

Interference in phase space and phase-only reconstruction

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We demonstrate that the Fourier transform of a function leads to interference effects in phase space which allow the reconstruction of the essential properties of the original function, even when we restrict the inverse Fourier transform to the phase factor rather than the complete Fourier transform. We analyze this phase-only reconstruction, known from signal processing, in quantum mechanics for the elementary example of a superposition of two real-valued Gaussians. We employ the Wigner function as well as techniques of semiclassical quantum mechanics to show that their key features, such as their separation and their width, are indeed encoded in the phase of the Fourier transform. For this example, the amplitude-only reconstruction performing the inverse Fourier transform of the absolute value of the Fourier transform fails.

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I. INTRODUCTION

A complex-valued signal is uniquely defined by an amplitude and a phase, both of which may seem to be equally important. However, for many signals and images the *phase* of the Fourier transform contains [1–4] most of the critical information.

Since in quantum mechanics [5] the wave functions in position and momentum space are related by a Fourier transform, we are led to the question: What can we learn from the phase of the momentum wave function?

In this article, we answer this question for the example of two real-valued Gaussian wave functions, and employ the Wigner function [6–8] to identify interference in phase space [8,9] as the origin of this phenomenon.

A. Phase-only and amplitude-only reconstructions

The central idea of Refs. [1–4] is summarized by a photograph such as the one shown in Fig. 1(a) depicting the Ulm Minster. When we translate [10] the different values of brightness into a two-dimensional density $f = f(x, y)$ and perform a two-dimensional Fourier transform $\mathcal{F}$ of f , we obtain a complex-valued function $\tilde{f} \equiv \mathcal{F}[f]$. Obviously, the inverse transform $\mathcal{F}^{-1}$ leads us back to f , since $\mathcal{F}^{-1}[\mathcal{F}[f]] = f$, as demonstrated in Fig. 1(b).

However, when we retain from $\tilde{f}$ only its phase φ and perform the inverse Fourier transform of the phase factor $e^{i\varphi} \equiv \tilde{f}/|\tilde{f}|$, we arrive at $f_p \equiv \mathcal{F}^{-1}[e^{i\varphi}]$, which in general is not identical to f . Nevertheless, as shown in Fig. 1(c) the characteristic features of f survive in the phase-only reconstruction f_p .

It is instructive to compare and contrast the *phase-only* reconstruction f_p with the *amplitude-only* reconstruction f_a . Here, we retain not the phase but the amplitude $|\tilde{f}|$ of the Fourier transform $\tilde{f}$ leading us to $f_a \equiv \mathcal{F}^{-1}[|\tilde{f}|]$.

In Fig. 1(d) we depict for the example of the Ulm Minster the amplitude-only reconstructed image and emphasize that the characteristic features which are still visible in the phase-only reconstruction have disappeared in noise. We point out that in the signal processing literature, the amplitude-only reconstruction leads to black pictures [1]. This sequence of images clearly demonstrates that the information about an image containing a lot of structure is encoded in the phase rather than the amplitude, as emphasized in Refs. [1–4].

In Fig. 2 we display in the center the familiar relation between the Fourier transform and its inverse, whereas on the bottom and the top, we depict the phase-only and the amplitude-only reconstruction, respectively. The connection between f and f_p or f and f_a defines the central question of this paper. We show that the deeper origin of the great similarity between f and f_p , highlighted in Fig. 1, is interference in phase space [9].

In order to bring out this fact most clearly, it suffices to recall that the Fourier transform $\tilde{f}$ is a continuous superposition of plane waves expressed by an integral over the coordinates. It is this superposition, which provides the interference and makes the phase-only reconstruction successful.

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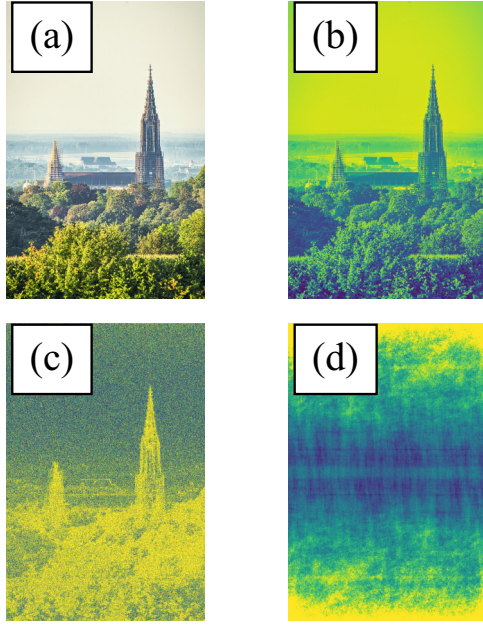


FIG. 1. Phase-only and amplitude-only reconstructions [1–4] illustrated by an image depicting the Ulm Minster (taken by Stefan Rau [11]). Whereas (a) shows the original picture, (b) is the result of first applying a fast Fourier transform (FFT) [12] and then the inverse (IFFT), reproducing the original image. The reconstruction in (c) is based on the IFFT of the phase factor of the FFT multiplied by the average amplitude; this image was brightened by a factor for better visibility. The phase-only reconstruction (c) still clearly shows the Ulm Minster. In comparison to (a) and (b), (c) has a higher contrast. The edges are highlighted, while the remaining information is lost. In the amplitude-only reconstruction of (d) the Ulm Minster has disappeared in noise.

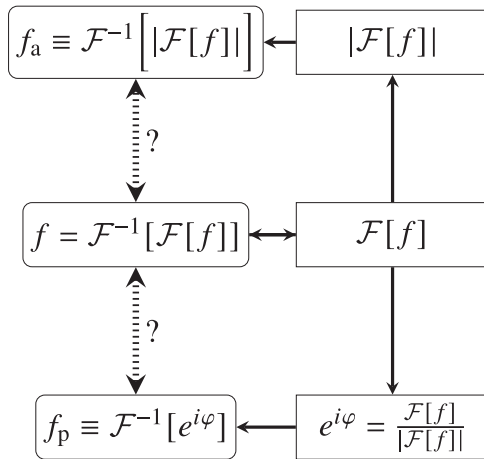


FIG. 2. Flowchart for obtaining the phase-only and amplitude-only reconstructions [1–4], consisting of two paths: For the phase-only reconstruction f_p (path going down from the central to the lower floor) we start from the function f and compute (central floor) the Fourier transform $\mathcal{F}[f]$, find its phase factor $e^{i\varphi}$ by dividing by the absolute value $|\mathcal{F}[f]|$, and obtain the inverse Fourier transform $f_p \equiv \mathcal{F}^{-1}[e^{i\varphi}]$. For the amplitude-only reconstruction f_a (path going up from the central to the top floor) we first calculate $|\mathcal{F}[f]|$ and then derive its inverse Fourier transform $f_a \equiv \mathcal{F}^{-1}[|\mathcal{F}[f]|]$. What is the connection between f and f_p , or f and f_a ?

B. Probability amplitudes in position and wave vector space

The Fourier transform also plays a prominent role in quantum mechanics [5], where it connects two different representations of a quantum state. However, in contrast to the problem of pattern recognition [1], which deals with the Fourier transform and the inverse Fourier transform of *positive-valued distributions*, in quantum mechanics we face *complex-valued probability amplitudes*.

For example, the probability amplitude $\psi(\mathbf{r})$ to find a particle at the coordinate $\mathbf{r}$ is related to the probability amplitude $\tilde{\psi}(\mathbf{k})$ to find the particle with a given wave vector $\mathbf{k}$ by a Fourier transform of the corresponding wave functions. Since the Fourier integral is a continuous superposition of waves, the wave function in wave number space contains interference. Throughout the article, we consider position $\mathbf{r}$ and wave vector $\mathbf{k}$ as conjugate variables, in contrast to standard treatments, relying on position $\mathbf{r}$ and momentum $\mathbf{p} \equiv \hbar\mathbf{k}$. Here, $\hbar$ denotes the reduced Planck constant.

Hence, common to the phase-only and amplitude-only reconstruction of an image based on a positive-valued distribution and of a probability amplitude in quantum mechanics is the Fourier transform which contains the interference information. In our article we illustrate the inner workings of both reconstructions from the point of view of quantum mechanics.

C. Partial quantum information

We emphasize that the problem of reconstruction from *partial* information is closely related to the question of quantum state reconstruction [13]. Here, the ultimate goal is to obtain from measured positive-valued distributions the underlying complex-valued wave function, that is, to find from *probabilities* the probability *amplitude*. It suffices to say that many algorithms for state reconstruction exist and have been realized experimentally [14–16]. However, a prominent example of such an algorithm is the Radon transform [17], which provides us with the help of infinitely many tomographic cuts representing probabilities, the underlying state.

When we want to avoid making infinitely many measurements we might wonder if the distributions in coordinate and wave number suffice to reconstruct the state. Already in the early days of quantum mechanics, Wolfgang Pauli [18] addressed this question in a footnote in his handbook article and referred to it as an unsolved problem. However, according to Hans Reichenbach [19], it was Valentine Bargmann who showed that there is no unique solution.

Hence, it is unlikely that the inverse Fourier transform of the square root of the wave number distribution, that is, the amplitude-only reconstruction, could be successful. From this point of view, it is even more surprising that the phase-only reconstruction does achieve this very goal.

We emphasize that the phase-only reconstruction is only one aspect of processing partial quantum information. Three examples suffice to illustrate the breadth and depth of this field.

Partial phase knowledge allows state-dependent cloners to surpass universal fidelity bounds [20]. Reusing projective measurement outcomes via feedback restores [21] temporal memory and enable time-series prediction despite destructive measurements. Quantum reservoir computing and its quantum

optical implementation [22] employs partial information to enhance learning and prediction.

These seemingly unrelated examples share a common feature with the problem addressed in the present paper: partial quantum information encoded either in the Fourier phases or the Wigner interference fringes allow us to distill quantum information stored in the system.

D. Overview

Our article is organized as follows: In Sec. II we introduce our model state consisting of the position representation of two real-valued Gaussians located at two different positions. We find the corresponding wave function in wave number space, and the Wigner function defined in quantum phase space. Here, we emphasize especially the domain of phase space, where the Wigner function assumes negative values. Moreover, we rederive the probability amplitude in wave number space from Wentzel-Kramers-Brillouin (WKB)-phase space using the concept of area-of-overlap [8,9].

We dedicate Sec. III to a discussion of the phase-only reconstruction and start our analysis with the example of two Gaussians of identical widths. In this case, we obtain an exact result for the phase-only reconstruction. We then concentrate exclusively on two Gaussians of different widths and first perform numerically the inverse Fourier transform of the phase factor. Moreover, we present an approximate but analytical approach toward the phase-only reconstruction utilizing the familiar generating function [23] of Legendre polynomials.

In Sec. IV we generalize our results to a superposition of arbitrary functions located at different positions. We show that the reproduction of the original positions is a consequence of the shift theorem of the Fourier transform and the fact that the division in the phase-only reconstruction by $|\hat{\psi}|$ modifies only the amplitude but not the phase factors. We discuss this modification of the amplitudes and obtain the resulting distribution in position space.

Finally we briefly address in Sec. V the amplitude-only reconstruction for our model state. In particular, we identify the origin of the failure to reproduce the central features of the original distribution.

We conclude in Sec. VI by comparing and contrasting the storage of key features in the phase of the Fourier transform with the encoding of phase information in the negative parts of the Wigner function [6–8]. We also provide a brief outlook.

In order to keep our article self-contained, yet concise, we have moved detailed calculations to three Appendixes. In Appendix A we summarize key properties of the Wigner function. Throughout our article, we have relied in our analysis of the phase-only reconstruction on obtaining the *phase factor* by dividing the wave number amplitude by its absolute value. We devote Appendix B to an approach where we first calculate the *phase angle* of the wave number amplitude using a technique familiar from complex analysis. We then perform the inverse Fourier integral using an approximate but analytical expression for the so-obtained phase and arrive at a formula for the phase-only reconstruction in terms of a Dirac delta function and two sinc functions.

In Appendix C we calculate the integral determining the phase-only reconstruction of a superposition of two *identical*

Gaussians located symmetrically with respect to the origin of the coordinate axis. We express this integral in terms of a sum of Dirac delta functions of alternating and decaying weights at positive and negative odd integers of the positions of the original Gaussians.

II. MODEL STATE IN DIFFERENT REPRESENTATIONS

In this section, we lay the foundations for our discussion of the phase-only reconstruction by introducing our model state $|\psi\rangle$ and analyzing it in four different representations. The mechanism of this interference-based reconstruction stands out most clearly for the example of two Gaussians located at two different positions and different widths. We present the wave functions in position and wave number space, and discuss the Wigner function corresponding to this state. In particular, we demonstrate that the information about the state is encoded in the interference term. This feature also comes to light in a wave function approach toward phase space based on an interpretation [9] of the scalar product in quantum mechanics.

For the sake of simplicity, we concentrate on a single coordinate denoted by x and the conjugate variable k representing the wave number. Both variables define the two-dimensional phase space.

A. Position representation

We illustrate the central idea of Refs. [1–4] using the superposition

$$\psi(x) \equiv \langle x|\psi \rangle \equiv N[G_1(x+x_0) + G_2(x-x_0)] \quad (1)$$

of two real-valued Gaussians

$$G_n(x) \equiv \frac{1}{\sqrt{\pi} \Delta x_n} \exp\left[-\frac{1}{2}\left(\frac{x}{\Delta x_n}\right)^2\right] \quad (2)$$

of widths Δx_n with $n = 1$ and 2 , located at the positions $x = -x_0$ and $x = +x_0$. Here, N is the normalization factor, and we emphasize that Δx_n does not represent the standard deviation, but rather the position, where the distribution G_n^2 has decayed to the value e^{-1} .

Moreover, the prefactor of the Gaussian in Eq. (2) is chosen such that

$$\int_{-\infty}^{\infty} dx G_n^2(x) = 1. \quad (3)$$

Throughout this article, we consider situations where the overlap between the two Gaussians is exponentially small, assuming that their widths Δx_n are much smaller than their separation, that is,

$$\Delta x_n \ll x_0. \quad (4)$$

In this case the two Gaussians are clearly distinguishable and we find from the normalization condition of the probability density

$$|\psi(x)|^2 = N^2[G_1^2(x+x_0) + G_2^2(x-x_0) + 2G_1(x+x_0)G_2(x-x_0)] \quad (5)$$

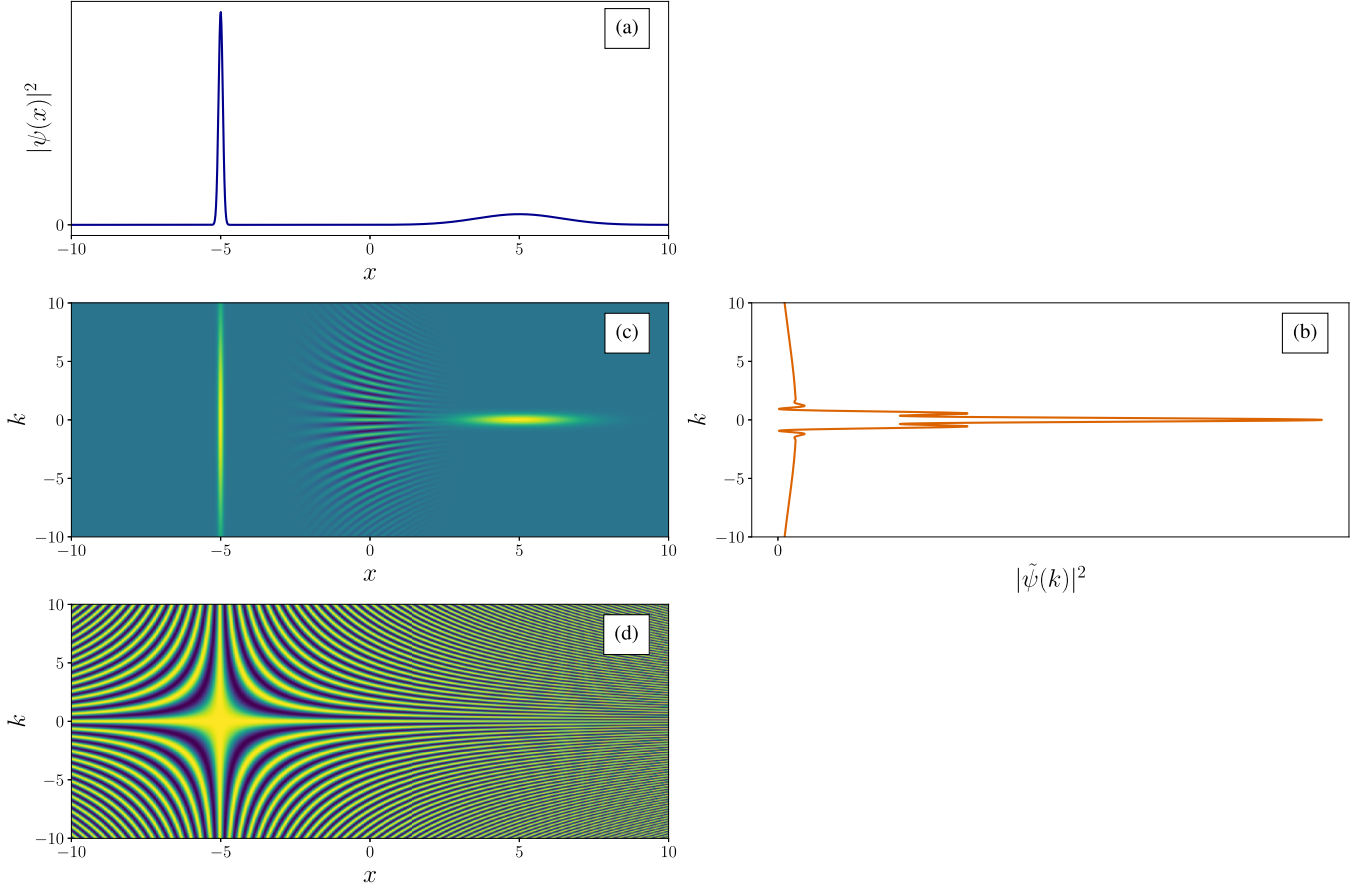


FIG. 3. Wigner function of the superposition ψ defined by Eq. (1), and its marginals, as well as the lines of constant phase of the interference term W_I . The two one-dimensional Gaussians G_1 and G_2 located at $x = -x_0$ and $x = x_0$ translate into two-dimensional Gaussians W_1 and W_2 in phase space, clearly depicted in (c). Integration of this Wigner function over the wave number k or the coordinate x yields the probability distributions $|\psi(x)|^2$ or $|\tilde{\psi}(k)|^2$ shown in (a) or (b). Between the peaks W_1 and W_2 , a hint of the interference pattern contained in W_I is observed. (d) demonstrates the hyperbolic behavior of the phase lines $\beta = \text{const.}$ given by Eq. (23). Due to the cosine being a symmetric function, β is symmetric with respect to $k = 0$ and $x = -2x_0/\kappa$. This point is an extremum of the hyperbolic function. The combination of the hyperbolic phase lines and the oscillatory nature of the cosine function creates the characteristic interference pattern caught between the two Gaussian Wigner functions. The values used are $x_0 = 5$, $\Delta x_1 = 0.1$, and $\Delta x_2 = 2$, which, according to Eq. (25), imply $\kappa \cong 2$.

with the well-known Gaussian integral relation

$$\int_{-\infty}^{\infty} dx e^{-ax^2 - bx} = \sqrt{\frac{\pi}{a}} \exp\left(\frac{b^2}{4a}\right), \quad (6)$$

valid for parameters a and b with $0 < \text{Re}\{a\}$, the explicit expression

$$N^2 = \frac{1}{2} \left\{ 1 + \mathcal{N} \exp\left[-\left(\frac{x_0}{\delta x}\right)^2\right] \right\}^{-1} \quad (7)$$

for the normalization constant. Here, we have introduced the constant

$$\mathcal{N} \equiv \sqrt{\frac{\Delta x_1 \Delta x_2}{\delta x^2}} \quad (8)$$

and the average

$$\delta x^2 \equiv \frac{1}{2} (\Delta x_1^2 + \Delta x_2^2) \quad (9)$$

of the squares of the two widths.

Hence, in the limit of Eq. (4) there is no interference between the two Gaussians in the coordinate space since the

overlap is exponentially small. In Fig. 3(a) we show the probability density $|\psi(x)|^2$ in the position space displaying the two Gaussians of different widths with no contribution in between.

B. Momentum or wave number representation

In quantum mechanics the conjugate variable to the position x is the momentum p . In order to simplify the equations containing Planck's constant $\hbar$, we consider the wave vector $k \equiv p/\hbar$ to be the conjugate variable to the position x .

Next, we consider the wave function

$$\tilde{\psi}(k) \equiv \langle k | \psi \rangle \equiv \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dx \psi(x) e^{ikx} \quad (10)$$

in wave number representation, which with the help of Eq. (1) reads

$$\tilde{\psi}(k) = N[\tilde{G}_1(k) e^{-ikx_0} + \tilde{G}_2(k) e^{ikx_0}], \quad (11)$$

with

$$\begin{aligned}\tilde{G}_n(k) &\equiv \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dx G_n(x) e^{ikx} \\ &\equiv \sqrt{\frac{1}{\pi \Delta k_n^2}} \exp \left[-\frac{1}{2} \left(\frac{k}{\Delta k_n} \right)^2 \right]\end{aligned}\quad (12)$$

and

$$\Delta k_n^2 \equiv \frac{1}{\Delta x_n^2}. \quad (13)$$

Hence, the probability density

$$|\tilde{\psi}(k)|^2 = N^2 [\tilde{G}_1^2(k) + \tilde{G}_2^2(k) + I(k)] \quad (14)$$

of finding the wave number k contains not only the Gaussian distributions $\tilde{G}_1^2(k)$ and $\tilde{G}_2^2(k)$ of width Δk_1 and Δk_2 , both located at $k = 0$, but also an interference term

$$I(k) \equiv 2A(k) \cos \alpha(k) \quad (15)$$

of amplitude

$$A(k) \equiv \tilde{G}_1(k) \tilde{G}_2(k) \equiv \sqrt{\frac{\Delta x_1 \Delta x_2}{\pi}} \exp(-\delta x^2 k^2) \quad (16)$$

and phase

$$\alpha(k) \equiv 2kx_0. \quad (17)$$

In Fig. 3(b), we show the probability distribution $|\tilde{\psi}(k)|^2$ as a function of the wave number k consisting of the sum of the Gaussians of different widths, both located at $k = 0$. The broad one, $\tilde{G}_1^2$, provides a pedestal for the narrow one, $\tilde{G}_2^2$, and corresponds to the narrow Gaussian G_1^2 at $x = -x_0$. The wide distribution G_2^2 at $x = x_0$ translates into the narrow Gaussian $\tilde{G}_2^2$ in k . Due to the different widths of the two Gaussians, the modulation caused by the interference term $I = I(k)$ does not create nodes.

C. Wigner phase space representation

Finally, we turn to a description of the state $|\psi\rangle$ given by the wave function Eq. (1) in quantum phase space spanned by the coordinate x and the wave number k based on the Wigner distribution [6–8], briefly introduced in Appendix A. Although this formulation of quantum mechanics is completely equivalent to the one in terms of wave functions, we gain deeper insight into interference phenomena.

The definition of the Wigner function

$$W(x, k) \equiv \frac{1}{2\pi} \int_{-\infty}^{\infty} dy e^{iky} \Phi^* \left(x + \frac{y}{2} \right) \Phi \left(x - \frac{y}{2} \right) \quad (18)$$

shows that W is always real. However, it is not always positive.

When we substitute Eq. (1) into the definition, Eq. (18), of the Wigner function, we find the expression

$$W(x, k) = N^2 [W_1(x + x_0, k) + W_2(x - x_0, k) + W_I(x, k)] \quad (19)$$

consisting of the superposition of the Gaussian Wigner functions

$$W_n(x, k) \equiv \frac{1}{\pi} \exp \left[-\left(\frac{x}{\Delta x_n} \right)^2 - \left(\frac{k}{\Delta k_n} \right)^2 \right] \quad (20)$$

with $n = 1$ and 2 corresponding to the two Gaussian wave functions at $x = -x_0$ and $x = x_0$, and the interference term

$$W_I(x, k) \equiv 2A(x, k) \cos \beta(x, k) \quad (21)$$

arising from the bilinearity of the Wigner function. Here, we have introduced the Gaussian Wigner function

$$\mathcal{A}(x, k) \equiv \frac{1}{\pi} \frac{1}{\sqrt{\delta x \delta k}} \exp \left[-\left(\frac{x}{\delta x} \right)^2 - \left(\frac{k}{\delta k} \right)^2 \right] \quad (22)$$

and the phase

$$\beta(x, k) \equiv k(\kappa x + 2x_0), \quad (23)$$

which, apart from the average δx^2 of the squares of the two widths defined by Eq. (9), involve the parameters

$$\delta k^2 \equiv \frac{\delta x^2}{\Delta x_1^2 \Delta x_2^2} = \frac{1}{2} \left[\frac{1}{\Delta x_1^2} + \frac{1}{\Delta x_2^2} \right] \quad (24)$$

and

$$\kappa \equiv \frac{\Delta x_2^2 - \Delta x_1^2}{\delta x^2}. \quad (25)$$

The remarkable property of the interference term is its dependence on the two widths Δx_1 and Δx_2 through the parameters δx^2 , δk^2 , and κ , and the location of the two peaks given by the separation $2x_0$.

Moreover, it is instructive to compare this expression to one in which the two widths are identical. In this case [24], the parameter κ defined by Eq. (25) vanishes, $\kappa = 0$, and the dependence of the phase β on the coordinate x disappears. Hence, the information about the two different widths is encoded in the interference fringes, which due to the product kx display a hyperbolic behavior in Wigner phase space.

In Fig. 3(c), we display the Wigner function of $|\psi\rangle$ given by Eq. (19), and note the two Gaussians at $x = \pm x_0$ determined by Eq. (20) and corresponding to the Gaussian wave functions of the superposition. Due to the uncertainty relation, the narrow Gaussian located at $x = -x_0$ displays a Wigner function that is elongated in the wave number variable k . In contrast, the wide Gaussian at $x = x_0$ has a Wigner function that is narrow in k .

The interference term given by Eq. (21) and located at the origin of phase space is barely visible. For this reason, we show in Fig. 3(d) the hyperbolic lines of constant phase defined by Eq. (23).

Finally, when we use the marginal properties, Eqs. (A2) and (A3), of the Wigner function, together with the integral relation, Eq. (6), we arrive at the distributions $|\psi(x)|^2$ and $|\tilde{\psi}(k)|^2$ given by Eqs. (5) and (14), shown in Figs. 3(a) and 3(b).

D. Area-of-overlap approach

In the preceding section we have shown that the negative parts of the Wigner function, most prominently located at

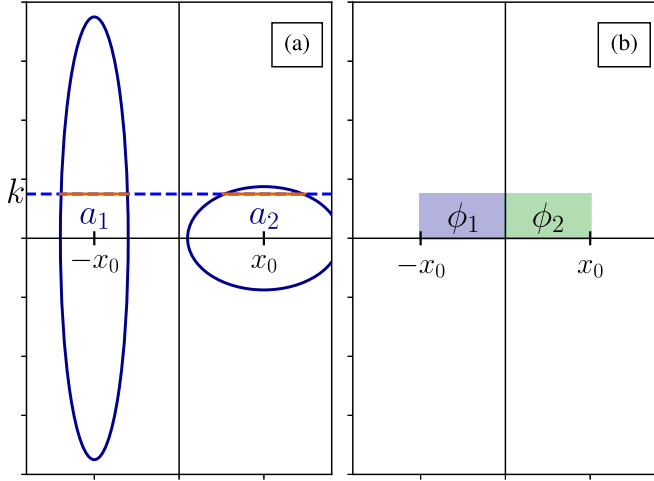


FIG. 4. Wave function $\tilde{\psi} = \tilde{\psi}(k)$ in wave number representation of the superposition state $|\psi\rangle$ derived from the concept of area-of-overlap in phase space, suggested by semiclassical quantum mechanics. In (a) we depict the two Gaussian phase space distributions $W_1(x, k) = G_1^2(x)\tilde{G}_1^2(k)$ and $W_2(x, k) = G_2^2(x)\tilde{G}_2^2(k)$ corresponding to the Gaussians $G_1(x)$ and $G_2(x)$ in position space and represented here by the blue elliptical lines of the exponential decay of W_1 and W_2 to the value e^{-1} . Their overlap with an infinitely thin line along $k = \text{const.}$ leads us to the two disjunct overlaps a_1 and a_2 , displayed here by red solid lines and given by Eq. (27). When we take the square root of these areas we find the two interfering probability amplitudes $\tilde{G}_1(k)$ and $\tilde{G}_2(k)$. The associated phases ϕ_1 and ϕ_2 are the blue and green areas of the rectangles shown in (b).

the origin of phase space, that is, between the two positive Gaussian Wigner functions corresponding to G_1 and G_2 , contain the information about the superposition state $|\psi\rangle$. Indeed, the bilinearity of the Wigner function in the wave function, corresponding to a description in terms of quasi-probabilities, requires the interference properties to be encoded in terms different from the classical ones.

However, a description of a quantum state in phase space in the language of wave functions, leading us straight to the wave function $\tilde{\psi} = \tilde{\psi}(k)$ given by Eq. (11), rather than quasi-probabilities, would be desirable. In this section we apply the concept of area-of-overlap [8,9] which is based on semiclassical quantum mechanics [8], to obtain $\tilde{\psi}$. We emphasize that in this elementary case the representation is even exact.

For this purpose, we recall that the wave function $\tilde{\psi}(k) \equiv \langle k|\psi\rangle$ in wave number representation is the scalar product between an eigenstate $|k\rangle$ of wave number and the state $|\psi\rangle$. Obviously $|k\rangle$ is an infinitely thin line parallel to the coordinate axis at the value k . Moreover, the state $|\psi\rangle$ is represented by two Gaussians at $x = -x_0$ and $x = x_0$ given by the product $G_n^2(x)\tilde{G}_n^2(k) = W_n$.

Since the scalar product $\langle k|\psi\rangle$ is the overlap [8,9] between the two states, that is, the line representing k and the two disjoint Gaussians, we find *two* overlaps a_1 and a_2 , shown in Fig. 4(a). Each represents a probability amplitude given by the square root of the overlap. Their phases ϕ_1 and ϕ_2 are given by the rectangles defined by the lines $k = \text{const.}$ and the coordinate axis, as well as the lines $x = \pm x_0$ and the wave number axis as depicted in Fig. 4(b).

Hence, we find

$$\tilde{\psi}(k) = N[\sqrt{a_1(k)} e^{i\phi_1(k)} + \sqrt{a_2(k)} e^{i\phi_2(k)}], \quad (26)$$

where

$$a_n(k) \equiv \int_{-\infty}^{\infty} dx G_n^2(x)\tilde{G}_n^2(k) = \tilde{G}_n^2(k) \quad (27)$$

with

$$\phi_1 \equiv -kx_0 \quad \text{and} \quad \phi_2 \equiv kx_0. \quad (28)$$

In the last step of Eq. (27) we have used Eq. (3).

In summary, the concept of the area-of-overlap in phase space [8,9] provides us immediately with the wave function $\tilde{\psi} = \tilde{\psi}(k)$ in wave number space. In particular, this wave function approach creates the two interfering probability amplitudes with the correct phase factors which contain the information about the locations of the two peaks.

III. PHASE-ONLY RECONSTRUCTION

In the preceding section, we have analyzed the model state $|\psi\rangle$ in different representations. We dedicate the present section to comparing and contrasting the original wave function in position space with the one obtained from the phase-only reconstruction. Although these calculations are based on a numerical analysis of the corresponding Fourier integral, we also apply approximate but analytical techniques.

A. Formulation of the problem

Due to the familiar integral representation

$$\delta(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dk e^{-ixk} \quad (29)$$

of the Dirac delta function, the inverse Fourier transform

$$\psi(x) \equiv \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dk \tilde{\psi}(k) e^{-ixk} \quad (30)$$

of $\tilde{\psi} = \tilde{\psi}(k)$ leads us back to $\psi = \psi(x)$.

However, we now consider the phase-only reconstruction

$$\psi_p(x) \equiv \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dk e^{i\varphi(k)} e^{-ixk}, \quad (31)$$

that is, the inverse Fourier transform of the phase factor

$$e^{i\varphi(k)} \equiv \frac{\tilde{\psi}(k)}{|\tilde{\psi}(k)|} \quad (32)$$

determined by the ratio of $\tilde{\psi}$ and $|\tilde{\psi}|$. For a different evaluation of the phase factor based on the phase *angle*, we refer to Appendix B.

When we compare the expressions, Eqs. (30) and (31), for the inverse Fourier transform of $\tilde{\psi}$ and for the phase-only function $\tilde{\psi}/|\tilde{\psi}|$, the only difference is the denominator given by $|\tilde{\psi}|$. Hence, we expect that both expressions provide similar results when $|\tilde{\psi}|$ is slowly varying as a function of the integration variable k compared to the product $\tilde{\psi}(k) \exp(-ixk)$.

Indeed, this property would allow us to factor $|\tilde{\psi}|$ out of the integral, and the resulting integral is $\psi = \psi(x)$. Hence, apart from a prefactor, we obtain ψ , and it would not be

surprising that we can reconstruct ψ . Unfortunately, the situation is more complicated, but also more interesting, as we demonstrate now.

B. Two identical Gaussians

For this purpose, we start by addressing a situation where the two widths are identical. On first sight, one might expect the corresponding analysis to be quite straightforward, since for $G_1 = G_2 \equiv G$ the wave number amplitude given by Eq. (11) reduces to

$$\tilde{\psi}(k) = 2N\tilde{G}(k) \cos(kx_0). \quad (33)$$

Moreover, the phase factor reads

$$e^{i\varphi(k)} = \frac{\cos(kx_0)}{|\cos(kx_0)|}, \quad (34)$$

where we have made use of the identity

$$|\tilde{G}| = \tilde{G} \quad (35)$$

following from Eq. (12).

We emphasize that this property is crucial in the derivation of Eq. (34), making the phase factor independent of $\tilde{G}$ and originating solely from the two phase factors in Eq. (11) which contain the information about the positions of the Gaussians in the superposition state, Eq. (1).

However, the phase factor $e^{i\varphi}$, given by Eq. (34), can only assume the two values $+1$ or -1 with the phase φ being integer multiples of 2π or odd integer multiples of π , depending on $\cos(kx_0)$ being positive or negative. As a result, the phase-only reconstruction, defined by Eq. (31), is nontrivial and reads

$$\psi_p(x) = \frac{1}{\sqrt{2\pi}} \frac{1}{x_0} I\left(\frac{x}{x_0}\right), \quad (36)$$

where we have introduced the abbreviation

$$I(\xi) \equiv \int_{-\infty}^{\infty} d\theta \frac{\cos \theta}{|\cos \theta|} e^{-i\xi\theta}. \quad (37)$$

In Appendix C, we perform this integral and thus find from Eq. (36) the expression

$$\psi_p(x) = \frac{1}{\sqrt{2\pi}} \sum_{n=-\infty}^{\infty} \frac{(-1)^n}{2n+1} \delta[x - (2n+1)x_0] \quad (38)$$

for the phase-only reconstruction of a superposition of two Gaussians of identical widths.

We emphasize that Eq. (38) constitutes an exact result which displays three characteristic features: (i) The phase-only reconstruction of two identical Gaussians located at $-x_0$ and x_0 is a sum of infinitely many weighted Dirac delta functions located at positive and negative odd integer multiples of x_0 ; (ii) for $n = -1$ and $n = 0$, we find two delta functions at $x = -x_0$ and $x = x_0$ of equal weight $+1$; and (iii) for increasing n , these weights decrease inversely with n , but the weights also assume negative values as shown in Fig. 5.

We conclude by slightly generalizing our result. Indeed, we have obtained Eq. (38) for two Gaussians of identical width. However, this expression is even more general.

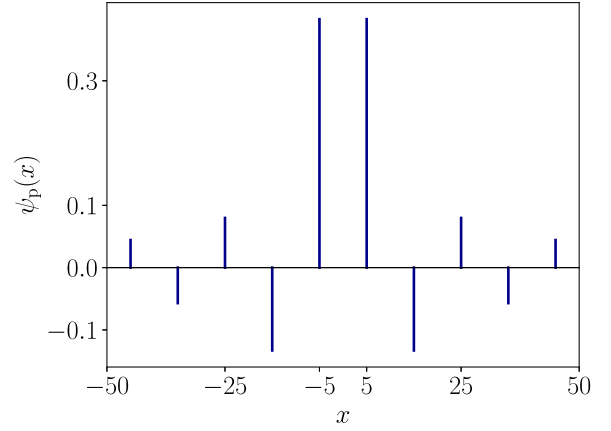


FIG. 5. Schematic representation of the phase-only reconstruction $\psi_p = \psi_p(x)$ corresponding to the superposition, Eq. (1), of two Gaussians of *identical* widths, located at $x = -x_0$ and $x = x_0$. According to Eq. (38), the original Gaussians are replaced by Dirac delta functions indicated here by lines. Apart from the peaks at $x = \pm x_0$ with weight $+1$, we find additional ones at positive and negative odd integers of x_0 , however, with positive and negative decaying weights. Here we have chosen $x_0 = 5$.

For example, the superposition

$$\psi(x) = N[\delta(x + x_0) + \delta(x - x_0)] \quad (39)$$

consisting of two Dirac delta functions which cannot be normalized, leads to the wave function

$$\tilde{\psi}(x) = N(e^{-ikx_0} + e^{ikx_0}) = 2N \cos(2kx_0), \quad (40)$$

and thus to the same phase factor as in the case of two Gaussians. As a result, the phase-only reconstruction of this superposition, Eq. (39), is also given by Eq. (38).

C. Emergence of characteristic features

We now analyze the influence of different widths of the two Gaussians on the phase-only reconstruction. For this purpose we first find, with the help of Eq. (14), the phase factor

$$e^{i\varphi(k)} = \frac{\tilde{G}_1(k)e^{-ikx_0} + \tilde{G}_2(k)e^{ikx_0}}{\sqrt{\tilde{G}_1^2(k) + \tilde{G}_2^2(k) + 2\tilde{G}_1(k)\tilde{G}_2(k)\cos(2kx_0)}}, \quad (41)$$

and then substitute it into the integral, Eq. (31), defining the phase-only reconstruction.

The blue line in Fig. 6 defines the phase-only reconstructed wave function $\psi_p = \psi_p(x)$ in position space resulting from a numerical evaluation of the phase-based Fourier transform, Eq. (31). We note that both peaks are located at the positions of the original Gaussians, depicted by the red line. Moreover, their widths match rather well.

Nevertheless, there are deviations between the two curves, which manifest themselves mainly in the neighborhood of the narrow peak, that is, at $x = -x_0$. Here, $\psi_p = \psi_p(x)$ assumes negative values.

Figure 7 shows $\psi_p = \psi_p(x)$ on a larger range of position space and demonstrates that the reduction of the wave function $\tilde{\psi}$ in wave number space to its phase factor also results in oscillations reminiscent of the Gibbs phenomenon.

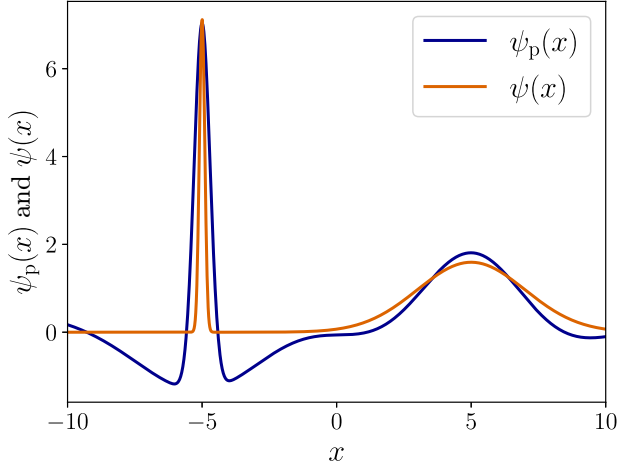


FIG. 6. Comparison between the phase-only reconstructed wave function $\psi_p = \psi_p(x)$, defined by Eq. (31), and the original superposition $\psi = \psi(x)$, Eq. (1), scaled by the average of the quotient of the peaks. Apart from the oscillation in ψ_p (blue line), the widths and the locations of the two original Gaussians (red line) are well reproduced. The numerical integration of Eq. (31) with Eq. (41) was performed by using the SciPy library [25].

D. Three domains of integration

In order to gain some insight into the deviations between ψ and ψ_p , we apply the familiar expansion [23]

$$\frac{1}{\sqrt{r^2 + R^2 - 2rR \cos \gamma}} = \sum_{m=0}^{\infty} \frac{r^m}{R^{m+1}} P_m(\cos \gamma), \quad (42)$$

in terms of Legendre polynomials P_m of order m to the denominator of $e^{i\varphi}$ given by $|\psi(k)|$, and apparent in Eq. (41). Next, we substitute this expression into the inverse Fourier integral,

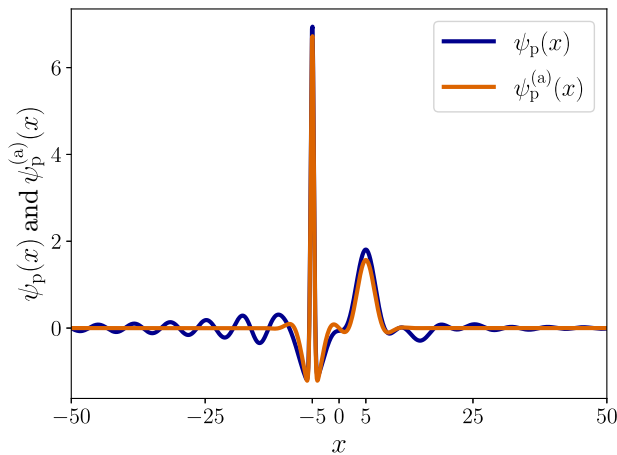


FIG. 7. The numerically obtained phase-only reconstruction, Eq. (31), of the superposition, Eq. (1), for an enlarged range, that is, for positions from -50 to 50 , and comparison to the approximate phase-only reconstruction $\psi_p^{(a)} = \psi_p^{(a)}(x)$, Eqs. (46) and (47), where we have replaced the cosine function in Eq. (41) by unity. The two curves differ in the oscillations present in ψ_p (blue line) but absent in $\psi_p^{(a)}$ (orange line).

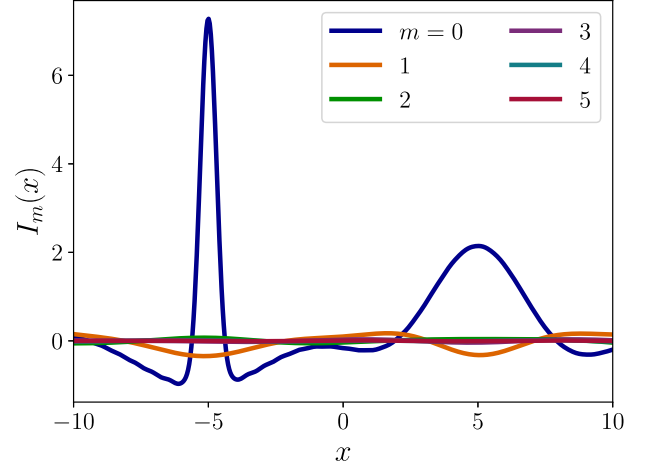


FIG. 8. Influence of the different orders m in the series expansion, Eq. (43), given by the integrals $I_m(x)$, Eq. (44), of the phase-only reconstruction ψ_p . For a better comparison, we do not sum the contributions $I_m(x)$ but display the individual orders by different colors for the first six orders. The dominant behavior of ψ_p originates from the term $I_0(x)$. The higher orders contribute less and less, as one would expect from any convergent series expansion.

Eq. (31), of $e^{i\varphi}$ and arrive at the expression

$$\psi_p(x) = \sum_{m=0}^{\infty} I_m(x), \quad (43)$$

where we have introduced the abbreviation

$$I_m(x) \equiv \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dk \frac{r^m}{R^{m+1}} P_m[\cos(2kx_0)] \times [\tilde{G}_1(k)e^{-ikx_0} + \tilde{G}_2(k)e^{ikx_0}]e^{-ixk}. \quad (44)$$

Next, we evaluate the integrals $I_m(x)$ numerically. In this process we need to take into account the fact that the distributions $\tilde{G}_1$ and $\tilde{G}_2$ in the square root depend on the integration variable k . Indeed, to ensure the convergence of the expansion, Eq. (42), it is necessary to have the smaller of the two parameters r and R in the numerator on the right side of Eq. (42). As a result, the k dependence of $\tilde{G}_1$ and $\tilde{G}_2$, and the nature of $|\tilde{\psi}|^2$, shown in Fig. 3(b) consisting of a narrow peak given by $\tilde{G}_2^2$ on top of a pedestal formed by $\tilde{G}_1^2$, decomposes the inverse Fourier integral, Eq. (31), of $e^{i\varphi}$ into three domains: (i) from negative infinity to the point $k = -k_0$, where $\tilde{G}_2(\pm k_0) = \tilde{G}_1(\pm k_0)$ in this domain $r = \tilde{G}_2$ and $R = \tilde{G}_1$; (ii) from $-k_0$ and k_0 , where $r = \tilde{G}_1$ and $R = \tilde{G}_2$; and (iii) from k_0 to plus infinity. In this range of the integration variable we have $r = \tilde{G}_2$ and $R = \tilde{G}_1$.

From the definition, Eq. (12), of the Gaussians $\tilde{G}_n$ we find the expression

$$k_0 \equiv \pm \frac{\Delta k_1^2 \Delta k_2^2}{\Delta k_2^2 - \Delta k_1^2} \ln \left(\frac{\Delta k_2}{\Delta k_1} \right) \quad (45)$$

for the point k_0 where the two Gaussians are identical.

We evaluate the three integrals numerically and display in Fig. 8 the influence of the different orders. We note that the main contribution to ψ_p arises from the lowest order, $m = 0$.

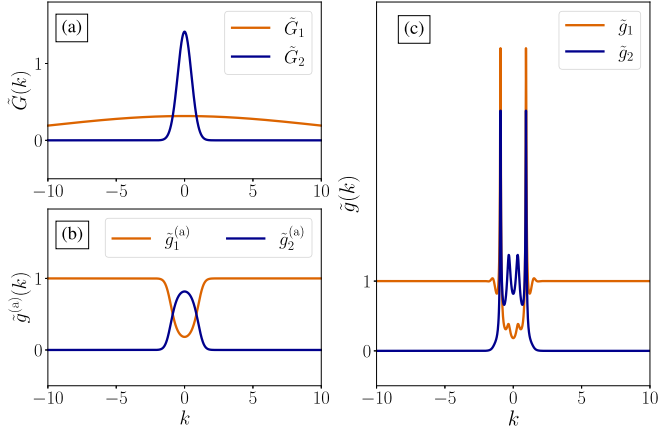


FIG. 9. Comparison between the amplitudes $\tilde{g}_n^{(a)} = \tilde{g}_n^{(a)}(k)$ and $\tilde{g}_n = \tilde{g}_n(k)$, defined by Eqs. (47) and (59), of the approximate and the exact phase-only reconstruction, Eqs. (46) and (57), together with the amplitudes $\tilde{G}_n = \tilde{G}_n(k)$, Eq. (12), of the complete inverse Fourier transform. The broad Gaussian $\tilde{G}_1$ [orange line in (a)] leads to an almost constant function $\tilde{g}_1^{(a)}$ of value unity [orange line in (b)], with a hole around $k = 0$ created by the maximum of the narrow Gaussian $\tilde{G}_2$ [blue line in (a)]. In contrast, $\tilde{G}_2$ gives rise to a function $\tilde{g}_2^{(a)}$ [blue line in (b)] which is zero almost everywhere, except at $k = 0$, where it assumes a maximum. In (c) we include in the amplitudes $\tilde{g}_1 = \tilde{g}_1(k)$ [orange line in (c)] and $\tilde{g}_2 = \tilde{g}_2(k)$ [blue line in (c)] the oscillations arising from the interference term in Eq. (41), which translate into modulations of the curves $\tilde{g}_n$ shown in (b). These modulations are the origin of the large-scale oscillations shown in Fig. 7.

Since $P_{m=0} = 1$, the dependence on $\gamma = 2kx_0$ does not enter in lowest approximation.

E. Three building blocks

The numerical evaluation of the integral, Eq. (31), associated with the phase-only reconstruction, as well as the integral resulting from the Legendre expansion, Eq. (42), demonstrate the claim of Refs. [1–3]. We dedicate the present section to an approximate but analytical treatment of these integrals, which provides deeper insight into the inner workings of the phase-only reconstruction. In particular, we show that in complete analogy to the three domains of integration discussed before, three integrals characterize the main features of ψ_p and can therefore be considered its building blocks.

In order to bring out this fact most clearly we now analyze the *approximate* phase-only reconstruction

$$\psi_p^{(a)}(x) \cong \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dk [\tilde{g}_1^{(a)}(k) e^{-i(x+x_0)k} + \tilde{g}_2^{(a)}(k) e^{-i(x-x_0)k}] \quad (46)$$

which results from Eq. (41) by replacing the cosine by unity, completing the square and taking the square root. Here, we have introduced the abbreviation

$$\tilde{g}_n^{(a)}(k) \equiv \frac{\tilde{G}_n(k)}{\tilde{G}_1(k) + \tilde{G}_2(k)}. \quad (47)$$

In Figs. 9(a) and 9(b) we depict the functions $\tilde{G}_1$ and $\tilde{G}_2$ as well as $\tilde{g}_1^{(a)}$ and $\tilde{g}_2^{(a)}$ in their dependence on k . We note that apart from their heights, $\tilde{g}_2^{(a)}$ and $\tilde{G}_2$ are rather similar.

However, $\tilde{g}_1^{(a)}$ and $\tilde{G}_1$ are dramatically different. Indeed, $\tilde{g}_1^{(a)}$ is constant with a hole around $k = 0$.

We now discuss these distinct behaviors of $\tilde{g}_1^{(a)}$ and $\tilde{g}_2^{(a)}$ in more detail and start with $n = 2$, recalling that $\tilde{G}_2$ is narrow compared to $\tilde{G}_1$. For values of $\Delta k_2 \ll |k|$, $\tilde{G}_2$ is exponentially small, but $\tilde{G}_1$, which is broad, is still nonzero.

In this domain we can neglect in the denominator of $\tilde{g}_2^{(a)}$, in Eq. (47), the term $\tilde{G}_2$ compared to $\tilde{G}_1$, leading us to the approximation

$$\tilde{g}_2^{(a)}(k) \cong \frac{\tilde{G}_2(k)}{\tilde{G}_1(k)} \cong 0 \quad (48)$$

valid for $\Delta k_2 \ll |k|$.

However, in the neighborhood of $k = 0$, $\tilde{G}_2$ is much larger than $\tilde{G}_1$ and we can neglect in the denominator of Eq. (47) the term $\tilde{G}_1$ compared to $\tilde{G}_2$, providing us with the approximate expression

$$\tilde{g}_2^{(a)}(k) \cong \frac{\tilde{G}_2(k)}{\tilde{G}_2(k)} = 1 \quad (49)$$

valid for $|k| \ll \Delta k_1$.

We can combine Eqs. (48) and (49) by the formula

$$\tilde{g}_2^{(a)}(k) \cong \exp \left[-\frac{1}{2} \left(\frac{k}{\Delta k_2} \right)^2 \right] = \sqrt[4]{\pi \Delta k_2^2} \tilde{G}_2(k), \quad (50)$$

which, apart from the normalization factor of Eq. (12), is identical to $\tilde{G}_2$.

Next, we turn to $\tilde{g}_1$ and recall that $\tilde{G}_1$ is broad compared to $\tilde{G}_2$. Hence, for $\Delta k_2 \ll |k|$ we can neglect in the denominator of Eq. (47) $\tilde{G}_2$ compared to $\tilde{G}_1$, leading us to the approximation

$$\tilde{g}_1^{(a)}(k) \cong \frac{\tilde{G}_1(k)}{\tilde{G}_1(k)} = 1, \quad (51)$$

valid for $\Delta k_2 \ll |k|$.

However, in the neighborhood of $k = 0$, that is, for $|k| < \Delta k_2$, the dominant contribution is $\tilde{G}_2$ and we find

$$\tilde{g}_1^{(a)}(k) \cong \frac{\tilde{G}_1(k)}{\tilde{G}_2(k)} < 1. \quad (52)$$

Again we can combine Eqs. (51) and (52) by the elementary approximation

$$\tilde{g}_1^{(a)}(k) \cong 1 - \tilde{g}_2^{(a)}(k), \quad (53)$$

leading us to the conclusion

$$\tilde{g}_1^{(a)}(k) + \tilde{g}_2^{(a)}(k) = 1, \quad (54)$$

obvious from Fig. 9(b).

When we substitute the approximations, Eqs. (50) and (53), into the approximate phase-only reconstruction, Eq. (46), we find

$$\psi_p^{(a)}(x) \cong \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dk [1 - \sqrt[4]{\pi \Delta k_2^2} \tilde{G}_2(k)] e^{-i(x+x_0)k} + \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dk \sqrt[4]{\pi \Delta k_2^2} \tilde{G}_2(k) e^{-i(x-x_0)k}, \quad (55)$$

or

$$\begin{aligned} \psi_p^{(a)}(x) = & \sqrt{2\pi} \delta(x + x_0) - \sqrt{\frac{\Delta k_2^2}{4}} G_2(x + x_0) \\ & + \sqrt{\frac{\Delta k_2^2}{4}} G_2(x - x_0). \end{aligned} \quad (56)$$

As a result, $\psi_p^{(a)}$ consists of three building blocks, two of which are located at $x = -x_0$, and one which is at $x = x_0$. The latter is again a Gaussian of width Δk_2 . However, its maximum is different, as expressed by Eq. (56).

Due to the fact that the phase factor $e^{i\varphi}$ has an absolute value of unity, there is a delta function at the location $x = -x_0$ of the thin Gaussian G_1 . Hence, the phase-only reconstruction makes the narrow distribution infinitely thin.

The hole of $\tilde{g}_1^{(a)}$ at $k = 0$, given by Eq. (53), is a reflection of $|e^{i\varphi}| = 1$ and provides us with a negative Gaussian G_2 located at $x = -x_0$. We emphasize that the width Δx_2 of this contribution is identical to the one of the Gaussian at $x = x_0$. This term is the origin of the negative values of the exact phase-only reconstruction ψ_p around $x = -x_0$.

F. The essence

We conclude our discussion of the phase-only reconstruction of the superposition, Eq. (1), of two Gaussians by highlighting the central results and by addressing the full problem.

Indeed, with the help of the expression, Eq. (11), for the amplitude $\tilde{\psi}$ in wave number space, the phase-only reconstructed amplitude following from Eqs. (31) and (41) reads

$$\psi_p(x) = N[g_1(x + x_0) + g_2(x - x_0)], \quad (57)$$

where we have introduced the abbreviation

$$g_n(x) \equiv \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dk \frac{\tilde{G}_n(k)}{|\tilde{\psi}(k)|} e^{-ikx}. \quad (58)$$

The fact that the two peaks g_1 and g_2 are again located at $x = -x_0$ and $x = x_0$ is a consequence of the information about the positions being stored in the phase factors of $\tilde{\psi}$, which, in contrast to their amplitudes $\tilde{G}_1$ and $\tilde{G}_2$, are not touched by the division of $|\tilde{\psi}(k)|$.

In the preceding section we have already found that the division by $|\tilde{\psi}|$ influences $\tilde{G}_1$ and $\tilde{G}_2$ differently. However, this analysis was based on replacing the cosine in Eq. (41) by unity. We now take into account the *complete* distribution $|\tilde{\psi}|$.

For this purpose we show in Fig. 9(c) the distributions

$$\tilde{g}_n(k) \equiv \frac{\tilde{G}_n(k)}{|\tilde{\psi}(k)|} \quad (59)$$

for $n = 1$ and 2 .

Again we obtain a narrow distribution and a flat distribution with the hole at $k = 0$, with characteristic oscillations. In contrast to Fig. 9(b), we have now taken into account the modulation of $|\tilde{\psi}|$ arising from the cosine term in Eq. (14).

This analysis brings out most clearly that at the very heart of the phase-only reconstruction are the two shift properties

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dx f(x + x_0) e^{ikx} = \tilde{f}(k) e^{-ikx_0} \quad (60)$$

and

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dk [\tilde{g}(k) e^{-ix_0 k}] e^{-ikx} = g(x + x_0) \quad (61)$$

of the Fourier transforms $\tilde{f}$ and $\tilde{g}$ of f and g .

It is interesting that in contrast to the derivation of Eq. (60), where we needed to introduce the new integration variable $y \equiv x + x_0$, Eq. (61) follows by just combining the arguments x and x_0 .

IV. GENERALIZATION

We are now in a position to summarize our insights into the phase-only reconstruction, and at the same time, generalize our results obtained for two Gaussians to a superposition

$$\Psi(x) \equiv N \sum_j H_j(x - x_j) \quad (62)$$

of functions $H_j = H_j(x)$ located at the positions $x = x_j$. Here, N is a normalization factor.

When we substitute this expression for Ψ into the definition, Eq. (10), of the Fourier transformation, we find

$$\tilde{\Psi}(k) = N \sum_j \tilde{H}_j(k) e^{ikx_j}, \quad (63)$$

where we have introduced the Fourier transform

$$\tilde{H}_j(k) \equiv \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dx H_j(x) e^{ikx} \quad (64)$$

of H_j .

Next, we determine the square root

$$|\tilde{\Psi}(k)| = N \left[\sum_{j,j'} \tilde{H}_j^*(k) \tilde{H}_{j'}(k) e^{-ik(x_j - x_{j'})} \right]^{\frac{1}{2}} \quad (65)$$

of the wave number distribution $|\tilde{\Psi}(k)|^2$, which yields the expression

$$e^{i\varphi(k)} = \frac{\tilde{\Psi}(k)}{|\tilde{\Psi}(k)|} = N \sum_j \frac{\tilde{H}_j(k)}{|\tilde{\Psi}(k)|} e^{ikx_j} \quad (66)$$

for the phase factor.

Finally, we substitute this formula into the phase-only reconstruction given by Eq. (31) to arrive at the result

$$\Psi_p(x) = \sum_j h_j(x - x_j), \quad (67)$$

where we have introduced the inverse Fourier transform

$$h_j(x) \equiv \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dk \tilde{h}_j(k) e^{-ikx} \quad (68)$$

of the modified coefficients

$$\tilde{h}_j(k) \equiv \frac{\tilde{H}_j(k)}{|\tilde{\Psi}(k)|}. \quad (69)$$

Hence, the phase-only reconstruction reproduces functions h_j centered at the positions x_j of the original wave functions H_j . This feature is a consequence of the shift property, Eq. (60), of the Fourier transform, combined with the fact that the division by $|\tilde{\Psi}|$ leaves the phase factor e^{ikx_j} in Eq. (63),

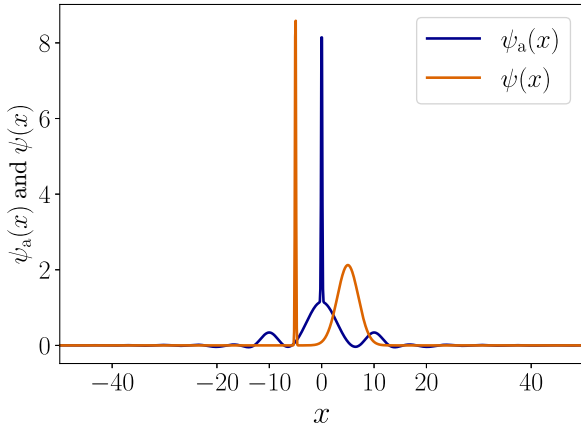


FIG. 10. Failure of the amplitude-only reconstruction $\psi_a = \psi_a(k)$ of the superposition state $|\psi\rangle$ given by Eq. (1). Indeed, the inverse Fourier transform of $|\tilde{\psi}(k)|$ defined by Eq. (70) creates a single peak made up of two overlapping Gaussians located at the origin (blue line) and has therefore nothing in common with the original wave function (orange line) consisting of two peaks located at two different positions.

where the information about the position is stored, untouched, but it alters the amplitudes $\tilde{H}_j(k)$ dramatically. Thin features in position become even thinner in the phase-only reconstruction. Broad distributions are almost untouched.

V. AMPLITUDE-ONLY RECONSTRUCTION

So far we have concentrated on the phase-only reconstruction and have demonstrated that it reproduces the key features of the original superposition state. We now briefly address the amplitude-only reconstruction summarized in the top row of Fig. 2, for the example of the two Gaussians given by Eq. (1).

In this case we perform the inverse Fourier transform

$$\psi_a(x) \equiv \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dk |\tilde{\psi}(k)| e^{-ikx} \quad (70)$$

of $|\tilde{\psi}|$, and display in Fig. 10 the resulting function $\psi_a = \psi_a(x)$. This distribution consists of a single narrow peak on an oscillatory background. Hence, the superposition of two Gaussians of different widths, located at two different positions in space, is lost.

Insight into this numerical result springs immediately from the approximation

$$|\tilde{\psi}(k)| \cong N[\tilde{G}_1(k) + \tilde{G}_2(k)] \quad (71)$$

of the amplitude of the wave function $\tilde{\psi} = \tilde{\psi}(k)$ in the wave number variable. Here, we have neglected again the oscillations in the interference term, that is, we have replaced the cosine function by unity.

When we substitute the approximation, Eq. (71), of $|\tilde{\psi}(k)|$ into the definition, Eq. (70), of the amplitude-only reconstruction, we arrive at the expression

$$\psi_a^{(a)}(x) = N[G_1(x) + G_2(x)] \quad (72)$$

consisting of the superposition of the two original Gaussians, but now both located at the same position, that is, at the origin. Thus, the information about the original locations of

the Gaussians is lost. It is interesting that the widths Δx_1 and Δx_2 are preserved.

Moreover, Fig. 10 shows the emergence of oscillations on a larger range of x values, similar to the ones of the phase-only reconstruction ψ_p depicted in Fig. 6. The approximate expression, Eq. (72), for $\psi_a^{(a)}$ is free of oscillations, which demonstrates that they are a consequence of the oscillations in the interference term.

VI. CONCLUSION AND OUTLOOK

In the present article, we have illuminated the phase-only reconstruction known from signal processing [1–4] using an elementary example from quantum mechanics. Our model wave function, which is a superposition of two Gaussians, with negligible overlap in position space, displays interference in the wave number space. This interference contains the essential properties of the original wave function and constitutes the deeper reason for the success of the phase-only reconstruction.

Interference manifests itself in the negative parts of the Wigner function located in phase space between the self-terms. The marginal properties immediately show that despite the interference term in phase space, there is no interference in position space. However, there is interference in wave number space.

Our Wigner function approach also teaches us that a superposition of two Gaussians aligned along the wave number, rather than the coordinate axis, displays interference in position, but not in wave number space. Hence, in this case the phase-only reconstruction is not successful unless the Fourier transform starts from wave number rather than position space.

Several ideas for applications and extensions of our insight offer themselves. Here, we only outline these “ideas for ideas” and postpone a detailed analysis to a future publication.

So far, we have studied a *time-independent* situation, but have already noticed that two superposition states either aligned along the position or the wave number axis allow for a successful or nonsuccessful phase-only reconstruction. Since the time evolution of an arbitrary quantum state in a harmonic oscillator corresponds to a rotation of the Wigner function in phase space, most of the time the phase-only reconstruction will fail.

Moreover, we have employed exclusively the Wigner function to understand the relative importance of the phase and amplitude of the wave function in wave number space. However, there exist other phase space representations with the same fundamental properties as the Wigner function, but whose interference terms differ significantly [26,27]. For instance, the Choi-Williams distribution [28,29] has a parameter that controls the behavior of the interference terms. A special case recovers the Wigner function, but for other values of the parameter the interference is spread throughout phase space. Other distributions include Zhao-Atlas-Marks time-frequency distribution (ZAM) [30], Baraniuk and Jones [31], and Margenau-Hill [32]. It would be interesting to investigate how our results may change by using different distributions.

Throughout our article, we have focused on the phase space spanned by coordinate and wave number; however, we could also work in the phase space formed by frequency and time

[26]. Since the order of the Gaussians is not changed in the phase-only reconstruction [33], we expect this property to also hold true for a sequence of pulses in time, that is, time ordering is preserved. This insight might lead toward a new approximation of the familiar time-ordered product emerging from the Schrödinger equation with explicitly time-dependent Hamiltonians.

Our findings underscore the fundamental role of the phase in Fourier-based transformations, offering new insights into quantum mechanics and potential applications in quantum information science [34].

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DATA AVAILABILITY

The data that support the findings of this article are not publicly available. The data are available from the authors upon reasonable request.

APPENDIX A: WIGNER FUNCTION ESSENTIALS

In this Appendix we summarize, for the sake of completeness, essential features of the Wigner distribution [6–8]

$$W(x, k) \equiv \frac{1}{2\pi} \int_{-\infty}^{\infty} dy e^{iky} \Phi^*\left(x + \frac{y}{2}\right) \Phi\left(x - \frac{y}{2}\right). \quad (\text{A1})$$

The Hudson-Piquet theorem [8] states that the only Wigner functions that are positive are those corresponding to Gaussian wave functions. For this reason, Wigner functions are not probability distributions. Nevertheless, they allow us to calculate quantum mechanical expectations values of symmetrically ordered operators, in a way familiar from classical statistical mechanics.

Moreover, we recall the marginal properties [6–8]

$$\int_{-\infty}^{\infty} dk W(x, k) = |\Phi(x)|^2 \quad (\text{A2})$$

or

$$\int_{-\infty}^{\infty} dx W(x, k) = |\tilde{\Phi}(k)|^2 \quad (\text{A3})$$

of the Wigner function, leading us to the probability distributions $|\Phi(x)|^2$ and $|\tilde{\Phi}(k)|^2$ in position or wave number space, where

$$\tilde{\Phi}(k) \equiv \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dx \Phi(x) e^{ikx}. \quad (\text{A4})$$

Most relevant for the present discussion is the fact that W is bilinear in the wave function. Hence, a superposition

$$\Phi(x) \equiv \mathcal{N}[\Phi_1(x) + \Phi_2(x)] \quad (\text{A5})$$

of two wave functions Φ_1 and Φ_2 leads to a Wigner function

$$W = W_{11} + W_{22} + 2\text{Re}[W_{12}], \quad (\text{A6})$$

where

$$W_{mn}(x, k) \equiv \frac{1}{2\pi} \int_{-\infty}^{\infty} dy e^{iky} \Phi_m^*\left(x + \frac{y}{2}\right) \Phi_n\left(x - \frac{y}{2}\right). \quad (\text{A7})$$

The terms $m = n$ are called self-terms and the terms $m \neq n$ arise due to the bilinearity of the Wigner function in the wave function.

APPENDIX B: PHASE RATHER THAN PHASE FACTOR

In the main body of our article, we have obtained the phase-only reconstruction by performing the inverse Fourier transform of the ratio of the wave number amplitude $\tilde{\psi}$ and its absolute value $|\tilde{\psi}|$, providing us directly with the phase factor $\exp[i\varphi]$. This approach has brought out most clearly that the phase factors $\exp[-ikx_0]$ and $\exp[ikx_0]$ which contain the information about the locations $-x_0$ and x_0 of the peaks are not influenced by the division by $|\tilde{\psi}|$. In this Appendix, we pursue a different approach and calculate first the phase angle φ of the complex-valued amplitude $\tilde{\psi}$ by employing the elementary rule of complex analysis. An approximate but analytical expression for φ immediately provides us with a formula for the phase-only reconstruction in terms of sinc functions.

We start by casting the wave number amplitude $\tilde{\psi} = \tilde{\psi}(k)$ given by Eq. (11) into the form

$$\tilde{\psi} = N[(\tilde{G}_1 + \tilde{G}_2) \cos(kx_0) - i(\tilde{G}_1 - \tilde{G}_2) \sin(kx_0)], \quad (\text{B1})$$

which leads us immediately to the expression

$$\varphi(k) = -\arctan[q(k) \tan(kx_0)] + l\pi, \quad (\text{B2})$$

where the quotient

$$q(k) \equiv \frac{\tilde{G}_1(k) - \tilde{G}_2(k)}{\tilde{G}_1(k) + \tilde{G}_2(k)} \quad (\text{B3})$$

involves the difference and the sum of the amplitudes $\tilde{G}_1$ and $\tilde{G}_2$, and l is an integer which takes into account the multivaluedness of $\arctan$.

In Fig. 11, we show the function q in its dependence on k , which assumes the value $+1$ for $\Delta k_2 \ll |k|$ and almost reaches -1 for $|k| \ll \Delta k_2$. Hence, we can approximate q by the difference

$$q^{(a)}(k) \cong \Theta(|k| - \Delta k_2) - \Theta(\Delta k_2 - |k|), \quad (\text{B4})$$

and the phase-only reconstruction, Eq. (31), assumes the form

$$\psi_p^{(p)} \cong \frac{1}{\sqrt{2\pi}} \left\{ \int_{-\infty}^{-\Delta k_2} dk e^{-i(x+x_0)k} + \int_{-\Delta k_2}^{\Delta k_2} dk e^{-i(x-x_0)k} + \int_{\Delta k_2}^{\infty} dk e^{-i(x+x_0)k} \right\}. \quad (\text{B5})$$

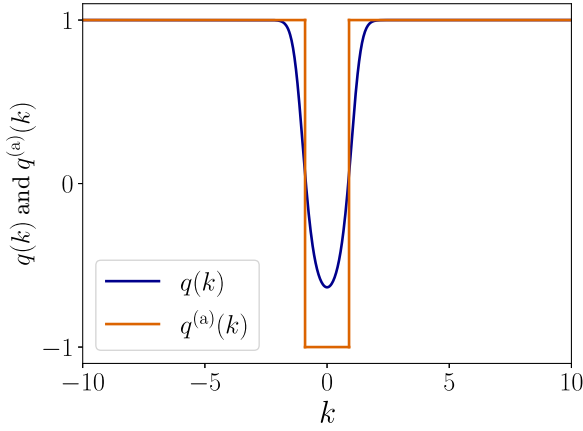


FIG. 11. Comparison between the exact quotient $q = q(k)$ given by Eq. (B3) and consisting of the difference and the sum of the two Gaussians, and the approximation $q^a(k)$ defined according to Eq. (B4) by two jumps between $+1$ and -1 separated by δk_2 . Due to the nonzero value of $\tilde{G}_1(k)$ at $k = 0$, the exact quotient (blue line) does not quite reach the value of -1 of the elementary approximation (orange line) given by Eq. (B4). Fortunately, the resulting phase-only reconstruction following from Eq. (B7) is not sensitive to this deviation.

Here, we have used the fact that in the two domains of k , we can replace q by $q^{(a)} = \pm 1$, which yields

$$-\arctan[\pm \tan(kx_0)] + l\pi = \mp kx_0. \quad (\text{B6})$$

We can immediately perform the integrations and find

$$\begin{aligned} \psi_p^{(p)} &\cong \sqrt{2\pi}[\delta(x + x_0) - \mathcal{D}(\Delta k_2, x + x_0) \\ &\quad + \mathcal{D}(\Delta k_2, x - x_0)], \end{aligned} \quad (\text{B7})$$

where we have introduced the function

$$\mathcal{D}(a, \xi) \equiv \frac{1}{2\pi} \int_{-a}^a dk e^{-i\xi k} = \frac{1}{\pi} \frac{\sin(a\xi)}{a} \quad (\text{B8})$$

familiar from diffraction theory.

When we compare and contrast the approximation, Eq. (B7), of ψ_p emerging from an elementary simplification, Eq. (B4), of the phase to the one [Eq. (56)] originating from neglecting the interference term in ψ , we find an identical structure with the Gaussians being replaced by the sinc functions. Since they display oscillations, we even find the oscillations emerging on the large scale of k values.

APPENDIX C: IDENTICAL GAUSSIANS

In this Appendix, we perform the integral

$$I(\xi) \equiv \int_{-\infty}^{\infty} d\theta \frac{\cos \theta}{|\cos \theta|} e^{-i\xi \theta} \quad (\text{C1})$$

determining the phase-only reconstruction of a superposition of two identical Gaussians located at $-x_0$ and x_0 .

For this purpose, we decompose the integration over the variable θ into domains where $\cos \theta$ is positive and where it is negative, taking into account its 2π periodicity. In this way, we find immediately the expression

$$I(\xi) = \sum_{j=-\infty}^{\infty} \left\{ \int_{-\pi/2+2\pi j}^{\pi/2+2\pi j} d\theta e^{-i\xi \theta} - \int_{\pi/2+2\pi j}^{3\pi/2+2\pi j} d\theta e^{-i\xi \theta} \right\}, \quad (\text{C2})$$

which, after integration, reads

$$I = \frac{2}{i\xi} \left(e^{i\xi\pi/2} - e^{-i\xi\pi/2} \right) \sum_l \delta(\xi - l). \quad (\text{C3})$$

Here, we have taken advantage of the relation

$$e^{-i\xi 3\pi/2} = e^{i\xi\pi/2} e^{-i\xi 2\pi} \quad (\text{C4})$$

and the identity

$$\sum_{j=-\infty}^{\infty} e^{-i2\pi\xi j} = \sum_{l=-\infty}^{\infty} \delta(\xi - l), \quad (\text{C5})$$

which is at the very heart of the Poisson summation formula [23].

Due to the property

$$f(\xi)\delta(\xi) = f(0)\delta(\xi) \quad (\text{C6})$$

of the Dirac delta function, the integral I given by Eq. (C3) reduces to

$$I = 4 \sum_{l=-\infty}^{\infty} \frac{\sin(l\pi/2)}{l} \delta(\xi - l). \quad (\text{C7})$$

Moreover, since

$$\sin(l\pi) = 0,$$

the sum over l reduces to the odd terms only, providing us with the final expression

$$I = 4 \sum_{n=-\infty}^{\infty} \frac{(-1)^n}{2n+1} \delta[\xi - (2n+1)] \quad (\text{C8})$$

for the integral I .

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