

Erratum: Embedding cyclic information-theoretic structures in acyclic space-times: No-go results for indefinite causality [Phys. Rev. A **110, 022227 (2024)]**

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Theorem 7 in Appendix F of our paper was not entirely correct. This theorem was only used in the proof of Theorem 3 and does not impact the results or conclusions of the paper in any other way. Here we provide an alternative proof of Theorem 3 which does not rely on Theorem 7, but instead uses a weaker version of the statement captured by Lemma 7 below. The main difference is that Lemma 7 refers to the signaling relation where $\{A_i^O\}$ signals to $\{A_j^I\}_{j \in \mathcal{S}} \cup \{A_i^I\}$ in the induced map $\mathcal{N}_{\mathcal{W}, \bar{i} \cup \mathcal{S}}$ (as opposed to where $\{A_i^O\}$ signals to $\{A_j^I\}_{j \in \mathcal{S}}$ in the induced map $\mathcal{N}_{\mathcal{W}, \bar{i} \cup \mathcal{S}}$ which Theorem 7 considered). This leads to the inclusion of an additional case (case 1) in the proof of Theorem 3, while the remaining proof (case 2) is essentially identical to the proof that was provided before. The equations and definitions referenced here, which are not included in this Erratum, refer to equations and definitions from the published version of the paper.

Lemma 7. Let W be a valid N -partite process matrix and $\mathfrak{N}_{\mathcal{W}, N}$ be the corresponding network in our framework. Let A_i be an agent, $A_{\mathcal{S}}$ be any subset of the remaining agents, and $\mathfrak{N}_{\mathcal{W}, \bar{i} \cup \mathcal{S}}$ be the subnetwork of $\mathfrak{N}_{\mathcal{W}, N}$ obtained from composing $\mathcal{W}$ with the extended local operations of all agents except those in $A_i \cup A_{\mathcal{S}}$. Then A_i probabilistically signals to $A_{\mathcal{S}}$ if and only if $\{A_i^O\}$ signals to $\{A_j^I\}_{j \in \mathcal{S}} \cup \{A_i^I\}$ in the induced map $\mathcal{N}_{\mathcal{W}, \bar{i} \cup \mathcal{S}}$.

Proof. For any agent A_i and another disjoint set of agents $A_{\mathcal{S}}$ involved in the process W under consideration, the subnetwork $\mathfrak{N}_{\mathcal{W}, \bar{i} \cup \mathcal{S}}$ which is formed by the composition of the process map $\mathcal{W}$ with the extended local operations of the agents $\{A_k\}_{k \notin i \cup \mathcal{S}}$ encodes all possible choices of local operations that can be performed by those agents. These choices can be instantiated by inputting a corresponding choice of classical settings for each of these agents, i.e., $\{a_k\}_{k \notin i \cup \mathcal{S}}$, into the classical input wire of the corresponding extended local operation. By Lemma 3, this implies that the subnetwork $\mathfrak{N}_{\mathcal{W}, \bar{i} \cup \mathcal{S}}$ encodes all possible reduced processes on the agents $A_i \cup A_{\mathcal{S}}$ that can arise from the given process W , by fixing the choices of local operations of the remaining agents. We will denote each such reduced process by $\tilde{W}(\{a_k\}_{k \notin i \cup \mathcal{S}})$ for short, corresponding to a choice of local operations $\{M_{a_k}^{A_i^O A_k^O}\}_{k \notin i \cup \mathcal{S}}$ [expressed in the Choi representation, following the notation of Eq. (14)].

Now it is easy to see that probabilistic signaling from A_i to $A_{\mathcal{S}}$ in the original process W is equivalent to saying that there exists a reduced process $\tilde{W}(\{a_k\}_{k \notin i \cup \mathcal{S}})$ of W where A_i probabilistically signals to $A_{\mathcal{S}}$. Consider any such reduced process $\tilde{W} := \tilde{W}(\{a_k\}_{k \notin i \cup \mathcal{S}})$ on $A_i \cup A_{\mathcal{S}}$ obtained from a particular choice of $\{a_k\}_{k \notin i \cup \mathcal{S}}$ of local operations of the remaining agents. It follows from Proposition III.2 of [1] that A_i does not probabilistically signal to $A_{\mathcal{S}}$ in $\tilde{W}$ if and only if this process satisfies $\tilde{W} = \text{Tr}_{A_i^O} \tilde{W} \otimes \frac{\mathbb{1}_{A_i^O}}{d_{A_i^O}}$.

The above condition on the process matrix $\tilde{W}$ is equivalent to the condition that $\{A_i^O\}$ does not signal to $\{A_j^I\}_{j \in \mathcal{S}} \cup \{A_i^I\}$ in the corresponding process map $\tilde{\mathcal{W}}$. This follows from well-known results in the causality literature (such as Refs. [29,31] in the published version of the paper) linking the absence of signaling in a completely positive and trace-preserving map (of which the process map $\tilde{\mathcal{W}}$ is an instance) to conditions on its Choi representation (here the process matrix $\tilde{W}$) and is explicitly articulated in Lemma 2.11 of [2]. The process map $\tilde{\mathcal{W}}$ of the process matrix $\tilde{W} := \tilde{W}(\{a_k\}_{k \notin i \cup \mathcal{S}})$ is precisely the induced map $\mathcal{N}_{\mathcal{W}, \bar{i} \cup \mathcal{S}}$ of the process subnetwork $\mathfrak{N}_{\mathcal{W}, \bar{i} \cup \mathcal{S}}$ acting on the inputs $\{a_k\}_{k \notin i \cup \mathcal{S}}$. As we have shown the above equivalence for any reduced process $\tilde{W} := \tilde{W}(\{a_k\}_{k \notin i \cup \mathcal{S}})$ of the original process W , it follows from taking the contrapositive statements, that probabilistic signaling from A_i to $A_{\mathcal{S}}$ in W is equivalent to signaling from $\{A_i^O\}$ to $\{A_j^I\}_{j \in \mathcal{S}} \cup \{A_i^I\}$ in $\mathcal{N}_{\mathcal{W}, \bar{i} \cup \mathcal{S}}$, as required. ■

Theorem 3. A process matrix W is not a fixed order process if and only if the signaling structure of the corresponding network $\mathfrak{N}_{\mathcal{W}, N}$ certifies the cyclicity of any causal structure $\mathcal{G}$ that it is compatible with.

Proof. In the process network $\mathfrak{N}_{\mathcal{W}, N}$, we have that $\{A_k^I\}$ signals to $\{A_k^O\}$ for every agent A_k since the extended local map for each agent encodes all possible operations between their input and output systems and in particular the identity channel which clearly signals from the input to the output. This implies that when we compatibly embed (Definition 14) the systems of $\mathfrak{N}_{\mathcal{W}, N}$ in a causal structure $\mathcal{G}$ with directed edges $\xrightarrow{\mathcal{G}}$, we must have a directed path

$$A_k^I \xrightarrow{\mathcal{G}} \dots \xrightarrow{\mathcal{G}} A_k^O \quad \forall k \in \{1, \dots, N\}. \quad (1)$$

Using Lemma 7 and the fact that every signaling relation present in a subnetwork is also present in the original network, we have that A_i probabilistically signals to $A_{\mathcal{S}}$ if and only if $\{A_i^O\}$ signals to $\{A_j^I\}_{j \in \mathcal{S}} \cup \{A_i^I\}$ in the network $\mathfrak{N}_{\mathcal{W}, N}$. By Definition 14,

compatibility of the signaling relation where $\{A_i^O\}$ signals to $\{A_j^I\}_{j \in \mathcal{S}} \cup \{A_i^I\}$ with an embedding in a causal structure $\mathcal{G}$ implies

$$\exists j \in \mathcal{S} \text{ or } j = i, \quad A_i^O \xrightarrow{\mathcal{G}} \dots \xrightarrow{\mathcal{G}} A_j^I, \quad (2)$$

where the edges $\xrightarrow{\mathcal{G}}$ capture the order induced by $\mathcal{G}$ on the embedded systems. We now consider two cases which exhaustively cover all possibilities entailed by the above equation, when applied to every probabilistic signaling relation (or equivalently, by Lemma 7, signaling relations of the process network) arising in the given process W .

Case 1: Choosing $j = i$ in Eq. (2) for at least one probabilistic signaling relation. In this case, we obtain a directed path $A_i^O \xrightarrow{\mathcal{G}} \dots \xrightarrow{\mathcal{G}} A_i^I$ by Eq. (2), which, along with Eq. (1), implies a directed cycle between A_i^O and A_i^I in any causal structure $\mathcal{G}$ which compatibly embeds the signaling relations. Note that this fact is independent of whether or not W is a fixed order process.

Case 2: Choosing $j \in \mathcal{S}$ in Eq. (2) for every probabilistic signaling relation. In this case Eq. (2) simplifies to the following requirement for every signaling relation where $\{A_i^O\}$ signals to $\{A_j^I\}_{j \in \mathcal{S}} \cup \{A_i^I\}$ in the process network, and for any causal structure $\mathcal{G}$ that compatibly embeds these signaling relations:

$$\exists j \in \mathcal{S}, \quad A_i^O \xrightarrow{\mathcal{G}} \dots \xrightarrow{\mathcal{G}} A_j^I. \quad (3)$$

Using Definitions 28 and 29, we can see that a process W is a fixed order process if and only if there exists a partial order on the input and output systems of all agents where A_i^O precedes A_j^I in the partial order for some $j \in \mathcal{S}$ whenever an agent A_i probabilistically signals to a set of agents $A_{\mathcal{S}}$, and A_k^I precedes A_k^O in this partial order for every agent A_k . Since every partially ordered set specifies a directed acyclic graph and vice versa (where A precedes B in the partial order corresponds to the existence of a directed path from A to B in the graph), this yields that W is a fixed order process if and only if there exists a directed acyclic graph $\mathcal{G}$ with nodes corresponding to the input and output systems $\{A_k^I, A_k^O\}_{k=1}^N$, where Eq. (1) holds generally and Eq. (3) holds for every probabilistic signaling relation where A_i signals to $A_{\mathcal{S}}$. Using Lemma 7 as before, we have probabilistic signaling from A_i to $A_{\mathcal{S}}$ if and only if we have a signaling relation where $\{A_i^O\}$ signals to $\{A_j^I\}_{j \in \mathcal{S}} \cup \{A_i^I\}$ in the process network. Combining the above arguments allows us to conclude that for every signaling relation where $\{A_i^O\}$ signals to $\{A_j^I\}_{j \in \mathcal{S}} \cup \{A_i^I\}$ of the process network, choosing $j \in \mathcal{S}$ in the directed path in Eq. (2) along with the already established Eq. (1) for such a network implies the presence of a directed cycle in any causal structure $\mathcal{G}$ which compatibly embeds the signaling relations of the network, if and only if W is not a fixed order process.

Finally, to complete the proof, note that whenever W is a fixed order process, resolving any signaling relation from A_i to $A_{\mathcal{S}}$ by choosing $j = i$ in Eq. (2) would lead to a directed cycle (according to case 1); however, choosing $j \in \mathcal{S}$ in every case will lead us to case 2, which guarantees that there is a way to compatibly embed the signaling relations of the process network in a causal structure $\mathcal{G}$ i.e., satisfy Eqs. (1) and (2) without having any directed cycles in $\mathcal{G}$ (in other words, $\mathcal{G}$ is a directed acyclic graph). On the other hand, whenever W is not a fixed ordered process, any way of resolving the conditions of Eq. (2) implies a directed cycle in $\mathcal{G}$. This implies, as required, that the signaling relations of the process network $\mathfrak{N}_{\mathcal{W},N}$ being compatible with an embedding in some causal structure $\mathcal{G}$ certify the cyclicity of $\mathcal{G}$ if and only if W is not a fixed order process. ■

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[1] O. Oreshkov and C. Giarmatzi, Causal and causally separable processes, *New J. Phys.* **18**, 093020 (2016).

[2] M. Sandfuchs, M. Haberland, V. Vilasini, and R. Wolf, Security of differential phase shift QKD from relativistic principles, *Quantum* **9**, 1611 (2025).