

# Multi-ion entangling gates mediated by spectrally unresolved modes

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(Received 13 December 2025; accepted 11 June 2026; published 8 July 2026)

High-fidelity entangling gates in trapped-ion processors are most commonly realized by spectrally isolating a single motional bus mode. As the ion chain grows, however, the coupling of each ion to any one mode weakens, making single-mode gates progressively slower. Existing multimode schemes offer a partial remedy but rely on perturbative driving and rotating-wave approximations, imposing trade-offs between speed and fidelity that become more severe with chain length. We introduce a nonperturbative magnetic-gradient scheme in which all axial motional modes participate collectively in mediating the entangling interaction. Exact spin-motion decoupling is enforced at the end of the protocol by construction, so that the achievable gate speed is limited only by the available gradient strength and control bandwidth. The framework supports both multiqubit gates and selective two-qubit gates between arbitrary pairs in a linear string, including simultaneous operation on multiple pairs, and remains effective as the ion number increases.

DOI: [10.1103/357p-y5qf](https://doi.org/10.1103/357p-y5qf)

## I. INTRODUCTION

Trapped-ion systems are among the most mature platforms for quantum information processing, yet realizing high-fidelity entangling gates remains a central challenge. Ions interact through their mutual Coulomb repulsion, but this interaction couples their motional rather than their electronic degrees of freedom. Entanglement between qubits must therefore be mediated indirectly: the shared motional modes serve as an interaction bus, and any gate protocol must ensure that the spins are cleanly disentangled from this bus by the time the operation concludes [1].

The tension between these two requirements becomes acute as ion chains grow. In the simplest and most widely adopted approach, a single collective motional mode is spectrally isolated to serve as the interaction bus, with the qubit-motion coupling generated by laser fields acting on optical or Raman transitions [2]. This strategy is conceptually clean and has underpinned many of the highest-fidelity two-qubit gates demonstrated to date, almost all of which involved just two ions in a single well [1]. Its scaling, however, is unfavorable: as the chain length increases, the coupling of each ion to any given normal mode decreases, so maintaining a fixed gate speed demands progressively stronger driving. Spectral addressability is fundamentally a weak-driving principle, and these two requirements are in direct conflict. Longer chains therefore face an inherent trade-off between gate speed and gate fidelity that single-mode schemes cannot resolve.

Multimode strategies offer a potential alternative [3], but existing realizations tend to operate in one of two limiting

regimes, each with its own drawbacks. Near-resonant schemes [4–7] allow several modes to participate, but the gate time must be long enough for all relevant phase-space trajectories to close simultaneously—a closure condition that becomes increasingly difficult to satisfy as the number of participating modes grows. Far-detuned schemes [2,3,5,8] avoid this problem by keeping the phonons only virtually excited, but the resulting interaction is correspondingly weaker, requiring longer times to accumulate the desired entangling phase. Both limits trade one problem for another, and neither aims at exploiting the full collective dynamics of the ion chain.

A qualitatively different route, complementary to optical approaches, is offered by microwave-driven gates that use magnetic-field gradients to generate qubit-state-dependent forces [9,10]. Rather than coupling the qubit transition directly to the motional sidebands via laser fields, these schemes modulate a static or oscillating gradient to impart spin-dependent displacements on the ion crystal. This makes them particularly attractive for scalability in surface-electrode trap architectures [11,12]: they do not require the Lamb-Dicke approximation, can tolerate higher motional temperatures, and are amenable to compact on-chip integration without optical access. Their main practical limitation is that achievable magnetic-field gradients are modest compared with optical fields, leading to weaker qubit-motion coupling and longer gate times [1,13–15]. Modulating the gradient at a frequency close to a targeted motional mode amplifies the effective qubit-qubit interaction by a factor of  $1/\delta$ , where  $\delta$  is the detuning [16], but this resonant enhancement reintroduces spectral selectivity and with it the same crosstalk and scaling limitations it was meant to avoid.

In this work, we take a different approach, one that does not require spectral resolution of individual modes and instead harnesses the collective dynamics of all axial motional modes simultaneously. The central idea is to design a control protocol based on a time-dependent magnetic-field gradient, in which

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all relevant modes participate in mediating the qubit-qubit interaction, while the closure conditions required for final spin-motion decoupling are enforced exactly by construction rather than perturbatively. Because the framework makes no use of the rotating-wave approximation, it remains valid for strong driving, and the achievable gate speed is limited in practice only by the maximum magnetic-field gradient and control bandwidth available in the experiment. The resulting scheme applies both to multiqubit entangling operations and to two-qubit gates between arbitrary pairs within a linear ion string, and its performance does not degrade as the chain length increases in the way single-mode protocols inevitably must.

## II. FRAMEWORK

The system we consider consists of  $N$  ions confined in a common potential well and coupled to  $N$  collective motion modes via a time-dependent magnetic-field gradient. The Hamiltonian describing this system is therefore taken to be

$$H(t) = \sum_{j=1}^N \frac{\omega_j(t)}{2} Z_j + \sum_{l=1}^N v_l a_l^\dagger a_l + f(t) \sum_{j,l} v_l \eta_{jl} Z_j (a_l^\dagger + a_l), \quad (1)$$

where  $Z_j$  is the Pauli-Z operator for the qubit encoded in ion  $j$  and  $\omega_j(t) = \omega_j^{(0)} + \delta\omega_j(t)$  is its instantaneous qubit transition frequency, including the bare splitting  $\omega_j^{(0)}$  and the local Zeeman shift  $\delta\omega_j(t)$  generated by the longitudinal magnetic field at the ion position. The operators  $a_l^\dagger$  and  $a_l$  create and annihilate excitations in the  $l$ th normal mode of motion with frequency  $v_l$ . The dimensionless coupling strength  $\eta_{jl}$  between ion  $j$  and motion in normal mode  $l$  is given by

$$\eta_{jl} = g_F m_F \mu_B \frac{z_0^{(l)}}{\hbar v_l} \chi_{jl} \frac{\partial B}{\partial z}, \quad (2)$$

where the scalars  $\chi_{jl}$  describe the displacement of ion  $j$  in terms of the normal modes  $l$ ,  $\mu_B$  is the Bohr magneton,  $g_F$  is the hyperfine Landé  $g$  factor,  $m_F$  is the magnetic quantum number, and  $z_0^{(l)} = \sqrt{\hbar/(2mv_l)}$  is the zero-point extent of mode  $l$  for an ion of mass  $m$ . The magnetic-field gradient  $\partial B/\partial z$  is assumed to be uniform across the ion chain.

Time-dependent modulations of the magnetic-field gradient are described by the envelope function  $f(t)$  [such that  $|f(t)| \leq 1$  at any time  $t$ ] in Eq. (1), which controls the temporal profile of the interaction strength. For example,  $f(t) = 1$  corresponds to a static gradient, while a monochromatic drive at frequency  $\omega$  would correspond to  $f(t) = \cos(\omega t)$ . The present approach exploits more general time-dependent profiles of  $f(t)$  in order to intentionally excite and manipulate the full collective motion of the ion chain [see Fig. 1(a) for a schematic overview].

### A. Effective interactions

Similar to the case of a static field gradient, the Hamiltonian can be diagonalized with a polaron transformation [9,17],

but a time-dependent gradient requires a time-dependent transformation. With the specific choice

$$U_P(t) = \exp \left( i \sum_{j,l} (g_l^*(t) a_l + g_l(t) a_l^\dagger) \eta_{jl} Z_j \right), \quad (3)$$

where the time-dependent functions  $g_l(t)$  satisfy the equations of motion

$$\dot{g}_l(t) + i v_l g_l(t) = v_l f(t) \quad (4)$$

of a driven harmonic oscillator, the transformed Hamiltonian

$$\tilde{H}(t) = U_P(t) H(t) U_P^\dagger(t) + i \dot{U}_P(t) U_P^\dagger(t)$$

reads

$$\tilde{H}(t) = \sum_j \frac{\omega_j(t)}{2} Z_j + \sum_l v_l a_l^\dagger a_l + \sum_{ij} \alpha_{ij}(t) Z_i Z_j, \quad (5)$$

with coupling elements  $\alpha_{ij}(t) = \sum_l \eta_{il} \eta_{jl} \Phi_l(t)$  given in terms of the real functions

$$\Phi_l(t) = v_l f(t) \text{Im}(g_l(t)), \quad (6)$$

where  $\text{Im}(g_l(t))$  denotes the imaginary part of  $g_l(t)$ .

Although the Hamiltonian  $\tilde{H}(t)$  is generally time-dependent, the commutator  $[\tilde{H}(t_1), \tilde{H}(t_2)] = 0$  holds for all  $t_1$  and  $t_2$ . This property ensures that the time-evolution operator  $\tilde{U}(t)$  induced by  $\tilde{H}(t)$  is of the simple form,

$$\tilde{U}(t) = \exp \left( -i \int_0^t d\tau \tilde{H}(\tau) \right),$$

eliminating the need for time ordering. The dynamics generate geometric phases  $\int_0^T dt \Phi_l(t)$  that depend on the phase-space trajectory of oscillator mode  $l$  [see Figs. 1(b) and 1(c)].

Given the ability to implement a suitable driving function  $f(t)$ , the dynamics induced by the Hamiltonian  $\tilde{H}(t)$  in Eq. (5) generate mode-dependent phase shifts

$$D_l = \int_0^T dt \Phi_l(t). \quad (7)$$

Collecting these into the diagonal matrix  $D = \text{diag}(D_1, \dots, D_N)$ , the propagator  $\tilde{U}(T)$  at time  $T$  takes the form

$$\tilde{U}(T) = \exp \left[ -i \left( \sum_j \phi_j Z_j + T \sum_l v_l a_l^\dagger a_l + \sum_{j \neq k} \Lambda_{jk} Z_j Z_k \right) \right], \quad (8)$$

where

$$\phi_j = \frac{1}{2} \int_0^T dt \omega_j(t) \quad (9)$$

are the accumulated local single-qubit phases, and the interaction matrix  $\Lambda$  has elements

$$\Lambda_{jk} = \sum_l \eta_{jl} \eta_{kl} D_l, \quad (10)$$

i.e.,  $\Lambda = \eta D \eta^\top$ .

Since  $Z_j^2 = \mathbb{1}$  for all  $j$ , any diagonal term of the form  $\Lambda_{jj} Z_j^2$  contributes only a global phase and may therefore be absorbed. Similarly, the local single-qubit phases  $\phi_j$ , which

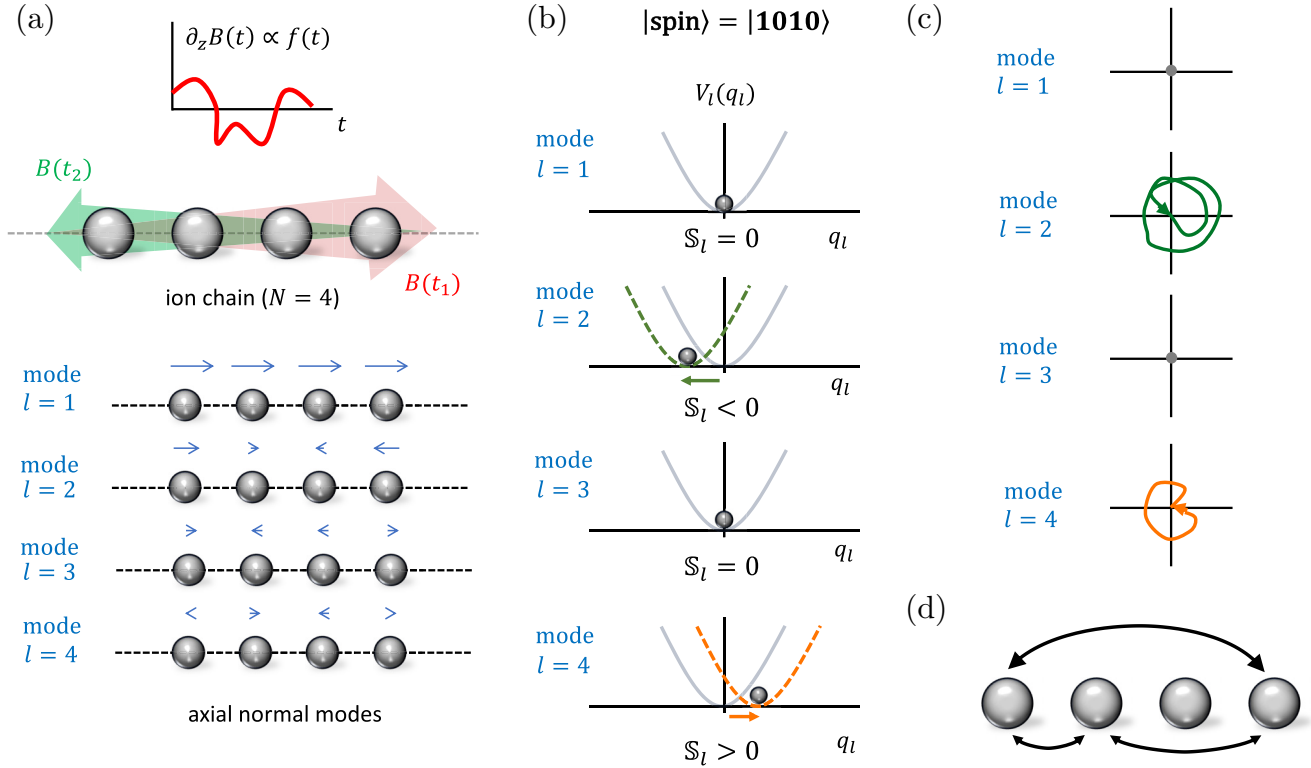


FIG. 1. Schematic overview of the magnetic-field-gradient gate mechanism. (a) A global, time-dependent magnetic-field gradient couples the internal states of all ions in the chain to all axial normal modes. Its time dependence is illustrated by two representative instants  $t_1$  and  $t_2$ , at which the gradient has opposite sign. The collective nature of the ion motion is encoded in the mode-participation factors  $\chi_{ji}$ . In the mode sketches, the arrows indicate the displacement pattern of each ion for a given mode: their direction gives the sign of the mode amplitude, and their length its relative magnitude. (b) For a fixed computational-basis spin branch (here  $|1010\rangle$  as an example), the gradient acts as a spin-dependent force on each mode, shifting the equilibrium position of the corresponding harmonic oscillator by an amount proportional to the branch-dependent overlap  $S_l = \sum_j \eta_{jl} z_j$ , with  $z_j = \pm 1$ . Modes with  $S_l = 0$  are not displaced; in this example, this occurs for the first and third modes. The polaron transformation in Eq. (3) defines a frame in which these mode-dependent displacements are absorbed explicitly. (c) The spin-dependent displacements generate phase-space trajectories for each mode. The control waveform is designed so that all trajectories close at the final time, thereby ensuring exact spin-motion decoupling. The enclosed areas determine the geometric phases accumulated during the gate. (d) Once all motional trajectories are closed, the remaining effect is a purely spin-dependent interaction described by the effective Ising coupling matrix  $\Lambda_{jk}$  in Eq. (10), synthesized from the combined contributions of all participating modes.

accumulate from the qubit frequencies and the Zeeman shifts of the longitudinal field, do not affect the entangling structure. They can be handled separately through phase tracking, calibration, or refocusing pulses. In the numerical examples below, these deterministic phases are tracked explicitly and absorbed into local  $Z$ -frame updates when comparing the implemented operation with the target entangling gate.

After removing these local and global contributions, the effective entangling interaction is entirely determined by the off-diagonal entries of  $\Lambda$ , which in turn depend only on the  $N$  mode-dependent phases  $D_l$  set by the modulation  $f(t)$ . Thus, for a system of  $N$  qubits, the global drive  $f(t)$  provides exactly  $N$  independent tunable parameters with which to synthesize the interaction geometry [see Fig. 1(d)].

This parameter counting makes explicit a fundamental constraint of the global-gradient setting. A general pairwise  $ZZ$  interaction geometry requires  $N(N-1)/2$  independent parameters, whereas a spatially uniform magnetic-field gradient provides access only to the  $N$  mode-dependent phases  $D_l$ . This is not a limitation of a particular drive waveform  $f(t)$ , but an intrinsic consequence of controlling the entire chain

through a single global field. In principle, local site-dependent gradients could lift this limitation, but generating strong spatially varying fields on the scale of the ion-ion separation in a confining trap is experimentally highly challenging. We therefore keep the global-gradient constraint and show next how spin-echo techniques can be used to extend the class of realizable effective interaction geometries.

### B. Spin echo

Interaction matrices  $\Lambda$  that cannot be obtained directly in the form of Eq. (10) can nevertheless be realized by combining the present framework with spin-echo techniques [18,19], which are well established in the context of static magnetic-field gradients [13,20].

This is based on the fact that applying  $\pi$  pulses  $\exp(-i\frac{\pi}{2}X)$  to a selected subset of ions before evolution under  $\tilde{H}(t)$  in Eq. (5), and the corresponding inverse pulses  $\exp(i\frac{\pi}{2}X)$  afterward, produces dynamics equivalent to evolution under a Hamiltonian of the same form but with the vector  $\vec{\omega}$  of resonance frequencies replaced by  $O\vec{\omega}$ , and the interaction matrix

$\alpha$  replaced by  $O\alpha O^\top$ , where the diagonal matrix  $O$  has entries  $-1$  for ions addressed by the  $\pi$  pulses and  $+1$  for any other ion.

A propagator of the form

$$\exp\left(i \sum_{jk} \Lambda_{jk} Z_j Z_k\right) \quad (11)$$

can thus be turned into the propagator

$$\exp\left(i \sum_{jk} [O\Lambda O^\top]_{jk} Z_j Z_k\right) \quad (12)$$

with a spin echo.

Since all propagators of this form commute with each other, a sequence of  $N_I$  periods of controlled dynamics interleaved with  $\pi$  pulses thus yields the effective interaction matrix

$$\Lambda = \sum_{p=1}^{N_I} O^{(p)} \Lambda^{(p)} (O^{(p)})^\top, \quad (13)$$

where  $\Lambda^{(p)}$  is the interaction matrix for period  $p$ , and  $O^{(p)}$  is the matrix for the corresponding  $\pi$  pulses. This lifts the restriction to the interaction matrices in Eq. (10), and the use of sufficiently many periods gives access to the most general interaction matrix  $\Lambda$  that one might aim at realizing.

The construction of elementary interaction matrices  $\Lambda^{(p)}$  and corresponding  $\pi$  pulses can be performed for any targeted gate. As shown in the following, the realization of an interaction of exactly one pair of ions can be achieved with  $N_I = 2$  periods of controlled dynamics. This construction directly yields any desired interaction geometry as a sequence of pairwise interactions, but bespoke constructions will likely yield constructions with fewer periods of controlled dynamics.

An interaction matrix  $\Lambda$  with a finite interaction for exactly one pair of ions can be obtained with two elementary interaction matrices  $\Lambda^{(p)}$  satisfying  $\Lambda^{(2)} = -\Lambda^{(1)}$ . There are no  $\pi$  pulses for the first period of controlled dynamics, but the second period is dressed by a  $\pi$  pulse on ion  $k$ , such that the matrix  $O^{(2)}$  has the elements  $O_{kk}^{(2)} = -1$  and  $O_{jj}^{(2)} = 1$  for  $j \neq k$ . With the transformation

$$[O^{(2)} \Lambda^{(2)} (O^{(2)})^\top]_{ij} = \Lambda_{ij}^{(2)} \quad (14)$$

for both  $i$  and  $j$  different than  $k$ , and for both  $i$  and  $j$  equal to  $k$ , and

$$[O^{(2)} \Lambda^{(2)} (O^{(2)})^\top]_{ik} = [O^{(2)} \Lambda^{(2)} (O^{(2)})^\top]_{ki} = -\Lambda_{ik}^{(2)} \quad (15)$$

for  $i \neq k$ , the total interaction matrix reads

$$\begin{aligned} \Lambda &= \Lambda^{(1)} + O^{(2)} \Lambda^{(2)} (O^{(2)})^\top \\ &= 2 \sum_{j, j \neq k} \Lambda_{kj}^{(1)} (|k\rangle\langle j| + |j\rangle\langle k|). \end{aligned} \quad (16)$$

This corresponds to an interaction geometry in which ion  $k$  interacts with any other ion, but the ions different from  $k$  do not interact with each other.

The remaining freedom lies in the choice of  $\Lambda^{(1)}$ , which is still parametrized by the  $N$  mode-dependent phases  $D_l$  [following Eq. (10)], determined by the modulation of the field

gradient via Eq. (7). These  $N$  free parameters are sufficient to realize a finite value for the interaction constant between ion  $k$  and some other selected ion while realizing a vanishing interaction between ion  $k$  and any other ion. Two periods of controlled dynamics combined with a simple spin echo thus suffice to realize a pure pairwise  $ZZ$  coupling on a chosen ion pair.

### C. Boundary conditions

The polaron transformation Eq. (3) describes the concept of dressing the bare qubit degrees of freedom with the motion. While the Hamiltonian  $H(t)$  in Eq. (1) captures the interaction of the bare qubits with the motional degrees of freedom, the Hamiltonian  $\tilde{H}(t)$  in Eq. (5) captures the interactions among the dressed qubits.

For the dressed-qubit picture to remain consistent throughout a sequence of several gates, one must impose boundary conditions on  $g_l(t)$  at the start ( $t = 0$ ) and end ( $t = T$ ) of each gate [see Fig. 1(c)]. The only requirement is that a gate's final condition coincides with the next gate's initial condition—but how one chooses those conditions depends on the gradient scheme in use.

To enable a direct comparison between the schemes developed here and earlier approaches employing static gradients [9] or oscillating gradients [10,16], we next discuss boundary conditions that are consistent with these two established methods.

#### 1. Consistency with static gradient

In the case of a static gradient, qubits are defined by dressed eigenstates in the presence of an always-on field gradient. Initialization and readout therefore occur with a nonzero gradient at both  $t = 0$  and  $t = T$ . Consistent with this picture, the boundary conditions can be chosen as

$$ig_l(0) - f(0) = ig_l(T) - f(T) = 0, \quad (17)$$

which, from Eq. (4), implies  $\dot{g}_l(0) = \dot{g}_l(T)$ . These conditions guarantee that the dressed-qubit basis at  $t = 0$  and  $t = T$  matches the static-gradient eigenbasis, and that the ions' motion is not altered as a result of the gate operation.

A key distinction, however, is that in the present scheme  $f(t)$  is time-dependent, so  $f(0)$  and  $f(T)$  need not be equal. As a result, the initial polaron transformation  $U_P(0)$  may differ from the final transformation  $U_P(T)$ , allowing the collective motional states to be displaced by different amounts at the start and end of the gate while remaining stationary at both times.

#### 2. Consistency with oscillating gradients

In the case of oscillating gradients, gates are implemented using finite gradients that are switched on and off instantaneously at the start and end of the gate operation, so that outside the gate window ( $t < 0$  or  $t > T$ ) the gradient vanishes, and qubits reduce to pure spin states.

Thus, to decouple the qubit-qubit entangling interaction from the initial phonon state, one enforces

$$g_l(T) = e^{-iT\nu_l} g_l(0), \quad (18)$$

which ensures that the net motional displacement vanishes at the end of the gate. This cleanly isolates the spin dynamics and yields a spin-spin entangling operation free from phonon-state dependence.

#### D. Gate synthesis

Building on Sec. II B, the synthesis of a general  $N$ -qubit entangling gate with prescribed interaction

$$\sum_{j,k} \mathbf{A}_{jk} Z_j Z_k, \quad (19)$$

where  $\mathbf{A}$  is the target coupling matrix, can be realized by combining global driving segments  $f_m(t)$  with optional refocusing  $\pi$ -pulse layers. For each segment  $m$ , we denote by  $\tilde{U}[f_m]$  the propagator generated by the gradient modulation during that interval, and by  $U_\pi^{(m)}$  the refocusing layer applied before and after the segment. When no refocusing is required,  $U_\pi^{(m)}$  is simply the identity. The net gate then takes the form

$$\tilde{U}(T) = \prod_m U_\pi^{(m)} \tilde{U}[f_m] U_\pi^{(m)}, \quad (20)$$

with the product understood to be time-ordered.

For sequences involving refocusing pulses, the natural construction is to impose the boundary conditions separately on each gradient-driven interval. Each waveform  $f_m(t)$  is therefore optimized so that the relevant spin-motion decoupling conditions are satisfied at the end of its segment, before the next refocusing layer is applied. The  $\pi$  pulses then modify only the spin sign pattern between consecutive closed blocks and do not interrupt open phase-space trajectories.

This instantaneous-pulse approximation is supported by the usual separation of timescales in trapped-ion experiments: single-qubit rotations can be performed on nanosecond to submicrosecond timescales, including microwave control on  $\approx 20$  ns timescales and addressed microwave gates with 600 ns  $\pi/2$  pulses [10,21], whereas entangling gates typically require microsecond to millisecond windows [13,22]. Finite pulse durations are therefore small on the scale of the entangling windows, justifying the treatment of  $U_\pi^{(m)}$  as instantaneous layers.

#### 1. Optimization formulation

Gate synthesis is naturally formulated as a mixed discrete-continuous optimization problem. The continuous degrees of freedom are the waveforms  $f_m(t)$  and segment durations  $T_m$ ; the discrete choices are whether a  $\pi$  pulse is applied to each ion in each interval. An iterative procedure jointly optimizes all three (detailed in Sec. II of the Supplemental Material [23]): the  $\pi$  pulses extend the reachable coupling classes, while the drives  $f_m(t)$  are shaped to satisfy two simultaneous objectives: reproducing the target pairwise phase shifts [Eq. (7)] and satisfying the boundary conditions (Sec. II C).

More specifically, the endpoint constraints—whether taken in the static-gradient convention of Eq. (17) or in the oscillating-gradient convention of Eq. (18)—impose two real conditions per motional mode, corresponding to the real and

imaginary parts of the closure condition. For a system with  $N$  modes, this amounts to a total of  $2N$  real constraints.

#### 2. Parametrization and scaling

To satisfy the  $2N$  boundary constraints while maintaining sufficient flexibility to shape the interaction geometry, the waveform must contain at least  $2N$  independent parameters, plus additional degrees of freedom for interaction design. In the explicit constructions used here, we expand the magnetic-gradient envelope in harmonic components,

$$f_m(t) = \sum_{j=1}^{2N+1} A_j \cos(\omega_j t + \vartheta_j), \quad (21)$$

where each component is specified by the amplitude, frequency, and phase ( $A_j$ ,  $\omega_j$ ,  $\vartheta_j$ ).

This parametrization is particularly natural here because the exact spin-motion propagator, mode displacements, and closure conditions all admit closed-form expressions for the harmonic ansatz (see Supplemental Material [23]), reducing the optimization to a finite-dimensional algebraic problem without repeated numerical propagation of the dynamics; general optimal-control methods such as gradient-ascent pulse engineering (GRAPE), Krotov, or chopped random basis (CRAB) [24–26] are therefore unnecessary.

For the few-ion examples, the choice of  $2N + 1$  harmonic components, corresponding to  $3(2N + 1)$  free parameters per segment, is sufficient to obtain high-fidelity gates. For larger systems, as discussed in Sec. IV, the infidelity can be further reduced by enlarging the parametrization. In practice, keeping the number of free parameters proportional to  $N$  provides a scalable compromise between flexibility, numerical tractability, and avoidance of unnecessary overparametrization.

### III. QUANTUM GATES

The gate construction is demonstrated explicitly for a four-ion chain, which is small enough to permit full simulation of the motional dynamics yet large enough to exhibit the essential features of the multimode structure. Gate design for larger systems is discussed in Sec. IV. Throughout, the time-dependent magnetic-field gradient is parametrized using the harmonic expansion of Eq. (21).

An important computational advantage of the present approach is that finding an explicit driving protocol—including the temporal modulation  $f(t)$  and the  $\pi$ -pulse sequence—requires no simulation of the system dynamics and is therefore not subject to the exponential scaling of Hilbert space with qubit and mode number. Detailed numerical validation of gate fidelity against existing schemes, by contrast, does require full simulation of the many-body dynamics; this is what restricts the explicit examples in this section to four ions. For a chain of this length, a decomposition involving nine frequency components is typically sufficient to meet the boundary conditions and engineer a broad class of effective couplings. More efficient parametrizations may exist, but this choice already provides ample flexibility.

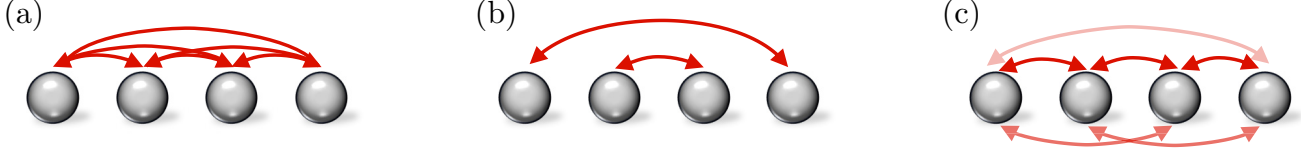


FIG. 2. Illustration of various interaction geometries, where arrows indicate interacting elements, and their color represents interaction strength (light for weak, and dark for strong). In panel (a), uniform coupling is shown across all ions, with equal interaction strength between each pair. In panel (b), a rainbow-style entanglement geometry is depicted, where symmetric pairs of ions are coupled with equal strength. In panel (c), distance-dependent coupling is represented, where the ion at site  $j$  interacts with ions at sites  $k > j$ , with interaction strength decreasing as a function of the distance between them. This interaction pattern forms the basis of the controlled-phase gates used in implementing the quantum Fourier transform.

### A. Homogeneous Ising interaction

As a concrete example, consider the generation of a homogeneous long-range Ising interaction of the form

$$H_{\text{Ising}} = J \sum_{i < j} Z_i Z_j, \quad (22)$$

where every pair of qubits interacts with the same strength  $J$ , regardless of their separation [see Fig. 2(a)]. This fully connected geometry is highly nonlocal and cannot be realized through nearest-neighbor couplings alone.

The target interaction corresponds to a matrix  $\mathbf{A}$  with elements  $A_{ij} = J$ . The resulting matrix  $\mathbf{D}$  [Eq. (10)] is diagonal, such that the desired interaction can be realized using a single global driving field  $f(t)$ .

#### 1. Existing schemes

Before introducing optimized waveforms, it is instructive to revisit two widely used limiting cases for generating the effective Ising interaction: static [9] and monochromatic drivings [16]. These approaches offer conceptual simplicity but face trade-offs between gate speed and fidelity.

(a) *Static gradient.* In the case of a static magnetic-field gradient, where  $f(t) = 1$ , the accumulated geometric phases grow linearly with time as  $D_l = -v_l T$ , leading to an effective interaction matrix

$$\Lambda_{jk} = -T \sum_l v_l \eta_{jl} \eta_{kl}. \quad (23)$$

This form constrains the possible interaction patterns: it naturally includes contributions from all motional modes, each weighted by its frequency  $v_l$ . While the center-of-mass (c.m.) mode indeed contributes equally to all qubit pairs, the presence of other modes with unequal couplings prevents the realization of a perfectly homogeneous interaction. As a result, a static global gradient alone cannot generate a fully connected Ising model with uniform couplings.

However, when combined with appropriately timed  $\pi$  pulses (see Sec. II B), it becomes possible to synthesize the desired interaction geometry. One can show that at least five interaction windows [ $N_l = 5$  in Eq. (13)] are required to approximate a homogeneous Ising interaction with vanishingly small infidelity; with fewer segments, the inhomogeneous coupling strengths cannot be compensated (see Sec. III of the Supplemental Material [23]).

Figure 3(a) shows the minimum total gate duration  $T$  (obtained by summing all interaction intervals and assuming

instantaneous  $\pi$  pulses), expressed in units of the COM-mode period  $2\pi/\nu_C$ , as a function of the c.m. coupling strength  $\eta_C$ , with the remaining mode couplings scaled accordingly.

Since the effective qubit-qubit interaction in Eq. (23) is quadratic in the spin-motion coupling, the gate timescales as  $T \propto 1/\eta_C^2 \nu_C$ . This is the fundamental bottleneck of the static approach: because the interaction is second order in the per-ion coupling, weaker gradients translate directly into slower gates. The corresponding dependence on ion number is analyzed separately in Sec. IV of the Supplemental Material [23].

(b) *Modulated gradient (single-tone) drive.* In contrast with the static case, a time-modulated magnetic-field gradient allows one to selectively enhance the contribution of a chosen motional mode  $C$ . The drive frequency is set close to the mode frequency,  $\omega \approx \nu_C$ , with detuning  $\delta = \nu_C - \omega$ . Under the rotating-wave approximation, the accumulated off-diagonal coupling over a gate duration  $T = 2\pi/\delta$  is

$$\Lambda_{jk} = 2\pi \frac{\Omega_{jC} \Omega_{kC}}{\delta^2}, \quad (24)$$

where  $\Omega_{jC} = \eta_{jC} \nu_C$  is the effective Rabi amplitude for ion  $j$  [16].

Homogeneous qubit-qubit coupling requires  $\Omega_{jC} = \Omega_{kC} \equiv \Omega_C$ , which can be achieved for generic ion chains only through the c.m. mode (or, equivalently, for the special case of two ions using the stretch mode). Imposing a uniform target strength  $J$  then fixes the detuning to

$$\delta = \Omega_C \sqrt{\frac{2\pi}{J}}, \quad (25)$$

which leads to a drive frequency and gate duration

$$\omega = \nu_C \left( 1 - \sqrt{\frac{\pi}{2J}} \eta_C \right), \quad T = \frac{2\sqrt{2\pi J}}{\eta_C \nu_C}. \quad (26)$$

For fixed ion number, the gate time therefore scales as  $T \propto \eta_C^{-1}$ , in contrast with the  $T \propto \eta_C^{-2}$  behavior of the static protocol. This parametric speedup is visible in Fig. 3(a), which shows up to an order-of-magnitude reduction in  $T$  in the small Lamb-Dicke parameter regime. The scaling with chain length is addressed separately in Sec. IV of the Supplemental Material [23].

The physical origin of this improvement is the resonant amplification of the c.m.-mode contribution by the factor  $\nu_C/\delta \gg 1$ , which effectively converts the second-order process into a first-order one for that mode. The price, however, is that the

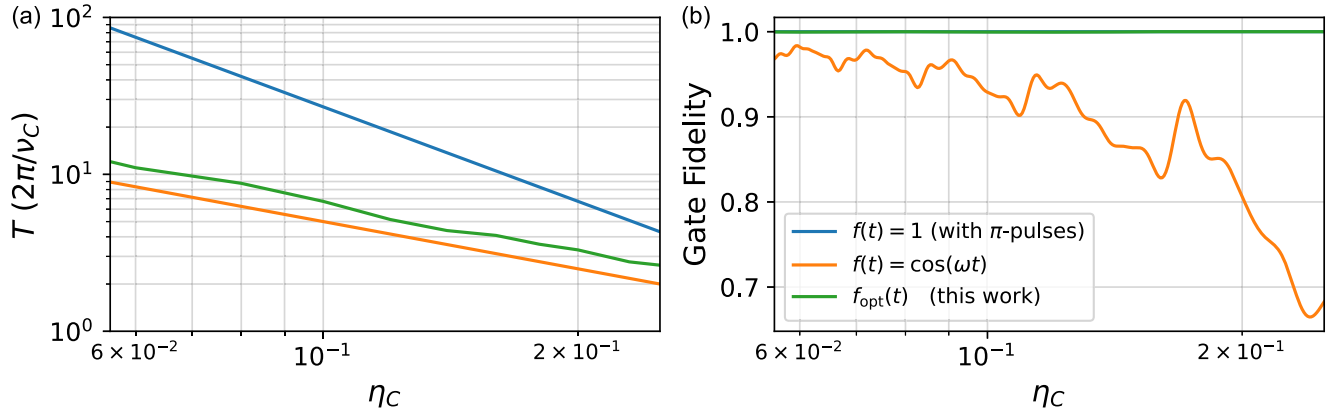


FIG. 3. (a) Required gate duration, expressed in units of the c.m. mode period, as a function of the Lamb-Dicke parameter for the c.m. mode (values for other modes follow accordingly), for a static magnetic field gradient (blue) a monochromatic oscillating gradient (orange), and the optimized drive  $f_{\text{opt}}(t)$  introduced in this work (green), which requires no  $\pi$  pulses. (b) Corresponding gate fidelities for the three protocols as a function of the Lamb-Dicke parameter, highlighting the trade-off between speed and fidelity in each case. While the fidelity of the monochromatic oscillating-gradient protocol decreases with increasing  $\eta_C$ , the optimized protocol remains essentially indistinguishable from the static protocol with  $\pi$  pulses, retaining near-unit fidelity throughout; accordingly, the blue curve is largely hidden beneath the green one. All results correspond to a four-ion chain and the generation of an effective long-range Ising interaction as in Eq. (22), with target coupling strength  $J = \pi/4$ . Fidelity is computed as described in Sec. V of the Supplemental Material [23].

same resonant enhancement also amplifies the sensitivity to off-resonant modes and to the approximations on which the derivation rests. The rotating-wave approximation discards terms oscillating at frequencies  $\approx \nu_l - \omega$  for  $l \neq C$ , and these become less negligible as  $\eta_C$  increases. As shown in Fig. 3(b), the resulting fidelity loss is modest in the weak-coupling regime but becomes severe precisely where the speed advantage is most attractive, namely, at large  $\eta_C$ . This is the central tension of the single-tone approach: the regime that enables fast gates is also the regime in which the approximations underlying the scheme break down.

## 2. Optimized protocol

The framework introduced here resolves this trade-off directly. Rather than targeting a single mode and treating the rest as a perturbation, the optimized drive  $f(t)$  is constructed to satisfy the spin-motion decoupling conditions for all modes simultaneously and exactly, with no recourse to the rotating-wave approximation. Several modes thereby contribute coherently to the entangling interaction, and exact closure is enforced by construction rather than achieved approximately.

As shown in Fig. 3(a), for the four-ion example considered here, the gate times obtained from our optimized solutions exhibit the same dependence on the Lamb-Dicke parameter as in the single-tone case, namely,  $T \propto \eta_C^{-1}$ , up to a finite constant offset (green line). This indicates that the optimized waveform retains the first-order, near-resonant character of the monochromatic protocol, rather than the second-order scaling of a purely static construction. At the same time, the gate is no longer built from a single resonantly enhanced mode: additional motional modes are brought into the dynamics and used to shape the geometric phases and enforce exact final spin-motion closure. As a result, contributions that appear as uncontrolled off-resonant errors in the monochromatic scheme are instead incorporated constructively into the gate

design. The corresponding benefit is that the gate infidelity remains below numerical precision ( $\approx 10^{-9}$ ) across the full range of coupling strengths shown in Fig. 3(b), including the strong-coupling regime where the monochromatic scheme breaks down.

Because the interaction is no longer confined to a single mode, the chain-length dependence of the optimized gate is qualitatively different from that of the monochromatic c.m. protocol. A dedicated analysis of the ion-number dependence is given in Sec. IV of the Supplemental Material [23].

The optimized solutions therefore inherit the speed advantage of the monochromatic protocol while retaining the exactness of the static construction, without relying on the approximations that limit either of them individually.

This is particularly valuable in the strong-coupling regime, precisely where conventional approaches become least reliable and where the prospect of fast gates is most compelling; see Fig. 3. As a concrete example, consider a magnetic-field gradient large enough to yield a COM-mode coupling of  $\eta_C = 0.3$ . Such operating points are consistent with a maximum gradient of  $\partial B/\partial z \simeq 250$  T/m and a trap frequency of  $\nu/2\pi \simeq 100$  kHz, values within current experimental reach [14,27].

For a target interaction strength of  $J = \pi/4$  [see Eq. (22)], the optimal drive depicted in Fig. 4(a) allows the gate to be completed in just two oscillation cycles of the c.m. mode.<sup>1</sup> This corresponds to a total gate time of  $T \simeq 2.3(2\pi/\nu_1) \simeq 23$   $\mu$ s, which is comparable to the fastest multiqubit gate implementations reported in trapped-ion systems [1]. The resulting gate infidelity is limited only by the numerical precision of the pulse shaping (set here to  $10^{-9}$ ), and is therefore

<sup>1</sup>In this example, we use the boundary conditions that match the monochromatic case; but analogous results hold for alternative boundary choices.

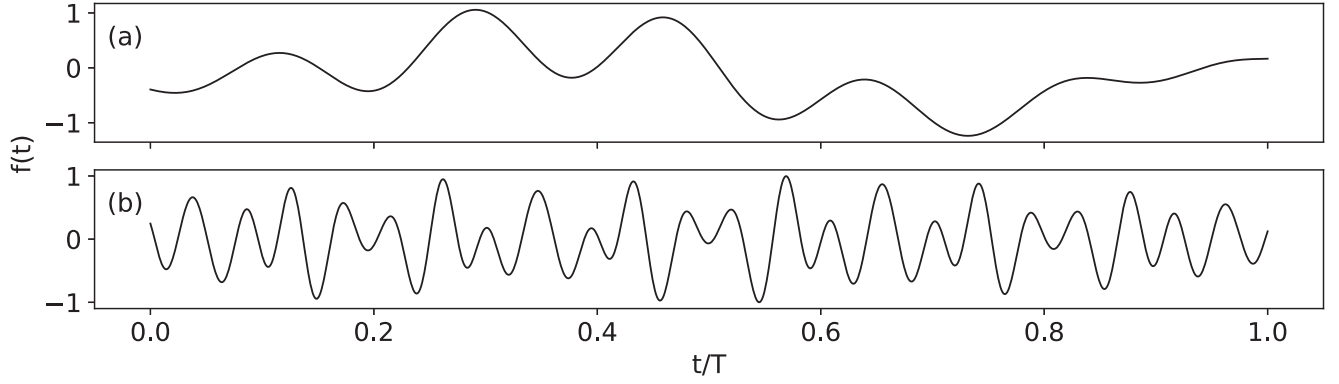


FIG. 4. Global modulation  $f(t)$  of the magnetic-field gradient as a function of time  $t$ , in fractions of the total gate time  $T$ . Panels (a) and (b) show distinct modulation profiles designed to implement specific quantum operations. Panel (a) shows the pulse shape used to realize a uniform Ising interaction with effective coupling  $J = \pi/4$  and gate time  $T = 2.321/2\pi\nu$ ; panel (b) shows the pulse for a rainbow coupling pattern [see Fig. 2(b)] with coupling  $J = \pi/4$  and a longer duration  $T = 8.125/2\pi\nu$ . The solution in panel (a) assumes a magnetic-field gradient of  $\simeq 250$  T/m and a trap frequency of  $\nu/2\pi = 100$  kHz, leading to a maximum qubit-motion coupling of  $\eta_{j1} \simeq 0.3$ . In panel (b), the magnetic field gradient is set to  $\simeq 125$  T/m, yielding  $\eta_{j1} \simeq 0.15$ .

several orders of magnitude smaller than the errors typically reported for existing multiqubit gate protocols.

The phase-space trajectories of all four motional modes during the gate are shown in Fig. 5, with the motional state initialized in the ground state  $|0\rangle^{\otimes 4}$  and each mode depicted in a separate panel. Time is encoded by a color gradient from blue to yellow, and the final point at gate time  $T$  is marked with a red cross. To highlight the qubit-dependent nature of the dynamics, we consider two distinct initial qubit configurations:  $|0010\rangle$  and  $|1010\rangle$ , corresponding to the first and second rows, respectively.

In the case of the qubit state  $|0010\rangle$ , all four motional modes are displaced during the gate, undergo closed

trajectories in phase space, and return precisely to the origin at the final time  $T$ . This ensures full qubit-motion decoupling and corresponds to a unitary evolution purely within the qubit subspace. The total geometric phase accumulated is proportional to the net area enclosed by the trajectories, summed over all modes. Notably, all four modes contribute nontrivially and constructively to the generation of the desired entangling operation.

For the alternative initial qubit state  $|1010\rangle$ , the symmetry of the Lamb-Dicke coefficients  $\eta_{jl}$  causes the net spin-dependent force on two of the four modes to vanish (the contributions from individual ions cancel by symmetry) so those modes remain stationary at the origin throughout the

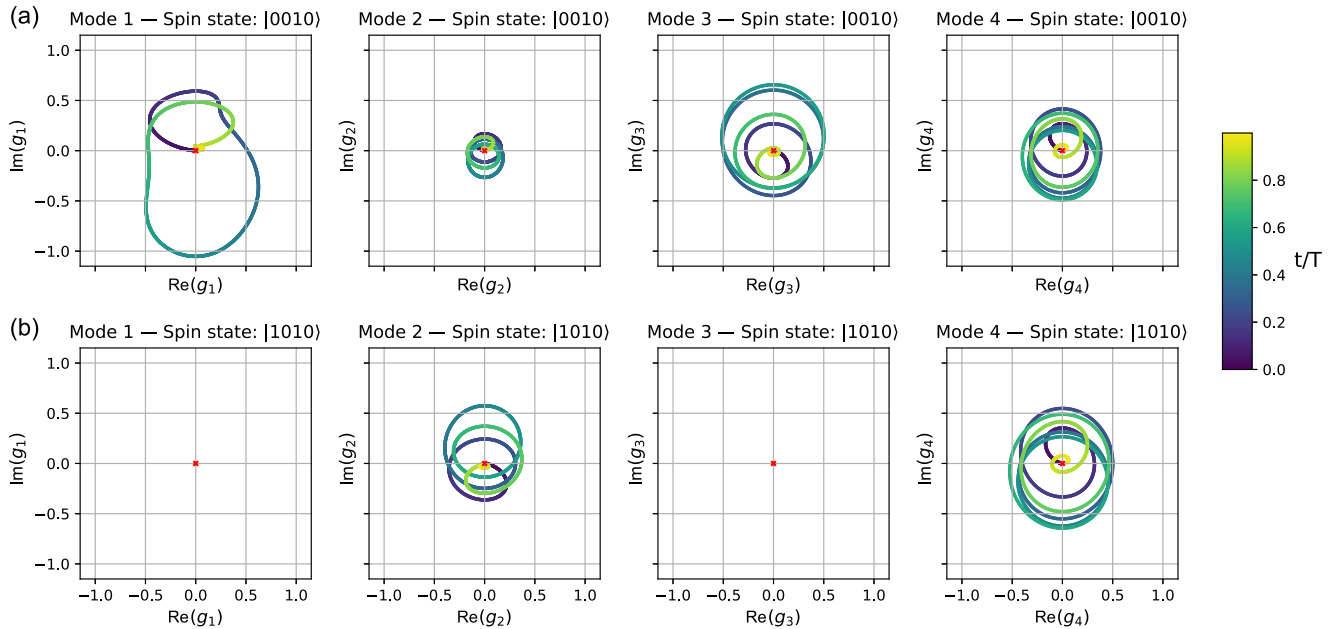


FIG. 5. Phase-space trajectories of the four motional modes in a four-ion chain during an effective Ising gate with target strength  $J = \pi/4$ . Columns correspond to  $l = 1, \dots, 4$  (left to right). In each panel, the real and imaginary parts of the displacement  $g_l(t)$  are shown, with time indicated by color from dark blue (start) to yellow (end); the red cross marks the displacement at the gate time  $T$ . Row (a) shows the evolution for the initial qubit state  $|0010\rangle$ , and row (b) for  $|1010\rangle$ , highlighting the qubit-dependent nature of the dynamics.

evolution while only the remaining two undergo nontrivial phase-space displacement. In both cases the excited modes close exactly at  $T$ , ensuring complete spin-motion decoupling. This state-dependent participation pattern is a general feature of the protocol: the collective structure of the drive naturally selects which modes contribute for a given spin configuration, with no individual mode addressing required. Generically all  $N$  modes of an  $N$ -ion chain participate, although mirror symmetry of the chain can impose parity selection rules that restrict the dynamics to the even- or odd-parity sector for suitably symmetric spin states.

The closure of the phase-space trajectories at  $t = T$  is not contingent on the motional state being initialized in the ground state. Since the displacement  $\Delta_I(t)$  depends only on the drive  $f(t)$  and the qubit configuration—and not on the initial phonon occupation—any initial Fock state  $n_I(0)$  shifts the trajectory by a constant offset without altering its shape. The phonon-number evolution during the gate is given by

$$\langle a_I^\dagger(t) a_I(t) \rangle = n_I(0) + |\Delta_I(t)|^2 \langle S_I^2 \rangle_{\text{qubit}}, \quad (27)$$

where  $\langle \cdot \rangle$ ,  $\langle \cdot \rangle_{\text{qubit}}$  denote expectation values over the full system and qubit subsystem, respectively. In a thermal state, the term  $|\Delta_I(t)|^2 \langle S_I^2 \rangle_{\text{qubit}}$  is added identically to each component. Since the protocol enforces  $\Delta_I(T) = 0$  (so that every phase-space trajectory closes), each state  $|n_I(t)\rangle$  returns to exactly  $n_I(0)$ . As a result, all qubit-motion entanglement vanishes at  $t = T$  regardless of the initial temperature, and no ground-state cooling is required. Crucially, this property holds for any gate constructed within our framework and is not specific to the example shown here.

### B. Rainbow interaction

Another interesting application of the present framework is the ability to engineer a global entangling gate of the form

$$U = \exp(iH_R), \quad H_R = J \sum_k Z_k Z_{N-k+1}, \quad (28)$$

where  $J$  sets the pairwise interaction strength between symmetrically located qubits. This gate can be realized using a single global drive  $f(t)$  engineered to induce ZZ-type interactions across reflection-symmetric qubit pairs. When combined with appropriate single-qubit rotations (implemented via standard techniques; see Sec. VI of the Supplemental Material [23]), it enables the deterministic preparation of a rainbow state [28]

$$|\Psi_{\text{rainbow}}\rangle = \bigotimes_{j=1}^{N/2} |\psi^-\rangle_{j, N-j+1}, \quad (29)$$

where  $|\psi^-\rangle_{a,b} = \frac{1}{\sqrt{2}}(|0\rangle_a |1\rangle_b - |1\rangle_a |0\rangle_b)$  is the singlet state between sites  $a$  and  $b$ . The resulting state, illustrated in Fig. 2(b), consists of qubits paired symmetrically across the chain's center, forming a structured network of long-range singlets. Notably, rainbow states exhibit volume law scaling of entanglement entropy [29], where the entropy grows linearly with subsystem size. This behavior stands in sharp contrast to the area law observed in the ground states of local Hamiltonians, making rainbow states a powerful tool for investigating

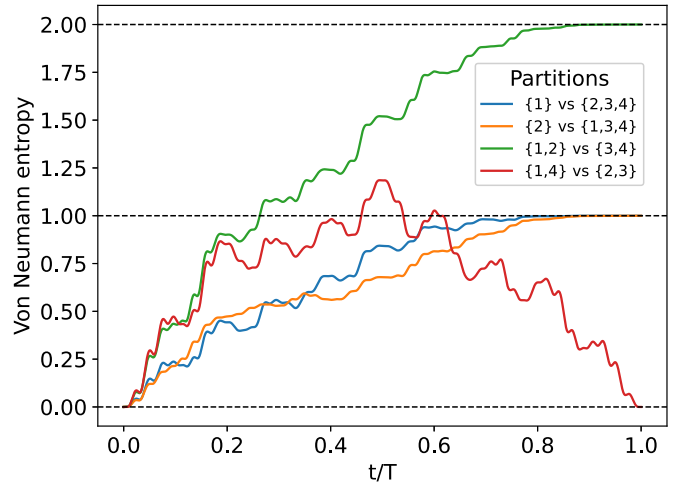


FIG. 6. Dynamics of the von Neumann entropy  $S$  for various qubit bipartitions in a four-ion system, shown as a function of the normalized gate time  $t/T$ . The results correspond to the application of an optimized global drive  $f(t)$  designed to generate the gate in Eq. (28).

quantum many-body entanglement and nonequilibrium dynamics [28,30,31].

To illustrate the approach, a four-qubit rainbow gate is implemented with  $J = \pi/4$ . The optimized driving profile  $f(t)$  is shown in Fig. 4(b), assuming a Lamb-Dicke parameter  $\eta_C = 0.15$  for the c.m. mode, with all other mode couplings adjusted consistently. Due to the weaker qubit-phonon interaction and the constraints imposed by the desired coupling geometry, the required gate duration is slightly longer, with  $T \simeq 8.1(2\pi/\nu_C) \simeq 81 \mu\text{s}$ .

The dynamics of entanglement generation under this gate are illustrated in Fig. 6, which displays the evolution of the von Neumann entropy  $S = -\text{Tr}(\rho \log \rho)$  for several qubit bipartitions throughout the gate operation. Each curve corresponds to a specific partition of the qubit register, either single-qubit subsystems (i.e., qubit  $\{1\}$  or  $\{2\}$ ) or two-qubit subsets (i.e., qubits  $\{1, 2\}$  or  $\{1, 4\}$ ). Time is normalized by the total gate duration  $T$ , and the system is initialized in the separable state  $|+\rangle^{\otimes 4} \otimes |0\rangle^{\otimes 4}$ , with qubits in superposition and motional modes in the ground states.

As the global drive  $f(t)$  induces phonon-mediated qubit-qubit interactions, entanglement builds up across the system in a reflection-symmetric pattern. The von Neumann entropy of qubit 1 increases during the protocol and saturates at  $S = 1$ , indicating that it becomes maximally entangled with the rest of the system. A similar trend is observed for qubit 2, although the temporal profile differs due to the specific values of its qubit-phonon coupling constants  $\eta_{jl}$ . The entropy of the pair  $\{1, 2\}$  reaches  $S = 2$ , reflecting maximal entanglement of both qubits with the remainder of the system.

The bipartition  $\{1, 4\}$ , which is expected to form a Bell pair in the final rainbow state, exhibits a notably nonmonotonic entropy evolution. In the first half of the protocol, the entropy of this pair rises above  $S = 1$ , indicating that qubits 1 and 4 become entangled not only with each other but also with the rest of the system, participating in more complex multi-qubit correlations. In the second half of the gate evolution,

however, the entropy steadily decreases and ultimately vanishes, confirming that qubits 1 and 4 become disentangled from the remainder of the register and form a pure, two-qubit state.

### C. Quantum Fourier transform

The preceding examples demonstrate how carefully tailored global magnetic-field gradients can produce specific qubit-qubit interaction patterns. Many unitaries, such as the quantum Fourier transform (QFT), however, require several steps of dynamics controlled by a magnetic-field gradient interleaved with single-qubit gates.

The QFT is realized by a sequence of Hadamard gates  $\frac{1}{\sqrt{2}}(X + Z)$ , a series of controlled phase gates after every single Hadamard gate, and a set of SWAP gates at the end of the circuit. The dynamics of all the SWAP gates reads  $\exp(i\frac{\pi}{4}H_S)$  with  $H_S = \sum_j X_j X_{N-j+1} + Y_j Y_{N-j+1} + Z_j Z_{N-j+1}$ . Since all the terms in the exponent commute, this can be realized as a sequence of unitaries induced by  $\sum_j X_j X_{N-j+1}$ , by  $\sum_j Y_j Y_{N-j+1}$  and by  $\sum_j Z_j Z_{N-j+1}$ , respectively. Together with single-qubit unitaries, this is the dynamics discussed above in Sec. III B. In the following, we thus take for granted that the SWAP operations can be implemented and focus on the remaining part of the quantum Fourier transform centered around controlled phase gates.

Even though all the controlled phase gates are applied after any Hadamard gate commute and can thus be realized simultaneously, most hardware platforms require a sequential realization. The present framework, however, permits a simultaneous realization in a rather natural fashion.

Recall that a controlled phase

$$\text{CP}(\varphi) = \exp(i\varphi |1\rangle\langle 1| \otimes |1\rangle\langle 1|) \quad (30)$$

differs from a ZZ interaction

$$\exp(i\theta Z \otimes Z) \quad (31)$$

only by single-qubit Z rotations and an overall phase, because of the identity

$$|1\rangle\langle 1| \otimes |1\rangle\langle 1| = \frac{1}{4}(\mathbb{1} \otimes \mathbb{1} - Z \otimes \mathbb{1} - \mathbb{1} \otimes Z + Z \otimes Z). \quad (32)$$

Thus, implementing the QFT reduces to realizing a pattern of ZZ couplings with angles  $\theta = \varphi/4$ ; and the accompanying single-qubit Z phases can be folded into the surrounding Hadamards.

With the convention that the QFT applies a Hadamard on qubit  $k$  and then controlled phases from qubit  $k$  to all later qubits  $j \in \{k+1, \dots, N\}$ , the required controlled-phase angles are

$$\varphi_{jk} = \frac{\pi}{2^{j-k}} \quad (j > k). \quad (33)$$

Equivalently, the corresponding ZZ phases in stage  $\ell$  define a target interaction matrix with entries

$$\Lambda_{jk}^{(\ell)} = \begin{cases} \frac{\pi}{2^{j-k+2}}, & \ell = \min(j, k), \quad \max(j, k) > \ell \\ 0, & \text{otherwise.} \end{cases} \quad (34)$$

For a four-qubit example, stages  $\ell \in \{1, 2, 3\}$  are required. In each case, the target  $\Lambda^{(\ell)}$  can be synthesized as a linear combination of two directly realizable interaction matrices

$\Lambda^{(\ell,p)}$  according to Eq. (13) [i.e.,  $\Lambda^{(\ell)} = O^{(1)}\Lambda^{(\ell,1)}(O^{(1)})^\top + O^{(2)}\Lambda^{(\ell,2)}(O^{(2)})^\top$ ], enabling all controlled phases  $\Lambda^{(\ell,p)}$  to be effected simultaneously. The full circuit is shown in Fig. S2 of the Supplemental Material [23].

We validate the construction by simulating the full time evolution—including all additional gates in the circuit—using tailored drive profiles  $f(t)$  that, at each stage, generate the effective couplings of Eq. (34). Figure 7 shows the resulting implementation of the QFT on a four-qubit register initialized in  $|1\rangle \equiv |0001\rangle$ . The vertical axis displays the computational basis states  $\{|\kappa\rangle\}_{\kappa=0}^{15}$ , while the horizontal axis shows the normalized time  $t/T$ . The color scale encodes the time-dependent phase of each basis state with finite occupation.

Long bars correspond to intervals of entangling evolution under the optimized global fields  $f_{\ell,p}(t)$  in each stage  $\ell$ , which generate the effective couplings  $\Lambda^{(\ell,p)}$ . The vertical dashed black lines mark the instants at which additional local gates are applied. These local operations (Hadamard gates,  $\pi$  pulses, and Z rotations) are treated as instantaneous on the timescale of the entangling dynamics and are indicated by different marker shapes (circle, square, triangle) on top of each dashed line.

The protocol starts with the state in  $|1\rangle$ , after which a Hadamard and a  $\pi$  pulse redistributes amplitude and phase between  $|1\rangle$  and  $|9\rangle$ . The first entangling window, driven by the optimized profile  $f_{1,1}(t)$ , then implements the effective coupling  $\Lambda^{(1,1)}$ , slightly reshaping the phase landscape. At  $t/T \simeq 0.16$  a further  $\pi$  pulse is applied, which now produces a clear phase contrast between  $|1\rangle$  and  $|9\rangle$ ; this is followed by a second entangling window that realizes  $\Lambda^{(1,2)} = -\Lambda^{(1,1)}$ , and a final  $\pi$  pulse. Importantly, the drive profiles associated with each matrix  $+\Lambda^{(\ell,p)}$  and  $-\Lambda^{(\ell,p)}$  are obtained from independent optimizations. Knowing a solution  $f_{\ell,p}(t)$  that realizes  $+\Lambda^{(\ell,p)}$  does not, in general, determine a corresponding transformation of  $f_{\ell,p}(t)$  that would produce  $-\Lambda^{(\ell,p)}$ ; this is reflected in the bars of different length  $T_{\ell,p}/T$ .

The same pattern repeats in the subsequent stages. Each Hadamard increases the number of computational-basis components that carry population, the following entangling interval imprints the appropriate collective phase pattern on this enlarged set, and intermediate single-qubit Z rotations modify relative phases on individual basis states. At the end of the protocol, a final Hadamard gate together with two SWAP gates (indicated by the solid white line), which restore the standard QFT ordering in  $\kappa$ , produce the desired output state

$$\text{QFT}|1\rangle = \frac{1}{\sqrt{2^N}} \sum_{\kappa=0}^{2^N-1} e^{\frac{2\pi i \kappa}{2^N}} |\kappa\rangle. \quad (35)$$

For  $N = 4$ , this corresponds to a uniform superposition with amplitudes  $1/4$  and phases  $e^{i\pi\kappa/8}$  across all basis states  $|\kappa\rangle$ .

Taken together, these results show that the engineered global-control sequence reproduces ideal QFT dynamics. The crucial advantage is simultaneity: all controlled phases associated with a given control qubit are generated in one shot. For large registers this replaces a long sequence of pairwise gates (scaling with qubit number  $N$ ) by a small number of global entangling windows (one per stage), reducing depth, easing scheduling, and improving scalability by concentrating the interaction into fewer, more coherent operations.

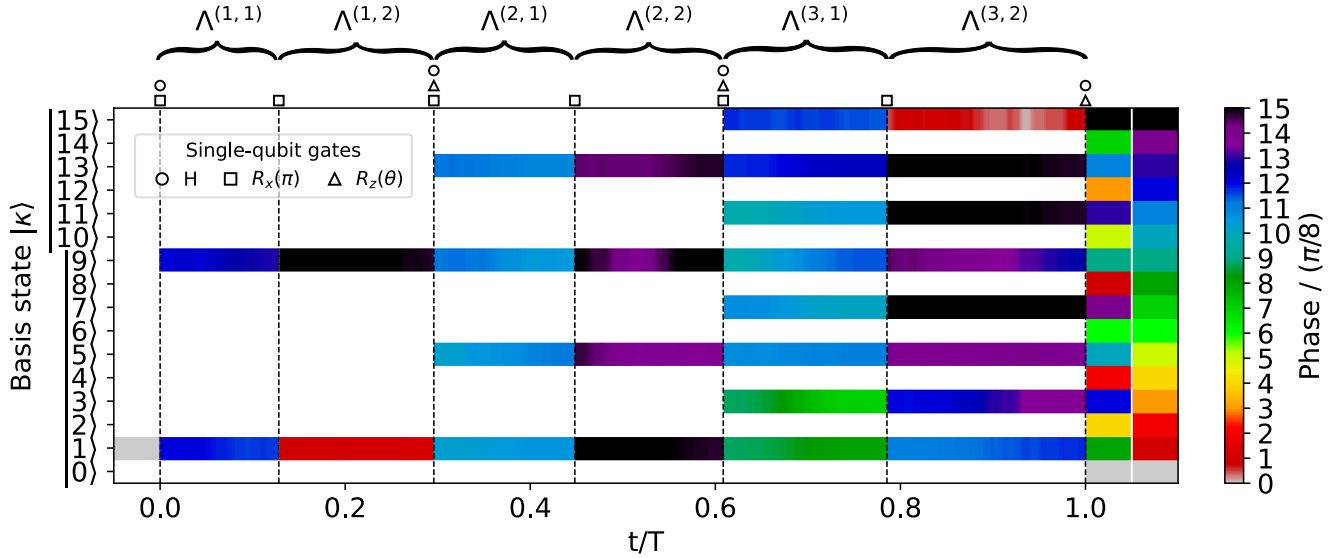


FIG. 7. Phase dynamics of levels with finite occupation during the implementation of the four-qubit quantum Fourier transform (QFT), simulated for the input state  $|1\rangle \equiv |0001\rangle$ . The horizontal axis shows the normalized time  $t/T$ , the vertical axis labels computational basis states  $|k\rangle$ , and the color encodes the instantaneous phase of each component in units of  $\pi/8$ . Basis states whose amplitude remains negligible throughout the evolution are shown in white, allowing phase-zero regions of unoccupied states to be distinguished from genuine phase-zero values of populated components. Vertical dashed black lines mark the instants at which single-qubit gates are applied; different marker shapes above each line indicate Hadamard,  $R_x(\pi)$ , and  $R_z(\theta)$  rotations on the qubits specified by the circuit in Fig. S1. The intervals between gate layers correspond to entangling evolution under the optimized global magnetic-field gradients  $f_{\ell,p}(t)$ , which implement the elementary interaction matrices  $\Lambda^{(\ell,p)}$ . Within each stage  $\ell$ , the combined effect of these segments yields the effective interaction matrix  $\Lambda^{(\ell)}$  defined in Eq. (34). Just before the final local layer, the phase pattern still appears to differ from the QFT target, but it is already in the correct form up to the concluding Hadamard and Z rotations, which produce the exact four-qubit QFT spectrum with phases  $\exp(ik\pi/8)$ . The final SWAP layer (solid white line), which can be implemented using the protocol of Sec. III B, simply reorders the outputs to obtain the standard QFT ordering. A small padding before  $t = 0$  and after  $t = T$  makes the initial and final phases visually explicit and separates the coherent evolution from the final SWAPs.

#### IV. QUANTUM GATES FOR LARGER QUBIT REGISTERS

A key advantage of the present framework is that gate synthesis does not require simulating the full system dynamics. Instead, the driving schemes are constructed directly from the harmonic dynamics of the individual modes. As a result, the method remains applicable even for systems where a full simulation of the joint qubit-motional evolution is infeasible due to the exponential growth of the Hilbert space.

Although the driving schemes themselves can be designed efficiently, a brute-force, state-vector-based evaluation of the resulting gate fidelity would still incur an exponential cost. This bottleneck can be avoided by using an efficiently computable lower bound on the worst-case gate fidelity, as we now describe.

##### A. Bound on gate fidelity

Within the present framework, an entangling gate is specified as  $U = \exp(-i \sum_{j \neq k} \Lambda_{jk} Z_j Z_k)$  in terms of a real symmetric matrix  $\Lambda$  with vanishing diagonal. Let  $\mathbf{A}$  denote the target coupling matrix and  $\mathbf{\Lambda}$  the one actually realized, and define the deviation  $\delta\mathbf{\Lambda} = \mathbf{\Lambda} - \mathbf{A}$ .

As shown in Sec. VII of the Supplemental Material [23], the worst case gate fidelity obeys the bound

$$F_{\min} \geq \cos^2(N \|\delta\mathbf{\Lambda}\|_2), \quad (36)$$

valid whenever  $N \|\delta\mathbf{\Lambda}\|_2 \leq \pi/2$ . In this regime the bound is faithful (it approaches unity as  $\delta\mathbf{\Lambda} \rightarrow 0$ ) and decreases monotonically with increasing error. Crucially, the operator 2-norm  $\|\delta\mathbf{\Lambda}\|_2$  can be evaluated with an effort polynomial in  $N$ , so Eq. (36) provides an efficiently computable and scalable certificate of gate performance, suitable as a postoptimization quality guarantee.

##### B. Homogeneous Ising interaction on a 20-qubit register

Gate synthesis for larger qubit registers and the assessment of gate fidelities in terms of the bound Eq. (36) is exemplified in the following with the target of a uniform all-to-all Ising interaction (see Sec. III A) and a chain of  $N = 20$  ions.

A representative solution is shown in Fig. 8 for an optimized solution with  $6N$  basis elements in the expansion  $f(t) = \sum_m A_m \cos(\omega_m t + \vartheta_m)$ . The left panel of Fig. 8 displays the spectrum of the optimal global control field, where the vertical stick marks each frequency component  $\omega_m$  (plotted in units of the trap frequency  $\nu_c$ ), its height encodes the relative amplitude  $|A_m|$ , and its color represents the phase  $\vartheta_m$ . The spectrum is dominated by tones within a few multiples of the lowest motional frequency  $\nu_c$ , even though the optimization also makes use of a sparse tail of higher-frequency components extending up to  $\simeq 20\nu_c$ . These higher tones primarily address the upper normal modes of the chain,

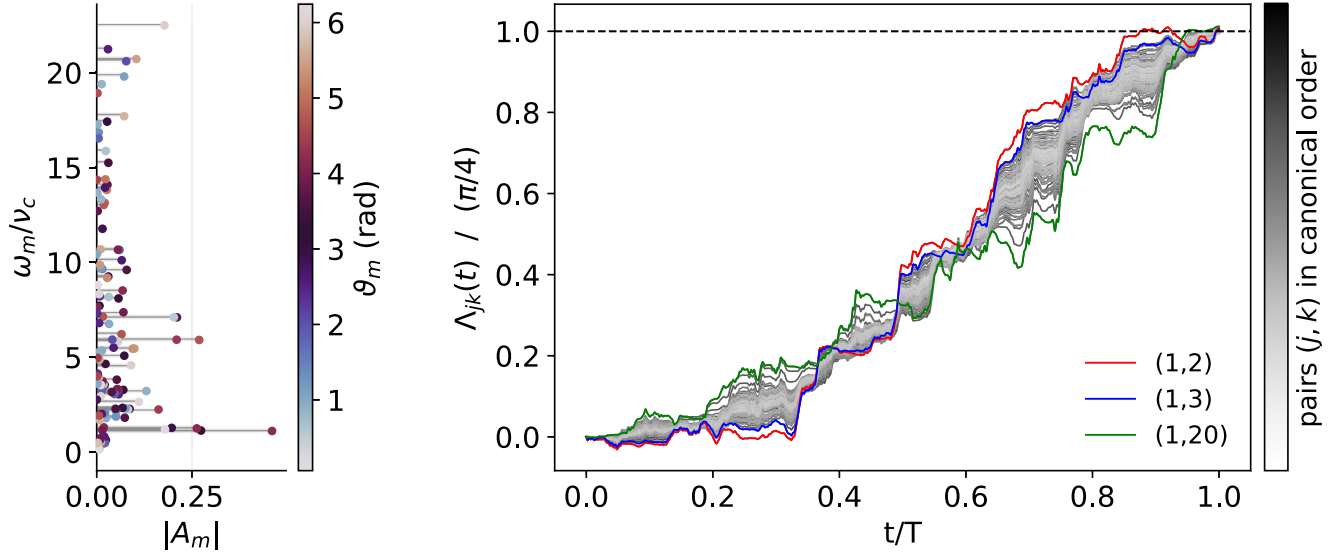


FIG. 8. (Left) Frequency-phase spectrum of the global drive  $f(t) = \sum_m A_m \cos(\omega_m t + \vartheta_m)$  used to implement a homogeneous Ising interaction with target pairwise phase  $J = \pi/4$  on a chain of  $N = 20$  ions (with  $\eta = 0.2$ ) in a total time  $T \simeq 8.95 \times (2\pi/\nu_c)$ . Each stick at  $\omega_m$  has height  $|A_m|$  and a marker color encodes the phase  $\vartheta_m$ . (Right) Time evolution of the implemented pairwise phases  $\Lambda_{jk}(t) = \int_0^t dt' \alpha(t')_{jk}$  as a function of normalized time  $t/T$  for the same  $N = 20$  ion chain. Curves are shown normalized by the target  $[\Lambda_{jk}(t)/J]$ , so the horizontal dashed line at unity marks the desired uniform coupling. A few representative pairs (1,2), (1,3), and (1,20) are highlighted in red, blue, and green. By the gate time  $t = T$ , all off-diagonal phases collapse onto the target line, certifying that the homogeneous interaction pattern is achieved. At gate time, motional displacements are closed for every mode, and the simulated gate infidelity is below  $10^{-4}$ .

whose frequencies reach up to  $\simeq 11\nu_c$ , while contributing only weakly to the dynamics.

Evaluating the norm-based bound on the worst-case gate fidelity, Eq. (36), yields  $F_{\min} > 0.96$ , so the protocol is certified to operate in the high-fidelity regime even without access to the full state dynamics. In this  $N = 20$  example, a direct evaluation of the gate performance is still feasible and shows that the actual average fidelity is substantially higher: computing the process fidelity  $F_{\text{pro}} = |\text{Tr}(U_r^\dagger U_T)|^2/d^2$  and the corresponding average gate fidelity  $F_{\text{avg}} = (dF_{\text{pro}} + 1)/(d + 1)$ , with  $d = 2^N$ , yields an average gate infidelity  $1 - F_{\text{avg}} \lesssim 10^{-4}$ . This confirms that the norm-based bound is conservative yet informative for the parameters of interest.

The right panel of Fig. 8 illustrates the corresponding time evolution of the implemented pairwise phases  $\Lambda_{jk}(t) = \int_0^t dt' \alpha_{jk}(t')$  plotted versus the normalized time  $t/T$  for every pair  $(j, k)$  in the 20-ion chain. Each trajectory is normalized by the target interaction strength  $J$ , so that the horizontal dashed line at unity marks the ideal homogeneous Ising phase. Gray curves show the phases for all pairs  $(j, k)$ , ordered according to the color gradient, while three representative cases—(1,2), (1,3), and the long-range pair (1,20)—are highlighted. Pairs involving neighboring ions, such as (1,2) and (1,3), have very similar phase trajectories  $\Lambda_{12}(t)$  and  $\Lambda_{13}(t)$ , in clear contrast with that of  $\Lambda_{1,20}(t)$  for the distant pair (1,20). This is consistent with the fact that, in larger ion chains, the qubit-motion couplings  $\eta_{jl}$  are very similar for nearby ions, while ions that are far apart exhibit more distinct participation patterns. Despite these microscopic variations, all  $\Lambda_{jk}(t)$  curves converge cleanly to the target value  $J$  at  $t = T$ , certifying that the gate implements a uniform Ising interaction across the entire chain.

## V. CONCLUSIONS

As the growth of registers of engineered qubits creates expectations to use quantum devices for practical computations, it becomes increasingly important to realize high-fidelity gates in such large registers. Careful gate design has resulted in impressive improvement of gate fidelities in devices with few (often two) qubits. Translating these achievements into the large registers that are necessary for practical applications is a highly nontrivial task.

The present approach achieves this in terms of an effective interqubit interaction that is mediated by all motional modes simultaneously instead of the common technique to spectrally select one single mode. Since the underlying optimal control problem does not require any numerical simulation, this approach is not limited by common restrictions to system size of few qubits.

As exemplified with the quantum Fourier transform, the present approach is also very well suited for gate parallelization. The ability to apply mutually commuting gates in parallel, has the potential to improve the scaling behavior of quantum algorithms when fully scalable hardware is available. In the current era of devices with limited coherence time, the ability to perform quantum gates in parallel can mean that an algorithm can be executed within the restrictions of finite coherence time, even if a fully sequential implementation would exceed these restrictions.

The resultant ability to implement multiqubit algorithms both faster and with higher accuracy offers a concrete route to extracting more value from today's imperfect hardware, while simultaneously laying the groundwork for the larger, fault-tolerant devices envisioned for the future.

The analytical tractability of the framework also opens a natural avenue for future work. The numerical examples presented here assume ideal conditions, but in realistic experiments, gate fidelity will be further limited by waveform distortion, finite control bandwidth, calibration errors in the magnetic-field gradient, and motional heating or dephasing. Because the mode displacements and closure conditions remain analytically accessible within the harmonic ansatz, robustness against such imperfections can in principle be incorporated directly into the synthesis objective—for instance, by penalizing deviations from the closure conditions over a finite window near the end of the gate, or by suppressing sensitivity to slow parameter drifts—without sacrificing the algebraic character of the optimization that makes the approach scalable.

## ACKNOWLEDGMENTS

We gratefully acknowledge stimulating discussions with Nguyen Le and Jungsang Kim, and the use of the Imperial College London Research Computing Service [32]. This work was supported by the U.K. Engineering and Physical Sciences Research Council through the EPSRC Hub in Quantum Computing and Simulation (EP/T001062/1).

## DATA AVAILABILITY

The data and code that support the findings of this study are available from the corresponding author upon reasonable request.

- [1] C. D. Bruzewicz, J. Chiaverini, R. McConnell, and J. M. Sage, Trapped-ion quantum computing: Progress and challenges, *Appl. Phys. Rev.* **6**, 021314 (2019).
- [2] A. Sørensen and K. Mølmer, Quantum computation with ions in thermal motion, *Phys. Rev. Lett.* **82**, 1971 (1999).
- [3] C. Monroe, W. C. Campbell, L.-M. Duan, Z.-X. Gong, A. V. Gorshkov, P. W. Hess, R. Islam, K. Kim, N. M. Linke, G. Pagano, *et al.*, Programmable quantum simulations of spin systems with trapped ions, *Rev. Mod. Phys.* **93**, 025001 (2021).
- [4] A. Sørensen and K. Mølmer, Entanglement and quantum computation with ions in thermal motion, *Phys. Rev. A* **62**, 022311 (2000).
- [5] K. Kim, M.-S. Chang, R. Islam, S. Korenblit, L.-M. Duan, and C. Monroe, Entanglement and tunable spin-spin couplings between trapped ions using multiple transverse modes, *Phys. Rev. Lett.* **103**, 120502 (2009).
- [6] T. Choi, S. Debnath, T. Manning, C. Figgatt, Z.-X. Gong, L.-M. Duan, and C. Monroe, Optimal quantum control of multimode couplings between trapped ion qubits for scalable entanglement, *Phys. Rev. Lett.* **112**, 190502 (2014).
- [7] Y. Lu, S. Zhang, K. Zhang, W. Chen, Y. Shen, J. Zhang, J.-N. Zhang, and K. Kim, Global entangling gates on arbitrary ion qubits, *Nature (London)* **572**, 363 (2019).
- [8] D. Porras and J. I. Cirac, Effective quantum spin systems with trapped ions, *Phys. Rev. Lett.* **92**, 207901 (2004).
- [9] F. Mintert and C. Wunderlich, Ion-trap quantum logic using long-wavelength radiation, *Phys. Rev. Lett.* **87**, 257904 (2001).
- [10] C. Ospelkaus, U. Warring, Y. Colombe, K. Brown, J. Amini, D. Leibfried, and D. J. Wineland, Microwave quantum logic gates for trapped ions, *Nature (London)* **476**, 181 (2011).
- [11] J. Chiaverini, R. B. Blakestad, J. Britton, J. D. Jost, C. Langer, D. Leibfried, R. Ozeri, and D. J. Wineland, Surface-electrode architecture for ion-trap quantum information processing, *Quantum Inf. Comput.* **5**, 419 (2005).
- [12] B. Lekitsch, S. Weidt, A. G. Fowler, K. Mølmer, S. J. Devitt, C. Wunderlich, and W. K. Hensinger, Blueprint for a microwave trapped ion quantum computer, *Sci. Adv.* **3**, e1601540 (2017).
- [13] T. Harty, M. Sepiol, D. Allcock, C. Ballance, J. Tarlton, and D. Lucas, High-fidelity trapped-ion quantum logic using near-field microwaves, *Phys. Rev. Lett.* **117**, 140501 (2016).
- [14] S. Weidt, J. Randall, S. Webster, K. Lake, A. Webb, I. Cohen, T. Navickas, B. Lekitsch, A. Retzker, and W. Hensinger, Trapped-ion quantum logic with global radiation fields, *Phys. Rev. Lett.* **117**, 220501 (2016).
- [15] V. Schäfer, C. Ballance, K. Thirumalai, L. Stephenson, T. Ballance, A. Steane, and D. Lucas, Fast quantum logic gates with trapped-ion qubits, *Nature (London)* **555**, 75 (2018).
- [16] C. Ospelkaus, C. E. Langer, J. M. Amini, K. R. Brown, D. Leibfried, and D. J. Wineland, Trapped-ion quantum logic gates based on oscillating magnetic fields, *Phys. Rev. Lett.* **101**, 090502 (2008).
- [17] S. Banerjee, *Open Quantum Systems Dynamics of Nonclassical Evolution, Texts and Readings in Physical Sciences* (Springer, Singapore, 2018), Vol. 20.
- [18] E. L. Hahn, Spin echoes, *Phys. Rev.* **80**, 580 (1950).
- [19] H. Y. Carr and E. M. Purcell, Effects of diffusion on free precession in nuclear magnetic resonance experiments, *Phys. Rev.* **94**, 630 (1954).
- [20] S. X. Wang, J. Labaziewicz, Y. Ge, R. Shewmon, and I. L. Chuang, Individual addressing of ions using magnetic field gradients in a surface-electrode ion trap, *Appl. Phys. Lett.* **94**, 094103 (2009).
- [21] A. Leu, M. Gely, M. Weber, M. Smith, D. Nadlinger, and D. Lucas, Fast, high-fidelity addressed single-qubit gates using efficient composite pulse sequences, *Phys. Rev. Lett.* **131**, 120601 (2023).
- [22] C. J. Ballance, T. P. Harty, N. M. Linke, M. A. Sepiol, and D. M. Lucas, High-fidelity quantum logic gates using trapped-ion hyperfine qubits, *Phys. Rev. Lett.* **117**, 060504 (2016).
- [23] See Supplemental Material at <https://link.aps.org/supplemental/10.1103/357p-y5qf> for additional derivations and numerical details.
- [24] N. Khaneja, T. Reiss, C. Kehlet, T. Schulte-Herbrüggen, and S. J. Glaser, Optimal control of coupled spin dynamics: Design of NMR pulse sequences by gradient ascent algorithms, *J. Magn. Reson.* **172**, 296 (2005).
- [25] T. Caneva, T. Calarco, and S. Montangero, Chopped random-basis quantum optimization, *Phys. Rev. A* **84**, 022326 (2011).
- [26] C. P. Koch, U. Boscain, T. Calarco, G. Dirr, S. Filipp, S. J. Glaser, R. Kosloff, S. Montangero, T. Schulte-Herbrüggen, D. Sugny, *et al.*, Quantum optimal control in quantum technologies. Strategic report on current status, visions and goals for research in Europe, *EPJ Quantum Technol.* **9**, 19 (2022).

- [27] I. Gerasin, I. Semerikov, and W. Zhang, Surface ion trap for fast microwave gates, *Chips* **4**, 41 (2025).
- [28] G. Ramírez, J. Rodríguez-Laguna, and G. Sierra, Entanglement over the rainbow, *J. Stat. Mech.* (2015) P06002.
- [29] G. Vitagliano, A. Riera, and J. I. Latorre, Volume-law scaling for the entanglement entropy in spin-1/2 chains, *New J. Phys.* **12**, 113049 (2010).
- [30] M. B. Hastings, An area law for one-dimensional quantum systems, *J. Stat. Mech.* (2007) P08024.
- [31] C. M. Langlett, Z.-C. Yang, J. Wildeboer, A. V. Gorshkov, T. Iadecola, and S. Xu, Rainbow scars: From area to volume law, *Phys. Rev. B* **105**, L060301 (2022).
- [32] M. Harvey, Imperial College Research Computing Service, <https://doi.org/10.14469/hpc/2232>.