

# Hadronic vacuum polarization effect in muonic atoms

Zoia A. Mandrykina<sup>✉,\*</sup>, Moritz Thierfeld<sup>✉</sup>, and Natalia S. Oreshkina<sup>✉†</sup>  
*Max-Planck-Institut für Kernphysik, Saupfercheckweg 1, 69117 Heidelberg, Germany*



(Received 3 September 2025; revised 5 March 2026; accepted 2 June 2026; published 15 June 2026)

The hadronic vacuum polarization (HVP) corrections to energy levels in muonic atoms are studied systematically across the periodic table. Two nuclear charge distribution models have been considered, with a partly analytical solution for a homogeneously charged sphere model and direct numerical integration for a Fermi nuclear charge distribution. Our comparison reveals a good agreement between both approaches with relative differences typically below a few percent. The magnitude of HVP corrections increases with nuclear charge, displaying irregularities that correlate directly with the nonmonotonic behavior of nuclear radii. Despite strong relativistic effects at high  $Z$ , the ratio of hadronic to muonic vacuum polarization remains remarkably stable, deviating by less than 10% from the established 67% nonrelativistic benchmark value.

DOI: [10.1103/3j45-ml4m](https://doi.org/10.1103/3j45-ml4m)

## I. INTRODUCTION

Precision spectroscopy of hydrogenlike and muonic systems provides a robust platform for testing QED and probing nuclear structure [1–3]. QED has proven to be one of the most precisely validated theories in physics, with remarkable agreement between theory and experiment for atomic systems [4,5]. As experimental techniques continue to advance, increasingly subtle quantum effects must be accounted for to maintain a corresponding theoretical accuracy [6].

While electronic systems serve as excellent benchmarks for self-energy correction, vacuum polarization (VP) effects dominate in muonic systems due to the muon's significantly larger mass and closer proximity to the nucleus [7–9]. These factors enhance the sensitivity of muonic systems to short-ranged quantum effects [10,11], including hadronic vacuum polarization (HVP), which involves virtual hadronic particles that interact via the strong force. In muonic atoms, the reduced Bohr radius significantly enhances HVP contributions, which become a measurable fraction of observables such as the Lamb shift and fine-structure splitting [12]. The calculation of HVP effects needs more complex theoretical approaches compared to those required for electronic or muonic VP due to the nonperturbative nature of quantum chromodynamics at low energies. High- $Z$  systems, in particular, exhibit enhanced relativistic and nuclear structure effects [13–15], further amplifying the impact of HVP correction.

Recently, Breidenbach *et al.* [16] have evaluated HVP in a semiempirical way for hydrogenlike ions and muonic

hydrogen. In our work, we extend this methodology to systematically compute HVP corrections across a wide range of nuclear charges  $Z$  in muonic atoms. We adopt a semiempirical, dispersion-relation-based approach, using a parametrized hadronic polarization function fitted to experimental  $e^+e^- \rightarrow$  hadrons cross-section data [17]. The resulting potential is used in fully relativistic energy shift calculations, including finite nuclear size effects with the two nuclear charge distribution models: well-founded two-parameter Fermi nuclear charge distribution and simpler but less realistic homogeneously charged sphere model. Our results provide improved theoretical predictions for low-lying energy levels in muonic systems compared to [18] and offer insight into the scaling behavior of HVP concerning nuclear charge and structure.

Relativistic muonic units ( $m_\mu = \hbar = c = 1$ ) and the Heaviside charge units ( $e^2 = 4\pi\alpha$ ) are used throughout the article. Here  $e = |e|$  denotes the elementary charge.

## II. THEORY

While the leptonic polarization function has an analytical expression, the hadronic case requires a semiempirical construction of its regularized real part due to the nonperturbative behavior of the strong interaction:

$$\text{Re}[\Pi_{\text{Had}}^R(q^2)] = A_i + B_i \ln(1 + C_i |q^2|), \quad (1)$$

with  $A_i$ ,  $B_i$ , and  $C_i$  defined piecewise for different energy intervals of  $q^2$ . Here  $q^2$  denotes the squared four-momentum transfer. The parameters corresponding to the low-energy domain (0.0–0.7 GeV) are particularly relevant in atomic physics calculations due to the comparable energy scale.

In this work, we restrict the analysis to the low-energy domain, which corresponds to the first parametrization region. The adopted values of the fit parameters are  $A_1 = 0$ ,  $B_1 = 0.002\,309\,2$ , and  $C_1 = 3.992\,537\,0 \text{ GeV}^{-2}$  [19]. With

\*Contact author: zoia.mandrykina@mpi-hd.mpg.de

†Contact author: Natalia.Oreshkina@mpi-hd.mpg.de

Published by the American Physical Society under the terms of the Creative Commons Attribution 4.0 International license. Further distribution of this work must maintain attribution to the author(s) and the published article's title, journal citation, and DOI. Open access publication funded by Max Planck Society.

this choice, the hadronic Uehling potential simplifies to

$$V_{\text{HVP}}(r) = -\frac{2e}{\pi} \int_0^{0.7 \text{ GeV}} dq j_0(qr) \tilde{\rho}(q) B_1 \ln(1 + C_1 q^2), \quad (2)$$

where  $j_0(qr)$  is the spherical Bessel function and  $\tilde{\rho}(q)$  is Fourier transformation of the nuclear charge distribution  $\rho(x)$  in coordinate space. In this equation,  $q$  represents the radial component of the three momentum  $\mathbf{q}$ .

Without a loss in the numerical accuracy we can extend the upper integration limit in Eq. (2) from 0.7 GeV to the infinity. Then this expression can be further simplified using the Fourier transform definition and represented in an easier form for computational purposes:

$$V_{\text{HVP}}(r) = -\frac{4\pi e B_1 \sqrt{C_1}}{r} \int_0^\infty dx x \rho(x) \times \left[ E_2\left(\frac{|r-x|}{\sqrt{C_1}}\right) - E_2\left(\frac{r+x}{\sqrt{C_1}}\right) \right], \quad (3)$$

$$V_{\text{HVP,out}}^{\text{sphere}}(r) = -\frac{3Z\alpha B_1 \sqrt{C_1}}{r R_{\text{sph}}^3} \left\{ \sqrt{C_1} R_{\text{sph}} \left[ E_3\left(\frac{|r-R_{\text{sph}}|}{\sqrt{C_1}}\right) + E_3\left(\frac{r+R_{\text{sph}}}{\sqrt{C_1}}\right) \right] - C_1 \left[ E_4\left(\frac{|r-R_{\text{sph}}|}{\sqrt{C_1}}\right) - E_4\left(\frac{r+R_{\text{sph}}}{\sqrt{C_1}}\right) \right] \right\}, \quad (5)$$

and inside it  $r \leq R_{\text{sph}}$ :

$$V_{\text{HVP,in}}^{\text{sphere}}(r) = -\frac{3Z\alpha B_1 \sqrt{C_1}}{r R_{\text{sph}}^3} \left\{ \sqrt{C_1} r + \sqrt{C_1} R_{\text{sph}} E_3\left(\frac{r+R_{\text{sph}}}{\sqrt{C_1}}\right) + C_1 E_4\left(\frac{r+R_{\text{sph}}}{\sqrt{C_1}}\right) - \frac{1}{6} e^{(r-R_{\text{sph}})/\sqrt{C_1}} \times \left[ 2C_1 + \sqrt{C_1}(r+2R_{\text{sph}}) + (r-R_{\text{sph}})(r+2R_{\text{sph}}) \right] - \frac{(r-R_{\text{sph}})^2(r+2R_{\text{sph}})}{6\sqrt{C_1}} E_1\left(\frac{R_{\text{sph}}-r}{\sqrt{C_1}}\right) \right\}. \quad (6)$$

The energy level correction in the first order of the perturbation theory for the bound muon state with the main quantum number  $n$ , the Dirac quantum number  $\kappa$ , and the projection of the total angular momentum  $m$ , can be represented as

$$\Delta E_{n\kappa m} = \langle n\kappa m | V | n\kappa m \rangle = \int_0^\infty dr V(r) (G_{n\kappa}^2(r) + F_{n\kappa}^2(r)), \quad (7)$$

where  $V$  is defined as described above depending on the nuclear model used. The functions  $G_{n\kappa}$  and  $F_{n\kappa}$  stand for large and small radial components of the bound state wave function.

For the calculation of the radial wave functions we employed the method of Ref. [20], which divides the radial domain into two regions. In the inner region, where nuclear effects are significant, the Dirac equation is solved numerically on a radial grid. In the outer region, where the nuclear potential reduces to the pure Coulomb form, the numerical solution is matched to the known analytical solutions for a pointlike nucleus. This hybrid approach maintains high accuracy across the full range of  $Z$  considered here, with the numerical precision set to  $1.2 \times 10^{-14}$ .

### III. RESULTS AND DISCUSSION

#### A. HVP energy corrections

The results for the HVP correction to the energies of the  $1s_{1/2}$ ,  $2s_{1/2}$ ,  $2p_{1/2}$ , and  $2p_{3/2}$  muonic states across a wide

the exponential integral being

$$E_n(x) = \int_1^\infty dt \frac{e^{-xt}}{t^n}. \quad (4)$$

In our work, we have considered two different nuclear charge distribution models to analyze the model dependence of the HVP contribution. The first is the well-known two-parameter Fermi model, for which it is computationally advantageous to transform the hadronic Uehling potential into coordinate space, avoiding repeated numerical evaluation of  $\tilde{\rho}(q)$ . The second is the less realistic but simpler homogeneous charged sphere model, where the corresponding radius is related to the root-mean-square (rms) nuclear charge radius ( $R = \sqrt{\langle r^2 \rangle}$ ) through  $R_{\text{sph}} = \sqrt{5/3 \langle r^2 \rangle}$  and Eq. (2) is evaluated directly using the analytical form of  $\tilde{\rho}(q)$ .

In the case of the spherical charge distribution, integral (3) can be further simplified and represented as a sum for two different regions, outside the nucleus ( $r > R_{\text{sph}}$ ):

range of nuclear charge values  $Z$  are presented in Fig. 1 and in Table I. Uncertainties, given in parentheses, reflect the combined systematic uncertainty discussed here in Sec. III B. For all four states, we observe a general monotonic increase in the magnitude of the energy shift with increasing  $Z$ ; however, in the high- $Z$  region ( $Z \geq 60$ ) we observe a spiky, slightly irregular behavior, especially for the  $1s$  state. Since these anomalies correlate with the nonmonotonic behavior of nuclear charge radii reported in Ref. [21], we repeated our calculations using a simple smooth approximation for the rms radius  $R = 1.2Z^{1/3}$  fm. The corresponding results are shown in the left panel of Fig. 1 with a black dashed line. This simple modification restores the monotonic profile, confirming that such irregularities correspond to fluctuations of the rms values.

TABLE I. HVP energy corrections (eV) for various muonic atoms with Fermi nuclear charge distribution.

|                  | rms (fm) | $1s_{1/2}$ | $2p_{1/2}$ | $2p_{3/2}$ | $2s_{1/2}$ |
|------------------|----------|------------|------------|------------|------------|
| $^{36}\text{Kr}$ | 4.1835   | 43(1)      | 0.62(2)    | 0.39(1)    | 6.1(2)     |
| $^{40}\text{Zr}$ | 4.2694   | 55(2)      | 1.12(3)    | 0.71(2)    | 8.0(2)     |
| $^{54}\text{Xe}$ | 4.7964   | 93(3)      | 5.3(2)     | 3.6(1)     | 14.3(4)    |
| $^{60}\text{Nd}$ | 4.9123   | 112(3)     | 8.7(3)     | 6.0(2)     | 17.6(5)    |
| $^{70}\text{Yb}$ | 5.3215   | 130(4)     | 16.6(5)    | 12.1(4)    | 21.3(6)    |
| $^{82}\text{Pb}$ | 5.5012   | 164(5)     | 30.4(9)    | 22.8(7)    | 27.7(8)    |
| $^{92}\text{U}$  | 5.8571   | 174(5)     | 43(1)      | 34(1)      | 30.3(9)    |

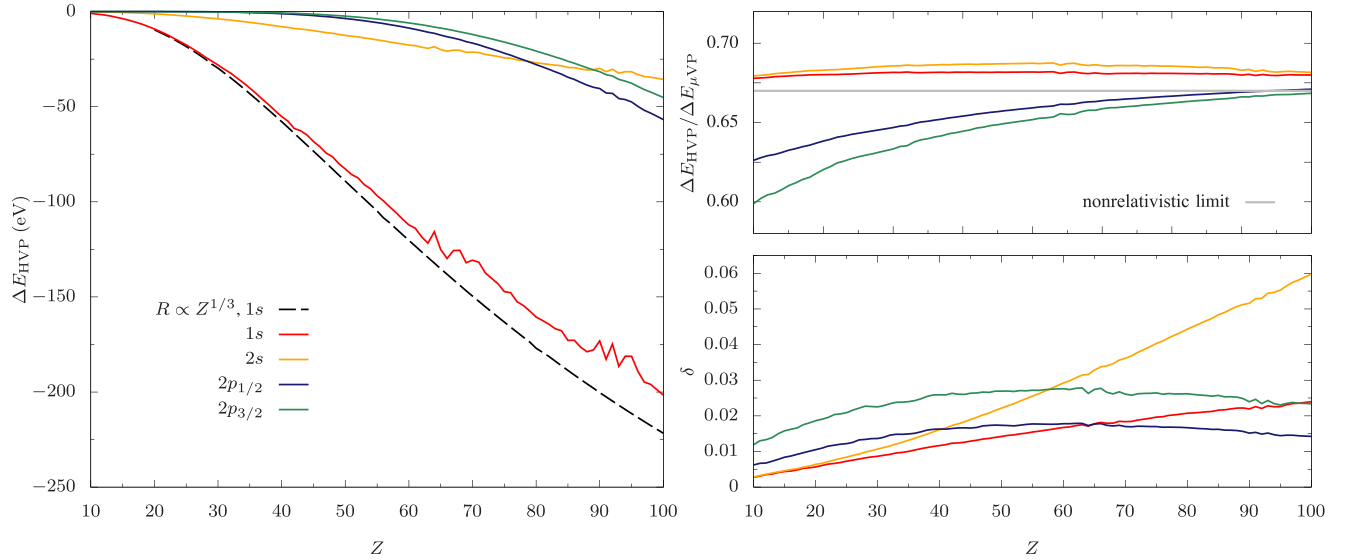


FIG. 1. Left panel: Hadronic vacuum polarization (HVP) energy shift for different muonic atom states using the two-parameter Fermi (2pF) charge distribution. Right top panel: Ratio of the HVP for the 2pF charge distribution energy shift to Uehling energy correction. rms values are taken from [21] unless it is stated explicitly otherwise. The right bottom panel: Relative difference between the 2pF and the homogeneous charged sphere nuclear charge distribution models defined by Eq. (8). All quantities are shown as functions of nuclear charge  $Z$ .

The  $1s_{1/2}$  state exhibits, as expected, the highest sensitivity to the HVP correction, as its wave function is predominantly localized in close proximity to the nucleus, where the HVP potential has its maximum effect. For muonic atoms, this effect is also significantly enhanced compared to electronic systems due to the muon's larger mass.

Next, we compare the HVP effect with the muonic vacuum polarization ( $\mu\text{VP}$ ) correction. Since an order of magnitude in the hierarchy of VP corrections is defined by the particle mass of the corresponding particle-antiparticle in the loop, this comparison is both natural and instructive. Additionally, by expressing the HVP correction as a fraction of the  $\mu\text{VP}$  term, we can better understand its relative significance within the overall QED correction framework and establish consistent scaling patterns across different atomic systems and energy levels. Previously, this ratio was calculated only nonrelativistically for the hydrogen case [22]. It was demonstrated that the HVP contribution amounts to approximately 67% of the  $\mu\text{VP}$  shift. This widely accepted proportion has been used as a standard reference in the field, but its applicability to higher- $Z$  muonic systems remains unexplored. Our systematic investigation provides a comprehensive analysis of how such relation changes with increasing nuclear charge, filling a significant gap in our understanding of vacuum polarization effects in exotic atoms.

The good agreement (differences of 2%–3%) validates both our numerical methods and the overall magnitude of HVP corrections. The small discrepancies primarily reflect improvements in hadronic data parametrization incorporating modern precision measurements from the analysis [19], whereas [23] used the older [24] parametrization.

The upper right panel of Fig. 1 reveals the  $Z$ -dependent evolution of the  $\Delta E_{\text{HVP}}/\Delta E_{\mu\text{VP}}$  ratio across the four lowest muonic states. All values lie within a narrow range (0.60–0.70), confirming its stability across the periodic table while revealing important state-dependent relativistic effects.

The  $s$  states maintain nearly identical and remarkably stable values around 0.675–0.68 throughout the entire  $Z$  range from 20 to 100. These states maintain the highest ratio values among all four considered states, consistently staying above the level of 67%. At the same time, both  $2p_{1/2}$  and  $2p_{3/2}$  states show qualitatively different behaviors, beginning with lower values of 0.60–0.63 at  $Z = 20$  and exhibiting a monotonic growth toward the nonrelativistic benchmark value. This demonstrates the importance of realistic nuclear modeling over simplified approximations, particularly in the low- and medium- $Z$  regions.

The model dependence has been investigated by introducing a relative difference obtained by adopting a potential of a two-parameter Fermi and of a homogeneous charged sphere:

$$\delta = \frac{\Delta E_{\text{HVP}}^{\text{Fermi}} - \Delta E_{\text{HVP}}^{\text{sphere}}}{\Delta E_{\text{HVP}}^{\text{Fermi}}}. \quad (8)$$

For most nuclei across the periodic table, the two approaches yield similar results, with relative differences typically below 0.03 for the  $1s$  state, confirming the reliability of both models. The calculations appear even more robust in the case of the  $p$  states (blue and green dashed lines): the delta values stay constantly below 0.03 throughout the entire  $Z$  range and show minimal variation even for high  $Z \gtrsim 70$ . The nearly identical behavior of the  $2p_{1/2}$  and  $2p_{3/2}$  states reflects their similar radial distributions and reduced nuclear penetration.

More significant discrepancies (up to 0.05–0.06) emerge only for heavy nuclei with  $Z \gtrsim 90$  in the  $2s$  state (see the bottom right panel of Fig. 1). These deviations suggest that the homogeneously charged sphere model may not fully adequately represent the nuclear charge distribution for these particular elements, especially when calculating the HVP correction. In fact, the same behavior can also be observed for  $\mu\text{VP}$  correction, suggesting a general trend for highly localized potentials.

### B. Uncertainty analysis

We have analyzed the systematic uncertainties affecting our HVP calculations using the  $1s_{1/2}$  state of muonic  $_{92}\text{U}$  as a representative example.

The dominant contributions arise from the choice of nuclear charge distribution model (2%), estimated by comparing Fermi and sphere models. Assuming a very conservative variation  $\pm 1\%$  in the half-density radius  $c$ , we would arrive to an uncertainty of 2% in HVP energy correction. The hadronic parametrization of Ref. [19] contributes approximately 0.9% uncertainty in the relevant low-energy region, while numerical integration errors ( $< 10^{-6}$ ) and contributions from energies above our primary cutoff at 0.7 GeV ( $\sim 0.03\%$ ) are negligible.

Combining these sources in quadrature yields a total systematic uncertainty of approximately 3% (5 eV for  $_{92}\text{U}$ ), which is mostly originated from nuclear model uncertainty and is well below typical experimental precision (100 eV Ref. [25]) in muonic atom spectroscopy. Although the HVP correction currently lies below the total theoretical uncertainty [26,27], it must be accounted for in precision analyses of muonic spectra, as future experimental and theoretical advances will demand increasingly complete treatments of all contributing effects.

### IV. CONCLUSIONS

We have calculated the hadronic vacuum polarization corrections to energy levels in muonic atoms across a wide range of nuclear charges from  $Z = 20$  to  $Z = 100$ , covering the four

states:  $1s_{1/2}$ ,  $2s_{1/2}$ ,  $2p_{1/2}$ , and  $2p_{3/2}$ . We verified the predicted scaling trends and found that while HVP corrections generally increase with nuclear charge following the expected theoretical behavior, pronounced deviations emerge in the high- $Z$  region that correlate directly with the nonmonotonic behavior of experimental nuclear charge radii.

We studied the model dependence by comparing results from the two-parameter Fermi nuclear charge distribution with those from the homogeneously charged sphere model, revealing excellent agreement for light and medium nuclei but significant discrepancies reaching up to 5%–6% for heavy elements.

For practical applications in high-precision spectroscopy of muonic atoms, this implies that the HVP correction becomes increasingly important alongside the  $\mu\text{VP}$  effect in heavy elements, where the HVP contribution reaches up to 68% of  $\mu\text{VP}$ . However, deviations from this trend can reach nearly 10%, highlighting the necessity of more rigorous calculations when high precision is required.

### ACKNOWLEDGMENT

The authors thank V. A. Yerokhin for fruitful discussions.

### DATA AVAILABILITY

The data that support the findings of this article are openly available [28], embargo periods may apply.

- 
- [1] A. Knecht, A. Skawran, and S. M. Vogiatzi, Study of nuclear properties with muonic atoms, *Eur. Phys. J. Plus* **135**, 777 (2020).
  - [2] A. Antognini, S. Bacca, A. Fleischmann, L. Gastaldo, F. Hagelstein, P. Indelicato, A. Knecht, V. Lensky, B. Ohayon, V. Pascalutsa, *et al.*, Muonic-atom spectroscopy and impact on nuclear structure and precision QED theory, *arXiv:2210.16929*.
  - [3] T. Y. Saito *et al.*, Muonic x-ray measurement for the nuclear charge distribution: The case of stable palladium isotopes, *Phys. Rev. C* **111**, 034313 (2025).
  - [4] J. Morgner *et al.*, Stringent test of QED with hydrogen-like tin, *Nature (London)* **622**, 53 (2023).
  - [5] T. Okumura *et al.*, Proof-of-principle experiment for testing strong-field quantum electrodynamics with exotic atoms: High precision x-ray spectroscopy of muonic neon, *Phys. Rev. Lett.* **130**, 173001 (2023).
  - [6] A. Antognini *et al.*, Measurement of the quadrupole moment of  $^{185}\text{Re}$  and  $^{187}\text{Re}$  from the hyperfine structure of muonic x rays, *Phys. Rev. C* **101**, 054313 (2020).
  - [7] S. G. Karshenboim, V. G. Ivanov, and E. Yu. Korzinin, Vacuum polarization in muonic atoms: The Lamb shift at low and medium  $Z$ , *Eur. Phys. J. D* **39**, 351 (2006).
  - [8] K. Pachucki, V. Lensky, F. Hagelstein, S. S. Li Muli, S. Bacca, and R. Pohl, Comprehensive theory of the Lamb shift in light muonic atoms, *Rev. Mod. Phys.* **96**, 015001 (2024).
  - [9] E. Borie, Lamb shift in light muonic atoms—Revisited, *Ann. Phys.* **327**, 733 (2012).
  - [10] M.-y. Chen, Nuclear polarization in muonic atoms of deformed nuclei, *Phys. Rev. C* **1**, 1176 (1970).
  - [11] I. A. Valiev and N. S. Oreshkina, Full leading-order nuclear polarization in highly charged ions, *Phys. Rev. A* **109**, 042811 (2024).
  - [12] E. Borie and G. A. Rinker, Improved calculation of the muonic-helium Lamb shift, *Phys. Rev. A* **18**, 324 (1978).
  - [13] V. A. Yerokhin and N. S. Oreshkina, QED calculations of the nuclear recoil effect in muonic atoms, *Phys. Rev. A* **108**, 052824 (2023).
  - [14] N. S. Oreshkina, Self-energy correction to the energy levels of heavy muonic atoms, *Phys. Rev. Res.* **4**, L042040 (2022).
  - [15] E. Borie and G. A. Rinker, The energy levels of muonic atoms, *Rev. Mod. Phys.* **54**, 67 (1982).
  - [16] S. Breidenbach, E. Dizer, H. Cakir, and Z. Harman, Hadronic vacuum polarization correction to atomic energy levels, *Phys. Rev. A* **106**, 042805 (2022).
  - [17] H. Burkhardt, F. Jegerlehner, G. Penso, *et al.*, Uncertainties in the hadronic contribution to the QED vacuum polarization, *Z. Phys. C: Part. Fields* **43**, 497 (1989).
  - [18] M. K. Sundaresan and P. J. S. Watson, Hadronic vacuum-polarization corrections in muonic atoms, *Phys. Rev. D* **11**, 230 (1975).
  - [19] H. Burkhardt and B. Pietrzyk, Update of the hadronic contribution to the QED vacuum polarization, *Phys. Lett. B* **513**, 46 (2001).

- [20] F. Salvat, J. M. Fernández-Varea, and W. Williamson, Accurate numerical solution of the radial Schrödinger and Dirac wave equations, *Comput. Phys. Commun.* **90**, 151 (1995).
- [21] I. Angeli and K. P. Marinova, Table of experimental nuclear ground state charge radii: An update, *At. Data Nucl. Data Tables* **99**, 69 (2013).
- [22] J. L. Friar, J. Martorell, and D. W. L. Sprung, Hadronic vacuum polarization and the Lamb shift, *Phys. Rev. A* **59**, 4061 (1999).
- [23] E. Borie, Hadronic vacuum polarization correction in muonic atoms, *Z. Phys. A: At. Nucl.* **302**, 187 (1981).
- [24] G. J. Gounaris and J. J. Sakurai, Finite-width corrections to the vector-meson-dominance prediction for  $\rho \rightarrow e^+e^-$ , *Phys. Rev. Lett.* **21**, 244 (1968).
- [25] P. Bergem, G. Piller, A. Rueetschi, L. A. Schaller, L. Schellenberg, and H. Schneuwly, Nuclear polarization and charge moments of  $^{208}\text{Pb}$  from muonic x rays, *Phys. Rev. C* **37**, 2821 (1988).
- [26] Z. Sun, K. A. Beyer, Z. A. Mandrykina, I. A. Valuev, C. H. Keitel, and N. S. Oreshkina,  $^{208}\text{Pb}$  nuclear charge radius revisited: Closing the fine-structure-anomaly gap, *Phys. Rev. Lett.* **135**, 163002 (2025).
- [27] K. A. Beyer, I. A. Valuev, Z. A. Mandrykina, Z. Sun, and N. S. Oreshkina, Relativistic recoil as a key to the fine-structure puzzle in muonic  $^{90}\text{Zr}$ , *arXiv:2511.22298*.
- [28] Z. A. Mandrykina, M. Thierfeld, and N. S. Oreshkina, Hadronic vacuum polarization effect in muonic atoms, Zenodo, 2025, <https://zenodo.org/records/17045083>.