

Thermodynamic constraints on the emergence of intersubjectivity in quantum systems

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Ideal quantum measurement requires divergent thermodynamic resources. This is a consequence of the third law of thermodynamics, which prohibits the preparation of the measurement pointer in a fully erased, pure state required for the acquisition of perfect, noiseless measurement information. In this work, we investigate the consequences of finite resources in the emergence of intersubjectivity as a model for measurement processes with multiple observers. Here, intersubjectivity refers to a condition in which observers agree on the observed outcome (agreement), and their local random variables exactly reproduce the original random variable for the system observable (probability reproducibility). While agreement and reproducibility are mutually implied in the case of ideal measurement, finite thermodynamic resources constrain each of them. Starting from the third law of thermodynamics, we derive how the achievability of ideal intersubjectivity is affected by restricted thermodynamic resources. Specifically, we establish a no-go theorem concerning perfect intersubjectivity and present a deviation metric to account for the influence of limited resources. We further present attainable bounds for the agreement and bias that are exclusively dependent on the initial state of the environment. In addition, we show that either by cooling or coarse graining, we can approximate ideal intersubjectivity even with finite resources. This work bridges quantum thermodynamics and the emergence of classicality in the form of intersubjectivity.

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I. INTRODUCTION

In quantum mechanics, measurement of any system has to be understood not only as a dynamical mechanism but also as a fundamental principle: the associated probabilities are dictated by the Born rule, and textbook projective measurement is postulated to result in a pure quantum state. This scenario has recently attracted attention due to the discrepancy between idealized projective measurements and the third law of thermodynamics [1–4]. Specifically, while the measurement postulate of quantum mechanics suggests that measurement leaves the system in a pure state, the third law of thermodynamics dictates the inherent impossibility of realizing such a pure state in practice [5]. Thermodynamics is primarily concerned with macroscopic systems consisting of multiple constituent parts and their interactions. For this reason, thermodynamic considerations are essential for modeling measurement and the transition from quantum to classical.

The first foundational efforts to explore such a quantum-to-classical transition involved the study of decoherence [6–11]. Over time, the framework of quantum Darwinism emerged from decoherence theory, shifting the discourse towards the view that measurement and the quantum-to-classical transition correspond to redundant, broadcasted information encoded in environmental macrofractions [12–14]. These concepts have recently found a more structured framework in operational objectivity and spectrum broadcast structures (SBS) [15,16]. This structure responds to the fact that quantum Darwinism, while ubiquitous [17], does not per se imply agreement about measurement outcomes among observers due to quantum discord [18,19], resulting in the introduction of strong quantum Darwinism [20–22]. These developments have provided a formalized approach to understanding the quantum-to-classical transition in terms of a specific state structure, where the salient features can be captured in terms of information broadcasting and almost perfect distinguishability of branch states [23].

Nonetheless, the framework developed thus far faces some limitations. First, it does not describe measurement as a process, but just as a target state, if not solely in entropic terms. Furthermore, the discussion is usually limited to projective measurement, implicitly omitting destructive measurement as a possibility. For this reason, in this article, we adopt a broader notion of objectivity known as intersubjectivity [24]. This can be understood as the minimal scenario for

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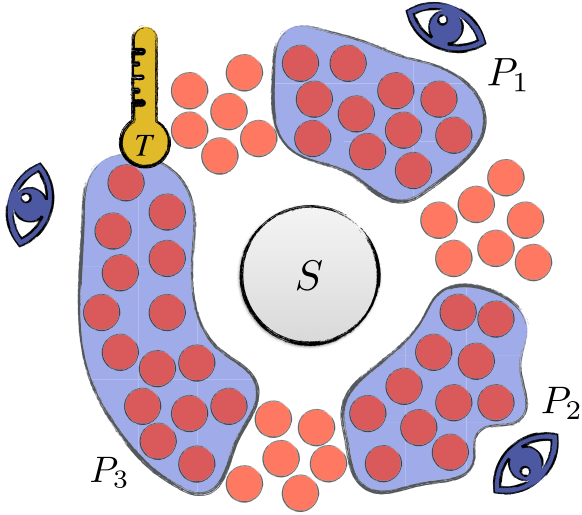


FIG. 1. Multiple macroscopic observers $\mathcal{P}_j$ monitor a central quantum system S in order to extract information about a specific central observable. The presence of a thermal environment at temperature T hinders the emergence of ideal intersubjectivity.

information broadcasting with multipartite agreement: multiple environments access local subsystems to extract information about the system's initial state. In this framework, additional properties, such as nondisturbance of the system or the environment, are not generally required. Only the broadcasting of information is essential [2], i.e., the copying of the classical random variable associated with a fixed measurement. This approach thereby generalizes the usual notion of ideal projective measurement and measurement schemes [25], by naturally including multipartite systems as environments.

The interplay between quantum measurement and thermodynamics has been extensively studied, in particular, the thermodynamic cost of acquiring information through measurement [26–33], and the resulting implication that limited thermodynamic resources constrain the precision with which quantum measurements can be performed [1,4,34–36]. In this context, the measurement process has been modeled as a thermal engine, whose efficiency can be directly compared to the precision of the measurement [37,38]. Within the framework of quantum Darwinism, the role of environmental mixedness [39], the compatibility between equilibrium states and spectrum broadcast structures (SBS) [40], measurement as a phase transition [41], the emergence of objectivity as a classical limit of information broadcasting [2], and the formation of SBS through equilibration processes [42–44] have been analyzed.

In this work, we apply these thermodynamic considerations to characterize the emergence of intersubjectivity in a thermodynamically consistent manner (see Fig. 1 for illustration). This is motivated by the observation that there is a conflict between ideal projective measurements and the third law of thermodynamics [1]: while the measurement postulate predicts a pure postmeasurement state, the third law forbids the creation of a pure, or rank reduced, state with finite resources. Moreover, thermodynamics limits the ability to create the redundancies necessary to broadcast information

encoded in a quantum system [2]. We investigate the constraints imposed by the third law of thermodynamics, and quantify the extent to which information can be broadcast and how consensus about observed outcomes can be achieved when available thermodynamic resources are finite. Our results thereby address the thermodynamic cost of building the type of correlations that result in intersubjectivity, adding to the broader understanding of the thermodynamic value of correlations [45–53].

We report the following central results: First, we quantify agreement between observers and limitations on it. When the system loses its original information, we observe that interactions aimed at maximizing agreement generally result in a proportional bias in the broadcasted information. Second, we discuss methods to overcome these limitations by coarse-graining environments into macrofractions. We show that the amount of achievable agreement between observers exponentially converges to unity with the size of the coarse-grained local pointers. Lastly, we compare the theoretical bound with a standard model for the emergence of objectivity: a star-spin model that results in pure dephasing.

The paper is structured as follows: In Sec. II, we present the intersubjectivity framework; Sec. III is dedicated to the relation between intersubjectivity and the third law, proving two no-go theorems; in Sec. IV, we quantify the deviation between imperfect and ideal intersubjectivity in terms of the thermal resources available initially, proving attainable bounds to agreement and bias; we present and discuss limits in which ideal intersubjectivity can be attained in Sec. V, e.g., by coarse graining; in Sec. VI, we compare our bounds to the agreement and bias in a standard model of quantum Darwinism, finding suboptimal values for agreement and bias which can be improved by coarse graining; we draw our final conclusions and perspectives in Sec. VII.

II. IDEAL INTERSUBJECTIVITY

Quantum objectivity was first formally articulated within the framework of quantum Darwinism [12], developed to encapsulate the essential features of the quantum-to-classical transition. More recently, the notion of objectivity has been refined into stricter formulations, such as strong quantum Darwinism [20] and spectrum broadcast structure [15], within an operational framework grounded in information-theoretic principles [16]. However, these definitions of objectivity face two principal limitations: First, quantum Darwinism and strong quantum Darwinism exclude the physically relevant case of destructive measurement, thereby imposing overly stringent conditions. Second, objectivity is characterized solely by the properties of the final output state, whereas quantum measurement should be regarded as a dynamical physical process. In this work, we address both limitations by introducing a more general, first-principles definition of objectivity, which we term *ideal intersubjectivity*.

We consider the system of interest S in an initial state ρ_S , and a collection of environments $\mathbf{P}$ which are initially uncorrelated as $\rho_{\mathbf{P}} = \bigotimes_{i=1}^{N_P} \rho_{P_i}$. Informally, after the joint evolution of system and environments, intersubjectivity is achieved when many observers can independently access information about the initial state of the system S , and all observers agree on

the observed outcome. In particular, the redundant encoding of the diagonal information of ρ_S to each environmental partition P_i can be considered as the broadcasting of the random variable X , on the system S , described by the probability distribution $\{p_S(x)\}_{x=0}^{d_S-1}$. The evolution is described by a unitary operator U that yields a final state $\tilde{\rho}_{S,P} = U \rho_S \otimes \rho_P U^\dagger$. More formally, we require that the following three properties be satisfied for ideal intersubjectivity.

Locality. There exists a joint probability distribution among the environmental subsystems (observers) in terms of N_P local measurements $\Pi^{(i)} = \{\Pi_x^{(i)}\}_{x=0}^{d_S-1}$, with $i = 1, \dots, N_P$:

$$\tilde{p}_P(x_1, \dots, x_{N_P}) = \text{Tr}\{\tilde{\rho}_P \Pi_x\}, \quad (1)$$

where the measurement operators $\Pi_x = \otimes_{i=1}^{N_P} \Pi_{x_i}^{(i)}$ describe the joint measurement. The marginal probabilities

$$\begin{aligned} \tilde{p}_{P_i}(x_i) &= \sum_{j \neq i} \sum_{x_j} \tilde{p}_P(x_1, \dots, x_{N_P}) \\ &= \text{Tr}\{\tilde{\rho}_{P_i} \Pi_{x_i}^{(i)}\}, \end{aligned} \quad (2)$$

correspond to local measurements by each observer.

Probability reproducibility. Each observer's independent measurement obeys the same probability distribution

$$\tilde{p}_{P_i}(x) = \tilde{p}_{P_j}(x) = p_S(x), \quad (3)$$

for all $x = 1, \dots, d_S$ and $i, j = 1, \dots, N_P$. This property encapsulates the joint unbiasedness of information broadcasting (to local degrees of freedom).

Agreement. The probability of observing distinct outcomes across the different environments vanishes, i.e.,

$$\tilde{p}_P(x_1, \dots, x_{N_P}) \neq 0 \iff x_1 = \dots = x_{N_P}. \quad (4)$$

Consequently, the final state of the environment $\tilde{\rho}_P$ is only supported on the agreement subspace $\mathcal{A} = \text{span}\{\otimes_{i=1}^{N_P} \Pi_x^{(i)}\}_{x=0}^{d_S-1}$. That is,

$$\sum_{x=0}^{d_S-1} \text{Tr}\{\tilde{\rho}_P \otimes_{i=1}^{N_P} \Pi_x^{(i)}\} = 1. \quad (5)$$

This condition thus ensures perfect correlations between the different environments.

Ideal intersubjectivity is defined by the simultaneous fulfillment of all three properties. Note that the second and third properties are theory independent and can be formulated without reference to quantum mechanics. We furthermore make no assumptions regarding the dimension of the Hilbert space, and all the following results hold even in the infinite-dimensional case. The only assumption concerns the number of measurement outcomes, which must be finite, a condition justified by the finite resolution inherent to any realistic measurement procedure.

It is worth noting that the three properties so defined are not independent. Indeed, Ozawa's theorem states that if a collection of measurements $\Pi^{(i)}$, $i = 1, \dots, N_P$, satisfies locality and probability reproducibility for a projective measurement on the system S , then it also satisfies the agreement condition [24]. A general proof of the theorem, which extends Ozawa's original to mixed states as required here, is provided in Appendix A. The theorem implies that if information about a projective measurement on S is perfectly broadcasted to

multiple environments, then anyone locally accessing these systems will always agree on the observation. In other words, ideal broadcasting implies agreement and, thus, intersubjectivity. However, broadcasting redundant information cannot be achieved with constrained thermodynamic resources [2], indicating the necessity to align with a thermodynamically consistent framework. Hence, the logical redundancy in the ideal case, as proved by Ozawa, does not directly extend to the finite resource regime, where the three properties need to be considered individually.

In this scenario, what prerequisites must be fulfilled for observers to reach *agreement*, considering the possible discrepancy between the copies and the original information? Does striving for maximum agreement inherently involve reducing local bias? We address these questions in the following sections.

III. THIRD LAW OF THERMODYNAMICS AND INTERSUBJECTIVITY

Ideal intersubjectivity, together with Ozawa's theorem introduced in the previous section, provides a framework for understanding the redundant broadcasting of information from a central quantum system to multiple observers. It ensures that all observers have access to consistent information about the system and that they reach agreement on the outcome of any independent measurement run.

On the other hand, the third law of thermodynamics implies that pure states can only be prepared at the cost of infinite resources such as time, complexity, and energy [5]. The same holds for any state with pure subspaces, that is, any state that is not full rank. In ideal intersubjectivity, pure states are implicitly assumed, but, as we will argue shortly, they are necessary to satisfy the agreement condition given in Eq. (5).

The agreement condition in Eq. (5) can only be satisfied if the final state is of sufficiently reduced rank, and such a rank-deficient state can only be unitarily reached from an initial state with that same rank. Hence, achieving ideal intersubjectivity requires starting from a severely rank-deficient initial state, a scenario that is physically unrealistic unless infinite resources are expended to cool the environments. For these reasons, thermodynamic constraints suggest that ideal intersubjective agreement cannot be exactly achieved in practice.

We note here that, within thermodynamics, unitary operations describe closed systems. They represent the most natural and optimal choice among those operations that preserve rank, and thus satisfy the third law of thermodynamics. While one might consider more general evolutions, such as completely positive trace-preserving (CPTP) maps that reduce rank, such operations are inconsistent with thermodynamic principles, as they explicitly violate the third law, any unitary dilation in turn requiring rank-deficient input states. Consequently, we restrict our attention to evolutions that do not decrease rank, with unitary dynamics being the most efficient within this class. Nevertheless, our approach is also consistent with more general nonunitary thermal operations [54,55], as auxiliary systems can be coupled to each pointer and subsequently discarded during the readout of the statistics.

The impossibility of achieving agreement under full-rank state preparation, as dictated by the third law, has deeper implications for the framework of intersubjectivity. Ozawa's theorem asserts that locality and probability reproducibility together imply agreement. Thus, the logical negation of this statement indicates that if agreement cannot be achieved, then either locality or probability reproducibility must be abandoned. Since locality is essential for intersubjectivity, allowing multiple observers to access information locally from their respective environments, we are forced to abandon or at least weaken the requirement of probability reproducibility, and, by extension, ideal classical broadcasting. We summarize these results in the following observation

Observation 1 (Impossibility of thermodynamically-consistent ideal intersubjectivity). Agreement [Eq. (5)] and probability reproducibility [Eq. (3)] cannot be satisfied simultaneously, and ideal intersubjectivity cannot be attained without the use of diverging thermodynamic resources.

This result raises an important question: Given a finite amount of resources, how closely can we approximate ideal intersubjectivity? In the next section, we address this question in depth, describing the fundamental limits of finite resources to agreement and probability reproducibility.

IV. FINITE RESOURCE-CONSISTENT INTERSUBJECTIVITY

As ideal intersubjectivity is in conflict with the third law of thermodynamics, we now turn to investigating how closely it can be approximated. Recalling the definitions of local measurement probabilities in Eqs. (1) and (2), we start by introducing a more permissive version of intersubjectivity, which we refer to as *biased joint information broadcasting*:

Definition 1 (Biased joint information broadcasting (BJIB)). A process leads to biased joint information broadcasting (BJIB) if many observers can access the same information locally, reproducing a noisy version of information about the initial state of the system S :

$$\tilde{p}_{P_j}(x) = \tilde{p}_{P_j}(x) \neq p_S(x), \quad (6)$$

for all $x = 0, \dots, d_S - 1$ and for all $i, j = 1, \dots, N_P$.

This definition agrees with semiclassical information broadcasting [2], where the information accessed locally is imperfect with respect to the one originally intended to be copied. Indeed, Eq. (6) can be understood as a more permissive and biased version of probability reproducibility in Eq. (3).

As stated in Observation 1, the ability to reproduce redundant information is naturally constrained by the thermodynamic resources in the environments. Here, a finite-resource quantum thermodynamic process is characterized by the unitary interaction of the system state ρ_S (on Hilbert space $\mathcal{H}_S$) with a thermal environment prepared as

$$\tau_\beta = \frac{e^{-\beta H}}{Z_\beta}, \quad (7)$$

with inverse temperature β , Hamiltonian $H = \sum_i E_i |E_i\rangle\langle E_i|$, collective environmental energy levels E_i , and partition function $Z_\beta = \text{Tr}\{e^{-\beta H}\}$. For noninteracting environmental subsystems, the set $\{|E_i\rangle\}$ includes all possible combinations

of local environmental energy eigenstates. At nonzero temperature, the state τ_β is full rank.

As the information about the system needs to be encoded in subspaces of the environment, we must decompose each local Hilbert space into d_S subspaces $\text{span}\{|E_i\rangle\langle E_i|\}_{i \in D_x}$, where d_S is the number of outcomes, D_x is a possibly infinite collection of indices that specify the orthogonal subspace corresponding to the x th outcome. In this way, we can rewrite our initial environment state as

$$\tau_\beta = \sum_{x=0}^{d_S-1} A_x, \quad (8)$$

with each orthogonal component A_x defined as

$$A_x = \sum_{i \in D_x} \frac{e^{-\beta E_i}}{Z_\beta} |E_i\rangle\langle E_i|. \quad (9)$$

We denote the trace of these orthogonal components as $a(x) = \text{Tr}\{A_x\}$, and we also define $\mathbf{a} = \{a(0), a(1), \dots, a(d_S - 1)\}$. This decomposition is, in principle, arbitrary, but in the following, we will show how to select and identify the one that allows for optimal intersubjectivity with finite resources. We stress that we have not assumed that the environment has a finite-dimensional Hilbert space, only that the number of outcomes is finite.

We proceed to quantify how close BJIB can get to ideal intersubjectivity. To do so, we define two quantities that capture the deviation from ideal intersubjectivity of a unitary process U . First, for the final state $\tilde{\rho}_{S:P} = U \rho_S \otimes \rho_P U^\dagger$, with the composed pointer prepared as $\rho_P = \tau_\beta^{\otimes N_P}$, the *agreement* is quantified by

$$\text{Agr}(U) = \sum_{x=0}^{d_S-1} \text{Tr} \left\{ \tilde{\rho}_{S:P} \mathbb{I} \bigotimes_{i=1}^{N_P} \Pi_x^{(i)} \right\}. \quad (10)$$

This quantity measures the support of the final state on the agreement subspace $\mathcal{A}$. The value of $\text{Agr}(U)$ reflects the probability on average of the observers to agree on their measurement output. Conversely, *disagreement* is naturally given by $\text{Dis}(U) = 1 - \text{Agr}(U)$.

We now have all the ingredients for introducing our main result:

Theorem 1 (Maximum agreement). For biased joint information broadcasting, as defined in Definition 1, the maximal agreement is

$$\max_U \text{Agr}(U) := \gamma_a^{(N_P)} = \sum_{x=0}^{d_S-1} a(x)^{N_P} \quad (11)$$

and there exists an evolution $U_{\text{BJIB}}^{\text{opt}}$ attains this maximum.

Any other unitary U that satisfies BJIB will have a smaller agreement, i.e., $\text{Agr}(U_{\text{BJIB}}) \leq \gamma_a^{(N_P)}$. The maximally attainable value is in turn definitionally upper bounded by unity: $\gamma_a^{(N_P)} \leq 1$.

The formal proof of Theorem 1 is provided in Appendix B, with an explicit construction of the optimal unitary which makes Eq. (11) achievable. We sketch its essence here: this relies on the decomposition of the initial state of the environment between the agreement subspace $\mathcal{H}_{\mathcal{A}}$ and its orthogonal part, the disagreement subspace $\mathcal{H}_{\mathcal{D}}$. In this way, the initial

state becomes

$$\rho_P = \rho_P^{\text{agr}} + \rho_P^{\text{dis}}. \quad (12)$$

Now, to have a maximal agreement, we need to focus on how to realize an unbiased information broadcast of the probability $\{p_S(x)\}_x$ with respect to the agreement subspace. This constraint is the unitary evolution on $\mathcal{H}_S \otimes \mathcal{H}_A$. To fix the unitary also on the disagreement subspace, we enforce the biased joint information broadcast condition in Eq. (6). This additionally forces the unitary evolution to be a specific form, eventually fixing the overall unitary evolution on the total Hilbert space.

Theorem 1 establishes an achievable upper bound on the maximum agreement given a fixed amount of resources, as captured by the quantity $\gamma_a^{(N_P)}$. The maximal agreement depends only on this function, which is independent of the initial state of the system ρ_S while being only a function of the initial state of the environments ρ_P and on the number of environments N_P that must agree. One can note that $\sum_{x=0}^{d_S-1} a(x) \equiv \text{Tr}\{\tau_\beta\} = 1$, therefore, it will only be equal to one, satisfying the ideal intersubjectivity if each local pointer's initial state has a single support subspace with dimension smaller than d_P/d_S , in other words $\text{rank}(\rho_P) < d_P/d_S$. This implies that each individual environment does not need to cool down its local pointers to a rank-1 state to obtain perfect intersubjective broadcasting. However, it must be cold enough to have a specific support subspace that fits all the system information. Furthermore, one can define the minimum observable disagreement as $\delta_a^{(N_P)} = 1 - \gamma_a^{(N_P)}$.

A comment on the dynamics that yield the maximum agreement is also necessary. Theorem 1 above provides a class of unitaries that realize the maximum agreement. An underlying Hamiltonian description can be obtained by taking the logarithm of the unitary operator. Rather than further relating such a procedure to basic, physical properties, we here give a simple circuit-inspired implementation of the required unitary itself: this can be achieved by applying a generalized SWAP gate between the system and one environment and then applying a sequence of generalized CNOT gate on the environments.

According to Observation 1, the information locally encoded in the environments will be biased, as the information cannot be completely copied to each pointer. The local bias can be quantified by the statistical distance between the probability distribution obtained by a local observer $\tilde{p}_{P_j}$ and the probability distribution p_S on the initial state of S that we originally aimed to broadcast,

$$\text{Bias}_j(U) = D_T(\tilde{p}_{P_j}, p_S), \quad (13)$$

where $D_T(p, q) = \sum_i |p_i - q_i|/2$ refers to the l_1 distance between $\{p(x)\}$ and $\{q(x)\}$. We note that for systems satisfying BJIB, the bias will not depend on j , so we drop it from our notation in the remainder of the paper and simply use $\text{Bias}(U)$.

In Theorem 1, we have only discussed the maximum agreement. We now address the bias of such optimal unitary that we have found, and summarize the result in the following theorem:

Theorem 2 (Bias for optimal BJIB). For the optimal unitary $U_{\text{BJIB}}^{\text{opt}}$, the local probabilities can be written as a convex combination of the original probability and a noisy

contribution

$$\tilde{p}_{P_j}(x) = \gamma_a^{(N_P)} p_S(x) + (1 - \gamma_a^{(N_P)}) \eta_P(x), \quad (14)$$

where

$$\eta_P(x) = \frac{a(x)[1 - a(x)^{N_P-1}]}{1 - \gamma_a^{(N_P)}}. \quad (15)$$

As a result, the bias is

$$\text{Bias}(U_{\text{BJIB}}^{\text{opt}}) = \delta_a^{(N_P)} D_T(p_S(x), \eta_P(x)) := \beta_a^{(N_P)}. \quad (16)$$

The detailed proof is provided in Appendix C. Here, one can observe that the bias of the optimal unitary depends on the probability distribution to be copied $\{p_S(x)\}_x$. On the other hand, it is proportional to the minimum disagreement, defined as $\delta_a^{(N_P)} = 1 - \gamma_a^{(N_P)}$. Consequently, as this quantity approaches zero, both perfect agreement and vanishing bias are attained, thereby realising ideal intersubjectivity.

Furthermore, we note that minimum disagreement $\delta_a^{(N_P)}$ can also be written as

$$\delta_a^{(N_P)} = (N_P - 1) S_{N_P}(\mathbf{a}), \quad (17)$$

where

$$S_q(\mathbf{p}) = \frac{1 - \sum_i p_i^q}{q - 1} \quad (18)$$

is the Tsallis entropy [56,57], which has applications in systems where energy and entropy are nonextensive [58]. We thus conclude that the disagreement is proportional to N_P and the Tsallis entropy of $\mathbf{a} = \{a(0), a(1), \dots, a(d_S - 1)\}$. Note that for a pure state of the environment (i.e., at zero temperature), the agreement converges to one, and the minimal bias diminishes to zero, as expected.

As the number of environments N_P increases, the upper bound $\gamma_a^{(N_P)}$ correspondingly decreases, reflecting the increased difficulty of establishing perfect correlations among a larger number of systems under fixed thermodynamic resource constraints. We also note that as $\gamma_a^{(N_P)}$ gets closer to 1, bias goes to zero, achieving ideal intersubjectivity. Conversely, for all nonoptimal unitaries in Eq. (6), the agreement is smaller.

As previously noted, enhancing agreement and reducing bias requires increasing the value of $\gamma_a^{(N_P)}$ as close to unity as possible. This can be achieved by appropriately selecting the dimensions and components of the subspaces D_x , as discussed in the text above. The optimal configuration can be constructed by the following procedure: first, assign the eigenvectors $|E_i\rangle\langle E_i|$ associated with the smallest eigenvalues E_i to the largest subspace D_x , where “largest” refers to the subspace with the greatest dimension d_x . Next, assign the eigenvectors corresponding to the second-smallest eigenvalues to the second-largest subspace, and so on. By sequentially filling all subspaces in this manner, we maximize a_0 , then a_1 (given the maximal a_0), and so on, ultimately yielding the highest possible value for $\gamma_a^{(N_P)}$. Notably, the grouping D_x is independent of N_P . This optimal arrangement can be realized when there is sufficient freedom to design the local measurement $\{\Pi_x^{(i)}\}_{x=0}^{d_S-1}$. However, in the presence of physical constraints, one may be limited to performing a fixed measurement $\{\Pi_x\}_x$. As a result, such restrictions also limit the ability to optimize over the

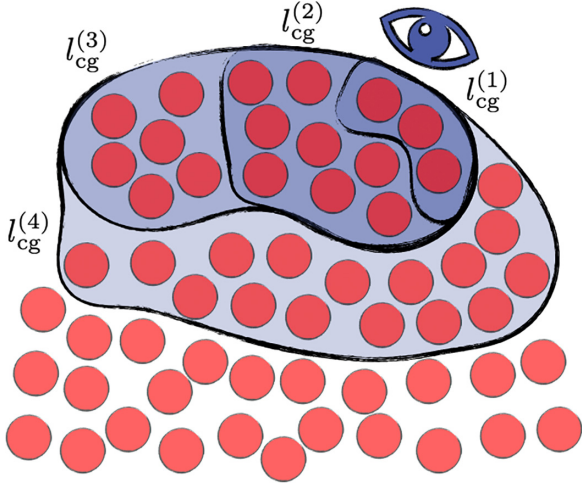


FIG. 2. Representation of coarse graining: The environments are grouped into larger and larger macrofractions of size $l_{cg}^{(x)}$, from which information is extracted.

choice of D_x , thereby imposing a corresponding constraint on the attainable value of $\gamma_a^{(N_p)}$.

Following this initial optimization step, the most natural strategy to further increase $\gamma_a^{(N_p)}$ is to increase the purity of the initial environmental states ρ_{P_i} . However, this approach may be unfeasible if the environments are insufficiently isolated or if the energy required is prohibitively high. For these reasons, in the next section, we introduce an alternative method to improve $\gamma_a^{(N_p)}$ without increasing the amount of initial resources.

V. COARSE GRAINING INTO MACROFRACTIONS

The coarse graining of the environments has been used to approach ideal objectivity, for instance, in Refs. [12,16]. Here, coarse graining refers to the grouping of subsystems into a larger composite system, commonly referred to as macrofractions [59,60]. We show that with this method, we can approximate ideal intersubjectivity even with a finite amount of thermodynamic resources. This is a feasible method for increasing $\gamma_a^{(N_p)}$ once the initial states and purities of the environments are fixed, thereby establishing coarse graining as a necessary condition for approaching ideal intersubjectivity. Our result generalizes previous model-dependent findings [42,61–63], highlighting the essential role of coarse graining in attaining ideal intersubjectivity within a model-independent framework.

We identify two different methods of coarse graining. In the first case, we coarse grain the system by simply incorporating subsystems together and keeping the total number of subsystems N_p constant. This method simply increases the size of the environments and, as we will show, makes $\gamma_a^{(N_p)}$ larger. This approach is particularly well suited for infinite-dimensional systems, such as bosonic systems, where analysis is restricted to a finite subset of modes. We adopt a similar strategy in the second case but maintain a fixed total Hilbert space dimension. In doing so, the grouping of subsystems into macrofractions effectively reduces the total number of environments, i.e., $N_p \rightarrow N_p^{cg}$. See Fig. 2 for a pictorial representation.

Whether we use the first or second coarse-graining method, coarse graining corresponds to a mapping of the original environments $\{P_i\}_{i=1}^{N_p}$ into a new set of coarse-grained environments $\{P_i^{l_{cg}}\}_i$. The total number of environments after coarse graining depends on the specific method used. However, regardless of the method, in our notation, each coarse-grained environment consists of l_{cg} original environments grouped together. Following this redefinition, the initial environmental state is replaced by $\tau_\beta^{\otimes l_{cg}}$ and, consequently, the decomposition introduced in Eq. (9) must also be updated accordingly. This results in the coarse-grained versions $A_x^{l_{cg}}$ and $a(x|l_{cg})$. Formally, when we coarse grain, the initial coarse-grained environment state is

$$\rho_{P_i^{l_{cg}}} = \bigotimes_{j \in D_i^{l_{cg}}} \rho_{P_j} = \sum_{x_1, \dots, x_{l_{cg}}} A_{x_1} \otimes \dots \otimes A_{x_{l_{cg}}}, \quad (19)$$

where $D_i^{l_{cg}}$ is a collection of indices corresponding to systems constituting the i th macrofraction. The dimension of the macrofraction is $l_{cg} = |D_i^{l_{cg}}|$, which we assume to be independent of i , since we consider all new $P_i^{l_{cg}}$ to have the same dimension.

We now want to decompose this state as

$$\rho_{P_i^{l_{cg}}} = \sum_x A_x^{l_{cg}} \quad (20)$$

in such a way that $a(x|l_{cg}) = \text{Tr}\{A_x^{l_{cg}}\}$ increases the coarse-grained optimal agreement $\gamma_{a,l_{cg}}^{(N_p)}$, as we increase l_{cg} . Regardless of the coarse-graining method applied, the problem simply consists in understanding how to construct $A_x^{l_{cg}}$ [or $a(x|l_{cg})$], from combinations of $A_{x_1} \otimes \dots \otimes A_{x_d}$ [or $a(x_1) \dots a(x_d)$]. After this reconstruction, the original bound $\gamma_a^{(N_p)}$ gets remapped to $\gamma_{a,l_{cg}}^{(N_p)} := \sum_x a(x|l_{cg})^{N_p}$, along with $\delta_{a,l_{cg}}^{(N_p)}$ and $\beta_{a,l_{cg}}^{(N_p)}$.

In general, there is no unique way to employ such mapping from original environments to coarse-grained ones. However, for $d_S = 2$, the following scaling holds.

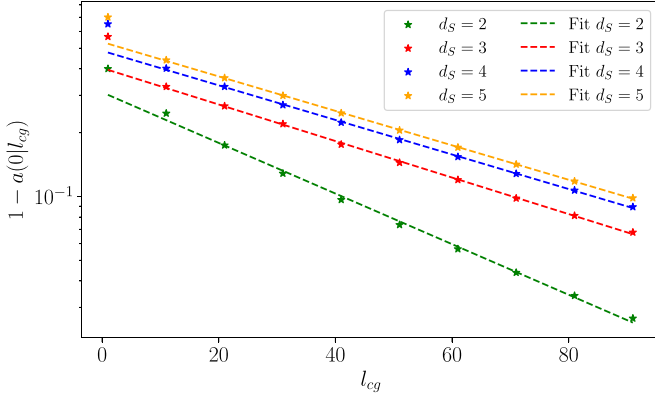
Theorem 3 (Approaching ideal intersubjectivity with coarse graining). For $d_S = 2$ outcomes, there exists a coarse graining for which the function $\gamma_{a,l_{cg}}^{(N_p)}$ scales as

$$\gamma_{a,l_{cg}}^{(N_p)} \simeq 1 - N_p \frac{e^{-D(a)(l_{cg}-1)}}{\sqrt{l_{cg}}} F(a) + O[a(l_{cg}|1)^2] \quad (21)$$

for l_{cg} large. Here, $D(a)$ and $F(a)$ are functions that depend solely on $\mathbf{a} = \{a(0), a(1), \dots, a(d_S - 1)\}$ and not on the size of the coarse graining l_{cg} .

This theorem is demonstrated in Appendix D, where we also provide a precise definition of the coarse-grained projectors $\{A_x^{l_{cg}}\}_x$. Notably, $\gamma_{a,l_{cg}}^{(N_p)}$ converges exponentially to 1 as a function of l_{cg} , indicating that ideal intersubjectivity can, in general, be recovered even under finite resources, provided that subsystems are grouped into sufficiently large macrofractions.

For $d_S > 2$, deriving an analytical formula becomes particularly challenging. Therefore, to determine whether $\gamma_{a,l_{cg}}^{(N_p)}$ approaches 1 exponentially fast as l_{cg} increases, we



d_S	R^2	c_0	c_1
2	0.99726	0.30992	-0.02747
3	0.99943	0.40351	-0.01982
4	0.99966	0.48630	-0.01875
5	0.99988	0.53518	-0.01869

FIG. 3. In this figure, we plot the exponential decay of $1 - a(0|l_{cg})$ to extend the result of Theorem 3. Top panel: logarithmic plot of $1 - a(0|l_{cg})$ as a function of l_{cg} . Stars correspond to values evaluated with the definition in Appendix D 2; dashed lines correspond to fit with $c_0 e^{c_1 l_{cg}}$. Table: results of the fit. We assess the efficiency of the fit with the coefficient of determination R^2 (see Appendix D 2 for details). We considered initial probability $\mathbf{a}_{d_S} = \{a(0) = 1/d_S + 0.1, a(1) = \dots = a(d_S - 1) = 1/d_S - 0.1/(d_S - 1)\}$, which is close to a maximally mixed initial configuration. The fit is performed for $l_{cg} > l_{cg}^{\min}$ to demonstrate the asymptotic behavior. In practice, we excluded the first point $l_{cg} = 1$.

numerically explore the scaling of $\gamma_{a,l_{cg}}^{(N_P)}$. More precisely, we investigate the behavior of $a(l_{cg}|0)$, that is, how the largest probability scales. In fact, to guarantee that $\gamma_{a,l_{cg}}^{(N_P)}$ increases with l_{cg} and approaches 1, it is sufficient for $a(l_{cg}|0)$ to also increase with l_{cg} and to converge toward 1.

Inspired by the case $d_S = 2$, we devise a similar strategy to group products of $a(x_i)$. We refer the reader to Appendix D 2 for details. In Fig. 3, we show $1 - a(l_{cg}|0)$ as a function of l_{cg} . A fit of the form $c_0 e^{c_1 l_{cg}}$ is also provided, along with the fit parameters listed in the table in Fig. 3. We observe that the model provides a good fit, and we argue that $\gamma_{a,l_{cg}}^{(N_P)}$ approaches 1 exponentially fast with respect to l_{cg} , even for $d_S > 2$. For completeness, we also include the case $d_S = 2$ in the plot. Additionally, we note that the rate of the exponential decay remains nearly constant across the cases $d_S > 2$, indicating that the functional dependence of c_0 and c_1 does not vary significantly.

We conclude this section with a remark regarding our result. In this section, we showed that coarse graining, if properly performed as described in Appendix D, can yield ideal intersubjectivity even with finite resources in a model-independent fashion. No other assumptions were made in that regard, apart from unitary evolution. We show in the next section that intersubjectivity can be achieved even without ideal unitary evolution. We conclude that coarse graining proves to be a sufficient condition to attain ideal intersubjectivity. On

the other hand, there are no other strategies that we can devise other than grouping environments together to increase $\gamma_a^{(N_P)}$. It thus becomes a necessary condition as well, without requiring full cooling of all pointers and the consequent use of infinite thermodynamic resources.

VI. COMPARISON WITH THE STANDARD MODEL OF OBJECTIVITY

Most of the results in quantum Darwinism and objectivity are Hamiltonian dependent. In this paper, however, we have established some general results using an approach that does not rely on a specific model and holds for arbitrary unitaries. In this section, we aim to connect our theoretical findings to a standard model of objectivity in order to show how agreement and bias in a specific model relate to our bounds.

We consider a central qubit system S coupled to N_P qubits. The resulting so-called pure-dephasing Hamiltonian can be written as

$$H_\star = \frac{1}{2} \sum_{i=1}^{N_P} \sigma_z^{(S)} \otimes \sigma_z^{(i)}, \quad (22)$$

with $\sigma_z^{(S)}$ the Pauli operator acting on the system, while $\sigma_z^{(i)}$ acting on P_i only. This model had been extensively studied in the literature [39,60,64], proving to result in quantum Darwinism and SBS structure in the coarse-grained limit.

The initial state of the system is given by $|\psi\rangle = \sqrt{p_0}|0\rangle + \sqrt{1-p_0}|1\rangle$, while each environment is initialized in a thermal state diagonal in the $\sigma_x^{(P_i)}$ basis. The joint evolution under the unitary $U_\star(t)$, generated by the Hamiltonian in Eq. (22), leads to the final state $\rho_{S,P}^t$. To extract the information that has been broadcast to the environments, we employ a local measurement $\{\Pi_x^{(i)}\}_{x=0,1}$ on each pointer, optimized to distinguish between the evolved branches of the system state. An analogous procedure is used in the coarse-grained scenario, where the measurement is now defined on the macrofractions, i.e., $\{\Pi_x^{(i,l_{cg})}\}_{x=0,1}$.

To simulate the evolution with a very large number of environmental spins N_P and large macrofractions, we employ a method developed in Ref. [64]. This approach takes advantage of the fact that dynamics generated by Eq. (22) can be efficiently described in terms of spin states, significantly reducing the numerical memory required for the simulation. Further details are provided in Appendix E, while here, we only provide a sketch of the method used.

Given the structure of the Hamiltonian, the unitary evolution can be written in terms of a conditional unitary

$$U_\star(t) = |0\rangle\langle 0| \otimes [V(t)^{\otimes N_P}] + |1\rangle\langle 1| \otimes [V(-t)^{\otimes N_P}] \quad (23)$$

which results in an environmental state at time t that reads as

$$\rho_P(t) = p_0 \tilde{\rho}_0(t)^{\otimes N_P} + p_1 \tilde{\rho}_1(t)^{\otimes N_P}, \quad (24)$$

where $\tilde{\rho}_0(t) = V(t)\rho_P V(t)^\dagger$ and $\tilde{\rho}_1(t) = V(-t)\rho_P V(-t)^\dagger$ are the local environments for the respective branch. The decomposition of the state into spin states allows us to represent the system in a block-diagonal form. Further, for a macrofraction

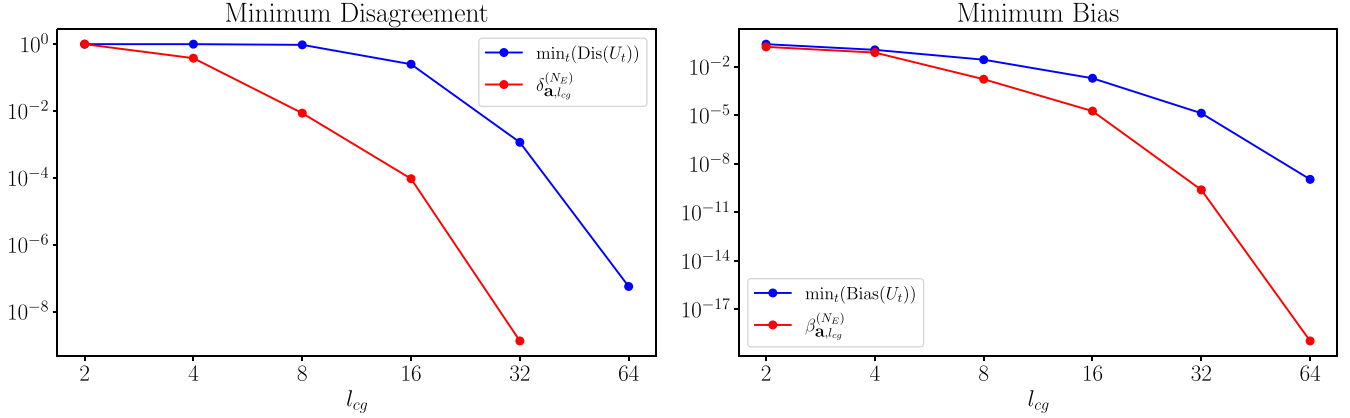


FIG. 4. We report minimum disagreement $\text{Dis}(U_t)$ and bias $\text{Bias}(U_t)$ on the left and right canvas, respectively, for a pure dephasing dynamics generated by a star-shape spin interaction [see Eq. (22)]. The quantities are evaluated for initial state $|\psi_0\rangle = \sqrt{p_0}|0\rangle + \sqrt{1-p_0}|1\rangle$ with $p_0 = 0.2$, while the environments are in a thermal state with inverse temperature $\beta = 1$ and the total number of environmental qubits is $N_P = 1024$. We evaluate the minimum of the two considered quantities over the time interval given as $t \in [0, 6]$, and compare them with the corresponding coarse-grained version of the bound, namely, $\delta_{a,l_{cg}}^{(N_P)}$ and $\beta_{a,l_{cg}}^{(N_P)}$. We observe a similar behavior in both the bound and pure dephasing models, with the only distinction being that the decrease is slower in the pure dephasing case. Nevertheless, it can still be characterized as an exponential decay. Note that the axis uses an exponential scaling in the l_{cg} . For $l_{cg} = 64$, the bound for $\delta_{a,l_{cg}}^{(N_P)}$ was too small to be represented in the plot.

of size l_{cg} we have that

$$\rho_{P_i^{l_{cg}}}(t) = \bigoplus_{j=0}^{l_{cg}/2} [p_0 N_j^{(0)}(t) + p_1 N_j^{(1)}(t)]^{\oplus B_j} \quad (25)$$

(see Appendix E for details and definitions of $N_j^{(x)}$ and B_j). Here N_j^x , $x = 0, 1$, are matrices whose dimension is up to $(2l_{cg} + 1) \times (2l_{cg} + 1)$, which means that we need to evaluate matrices with dimensions that scale linearly with l_{cg} rather than exponentially.

Using this structure, we can efficiently simulate the evolution of the total environment and evaluate the optimal measurement for discriminating between the two branches based on optimal discrimination theory [65,66]. Notably, the optimal discrimination measurement can also be expressed in a block-diagonal form, further enhancing computational efficiency.

We simulate a pure-dephasing model with up to 1024 environmental spins and evaluate agreement and bias, respectively, defined in Eqs. (10) and (13), and compare it with respect to our bounds. More specifically, we compute the coarse-grained version of agreement and bias as a function of l_{cg} and evaluate their minimum on a typical timescale of the Hamiltonian evolution. We compare these values to the coarse-grained version of our bounds, namely, $\gamma_{a,l_{cg}}^{(l_{cg})}$ and $\beta_{a,l_{cg}}^{(N_P)}$. We report disagreement and bias in Fig. 4: we observe that, as the coarse-graining size l_{cg} increases, not only does $\delta_{a,l_{cg}}^{(N_P)}$ approach zero, consistent with the results of the previous section, but both the minimum disagreement for the pure-dephasing model and the bias also exhibit an exponential decay toward zero, albeit at a slower rate.

We also observe that there is always a gap between our bound and the agreement and bias evaluated for the pure dephasing Hamiltonian. This discrepancy is mainly due to the fact that the pure dephasing evolution described by the Hamiltonian in Eq. (22) corresponds to a local interaction. In

contrast, the optimal unitary evolution might not be local and could require additional fine tuning. However, the values ultimately approach 0 as the size of the coarse graining increases. This indicates that locality plays a role in determining the rate at which we approach ideal intersubjectivity, but not whether ideality is ultimately attained by coarse graining. This further suggests that coarse-graining methods are crucial for universal objectivity, regardless of the model used.

As a final comment on the central spin model, we emphasize that the coarse-graining method also influences the agreement time, i.e., the minimum time required to reach an ϵ -close approximation to the ideal values of bias and agreement. In Appendix E 1, we plot the agreement and bias as functions of time for various values of l_{cg} . While the maximum (minimum) agreement (bias) stabilizes beyond a certain coarse-graining size, increasing l_{cg} still has a significant effect on the time required to reach these extremal values (see Fig. 5). Specifically, larger coarse-graining sizes lead to a reduction in the minimum time needed to achieve the desired level of precision. Intuitively, this is due to the increasing Hilbert space dimension allocated the output subspaces, thereby making different branches easier to distinguish between one another and more quickly so as the joint system evolves. This suggests that coarse graining impacts not only the asymptotic values of agreement and bias but also the dynamics of convergence, potentially accelerating the measurement process, a direction we leave open for future investigation.

VII. DISCUSSIONS AND CONCLUSIONS

The interconnection between measurement, thermodynamics, and the emergence of classicality has been of central research interest in quantum thermodynamics and the foundations of quantum theory. Particularly the clash between textbook projective measurement and the third law of thermodynamics has received attention in recent years. We address

this question by appealing to quantum intersubjectivity, a more permissive form of objectivity that does not require the system to be preserved when observers gather information about it. Here, ideal intersubjectivity is defined by the requirements of locality, probability, reproducibility, and agreement (between different observers). Ozawa's theorem [24] revisited in Sec. II and in Appendix A established agreement as a necessary result of the other two properties.

We first showed that ideal intersubjectivity cannot be unitarily produced between observers that are initially in local thermal equilibrium (Observation 1). This first observation can be directly intuited from the fact that agreement implies a rank-reduced final state compared to the generally full-rank initial state. It naturally leads to the questions of how close we can get to ideal intersubjectivity, if it can be approximated, and at what cost.

Following this observation, our first main result (Theorem 1) gives a saturable upper bound for the maximal amount of agreement that can be achieved with an initially thermal environment. While the optimal unitary operation that does so is not unique, an explicit construction is provided in Appendix B. The second main result was then to show the unavoidable tradeoff incurred when optimizing for agreement: a bias is introduced vis-à-vis the original statistics encoded in the system. Theorem 2 quantifies this bias for the case of a maximal-agreement unitary. As the last of our three main results, we showed that coarse graining (i.e., combining parts of the environment into larger macroenvironments) lets us recover perfect agreement arbitrarily closely. This is stated precisely in Theorem 3. Therefore, ideal intersubjectivity is recovered not only in the limiting case of cooling the environment to absolute zero but also at finite temperatures, provided the environment undergoes appropriate coarse graining. This finding reconciles Ozawa's conclusions with the conventional macroscopic limit, here formalized as a coarse-graining procedure essential for a consistent quantum-to-classical transition. It parallels a similar recent use of coarse graining for obtaining objectivity under equilibration of a closed quantum system [42]. We finished by illustrating our results in the relevant limits of the paradigmatic central-spin system in Sec. VI.

Extending this work, it would be interesting to investigate how the emergence of intersubjectivity is further constrained by considering a smaller set of processes, such as the ones due to microscopic energy conservation and thermal operations [54,55,67,68].

Finally, we observed that the speed of measurement, defined as the timescale at which we approach ideal intersubjectivity, strongly depends on the level of coarse graining applied to the environments. Quantifying this dependence constitutes a tantalizing avenue for future research.

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APPENDIX A: GENERALIZATION OF OZAWA'S THEOREM

We present a generalized proof of Ozawa's theorem [24], extending the original result from pure states to arbitrary density matrices. For clarity, we first consider the minimal case of two environmental subsystems ($N_p = 2$). The extension to $N_p > 2$ follows naturally through identical reasoning. This generalization preserves the theorem's core insight while accommodating the mixed-state scenarios inherent in thermodynamic contexts.

We recall that Ozawa's theorem states that if locality and probability reproducibility are satisfied, then the agreement condition is automatically fulfilled. In this context, locality implies the existence of local measurements $\{\Pi_x^{(i)}\}_{x=0}^{d_S-1}$ and local probability distributions $\tilde{p}_P(x_i)$, while probability reproducibility means that each observer's measurement obeys the same probability distribution originally derived from the system $\tilde{p}_{P_i}(x) = \tilde{p}_P(x) = p_S(x)$. We can write the probabilities in the Heisenberg picture as

$$\begin{aligned}\tilde{p}_{P_i}(x) &= \text{Tr}\{\mathbb{I}_S \otimes \Pi_x^{(1)} \otimes \mathbb{I}_{P_2} \tilde{\rho}_{S:P_1,P_2}\} \\ &= \text{Tr}\{\Pi_x^{(1)}(t) \rho_S \otimes \rho_{P_1} \otimes \rho_{P_2}\},\end{aligned}\quad (\text{A1})$$

where $\Pi_x^{(1)}(t) = U^\dagger(\mathbb{I}_S \otimes \Pi_x^{(1)} \otimes \mathbb{I}_{P_2})U$ is the operator in the Heisenberg picture. Similarly, this must hold also for $\tilde{p}_{P_2}(x)$ and $\Pi_x^{(2)}(t)$. Assuming now probability reproducibility, we have that

$$\text{Tr}\{\Pi_x^{(i)}(t) \rho_S \otimes \rho_{P_1} \otimes \rho_{P_2}\} = \langle x | \rho_S | x \rangle \quad (\text{A2})$$

for $i = 1, 2$. The last equation must hold for all initial factorized states and, in particular, for any arbitrary pure state $|\psi^0\rangle = |\psi_S^0\rangle |\psi_{P_1}^0\rangle |\psi_{P_2}^0\rangle$. Given that $A_x^{(S)}(0) = |x\rangle\langle x|$, it follows that $\Pi_x^{(1)}(t)$ and $\Pi_x^{(2)}(t)$ are all orthogonal projectors and we have that Eq. (A2) can be written as

$$A_x^{(S)}(0) |\psi_0\rangle = \Pi_x^{(1)}(t) |\psi_0\rangle = \Pi_x^{(2)}(t) |\psi_0\rangle. \quad (\text{A3})$$

This means that, by linearity, it also holds for any separable density matrix $\rho_0 = \sum_j q_j |\psi_0^{(j)}\rangle\langle\psi_0^{(j)}|$ (and thus also for any factorized state), implying

$$\text{Tr}\{\Pi_x^{(1)}(t) \Pi_y^{(2)}(t) \rho_0\} = \delta_{xy} \text{Tr}\{A_x^{(S)}(0) \rho_0\} \quad (\text{A4})$$

$$= \delta_{xy} \langle x | \rho_S | x \rangle, \quad (\text{A5})$$

which is nothing but the agreement condition for two environments. We have thus proved that locality and probability reproducibility imply agreement even in the case of a mixed initial state, extending Ozawa's result to this scenario.

APPENDIX B: PROOF OF THEOREM 1

In this appendix, we provide the formal proof of our main Theorem 1. The initial state of the system can be written as

$$\rho_S = \sum_{x=0}^{d_S-1} p_S(x) |x\rangle\langle x| + \rho_S^{\text{off}}, \quad (\text{B1})$$

where ρ_S^{off} is the off-diagonal part of the initial system, which is irrelevant to our information broadcasting analysis. The initial state of the environment is the tensor product

$$\rho_P = \tau_\beta^{\otimes N_P}, \quad (\text{B2})$$

where τ_β is the thermal state at temperature β . According to the discussion in Sec. IV, we can split it into d_S orthogonal components

$$\tau_\beta = \sum_{x=0}^{d_S-1} A_x = \bigoplus_{x=0}^{d_S-1} A_x, \quad (\text{B3})$$

where we stressed the orthogonality of the $\{A_x\}$ with the last notation, and $A_x = \sum_{i \in D_x} p(E_k^{(x)}) |E_i\rangle\langle E_i|$, and with the Boltzmann factor $p(E_k^{(y)}) = \frac{e^{-\beta E_k^{(y)}}}{Z_\beta}$. Following this decomposition, we can further decompose the initial state of the environment as

$$\rho_P = \rho_P^{\text{agr}} \oplus \rho_P^{\text{dis}}, \quad (\text{B4})$$

where ρ_P^{agr} is the component supported on the agreement subspace $\mathcal{A} = \text{span}\{\bigotimes_{i=1}^{N_P} \Pi_x^{(i)}\}_{x=0}^{d_S-1}$, and thus can be written as

$$\rho_P^{\text{agr}} = \bigoplus_{x=0}^{d_S-1} (A_x)^{\otimes N_P}, \quad (\text{B5})$$

while ρ_P^{dis} is the orthogonal component. The initial state of the system and environments is then

$$\begin{aligned} \rho_{S:P} &= \sum_{x=0}^{d_S-1} p_S(x) |x\rangle\langle x| \otimes \rho_P^{\text{agr}} + \sum_{x=0}^{d_S-1} p_S(x) |x\rangle\langle x| \otimes \rho_P^{\text{dis}} \\ &+ \rho_S^{\text{off}} \otimes \rho_P. \end{aligned} \quad (\text{B6})$$

The first term

$$\rho_{S:P}^{\text{agr}} = \rho_S^{\text{dia}} \otimes \rho_P^{\text{agr}} \quad (\text{B7})$$

is the relevant part for intersubjectivity since it is supported on diagonal components of the system ρ_S^{dia} and the agreement subspace ρ_P^{agr} . To achieve maximum agreement, it is sufficient to evaluate the optimal broadcast between the system and $\mathcal{A}$, given that sending information to the orthogonal subspace $\mathcal{A}_\perp$ will only reduce the agreement and that the off-diagonal part ρ_S^{off} of the system does not contain the information to be broadcast.

Since we focus on the evolution within the subspace $\mathcal{H}_S \otimes \mathcal{H}_A$, where $\mathcal{H}_A$ is the Hilbert space corresponding to the vector space $\mathcal{A}$, we will not obtain a unique unitary on the whole Hilbert space composed of system and environments. Instead, there exists a class $U_{\text{BJIB}}^{\text{opt}}$ of unitaries that maximizes agreement, and we will provide a specific example of such a

unitary later. Hence, our starting point is the state

$$\rho_{S:P}^{\text{agr}} = \sum_{x=0}^{d_S-1} p_S(x) |x\rangle\langle x| \otimes \bigoplus_{y=0}^{d_S-1} \tilde{A}_y, \quad (\text{B8})$$

where we have defined $\tilde{A}_y = A_y^{\otimes N_P}$. Given that we have to satisfy the condition of joint information broadcasting, we can use Lemma 2 from [1], which we restate here for the current context:

Lemma B1. [1] Let $\mathcal{M}_U(\beta)$ be a finite resource measurement procedure consisting of a unitary correlating step between a system and $\mathcal{H}_P$. If the initial state can be written as

$$\rho_{SP} = \rho_S \otimes \sum_{y=0}^{d_S-1} A_y, \quad (\text{B9})$$

then the unitary map U that realizes the correlating step and satisfies unbiasedness can be decomposed as two distinct unitaries $\tilde{U}$ and V :

$$\tilde{U} = \sum_{x=0}^{d_S-1} |x\rangle\langle x| \otimes \tilde{U}_x, \quad (\text{B10})$$

$$V = \sum_{x,y} \sum_m |y\rangle\langle x| \otimes |\psi_m^{(x)}\rangle\langle\psi_m^{(y)}|, \quad (\text{B11})$$

with $\tilde{U}_x$ arbitrary unitary on $\mathcal{H}_P$, while $\{|\psi_m^{(x)}\rangle\}_m$ span the support of A_x for all $x = 0, \dots, d_S - 1$.

The theorem is given for the case in which all the A_x have the same dimension, but this can be relaxed at the cost of a more complicated V . We note the analogy with our reduced problem of biased joint information broadcasting, in particular the resemblance of Eqs. (B8) and (B9). Furthermore, the unbiasedness at the level of the lemma can be translated to the maximization of agreement, i.e., the broadcasting of information on the agreement subspace. Indeed, formally, our problem on $\mathcal{H}_S \otimes \mathcal{H}_A$ can be recast exactly in the language of Lemma B1. More specifically, we can identify A_y with $\tilde{A}_y$, and then apply the theorem. Even though $\rho_{S:P}^{\text{agr}}$ is not a normalized state, the proof of the Lemma above still holds. This means that the optimal unitary on $\mathcal{H}_S \otimes \mathcal{H}_A$ that achieve the unbiasedness on the agreement subspace is given by an arbitrary step $\tilde{U}$ as in Eq. (B10) (that does not change the agreement and the bias, and thus will be neglected in the following), and a broadcasting step given by

$$V = \sum_{x,y=0}^{d_S-1} \sum_m |y\rangle\langle x| \otimes |\tilde{\psi}_m^{(x)}\rangle\langle\tilde{\psi}_m^{(y)}|, \quad (\text{B12})$$

where now $\{|\tilde{\psi}_m^{(i)}\rangle\}_m$ span the support of $\tilde{A}_i$. We can further expand on the the $\{|\tilde{\psi}_m^{(i)}\rangle\}$. Indeed, we have that

$$\begin{aligned} \tilde{A}_y &= \bigotimes_{i=1}^{N_P} A_y = \bigotimes_{i=1}^{N_P} \sum_{k_i \in D_y} p(E_{k_i}^{(y)}) |E_{k_i}\rangle\langle E_{k_i}| \\ &= \sum_{k_1, \dots, k_{N_P} \in D_y} \bigotimes_{i=1}^{N_P} p(E_{k_i}^{(y)}) |E_{k_i}\rangle\langle E_{k_i}|. \end{aligned} \quad (\text{B13})$$

We thus identify m as the multi-indexed vector $(k_1, \dots, k_{N_p})$ with $k_i \in D_y$ and

$$|\tilde{\psi}_m^{(y)}\rangle = |E_{k_1} E_{k_2} \dots E_{k_{N_p}}\rangle, \quad k_1, \dots, k_{N_p} \in D_y. \quad (\text{B14})$$

In this way, we can write

$$\begin{aligned} \sum_m |\tilde{\psi}_m^{(x)}\rangle \langle \tilde{\psi}_m^{(y)}| &= \sum_{k_1} |E_{k_1}^{(x)}\rangle \langle E_{k_1}^{(y)}| \otimes \dots \otimes \sum_{k_{N_p}} |E_{k_{N_p}}^{(x)}\rangle \langle E_{k_{N_p}}^{(y)}| \\ &= T_{x,y} \otimes \dots \otimes T_{x,y} = T_{x,y}^{\otimes N_p}, \end{aligned} \quad (\text{B15})$$

where we have defined $T_{x,y}$ as the transition between the subspace y to subspace x on a single environment, and set $|E_k^{(y)}\rangle$ equivalent to $|E_k\rangle$ for $k \in D_y$. This leads to an operator in the form of

$$V = \sum_{x,y=0}^{d_S-1} |y\rangle \langle x| \otimes T_{x,y}^{\otimes N_p}. \quad (\text{B16})$$

To ensure that V is a unitary operator, it must be extended to act on $\mathcal{H}_S \otimes \mathcal{H}_D$, where some degree of arbitrariness exists in the choice of this extension. This could, in principle, affect the bias, but not the agreement, which is evaluated only on the subspace where V acts. Thus, we postpone the discussion of the extension V_D until we address the bias in Appendix C.

We are now ready to evaluate the maximum agreement. The post-interaction state on the agreement subspace only is

$$\begin{aligned} \tilde{\rho}_{S:P}^{\text{agr}} &= V \left(\sum_{x=0}^{d_S-1} p_S(x) |x\rangle \langle x| \otimes \sum_{y=0}^{d_S-1} (A_y)^{\otimes N_p} \right) V^\dagger \\ &= \left(\sum_{x',y'} |y'\rangle \langle x'| \otimes T_{x',y'}^{\otimes N_p} \right) \\ &\quad \times \left(\sum_{x=0}^{n-1} p_S(x) |x\rangle \langle x| \otimes \sum_{y=0}^{d_S-1} (A_y)^{\otimes N_p} \right) \\ &\quad \times \left(\sum_{x'',y''} |x''\rangle \langle y''| \otimes T_{y'',x''}^{\otimes N_p} \right) \\ &= \sum_{x,x',x''} \sum_{y,y',y''} \delta_{x,x'} \delta_{x,x''} p_S(x) |y'\rangle \langle y''| \otimes (T_{x',y'} A_y T_{y'',x''})^{\otimes N_p}. \end{aligned} \quad (\text{B17})$$

$$\quad (\text{B18})$$

$$\quad (\text{B19})$$

However, as the subspaces are orthogonal, we have

$$\begin{aligned} T_{x',y'} A_y T_{y'',x''} &= \sum_k |E_k^{(x')}\rangle \langle E_k^{(y')}| \sum_{k'} p(E_{k'}^{(y)}) |E_{k'}^{(y)}\rangle \langle E_{k'}^{(y)}| \\ &\quad \times \sum_{k''} |E_{k''}^{(y'')}\rangle \langle E_{k''}^{(x'')}| \end{aligned} \quad (\text{B20})$$

$$= \delta_{y',y} \delta_{y'',y} \sum_k p(E_k^{(y)}) |E_k^{(x')}\rangle \langle E_k^{(x'')}|. \quad (\text{B21})$$

Using this expression in the state evolution and defining $A_{y,x} = \sum_k p(E_k^{(y)}) |E_k^{(x)}\rangle \langle E_k^{(x)}|$, we get

$$\begin{aligned} \tilde{\rho}_{S:P}^{\text{agr}} &= \sum_{x,x',x''} \sum_{y,y',y''} \delta_{x,x'} \delta_{x,x''} \delta_{y',y} \delta_{y'',y} p_S(x) |y'\rangle \langle y''| \\ &\quad \times \otimes \left(\sum_k p(E_k^{(y)}) |E_k^{(x')}\rangle \langle E_k^{(x'')}| \right)^{\otimes N_p} \end{aligned} \quad (\text{B22})$$

$$= \sum_{x,y} p_S(x) |y\rangle \langle y| \otimes A_{y,x}^{\otimes N_p}. \quad (\text{B23})$$

As the matrices $A_{y,x}$ have orthogonal support to each other,

$$\text{Tr}\{\Pi_z A_{y,x}\} = \delta_{z,x} a(y), \quad (\text{B24})$$

with $a(y) = \text{Tr}\{A_{y,x}\} = \text{Tr}\{A_y\}$, we can calculate the agreement $\text{Agr}(U_{\text{BJIB}}^{\text{opt}})$ (where we have identified $U_{\text{BJIB}}^{\text{opt}}$ as the class of evolutions that maximize the agreement as described above) as follows:

$$\begin{aligned} \text{Agr}(U_{\text{BJIB}}^{\text{opt}}) &= \sum_z \text{Tr}\{\mathbb{I}_S \otimes \Pi_z^{\otimes N_p} \tilde{\rho}_{S:P}^{\text{agr}}\} \\ &= \sum_z \text{Tr}\{\mathbb{I}_S \otimes \Pi_z^{\otimes N_p} \tilde{\rho}_{S:P}^{\text{agr}}\} \end{aligned} \quad (\text{B25})$$

$$= \sum_z \sum_{x,y} p_S(x) \text{Tr}\{\Pi_z^{\otimes N_p} A_{y,x}^{\otimes N_p}\} \quad (\text{B26})$$

$$= \sum_z \sum_{x,y} p_S(x) \text{Tr}\{\Pi_z A_{y,x}\}^{N_p} \quad (\text{B27})$$

$$= \sum_z \sum_{x,y} p_S(x) \delta_{z,x} a(y)^{N_p} \quad (\text{B28})$$

$$= \sum_y a(y)^{N_p} \equiv \gamma_a^{(N_p)}. \quad (\text{B29})$$

This proves Theorem 1 since any other unitary U will have smaller agreement.

APPENDIX C: PROOF OF THEOREM 2

In this appendix, we prove Theorem 2, which regards the bias for the optimal unitary $U_{\text{BJIB}}^{\text{opt}}$ discussed in the previous appendix.

As already discussed, the optimal unitary represents a class of unitaries since the extension of V to the whole Hilbert space is not uniquely determined by the maximization of agreement. However, as we will show here, most of the arbitrariness can be eliminated by imposing the condition of BJIB given in Eq. (6). The only arbitrariness left is the one in $\tilde{U}$, which does not affect any property we are looking at.

The extension of the unitary V given in Eq. (B16) to the entire Hilbert space is achieved by introducing an additional term W that acts on the orthogonal subspace $\mathcal{H}_S \otimes \mathcal{H}_D$, such that the operator $U = V + W$ remains unitary. Since W acts solely on the orthogonal subspace, we have $VW = 0$. Consequently, W must be unitary within the orthogonal subspace alone, and we can study its properties by looking only at how

it acts on $\rho_S \otimes \rho_P^{\text{dis}}$. More in detail, this reads as

$$\rho_P^{\text{dis}} = \bigotimes_{i=1}^{N_P} \sum_{y_i=0}^{d_S-1} A_{y_i} - \sum_{y=0}^{d_S-1} \bigoplus_{i=1}^{N_P} A_y = \sum_{\{y_1, \dots, y_{N_P}\}} \bigotimes_{i=1}^{N_P} A_{y_i}, \quad (\text{C1})$$

where we have introduced the notation $\sum_{\{y_1, \dots, y_{N_P}\}}$ to denote the sum over the indices $\{y_1, \dots, y_{N_P}\}$, excluding the case where $y_1 = y_2 = \dots = y_{N_P}$ (that would correspond to the agreement subspace).

Applying the constraint of BJIB given in Eq. (6), we need a symmetric broadcast of information across all environments. Given the form of the initial state on $\mathcal{H}_S \otimes \mathcal{H}_D$, this is not possible. Thus, in order to satisfy the BJIB, we must avoid acting on $\mathcal{H}_S \otimes \mathcal{H}_D$ in order to prevent an asymmetric distribution of the information. We thus conclude that

$$W = \mathbb{I}_S \otimes \mathbb{I}_{\mathcal{H}_D}. \quad (\text{C2})$$

In this way, the global evolution that maximizes agreement and satisfies BJIB is

$$U = V + W = \sum_{x,y=0}^{d_S-1} |y\rangle\langle x| \otimes T_{x,y}^{\otimes N_P} + \mathbb{I}_S \otimes \mathbb{I}_{\mathcal{H}_D}. \quad (\text{C3})$$

We can now evaluate the bias, completing the proof of Theorem 2. The action of the whole unitary can be written as

$$\tilde{\rho}_{S:P} = U(\rho_S^{\text{dia}} \otimes \rho_P^{\text{agr}} + \rho_S^{\text{dia}} \otimes \rho_P^{\text{dis}} + \rho_S^{\text{off}} \otimes \rho_P)U^\dagger \quad (\text{C4})$$

$$= U\rho_S^{\text{dia}} \otimes \rho_P^{\text{agr}}U^\dagger + U\rho_S^{\text{dia}} \otimes \rho_P^{\text{dis}}U^\dagger + U\rho_S^{\text{off}} \otimes \rho_PU^\dagger \quad (\text{C5})$$

$$= V\rho_S^{\text{dia}} \otimes \rho_P^{\text{agr}}V^\dagger + W\rho_S^{\text{dia}} \otimes \rho_P^{\text{dis}}W^\dagger + U\rho_S^{\text{off}} \otimes \rho_PU^\dagger \quad (\text{C6})$$

$$= \tilde{\rho}_{S:P}^{\text{agr}} + \rho_S^{\text{dia}} \otimes \rho_P^{\text{dis}} + \tilde{\rho}_{S:P}^{\text{off}}, \quad (\text{C7})$$

where we have used the fact that V acts only on $\rho_S^{\text{dia}} \otimes \rho_P^{\text{agr}}$ while W acts only on $\rho_S^{\text{dia}} \otimes \rho_P^{\text{dis}}$. Moreover, $\tilde{\rho}_{S:P}^{\text{off}}$ is irrelevant since it is off diagonal. Evaluating the local probability yields (given the symmetry, the environment to be measured is irrelevant)

$$\begin{aligned} \tilde{p}_P(z) &= \text{Tr}\{\mathbb{I}_S \otimes \Pi_z \otimes \mathbb{I}_P^{\otimes N_P-1} \tilde{\rho}_{S:P}\} \\ &= \text{Tr}\{\mathbb{I}_S \otimes \Pi_z \otimes \mathbb{I}_P^{\otimes N_P-1} \tilde{\rho}_{S:P}^{\text{agr}}\} \\ &\quad + \text{Tr}\{\mathbb{I}_S \otimes \Pi_z \otimes \mathbb{I}_P^{\otimes N_P-1} \rho_S^{\text{dia}} \otimes \rho_P^{\text{dis}}\}. \end{aligned} \quad (\text{C8})$$

The first term yields

$$\begin{aligned} &\text{Tr}\{\mathbb{I}_S \otimes \Pi_z \otimes \mathbb{I}_P^{\otimes N_P-1} \tilde{\rho}_{S:P}^{\text{agr}}\} \\ &= \text{Tr}\left\{\mathbb{I}_S \otimes \Pi_z \otimes \mathbb{I}_P^{\otimes N_P-1} \sum_{x,y} p_S(x)|y\rangle\langle y| \otimes A_{y,x}^{\otimes N_P}\right\} \end{aligned} \quad (\text{C9})$$

$$= \sum_{x,y} p_S(x) \text{Tr}\{\Pi_z A_{y,x}\} \text{Tr}\{A_{y,x}\}^{N_P-1} \quad (\text{C10})$$

$$= \sum_{x,y} p_S(x) \delta_{z,x} a(y) a(y)^{N_P-1} \quad (\text{C11})$$

$$= p_S(z) \gamma_a^{(N_P)}, \quad (\text{C12})$$

while the second yields

$$\text{Tr}\{\mathbb{I}_S \otimes \Pi_z \otimes \mathbb{I}_P^{\otimes N_P-1} \rho_S^{\text{dia}} \otimes \rho_P^{\text{dis}}\} = \text{Tr}\left\{\mathbb{I}_S \otimes \Pi_z \otimes \mathbb{I}_P^{\otimes N_P-1} \sum_x p_S(x)|x\rangle\langle x| \otimes \sum_{\{y_1, \dots, y_{N_P}\}} \bigotimes_{i=1}^{N_P} A_{y_i}\right\} \quad (\text{C13})$$

$$= \text{Tr}\left\{\Pi_z \otimes \mathbb{I}_P^{\otimes N_P-1} \left(\sum_{y_1, \dots, y_{N_P}} \bigotimes_{i=1}^{N_P} A_{y_i} - \sum_y A_y^{\otimes N_P}\right)\right\} \quad (\text{C14})$$

$$= \sum_{y_1, \dots, y_{N_P}} \text{Tr}\{\Pi_z A_{y_1}\} \prod_{i=2}^{N_P} \text{Tr}\{A_{y_i}\} - \sum_y \text{Tr}\{\Pi_z A_y\} \text{Tr}\{A_y\}^{\otimes N_P-1} \quad (\text{C15})$$

$$= \sum_{y_1} \delta_{z,y_1} a(y_1) \sum_{y_2, \dots, y_{N_P}} \prod_{i=2}^{N_P} a(y_i) - \sum_y \delta_{z,y} a(y) a(y)^{N_P-1} \quad (\text{C16})$$

$$= a(z) \sum_{y_2, \dots, y_{N_P}} \prod_{i=2}^{N_P} a(y_i) - \sum_y \delta_{z,y} a(y) a(y)^{N_P-1}. \quad (\text{C17})$$

Given that

$$\begin{aligned} \sum_{y_2, \dots, y_{N_P}} \prod_{i=2}^{N_P} a(y_i) &= \sum_{y_2} a(y_2) \sum_{y_3, \dots, y_{N_P}} \prod_{i=3}^{N_P} a(y_i) \\ &= \sum_{y_4, \dots, y_{N_P}} \prod_{i=3}^{N_P} a(y_i) = \dots = 1, \end{aligned} \quad (\text{C18})$$

we have that

$$\text{Tr}\{\mathbb{I}_S \otimes \Pi_z \otimes \mathbb{I}_P^{\otimes N_P-1} \rho_S^{\text{dia}} \otimes \rho_P^{\text{dis}}\} = a(z) - a(z)^{N_P}, \quad (\text{C19})$$

which eventually leads to

$$\tilde{p}_P(z) = p_S(z) \gamma_a^{(N_P)} + a(z) - a(z)^{N_P}, \quad (\text{C20})$$

which can be rewritten as a mixture of probabilities

$$\tilde{p}_P(z) = \gamma_a^{(N_P)} p_S(z) + (1 - \gamma_a^{(N_P)}) \eta_P(z), \quad (\text{C21})$$

where

$$\eta_P(z) = \frac{a(z) - a(z)_P^N}{1 - \gamma_a^{(N_P)}}. \quad (C22)$$

We can eventually evaluate the bias

$$\text{Bias}(U_{\text{BJIB}}^{\text{opt}}) = D_T(p_S(x), \tilde{p}(x)) \quad (C23)$$

$$= \frac{1}{2} \sum_x |p_S(x) - \gamma_a^{(N_P)} p_S(x) - (1 - \gamma_a^{(N_P)}) \eta_P(x)| \quad (C24)$$

$$= \delta_a^{(N_P)} D_T(p_S(x), \eta_P(x)). \quad (C25)$$

This completes the proof of Theorem 2.

APPENDIX D: EXPONENTIAL SCALING OF $\gamma_{a,l_{\text{cg}}}^{(N_P)}$

Our aim is to demonstrate the exponential approach of $\gamma_{a,l_{\text{cg}}}^{(N_P)}$ to 1 in terms of the dimension of the coarse graining l_{cg} . We recall the definition of $\gamma_a^{(N_P)}$

$$\gamma_a^{(N_P)} = \sum_{x=0}^{d_S-1} a(x)^{N_P}. \quad (D1)$$

The coarse-grained version of the bound $\gamma_{a,l_{\text{cg}}}^{(N_P)}$ means that we have redefined $\{a(x|l_{\text{cg}})\}_{x=0}^{d_S-1}$ in terms of products of $a(x_i)$. In order to increase $\gamma_{a,l_{\text{cg}}}^{(N_P)}$, we need that $a(0|l_{\text{cg}})$ grows to 1 when l_{cg} increases, while $a(1|l_{\text{cg}}), \dots, a(d_S-1|l_{\text{cg}})$ must decrease with l_{cg} , in order to concentrate the largest eigenvalues in the $A_0^{l_{\text{cg}}}$ partition. We can understand this strategy as the one that mimics the closest thing to a pure state.

The $a(x|l_{\text{cg}})$ must be defined as the sum over $a(x_0)^{k_0} \dots a(x_{d_S-1})^{k_{d_S-1}}$ with $k_0 + \dots + k_{d_S-1} = l_{\text{cg}}$, given that our initial state is given simply as in Eq. (19), that we report here again

$$\rho_{P_i^{\text{cg}}} = \rho_P^{\otimes l_{\text{cg}}} = \sum_{x_1, x_2, \dots, x_{l_{\text{cg}}}} A_{x_1} \otimes \dots \otimes A_{x_{l_{\text{cg}}}}. \quad (D2)$$

The available eigenvalues to be collected in this $a(x|l_{\text{cg}})$ are given from the following expansion:

$$\text{Tr}\{\rho_P^{\otimes l_{\text{cg}}}\} = \sum_{x_1, x_2, \dots, x_{l_{\text{cg}}}} \prod_{i=1}^{l_{\text{cg}}} \text{Tr}\{A_{x_i}\} = \sum_{x_1, x_2, \dots, x_{l_{\text{cg}}}} \prod_{i=1}^{l_{\text{cg}}} a(x_i) \quad (D3)$$

$$= \sum_{k_0 + \dots + k_{d_S-1} = l_{\text{cg}}} \binom{l_{\text{cg}}}{k_0, \dots, k_{d_S-1}} \times a(0)^{k_0} \dots a(d_S-1)^{k_{d_S-1}}, \quad (D4)$$

where in the last step we have used the multinomial theorem, and where

$$\binom{l_{\text{cg}}}{k_0, \dots, k_{d_S-1}} = \frac{l_{\text{cg}}!}{k_0! \dots k_{d_S-1}!} \quad (D5)$$

is the multinomial coefficient. We need to use the elements in the sum in Eq. (D4) to define the coarse grained $a(x|l_{\text{cg}})$. As one can imagine, dealing with such an expression is not easy. We thus analytically study the simplest scenario with $d_S = 2$

in Appendix D 1. We provide the method to study the problem numerically for $d_S > 2$ in Appendix D 2

1. Proof of Theorem 3 (the case of $d_S = 2$)

In this case, the expression for the expansion simplifies to

$$1 = \sum_{k=0}^{l_{\text{cg}}} \binom{l_{\text{cg}}}{k} a(0)^{l_{\text{cg}}-k} a(1)^k := a(l_{\text{cg}}|0) + a(l_{\text{cg}}|1). \quad (D6)$$

In order to have neater expressions, we consider only odd l_{cg} , but the results that follow are easily generalizable to even l_{cg} . In Eq. (D6), we assign the first half to $a(0|l_{\text{cg}})$ in such a way that

$$a(0|l_{\text{cg}}) = \sum_{k=0}^{\frac{l_{\text{cg}}-1}{2}} \binom{l_{\text{cg}}}{k} a(0)^{l_{\text{cg}}-k} a(1)^k. \quad (D7)$$

Now, we can expand this as follows:

$$\sum_{k=0}^{\frac{l_{\text{cg}}-1}{2}} \binom{l_{\text{cg}}}{k} a(0)^{l_{\text{cg}}-k} a(1)^k = 1 - \sum_{k=\frac{l_{\text{cg}}-1}{2}+1}^{l_{\text{cg}}} \binom{l_{\text{cg}}}{k} a(0)^{l_{\text{cg}}-k} a(1)^k. \quad (D8)$$

Making the change of variable $k' = k - (l_{\text{cg}} + 1)/2$, we have that

$$\begin{aligned} & \sum_{k=0}^{\frac{l_{\text{cg}}-1}{2}} \binom{l_{\text{cg}}}{k} a(0)^{l_{\text{cg}}-k} a(1)^k \\ &= 1 - \sum_{k'=0}^{\frac{l_{\text{cg}}-1}{2}} \binom{l_{\text{cg}}}{k' + \frac{l_{\text{cg}}+1}{2}} a(0)^{l_{\text{cg}}-k' - \frac{l_{\text{cg}}+1}{2}} a(1)^{k' + \frac{l_{\text{cg}}+1}{2}} \end{aligned} \quad (D9)$$

$$= 1 - a(0)^{\frac{l_{\text{cg}}-1}{2}} a(1)^{\frac{l_{\text{cg}}+1}{2}} \sum_{k'=0}^{\frac{l_{\text{cg}}-1}{2}} \binom{l_{\text{cg}}}{k' + \frac{l_{\text{cg}}+1}{2}} \left(\frac{a(1)}{a(0)}\right)^{k'}. \quad (D10)$$

We further have that

$$\binom{l_{\text{cg}}}{k' + \frac{l_{\text{cg}}+1}{2}} = \frac{l_{\text{cg}}!}{(k' + \frac{l_{\text{cg}}+1}{2})! (l_{\text{cg}} - k' - \frac{l_{\text{cg}}+1}{2})!} \quad (D11)$$

$$= \frac{l_{\text{cg}}!}{(k' + \frac{l_{\text{cg}}+1}{2})! (\frac{l_{\text{cg}}-1}{2} - k')!} \quad (D12)$$

$$= \frac{l_{\text{cg}}!}{(\frac{l_{\text{cg}}+1}{2})! (\frac{l_{\text{cg}}-1}{2})!} \frac{(\frac{l_{\text{cg}}+1}{2})! (\frac{l_{\text{cg}}-1}{2})!}{(\frac{l_{\text{cg}}+1}{2} + k')! (\frac{l_{\text{cg}}-1}{2} - k')!} \quad (D13)$$

$$= \binom{l_{\text{cg}}}{\frac{l_{\text{cg}}+1}{2}} \frac{(\frac{l_{\text{cg}}-1}{2})!}{(\frac{l_{\text{cg}}-1}{2} - k')!} \frac{(\frac{l_{\text{cg}}+1}{2})!}{(\frac{l_{\text{cg}}+1}{2} + k')!}, \quad (D14)$$

which leads to

$$a(0|l_{\text{cg}}) = 1 - a(0)^{\frac{l_{\text{cg}}-1}{2}} a(1)^{\frac{l_{\text{cg}}+1}{2}} \left(\frac{l_{\text{cg}}}{2} \right) \times \sum_{k'=0}^{\frac{l_{\text{cg}}-1}{2}} \frac{\left(\frac{l_{\text{cg}}-1}{2} \right)!}{\left(\frac{l_{\text{cg}}-1}{2} - k' \right)!} \frac{\left(\frac{l_{\text{cg}}+1}{2} \right)!}{\left(\frac{l_{\text{cg}}+1}{2} + k' \right)!} \left(\frac{a(1)}{a(0)} \right)^{k'}. \quad (\text{D15})$$

Considering that the hypergeometric function can be written as [69]

$${}_2F_1(b, -m; c; z) = \sum_{n=0}^m (-1)^n \binom{m}{n} \frac{(b)_n}{(c)_n} z^n, \quad (\text{D16})$$

where m is a positive integer, and $|z| < 1$, and $(p)_n$ is the Pochhammer symbol

$$(p)_n = \begin{cases} 1, & n = 0 \\ p(p+1) \dots (p+n-1), & n > 0. \end{cases} \quad (\text{D17})$$

In our case, we identify $b = 1$, $m = (l_{\text{cg}} - 1)/2$, $c = (l_{\text{cg}} + 3)/2$, and $z = -a(1)/a(0)$. Since $(1)_n = n!$ and

$$\left(\frac{l_{\text{cg}} + 3}{2} \right)_k = \frac{\left(\frac{l_{\text{cg}}+3}{2} + k - 1 \right)!}{\left(\frac{l_{\text{cg}}+3}{2} - 1 \right)!}. \quad (\text{D18})$$

We eventually get

$${}_2F_1\left(1, \frac{1-l_{\text{cg}}}{2}; \frac{l_{\text{cg}}+3}{2}; -\frac{a(1)}{a(0)}\right) = \sum_{n=0}^{(l_{\text{cg}}-1)/2} (-1)^n \frac{\left(\frac{l_{\text{cg}}-1}{2} \right)!}{n! \left(\frac{l_{\text{cg}}-1}{2} - n \right)!} \frac{n!}{\left(\frac{l_{\text{cg}}+3}{2} + k - 1 \right)!} \left(\frac{l_{\text{cg}}+3}{2} - 1 \right)! \left(-\frac{a(1)}{a(0)} \right)^n \quad (\text{D19})$$

$$= \sum_{n=0}^{(l_{\text{cg}}-1)/2} \frac{\left(\frac{l_{\text{cg}}-1}{2} \right)!}{\left(\frac{l_{\text{cg}}-1}{2} - n \right)!} \frac{\left(\frac{l_{\text{cg}}+1}{2} \right)!}{\left(\frac{l_{\text{cg}}+1}{2} + n \right)!} \left(\frac{a(1)}{a(0)} \right)^n \quad (\text{D20})$$

from which we have that

$$a(0|l_{\text{cg}}) = 1 - a(0)^{\frac{l_{\text{cg}}-1}{2}} a(1)^{\frac{l_{\text{cg}}+1}{2}} \left(\frac{l_{\text{cg}}}{2} \right) {}_2F_1\left(1, \frac{1-l_{\text{cg}}}{2}; \frac{l_{\text{cg}}+3}{2}; -\frac{a(1)}{a(0)}\right). \quad (\text{D21})$$

Since we are interested in large l_{cg} , thanks to Stirling's expansion, we have that

$$\left(\frac{l_{\text{cg}}}{2} \right) \simeq \frac{2^{l_{\text{cg}}-1/2}}{\sqrt{\pi l_{\text{cg}}}}. \quad (\text{D22})$$

The hypergeometric function defined above has the following asymptotic behavior [70–72]:

$${}_2F_1(b, a + \epsilon\lambda; c + \lambda; z) \sim \frac{1}{(1 - \epsilon z)^b}, \quad |\lambda| \rightarrow +\infty. \quad (\text{D23})$$

In our case we identify $b = 1$, $a = \frac{1}{2}$, $\epsilon = -1$, $c = \frac{3}{2}$, and we rescale $\lambda \rightarrow l_{\text{cg}}/2$. In this way, we get that

$$\begin{aligned} {}_2F_1\left(1, \frac{1-l_{\text{cg}}}{2}; \frac{l_{\text{cg}}+3}{2}; -\frac{a(1)}{a(0)}\right) &\sim \frac{1}{1 - \frac{a(1)}{a(0)}} \\ &= \frac{a(0)}{a(0) - a(1)} \\ &= C(\mathbf{a}). \end{aligned} \quad (\text{D24})$$

As a last step, we consider

$$\begin{aligned} a(0)^{\frac{l_{\text{cg}}-1}{2}} a(1)^{\frac{l_{\text{cg}}+1}{2}} \frac{2^{l_{\text{cg}}-1/2}}{\sqrt{\pi l_{\text{cg}}}} &= a(1)[a(0)a(1)]^{\frac{l_{\text{cg}}-1}{2}} 2^{l_{\text{cg}}-1} \frac{\sqrt{2}}{\sqrt{\pi l_{\text{cg}}}} \\ &= a(1)(2\sqrt{a(0)a(1)})^{l_{\text{cg}}-1} \frac{\sqrt{2}}{\sqrt{\pi l_{\text{cg}}}}. \end{aligned} \quad (\text{D25})$$

This quantity decreases as l_{cg} increase if $2\sqrt{a(0)a(1)} \leq 1$. Without loss of generality, we can write $a(0) = \frac{1}{2} + \varepsilon$ and

$a(1) = \frac{1}{2} - \varepsilon$, with $\varepsilon \in [0, \frac{1}{2}]$. In this way

$$2\sqrt{a(0)a(1)} = 2\sqrt{\frac{1}{4} - \varepsilon^2} = \sqrt{1 - 4\varepsilon^2} \leq 1. \quad (\text{D26})$$

We have equality iff $\varepsilon = 0$, but even in this case the overall expression decreases as l_{cg} thanks to a term proportional to $l_{\text{cg}}^{-1/2}$ [see Eq. (D25)]. We thus define the positive quantity

$$D(\mathbf{a}) = -\ln 2\sqrt{a(0)a(1)}, \quad (\text{D27})$$

in such a way that

$$a(0)^{\frac{l_{\text{cg}}-1}{2}} a(1)^{\frac{l_{\text{cg}}+1}{2}} \frac{2^{l_{\text{cg}}-1/2}}{\sqrt{\pi l_{\text{cg}}}} = a(1)e^{-D(\mathbf{a})(l_{\text{cg}}-1)} \sqrt{\frac{2}{\pi l_{\text{cg}}}}. \quad (\text{D28})$$

Thus, we conclude that

$$\begin{aligned} a(0|l_{\text{cg}}) &\simeq 1 - \frac{e^{-D(\mathbf{a})(l_{\text{cg}}-1)}}{\sqrt{l_{\text{cg}}}} a(1) \sqrt{\frac{2}{\pi}} C(\mathbf{a}) \\ &= 1 - \frac{e^{-D(\mathbf{a})(l_{\text{cg}}-1)}}{\sqrt{l_{\text{cg}}}} F(\mathbf{a}), \end{aligned} \quad (\text{D29})$$

where $F(\mathbf{a})$ is simply a finite constant

$$F(\mathbf{a}) = \sqrt{\frac{2}{\pi}} \frac{a(1)a(0)}{a(0) - a(1)}. \quad (\text{D30})$$

At the same time, we have that

$$a(1|l_{\text{cg}}) = \frac{e^{-D(\mathbf{a})(l_{\text{cg}}-1)}}{\sqrt{l_{\text{cg}}}} F(\mathbf{a}). \quad (\text{D31})$$

Eventually, using the fact that $(1 - x)^N \sim 1 - Nx + O(x^2)$ for small x , we can consider $\gamma_{a, l_{cg}}^{(N_P)}$:

$$\gamma_{a, l_{cg}}^{(N_P)} = a(0|l_{cg})^{N_P} + a(1|l_{cg})^{N_P} \quad (D32)$$

$$\simeq 1 - N_P a(1|l_{cg}) + a(1|l_{cg})^{N_P} + O[a(1|l_{cg})^2] \quad (D33)$$

$$\simeq 1 - N_P \frac{e^{-D(a)(l_{cg}-1)}}{\sqrt{l_{cg}}} F(a) + O[a(1|l_{cg})^2]. \quad (D34)$$

As expected, $\gamma_{a, l_{cg}}^{(N_P)}$ converges to 1 exponentially fast with the dimension of the coarse graining l_{cg} . We also see the interplay between N_P and l_{cg} here: a larger N_P increases the gap with respect to the ideal 1, but the effect is merely linear, and the exponential convergence is not affected by the value of N_P .

2. The case of $d_S > 2$

When considering $d_S > 2$, deriving an explicit analytical expression for the $a(x|l_{cg})$ proves to be challenging. For this reason, we provide numerical evidence of such an exponential scaling to 1, but only for $a(0|l_{cg})$. In this case, the latter quantity is defined from the multinomial expansion as

$$\begin{aligned} a(0|l_{cg}) &= \sum_{\substack{k_0 + \dots + k_{d_S-1} = l_{cg} \\ k_0 \geq k_j \forall j \neq 0}} \binom{l_{cg}}{k_0, \dots, k_{d_S-1}} a(0)^{k_0} \dots a(d_S - 1)^{k_{d_S-1}}. \end{aligned} \quad (D35)$$

This mimics the behavior of the decomposition that we employed for $d_S = 2$ in Eq. (D7) but for larger d_S .

We explore numerically the conjecture that even in this case the $\gamma_{a, l_{cg}}^{(N_P)}$ approaches 1 exponentially fast with l_{cg} . To avoid any complications, we only deal with the scaling $1 - a(l_{cg}|0)$, which proves to be sufficient to prove the scaling of $\gamma_{a, l_{cg}}^{(N_P)}$ with some additional rescaling in terms of N_P , as in the case of $d_S = 2$.

For this reason, we want to test if $1 - a(0|l_{cg})$ behave as a function of the form $c_0 e^{c_1 l_{cg}}$. In order to perform the fit, we linearize the function by taking the logarithm, i.e., by considering $\ln c_0 + c_1 l_{cg}$. For fixed d_S , we evaluate the $\ln[1 - a(0|l_{cg})]$ using Eq. (D35) for different values of $\{l_{cg}^{(i)}\}_i$, obtaining a vector of data $\{l_i\}_i$. We then use these data to fit the function and extrapolate $\ln c_0$ and c_1 . We assess the precision of the fit with the coefficient of determination R^2 . If $\{f_i\}_i$ are the values that the fitted function takes at the points $l_{cg}^{(i)}$, we define the residuals as $r_i = l_i - f_i$. Then, R^2 is

$$R^2 = 1 - \frac{S_{\text{res}}}{S_{\text{tot}}}, \quad (D36)$$

where

$$S_{\text{res}} = \sum_i r_i^2, \quad (D37)$$

$$S_{\text{tot}} = \sum_i (l_i - \bar{l})^2, \quad (D38)$$

and where $\bar{l}$ is the average of $\{l_i\}_i$. Results are reported in Fig. 3 in the main text, with the fit parameters in the table below the figure.

APPENDIX E: DETAILS ON PURE DEPHASING HAMILTONIAN

In this appendix, we provide the methods for the simulation of a star-shaped quantum system specified by the Hamiltonian

$$H_\star = \frac{g}{2} \sum_{i=1}^{N_P} \sigma_z^{(S)} \otimes \sigma_z^{(i)}. \quad (E1)$$

To efficiently simulate this system for large N_P , we adopt the approach developed in Ref. [64], which utilizes the symmetry of the interaction. In addition to their derivation, here we provide the method we used to efficiently evaluate the optimal discrimination measurement between the two branches, exploiting the block-diagonal form of the density matrix.

Considering an initial product state $\rho(0) = \rho_S \otimes_i \rho_P$, the evolution is given by $U(t) = \exp(-iH_\star t)$, which can be expanded as a conditioned unitary

$$U(t) = |0\rangle\langle 0| \otimes [V(t)^{\otimes N_P}] + |1\rangle\langle 1| \otimes [V(-t)^{\otimes N_P}], \quad (E2)$$

where $V(t)$ is a unitary matrix given as $V(t) = \exp(-itg\sigma_z/2)$. We are mainly interested in the evolution of the environments, which can be easily written as

$$\rho_P(t) = p_0 \tilde{\rho}_0(t)^{\otimes N_P} + p_1 \tilde{\rho}_1(t)^{\otimes N_P}, \quad (E3)$$

where

$$\tilde{\rho}_0(t) = V(t) \rho_P V(t)^\dagger \quad (E4)$$

$$\tilde{\rho}_1(t) = V(-t) \rho_P V(-t)^\dagger \quad (E5)$$

is the evolution of a single environment state, while ρ_P corresponds to its initial state. Further, the state of a coarse-grained macrofraction has the same structure, i.e., for any coarse graining of size l_{cg} we have that

$$\rho_{P_i}^{l_{cg}}(t) = p_0 \tilde{\rho}_0(t)^{\otimes l_{cg}} + p_1 \tilde{\rho}_1(t)^{\otimes l_{cg}}. \quad (E6)$$

The strategy for evaluating such an evolution involves rewriting the two branch states $\tilde{\rho}_x(t)^{\otimes N}$, $x = 0, 1$, as direct sums of total spin states. This reformulation allows the matrix to be expressed in a block-diagonal form, significantly reducing the memory cost of the computation. Specifically, the largest block to handle has dimensions $(2N + 1) \times (2N + 1)$, compared to the full density matrix of the same fraction, which scales as $2^N \times 2^N$. This simplification is crucial in our analysis, as it enables us to efficiently determine eigenvalues and the optimal discrimination measurement, and thus investigate how agreement and bias scale with the increasing size of the macrofraction.

The first step to write $\rho_{P_i}^{l_{cg}}(t)$ in block-diagonal form consists of diagonalizing the local density matrix $\tilde{\rho}_x(t)$. The diagonalization can be understood as a rotation of the two-level system, to change the representation from σ_z to $\sigma_{\vec{n}}$, where $\vec{n}$ is the vector in the Bloch sphere along which the density matrix is diagonal. The rotation can be written as

$$R(\alpha, \beta, \gamma) = e^{-i\alpha J_z} e^{-i\beta J_y} e^{-i\gamma J_z}, \quad (E7)$$

where J_y and J_z are the components of the angular momentum in the fundamental representation, i.e., proportional to the Pauli matrices. Therefore, we have that

$$R(\alpha, \beta, \gamma) \tilde{\rho}_0(t) R(\alpha, \beta, \gamma)^\dagger = \text{Diag}[\lambda_+, \lambda_-], \quad (E8)$$

and similarly for the other branch

$$R(-\alpha, -\beta, -\gamma)\tilde{\rho}_1(t)R(-\alpha, -\beta, -\gamma)^\dagger = \text{Diag}[\lambda_-, \lambda_+]. \quad (\text{E9})$$

Following [64], the rotation angles can be written as

$$\alpha = \gamma = t, \quad (\text{E10})$$

$$\sin(\beta/2) = -\frac{\rho_{00} - \lambda_+}{\sqrt{|\rho_{01}|^2 + (\rho_{00} - \lambda_+)^2}}, \quad (\text{E11})$$

$$\cos(\beta/2) = \frac{\rho_{01}}{\sqrt{|\rho_{01}|^2 + (\rho_{00} - \lambda_+)^2}}, \quad (\text{E12})$$

where $\rho_{00} = \langle 0|\rho_P|0\rangle$ and $\rho_{01} = \langle 0|\rho_P|1\rangle$. We note that β is time independent, as well as $\lambda_\pm$, and thus can be computed at the start of the simulation.

At this point, we can write $\text{Diag}[\lambda_+, \lambda_-]^{\otimes l_{\text{cg}}}$ as the direct sum of total spin states using the Clebsch-Gordan coefficients

$$\text{Diag}[\lambda_+, \lambda_-]^{\otimes l_{\text{cg}}} = \bigoplus_{j=0}^{l_{\text{cg}}/2} [M_j^{(0)}]^{\oplus B_j}, \quad (\text{E13})$$

where we have defined

$$M_j^{(0)} = \text{Diag}[\lambda_+^{l_{\text{cg}}/2+j} \lambda_-^{l_{\text{cg}}/2-j}, \dots, \lambda_+^{l_{\text{cg}}/2-j} \lambda_-^{l_{\text{cg}}/2+j}], \quad (\text{E14})$$

and

$$B_j = \binom{l_{\text{cg}}}{l_{\text{cg}}/2 - j} - \binom{l_{\text{cg}}}{l_{\text{cg}}/2 - j - 1}. \quad (\text{E15})$$

The very same step can be performed for the second branch, just by swapping λ_+ and λ_- , obtaining $M_j^{(1)}$. In this representation, the basis of the density matrix $\tilde{\rho}_0(t)$ corresponds to $\{|j, m\rangle_{\tilde{n}}\}$, while for the $\tilde{\rho}_1(t)$ corresponds to $\{|j, m\rangle_{\tilde{n}'}\}$. In order to have a block-diagonal representation of $\rho_{P_{l_{\text{cg}}}}(t)$, we need to rotate back the two branch states in the same basis $\{|j, m\rangle_z\}$. To do so, we rotate each block with the rotation $R^j(-\gamma, \beta, \alpha)$, where j corresponds to the representation of the corresponding block. In other words

$$R^j(-\gamma, -\beta, -\alpha) = e^{i\gamma J_z} e^{i\beta J_y} e^{i\alpha J_z}, \quad (\text{E16})$$

where now J_y and J_z are the spin in the j th representation. In this way, we get into the new representation

$$\tilde{\rho}_x(t)^{\otimes l_{\text{cg}}} = \bigoplus_{j=0}^{l_{\text{cg}}/2} [N_j^{(x)}(t)]^{\oplus B_j}, \quad (\text{E17})$$

with $x = 0, 1$, and where

$$N_j^{(0)}(t) = e^{i\gamma J_z} e^{i\beta J_y} e^{i\alpha J_z} M_j^{(0)} e^{-i\gamma J_z} e^{-i\beta J_y} e^{-i\alpha J_z}, \quad (\text{E18})$$

and similarly for $N_j^{(1)}(t)$. In this way, both branch states are written on the same basis and the total state can be written as

$$\rho_{P_{l_{\text{cg}}}}(t) = \bigoplus_{j=0}^{l_{\text{cg}}/2} [p_0 N_j^{(0)}(t) + p_1 N_j^{(1)}(t)]^{\oplus B_j}. \quad (\text{E19})$$

Given that we want to optimally discriminate between the two branches, we need to find a measurement $\{\Pi_0(t), \Pi_1(t)\}$ that optimally discriminates between $\tilde{\rho}_0(t)^{\otimes l_{\text{cg}}}$ and $\tilde{\rho}_1(t)^{\otimes l_{\text{cg}}}$. To do so, we use optimal discrimination theory [66] and we identify the optimal measurement for discrimination as the

one to evaluate bias and agreement as per the definition in Eqs. (13) and (10), respectively.

For binary discrimination of two equiprobable hypotheses ρ_0 and ρ_1 , the strategy to find the optimal measurement is based on the operator

$$\Lambda = \frac{1}{2}(\rho_0 - \rho_1). \quad (\text{E20})$$

The optimal measurement can be found by diagonalizing this operator Λ . If we denote $\{\lambda_k, |\phi_k\rangle\}_k$ the eigenvalues and eigenvectors, we can number them in such a way that

$$\lambda_k < 0 \quad \text{for} \quad 1 \leq k < k_0, \quad (\text{E21})$$

$$\lambda_k > 0 \quad \text{for} \quad k_0 \leq k \leq D, \quad (\text{E22})$$

$$\lambda_k = 0 \quad \text{for} \quad D < k \leq D_\Lambda, \quad (\text{E23})$$

where D_Λ is the dimension of Λ . Then the optimal measurement for the discrimination is given as

$$\Pi_1 = \sum_{k=1}^{k_0-1} |\phi_k\rangle\langle\phi_k|, \quad (\text{E24})$$

$$\Pi_0 = \sum_{k=k_0}^{D_\Lambda} |\phi_k\rangle\langle\phi_k|. \quad (\text{E25})$$

This means that in the one-shot scenario, if we perform measurement $\{\Pi_1, \Pi_0\}$ and we observe outcome $x = 0, 1$, we can infer that the state is ρ_x with some probability of error given as $P_e = 1/2(1 - \text{Tr}\{|\Lambda|\})$.

In our case, the two hypotheses correspond to $\tilde{\rho}_0(t)^{\otimes l_{\text{cg}}}$ and $\tilde{\rho}_1(t)^{\otimes l_{\text{cg}}}$, and to compute the optimal measurement to discriminate them, we need to evaluate Λ and its eigenvalues and eigenvectors. Thanks to the block-diagonal matrix representation, this can be computed efficiently as follows. Given the representation in the spin basis $\{|j, m\rangle\}$ is given as Eq. (E17), we have that

$$\Lambda = \frac{1}{2} \bigoplus_{j=0}^{l_{\text{cg}}/2} [N_j^{(0)}(t) - N_j^{(1)}(t)]^{\oplus B_j}. \quad (\text{E26})$$

We clearly see here that we do not need to diagonalize the full Λ , but only each block $N_j^{(0)}(t) - N_j^{(1)}(t)$, whose maximum dimension scales as $2l_{\text{cg}} + 1 \times 2l_{\text{cg}} + 1$, i.e., linearly with l_{cg} . To each block corresponds the pair of eigenvalues and eigenvectors $\{\lambda_k^{(j)}, |\lambda_k^{(j)}\rangle\}$, and if $\lambda_k^{(j)} < 0$ then the eigenvector will correspond to outcome $x = 1$, and to outcome $x = 0$ if $\lambda_k^{(j)} \geq 0$. By denoting D_j^0 the set of k_j for which $\lambda_{k_j}^{(j)} \geq 0$ and D_j^1 the set of k_j for which $\lambda_{k_j}^{(j)} < 0$, we have that on a specific block $\Pi_j^{(0)} = \sum_{k_j \in D_j^0} |\lambda_{k_j}^{(j)}\rangle\langle\lambda_{k_j}^{(j)}|$ and $\Pi_j^{(1)} = \sum_{k_j \in D_j^1} |\lambda_{k_j}^{(j)}\rangle\langle\lambda_{k_j}^{(j)}|$. We eventually get the optimal discrimination measurement as

$$\Pi^{(x)} = \bigoplus_{j=0}^{l_{\text{cg}}/2} [\Pi_j^{(x)}]^{\oplus B_j}, \quad x = 0, 1. \quad (\text{E27})$$

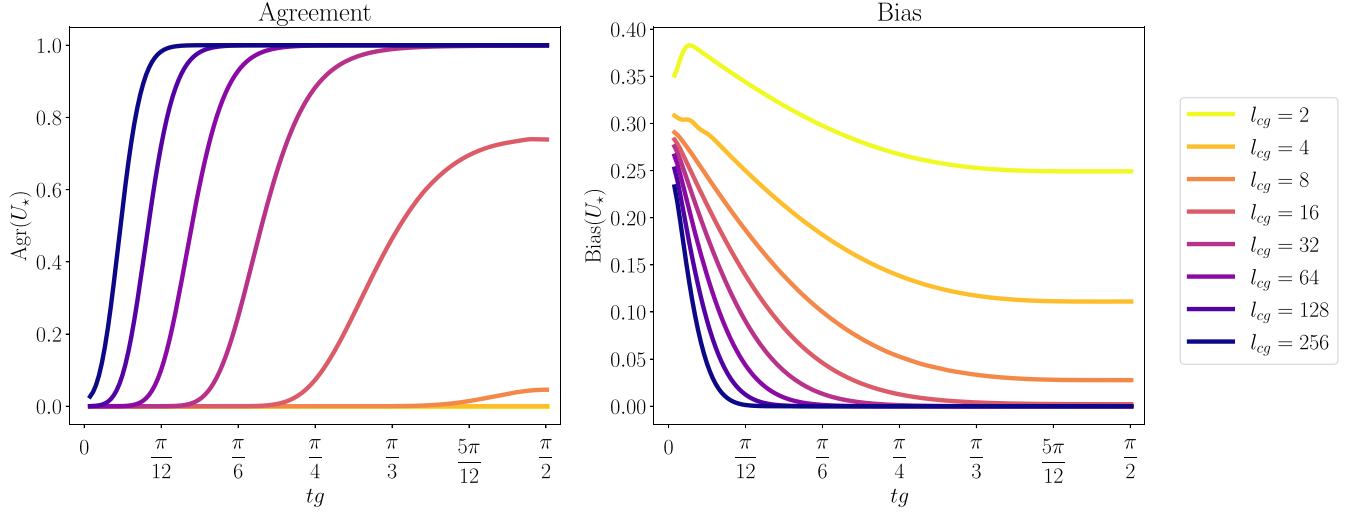


FIG. 5. Plot of agreement and bias as a function of rescaled time tg , for an initial state of the system $|\psi_0\rangle = \sqrt{p_0}|0\rangle + \sqrt{1-p_0}|1\rangle$ with $p_0 = 0.2$, while the environments are in a thermal state with inverse temperature $\beta = 1$. Different colors correspond to different coarse grainings l_{cg} . Here, $N_p = 1024$ spins. The timescale $[0, \pi/(2g)]$, where the relevant dynamics occur, corresponds to half of the recurrence time $t_r = \pi/g$.

This means that if we perform this pair of measurements, we would obtain

$$\begin{aligned} p_t(x|l_{cg}) &= \text{Tr}\{\Pi^{(x)}\rho_{P^{l_{cg}}}(t)\} \\ &= p_0\text{Tr}\{\Pi^{(x)}\tilde{\rho}_0(t)^{\otimes l_{cg}}\} + p_1\text{Tr}\{\Pi^{(x)}\tilde{\rho}_1(t)^{\otimes l_{cg}}\} \end{aligned} \quad (\text{E28})$$

$$= p_0 p_t[x|\rho_0(t)^{\otimes l_{cg}}] + p_1 p_t[x|\rho_1(t)^{\otimes l_{cg}}], \quad (\text{E29})$$

where

$$p_t[x|\rho_y(t)^{\otimes l_{cg}}] = \text{Tr}\{\Pi^{(x)}\tilde{\rho}_y(t)^{\otimes l_{cg}}\} \quad (\text{E30})$$

$$= \sum_{j=0}^{l_{cg}/2} B_j \text{Tr}\{\Pi_j^{(x)} N_j^{(y)}(t)\} \quad (\text{E31})$$

$$= \sum_{j=0}^{l_{cg}/2} B_j \sum_{k_j \in D_j^x} \langle \lambda_{k_j}^{(j)} | N_j^{(y)}(t) | \lambda_{k_j}^{(j)} \rangle. \quad (\text{E32})$$

Using the latter form, we can thus evaluate efficiently the probabilities for the bias and agreement by just diagonalizing the blocks $N_j^{(0)}(t) - N_j^{(1)}(t)$ and computing the latter formula. The bias thus becomes

$$\text{Bias}[U_*(t)] = \frac{1}{2}[|p_t(0|l_{cg}) - p_0| + |p_t(1|l_{cg}) - p_1|], \quad (\text{E33})$$

while the agreement is given as

$$\begin{aligned} \text{Agr}[U_*(t)] &= \sum_{x=0}^1 \text{Tr}\{\Pi^{(x)}\}^{\otimes \frac{N_p}{l_{cg}}} \rho_P(t) \\ &= p_0 p_t(0|l_{cg})^{\frac{N_p}{l_{cg}}} + p_1 p_t(1|l_{cg})^{\frac{N_p}{l_{cg}}}. \end{aligned} \quad (\text{E34})$$

We report these two quantities as a function of time t in Fig. 5.

Coarse graining and time of measurement in the star spin model H_*

In this subsection, we comment on the time evolution of agreement and bias for the H_* model, as reported in Fig. 5. For any tolerable error ϵ the agreement time t_ϵ is the minimum time for which $\text{Agr}(U_*) \geq 1 - \epsilon$.

As we see from this Fig. 5, coarse graining significantly influences the evolution of the two quantities in a similar manner. As l_{cg} increases, both the maximum value and the duration for which we stay close to it increase. After some threshold value of l_{cg} , there is no visible difference in the maximum or minimum values. That difference becomes evident and can only be appreciated in the logarithmic plot shown in Fig. 4.

However, even after reaching such a threshold, increasing l_{cg} continues to have some effect, particularly on the threshold time at which the plateau is reached. Specifically, larger coarse graining enables the two quantities to attain their ideal value more quickly and maintain it for a longer time, forming a stable plateau. This suggests that coarse graining also impacts the speed of measurement, and we conjecture that there exists a threshold value of l_{cg} after which the minimum time required to reach the stable plateau does not decrease as we increase l_{cg} . Investigating this behavior in detail is left for future work.

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