where $a_{\alpha, j}$ ($a_{\alpha, j}^{\dagger}$) is the annihilation (creation) operator at site j of layer $\alpha = \{1, 2\}$, $\hat{n}_{\alpha, j} = \hat{a}_{\alpha, j}^{\dagger} \hat{a}_{\alpha, j}$, U characterizes the on-site repulsive interactions, assumed to be the same in both layers, and μ is the chemical potential. Note that whereas intralayer hopping to nearest neighbors is given by a fixed hopping rate t , tunneling between a site j in one layer and site j' in the opposite one is characterized by the variable rate $t_{\perp}(j, j')$. The repulsive interlayer interactions are given by the couplings $V_{\perp}(j, j')$. In the considered scenario using synthetic dimensions, intralayer hopping results from either a microwave or two-photon optical Raman coupling between the two internal levels, whereas interlayer interactions result from contactlike interactions between the two internal components. For a sufficiently deep optical lattice, the on-site Wannier functions (in each layer) can be approximated as a Gaussian of width $l_0 = \frac{a}{\pi s^{1/4}}$, where a is the lattice spacing characterizing the square lattice in both layers, and s is the lattice depth in recoil units. As a result, the distance-dependent interlayer hopping and interaction acquire, respectively, the form [12] $t_{\perp}(j, j') = t_{\perp} e^{-|R_{1j} - R_{2j'}|^2 / 4l_0^2}$, and $V_{\perp}(j, j') = V_{\perp} e^{-|R_{1j} - R_{2j'}|^2 / 2l_0^2}$. We have chosen $l_0 = 0.15a$ ($s \simeq 20$) throughout the paper. Note that $t_{\perp}$ can be controlled by the strength and detuning of the coupling between internal

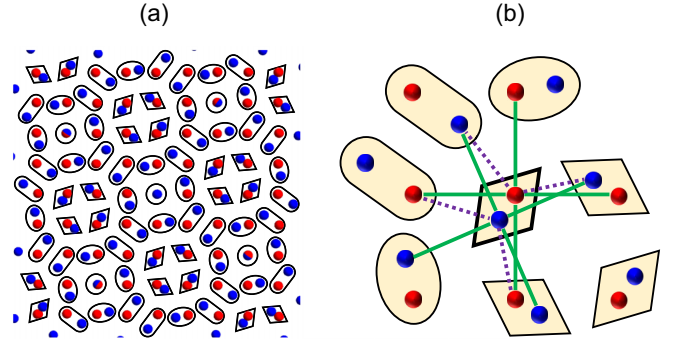


FIG. 1. (a) Twisted-bilayer lattice for a moiré angle $\theta = \theta(3, 2)$. Sites of layer 1 (2) are indicated by filled red (blue) circles. We indicate with different forms the different cluster types: cluster 1 (circle), 2 (rhombus), 3 (oval), and 4 (rounded rectangle). These clusters are characterized by different interlayer couplings (see text). For this lattice there are no unclustered sites. (b) Hopping between neighboring clusters results from intralayer (interlayer) tunneling indicated with solid-green (dashed-purple) lines.

levels, whereas $V_{\perp}$ depends on the contactlike intercomponent interactions.

B. Site clusterization

Due to the twisting between the two lattices, and the Gaussian dependence of the interlayer hopping, sites at different positions present a markedly different coupling to sites in the opposite layer. In particular, pairs of neighboring sites i and j may present a significant interlayer hopping $t_{\perp}(i, j)$, much stronger than the interlayer coupling to any other site. Moreover, for sufficiently low t , $t_{\perp}(i, j)$ may overwhelm the intralayer tunneling as well. When this occurs, in first approximation, those two sites could be then considered as a cluster decoupled from the rest of the lattice, forming an effective two-well potential. As a result, a single particle in those two sites would occupy in the ground state the symmetric superposition $(|i\rangle + |j\rangle)/\sqrt{2}$ with energy $-t_{\perp}(i, j)$. Although this decoupling is generally not exact, and intercluster hopping plays a relevant role [as depicted in Fig. 1(b)], site clusterization turns out to be crucial to understand the ground-state physics of the interacting Bose gas in the twisted-bilayer lattice.

We define a cluster c as formed by two sites c_1 and c_2 , in layers 1 and 2, respectively, such that $t_{\perp}(c_1, c_2) > t_{\perp, cr}$. In the following, we set $t_{\perp, cr} = 0.03$ (0.06) $t_{\perp}$ for commensurate (incommensurate) twist angles. We have checked that lower values of $t_{\perp, cr}$ do not appreciably change the results discussed below. Note that due to the Gaussian dependence of the interlayer tunneling, for all values of $t_{\perp}$ discussed in this paper, clusters are always limited to two sites. In addition to the clustered (cl) states, there may also be sites without a significant coupling to any neighboring site in the opposite ladder. They do not belong to a cluster, and are referred to below as nonclustered (ncl) sites. For the case of a given moiré angle, the clusters can be classified in well-defined types, characterized by a different interlayer coupling. The case of $\theta = \theta(3, 2)$ is shown in Fig. 1(a), where we indicate the different cluster types. As discussed below, coupling

between neighboring clusters via intra- and interlayer tunneling, illustrated in Fig. 1(b), plays a key role in the mobility of particles in the lattice.

Motivated by the previous discussion, we introduce the Gutzwiller-like ansatz $|\Psi\rangle = |\Psi_{\text{cl}}\rangle \otimes |\Psi_{\text{ncl}}\rangle$. Clustered sites are described by $|\Psi_{\text{cl}}\rangle = \bigotimes_{c \in \text{cl}} |\rho_c^{\text{cl}}\rangle$, with $|\rho_c^{\text{cl}}\rangle = \sum_{n_1, n_2} g_c(n_1, n_2) |n_1\rangle_{c_1} |n_2\rangle_{c_2}$, where $g_c(n_1, n_2)$ determines the amplitude of probability of finding n_1 (n_2) particles in site c_1 (c_2) of cluster c . Nonclustered sites are characterized by $|\Psi_{\text{ncl}}\rangle = \bigotimes_{\alpha, j \in \text{ncl}} |\rho_{\alpha, j}\rangle$, with $|\rho_{\alpha, j}\rangle = \sum_n f_{\alpha, j}(n) |n\rangle_{\alpha, j}$, where $f_{\alpha, j}(n)$ is the amplitude of probability of finding n particles in the nonclustered site j of layer α . The coefficients fulfill the normalization conditions: $\sum_n |f_{\alpha, j}(n)|^2 = \sum_{n_1, n_2} |g_c(n_1, n_2)|^2 = 1$.

An uncoupled nonclustered site (α, j) is characterized by an energy

$$\tilde{E}_{(\alpha, j)} = \sum_n \epsilon(n) |f_{\alpha, j}(n)|^2, \quad (2)$$

with $\epsilon(n) = \frac{U}{2}n(n-1) - \mu n$, whereas the energy of an uncoupled cluster c is

$$E_c = \sum_{n_1, n_2} \xi_c(n_1, n_2) |g_c(n_1, n_2)|^2 - t_{\perp, c} \times \sum_{n_1, n_2} (\sqrt{n_1(n_2+1)} g_c^*(n_1, n_2) g_c(n_1-1, n_2+1) + \text{c.c.}), \quad (3)$$

with $t_{\perp, c}$ and $V_{\perp, c}$ characterizing the interlayer couplings in the cluster, and $\xi_c(n_1, n_2) = \sum_{\sigma=1,2} \epsilon(n_\sigma) + V_{\perp, c} n_1 n_2$.

Neglecting interactions between different clusters or unclustered sites (which is extremely small under the conditions discussed below), coupling between clusters is only given by hopping. Intralayer hopping between neighboring clusters c and c' result in the energy $T_{c, c'}^{\text{intra}} = -t \sum_{\alpha} (b_{\alpha, c}^* b_{\alpha, c'} + \text{c.c.})$, if c_α and c'_α are nearest neighbors in layer α , whereas interlayer tunneling results in the contribution $T_{c, c'}^{\text{inter}} = -t_{\perp}(c_\alpha, c'_\alpha) (b_{\alpha, c}^* b_{\alpha, c'} + \text{c.c.})$. Similarly, the hopping between two neighboring ncl sites contributes with $T_{(\alpha, i), (\alpha, j)}^{\text{intra}} = -t b_{\alpha, i}^* b_{\alpha, j} + \text{c.c.}$, and $T_{(1, i), (2, j)}^{\text{inter}} = -t_{\perp}(i, j) b_{1, i}^* b_{2, j} + \text{c.c.}$. Finally, hops between clustered and nonclustered sites result, respectively, in couplings $T_{(\alpha, c), (\alpha, j)}^{\text{intra}} = -t b_{\alpha, c}^* b_{\alpha, j} + \text{c.c.}$, where c_α and j are nearest neighbors, and $T_{(\alpha, c), (\alpha, j)}^{\text{inter}} = -t_{\perp}(c_\alpha, j) (b_{\alpha, c}^* b_{\alpha, j} + \text{c.c.})$. In the previous expressions we have introduced the mean fields $b_{\alpha, c \in \text{cl}} \equiv \langle \hat{a}_{\alpha, c} \rangle$ and $b_{(\alpha, j) \in \text{ncl}} \equiv \langle \hat{a}_{\alpha, j} \rangle$, which are determined from the Gutzwiller coefficients:

$$b_{1, c} = \sum_{n_1, n_2} \sqrt{n_1+1} g_c^*(n_1, n_2) g_c(n_1+1, n_2), \quad (4)$$

$$b_{2, c} = \sum_{n_1, n_2} \sqrt{n_2+1} g_c^*(n_1, n_2) g_c(n_1, n_2+1), \quad (5)$$

$$b_{\alpha, j} = \sum_n \sqrt{n+1} f_{\alpha, j}^*(n) f_{\alpha, j}(n+1). \quad (6)$$

Adding up all the energy terms, we obtain the overall energy E , which we minimize obtaining the coefficients $g_c(n_1, n_2)$ and $f_{\alpha, j}(n)$.

C. Percolation analysis

In Bose-Hubbard models in uniform square lattices, the superfluid (SF) phase is characterized by a nonzero mean field $\langle \hat{a}_j \rangle$ in all sites j , whereas a Mott-like insulator present a zero mean field and an integer $\langle \hat{n}_j \rangle$. In the twisted-bilayer scenario, sites belonging to different clusters (or sites in noncluster states) typically present different fillings and different values of the mean fields $b_{\alpha, s}$. As discussed below, this leads to a rich landscape of possible phases.

Due to the inherent inhomogeneity of the interlayer couplings in twisted-bilayer lattices, some sites are efficiently coupled to their neighbors, and hence present a significant mean field, whereas for others the coupling is exponentially small, and hence present a very small mean field, being in practice decoupled for any reasonable experimental lifetimes in experiments. In order to account for this experimentally relevant fact, we establish a cutoff $|b_{\alpha, s}| > 0.05$ to determine in our analysis whether a site has a significant mean field, and hence it is efficiently coupled to other sites. Note that, slightly different choices of the cutoff could be made, without changing the overall qualitative picture. We associate the parameter $S = 1$ (0) to each site with nonvanishing (vanishing) mean field.

We define a Mott-like phase as having $S = 0$ for all sites, and hence clusters with an integer filling. Note, however, that different clusters can have different integer fillings, hence resulting rather in a density wave. Due to the inhomogeneity of the intersite and intercluster couplings, some sites may acquire a significant mean field ($S = 1$), whereas other sites remain in practice decoupled from the rest of the lattice ($S = 0$). As discussed below, mobility islands may occur, formed by neighboring sites among which particles can efficiently move, but that are decoupled from the rest of the sites (i.e., regions of $S = 1$ sites, fully surrounded by $S = 0$ sites). This may occur even in the case of commensurate moiré angles. Following Refs. [24,29,30], we denote as the superfluid phase that in which $S = 1$ sites percolate, i.e., particles may efficiently move from one side of the system to the other in a finite-size lattice.

We determine the network of sites formed by connecting all the nearest-neighboring (both intra- and interlayer) sites with $S = 1$. We consider two $S = 1$ sites as connected if their coupling is larger than $10^{-3} t_{\perp}$. A slight change of this value does not affect any of our results. We then employ a percolation analysis, based on the Hoshen-Kopelman algorithm [31], to determine whether there is at least one chain of $S = 1$ sites that percolates from one side to the other of our system.

Motivated by the previous discussion, and following Refs. [24,29], we introduce two further parameters: the fraction of $S = 1$ sites $F \equiv N_{\phi}/N$, and the percolation probability $P \equiv N_{\text{span}}/N$, where N_{span} , N_{ϕ} , and N denote, respectively, the number of $S = 1$ sites in a percolating chain, the total number of $S = 1$ sites, and the total number of sites in the two layers. Mott-like phases are characterized by $F = P = 0$, whereas for the SF phase $F, P > 0$. Nonpercolated phases with $\langle S \rangle > 0$ present $P = 0$ but $F > 0$. As previously mentioned, we denote as BG-like phases those with isolated mobility islands, due to their resemblance with the BG phase known in disorder Bose-Hubbard models [21]. Although in an actual BG

phase S would rigourously vanish everywhere, our Gutzwiller mean-field formalism properly takes into consideration the experimentally relevant fact that in the BG-like regime particles would be localized within the mobility islands for any relevant timescales.

III. COMMENSURATE TWIST ANGLE

We consider first the case of a commensurate twist angle $\theta = \theta(m, n)$. We may then simplify our analysis by using periodic boundary conditions, focusing on a single unit cell of the effective moiré lattice. We illustrate the possible ground-state phases of the interacting Bose gas with the case of $\theta = \theta(3, 2)$, since it already presents a complex landscape of clusters, but the different phases are still intuitively easy to visualize. The lattice presents four different cluster types, $c = 1 \dots 4$ (see Fig. 1). Clusters 1 (circles) are formed by two sites on top of each other, and hence $t_{\perp, c=1}/t_{\perp} = 1$. The rest of the clusters present a lower interlayer coupling: cluster 2 (rhombuses) present $t_{\perp, 2}/t_{\perp} \simeq 0.425$, cluster 3 (ovals) $t_{\perp, 3}/t_{\perp} \simeq 0.181$, and cluster 4 (rounded rectangles) $t_{\perp, 4}/t_{\perp} \simeq 0.033$.

Figure 2(a) shows the phase diagram for $t_{\perp} = U$, and $V_{\perp} = 4U$ (other choices lead to qualitatively similar phase diagrams). The Mott-like phases (indicated in blue) are characterized by an integer, generally different, population N_c of the different cluster types. We denote these phases in the form (N_1, N_2, N_3, N_4) .

At low chemical potential, the lattice is first filled with one particle in each cluster 1, due to the strong interlayer hopping rate in these clusters. Increasing the chemical potential results in the filling of other clusters, starting with cluster 2. Note, however, that the transition between the $(1,0,0,0)$ and $(1,1,0,0)$ phases occurs at low t/U via an intermediate phase, illustrated in Fig. 2(b), in which the dc sites have unit filling, but clusters 2 form ringlike mobility islands, which resemble the quantum wheels recently reported in the context of quasicrystal potentials [25]. These islands remain for any practical purposes isolated, without percolating through the lattice. As a result the system is in a BG-like phase (indicated in yellow), with $P = 0$ but $F > 0$.

At larger t/U the coupling with clusters 3 eventually results in a percolating network of clusters 2 and 3, and hence the system enters the SF phase (red). Larger values of the chemical potential result in a nontrivial series of Mott-like phases, in which the filling is not necessarily sequential, e.g., $(1,1,1,0)$ is followed by $(1,1,1,2)$, with an appreciable gap in between. Concerning that gap, note that even if $t = 0$, mobility can be established exclusively via interlayer hops, explaining the absence of Mott phases at $t/U = 0$ when $(\mu + t_{\perp})/U \simeq 1$.

IV. INCOMMENSURATE TWIST ANGLE

We consider next the case of incommensurate twist angles. Since the lattice lacks periodicity, we cannot employ periodic boundary conditions and must hence work with finite-size systems. Interlayer clusters of two neighboring sites may still be formed. Sites that experience an interlayer coupling $t_{\perp, c} > 0.06t_{\perp}$ are considered as belonging to the same cluster.

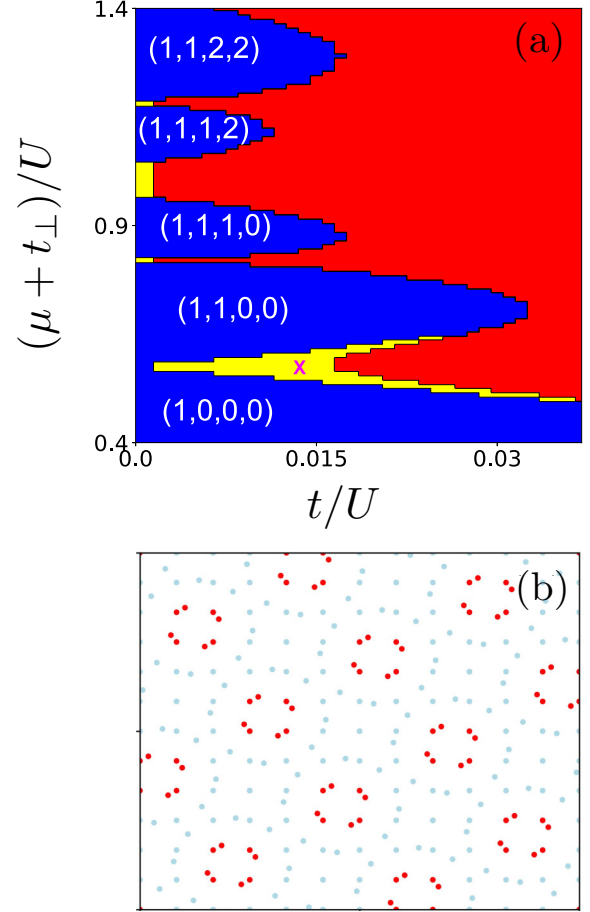


FIG. 2. (a) Phase diagram for $\theta = \theta(3, 2)$, $l_0 = 0.15a$, $t_{\perp} = U$, and $V_{\perp} = 4U$. Insulating Mott-like phases are indicated in blue, SF phases in red, and BG-like phases in yellow. The Mott-like phases are labeled in the form (N_1, N_2, N_3, N_4) , where N_c is the population of cluster c (see Fig. 1). (b) Distribution of sites with $S = 1$ (red) and $S = 0$ (light blue) characteristic of the BG-like phase in between $(1,0,0,0)$ and $(1,1,0,0)$ [for the case indicated with a cross in (a)]. Note the appearance of ringlike mobility islands.

This cutoff value prevents in the cases discussed below the existence of clusters of more than two sites, without compromising the generality of the results. Sites that experience an interlayer coupling below that cutoff are considered non-clustered sites. Our numerical simulation is performed on a 40×40 twisted-bilayer optical lattice.

In the following, we illustrate the possible ground-state phases for the case of incommensurate twist angles, using the relatively simple case of $\theta = \theta(2, 1) + 3^\circ$ (see Fig. 3). By construction, there is one site in the middle of each layer forming a single cluster with coupling $t_{\perp}$. In contrast to the commensurate case, in the incommensurate case, due to the aperiodicity of the lattice, the other two-site clusters present generally different interlayer couplings. In order to simplify the analysis, we classify the clusters into four families, with $t_{\perp, c=1\dots 4}/t_{\perp} \in [0.75, 1)$, $[0.5, 0.75)$, $[0.2, 0.5)$, and $[0.06, 0.2)$, respectively. We then assign to the clusters in each family the average value within the family, i.e., $t_{\perp, c=1\dots 4}/t_{\perp} = 0.875$, 0.625 , 0.35 , and 0.013 . Slightly different choices of these

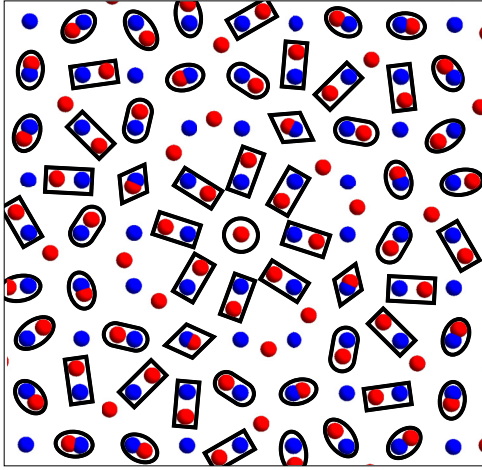


FIG. 3. Spatial distribution of sites in layer 1 (blue) and layer 2 (red) for a lattice with an angle $\theta(2, 1) + 3^\circ$. For simplicity, in addition to the two central overlapping sites (circle), we classify the rest of the clusters in four families: 1 (rhombuses), 2 (ovals), 3 (rounded rectangles), and 4 (rectangles). Note also the presence of nonclustered sites.

cluster families, or even considering a larger number of families, do not change the qualitative picture discussed below.

We consider first the case $V_\perp = 0$ and $t_\perp = U$ (see Fig. 4). The ground-state phase diagram in this case is characterized by a broad Mott insulator regime with one particle in all sites, separated from the SF regime by a BG-like region. For larger chemical potentials, we observe the appearance of a wide BG-like regime

at low t/U , which almost completely wipes out any other Mott-like insulating lobes. As for the previous section, that regime is characterized at low t/U by the formation of small isolated mobility islands, formed by sites in clusters $c = 4$ and unclustered sites [see Fig. 4(c)]. Note that even at $t = 0$, there may be pockets of sites in which particles can move. This is because motion between sites may be established solely by interlayer hops. When t/U increases, inside the BG-like phase, larger (but still disconnected) regions with $S = 1$ form [see Fig. 4(d)]. For an even larger t/U the system eventually

enters the SF phase with percolated $S = 1$ sites (a more clear example of this is discussed below).

We discuss at this point a different scenario, in which the layers are not coherently coupled, i.e., there is no interlayer hopping ($t_\perp = 0$). We consider, however, $V_\perp \neq 0$. Note that for commensurate twist angles, unless they are very small, interlayer interactions, which decay much faster than the interlayer hopping with distance, are typically very weak, except for the case of sites placed on top of each other. As a result, the layers behave as almost disconnected in the absence of interlayer hopping, except for the sites that are on top of each other, in which particles may become disconnected from the rest of the lattice if $V_\perp \gg t$. Excluding these sites, however, does not result in disconnected regions in each layer, and hence it does not affect the overall mobility in the lattice. As a result, the BG-like phase is not observed. We note that for very small twist angles, interactions may outproject more sites in the lattice at each layer, and as a result a BG-like phase may occur, as recently reported in Ref. [28]. In contrast, even for large twisting angles, incommensurate twist angles result in sites that may eventually be close enough to experience significant interlayer interactions, and as result, for aperiodic lattices interlayer interactions do significantly affect the phase diagram, even for $t_\perp = 0$, resulting in a rich ground-state physics. We illustrate this physics again for the case of $\theta(2, 1) + 3^\circ$, with $t_\perp = 0$ and $V_\perp/U = 1$ [see Fig. 5(a)].

In the absence of interlayer coupling, we may carry on with the idea of clusterization but classifying the clusters (still formed by only two sites) according to the interlayer interaction strength between their sites. Since pairs of neighboring sites in each layer present very different interlayer interactions across the lattice, we split the sites, according to their interlayer interactions, into five cluster families with $V_{\perp,c=1\dots 5}/V_\perp \in 1, [0.75, 1), [0.5, 0.75), [0.2, 0.5),$ and $[0.06, 0.2)$. As for the case with $t_\perp > 0$, we characterized the phases with the parameters F and P , discussed above. For the case considered we observe several Mott-like insulating phases, which we characterized by the number of particles per site in the different clusters. The three largest Mott-like regions are characterized, from bottom to top, by populations $(1, 1, 1, 2, 2)$, $(1, 1, 2, 2, 2)$, and $(1, 2, 2, 2, 2)$. Note that the clusters with lower interlayer interactions are more populated, as

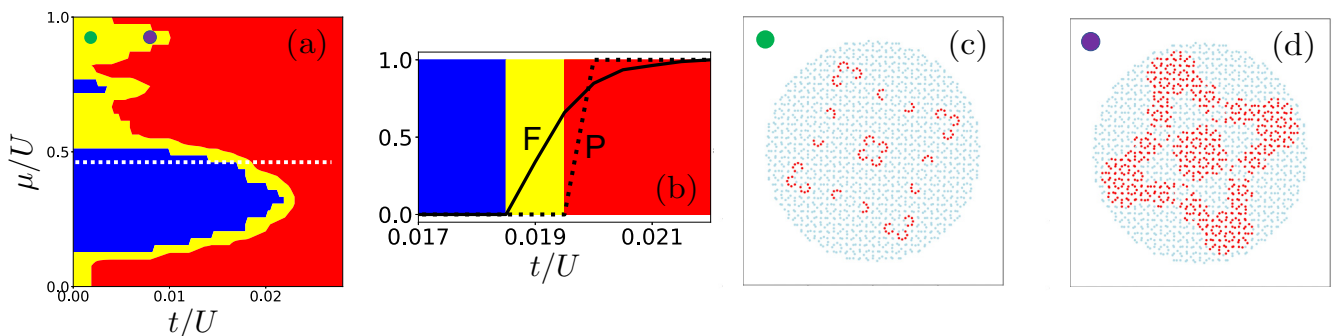


FIG. 4. (a) Phase diagram for $\theta = \theta(2, 1) + 3^\circ$, $t_\perp/U = 1$, and $V_\perp = 0$. Insulating Mott-like phases are indicated in blue, BG-like phases in yellow, and superfluid phases in red. (b) the F (solid line) and P (dashed line) parameters (see text) as a function of t/U for $\mu/U = 0.45$. These parameters are employed to characterize the phases. (c),(d) distribution of sites with $S = 1$ (red) and $S = 0$ (light blue) for the cases indicated in (a). We choose 40×40 sites in each layer.

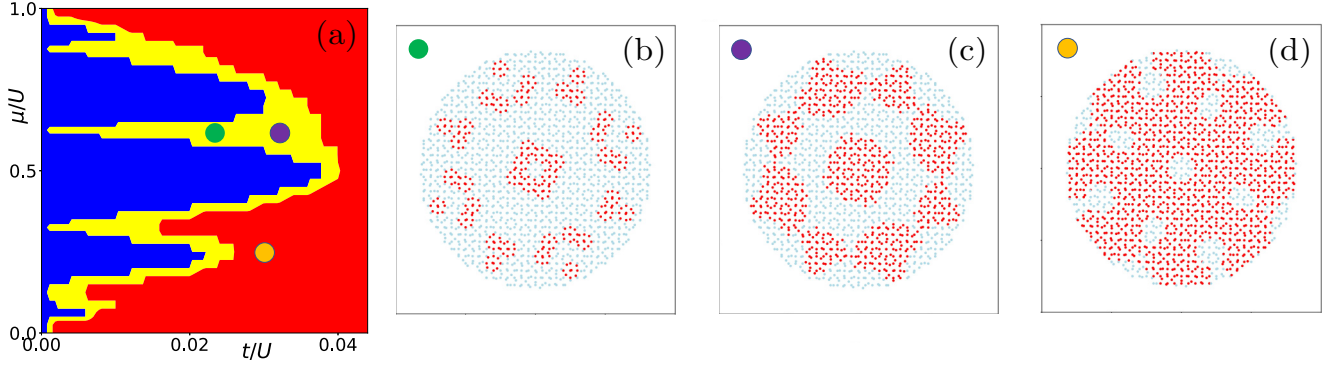


FIG. 5. (a) Phase diagram for $\theta = \theta(2, 1) + 3^\circ$, $V_\perp/U = 1$, and $t_\perp = 0$. Insulating Mott-like phases are indicated in blue, BG-like phases in yellow, and superfluid phases in red. (b)–(d) Distribution of the sites with $S = 1$ (red) and $S = 0$ (blue), for the cases indicated in (a) by the colored circles.

expected due to the repulsive character of the interactions. The BG-like regime completely surrounds the Mott-like regions, separating them from the SF regime. Note that mobility results exclusively from intralayer hops, and hence in contrast to the case of finite $t_\perp$, at $t = 0$ no mobility is possible and the phase diagram is dominated by the Mott-like regions. Figures 5(b) and 5(c) show the spatial distribution of $S = 1$ sites within the BG-like regime. For growing t/U the mobility islands become larger, although still disconnected. In the SF regime, the regions with $S = 1$ percolate, but islands of insulating sites (which do not participate in particle transport) remain in the system [see Fig. 5(d)]. The latter case is characterized by $P = 1$ (i.e., percolation), but $F < 1$, i.e., not all sites have $S = 1$.

V. CONCLUSIONS

Cold atoms in twisted-bilayer optical potentials constitute an excellent platform for the study of the interplay of interactions and the twisted geometry, since they present vastly tunable parameters, a feature typically not shared by the solid-state implementation of similar setups. Our results illustrate the rich and highly nontrivial ground-state physics introduced by interactions in twisted-bilayer-like optical lattices, which results from the interplay between inter- and intralayer hopping and interactions. Crucially, interlayer hopping induces the formation of two-site clusters, which we have properly taken into account by means of a specifically tailored cluster Gutzwiller approach. For commensurate twist angles

(periodic lattices) regular cluster families with different interlayer couplings are formed. Cluster formation results in a rich variety of Mott-like phases. Moreover, even for periodic lattices, Bose-glass-like phases occur in which particles can move in between specific sites forming isolated pockets, without establishing an overall superfluidity. The BG-like regime is significantly more relevant in the case of incommensurate twistings (aperiodic lattice). Interestingly, for that case (but not for the commensurate one, at least for large angles), highly nontrivial BG-like and superfluid phases may result exclusively from interlayer interactions, even in the absence of interlayer hopping. Although we have focused on ground-state properties, a highly nontrivial dynamics is also expected in these setups, which may be studied as well using the introduced cluster Gutzwiller formalism.

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DATA AVAILABILITY

The data that support the findings of this article are not publicly available. The data are available from the authors upon reasonable request.

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