

Quantum theory of photon emission during strong-laser-field-induced ionization.

II. Continuum dynamics

D. B. Milošević^{1,2,3} and W. Becker³¹*Faculty of Science, University of Sarajevo, Zmaja od Bosne 35, 71000 Sarajevo, Bosnia and Herzegovina*²*Academy of Sciences and Arts of Bosnia and Herzegovina, Bistrik 7, 71000 Sarajevo, Bosnia and Herzegovina*³*Max-Born Institute for Nonlinear Optics and Short Pulse Spectroscopy, Max-Born-Straße 2A, D-12489 Berlin, Germany*

(Received 27 March 2026; accepted 1 July 2026; published 14 July 2026)

Emission of a photon during the process of above-threshold ionization is investigated. In previous work [D. B. Milošević, *Phys. Rev. A* **108**, 033110 (2023)] one of the present authors studied this process in the leading order of the strong-field approximation for the matrix element that describes direct emission of a photon during the transition from the bound state to the continuum. We now show that the correction that corresponds to photon emission while the electron is propagating in the continuum, very significantly modifies the process for all but some intermediate photon energies. In contrast to the lowest-order term of the previously considered expansion, the continuum correction is infrared divergent and resonant when the photon energy agrees with the energy of a driving photon. If the energy of the emitted photon is much larger than the energy of a driving photon, then the continuum correction is very close to the leading-order direct contribution and the two terms interfere constructively. These statements hold true for a variety of laser-field configurations; we investigated various bichromatic linearly and circularly polarized fields including bicircular fields.

DOI: [10.1103/2f89-hhpb](https://doi.org/10.1103/2f89-hhpb)

I. INTRODUCTION

In any process involving charged particles in the initial or final state, an additional photon may be emitted into the final state. Typically, the corresponding rate will be smaller than the rate without this photon. However, this is not necessarily true [1], and the rate of emission may even diverge, when the energy of the additional photon becomes smaller and smaller. This is relevant for several reasons. The photon may be of interest in itself, because it is in a frequency range that is in need of novel methods for photon generation as is the case for THz generation [2,3]. Or, from a different angle, the existence of this additional process must be kept in mind because it may interfere with the observation of the original process of interest. Namely, due to the limited sensitivity of any detector, emission of an additional photon that escaped detection can never be ruled out [4,5]. Here, we will address the additional emission of a photon in the case of laser-induced above-threshold ionization (ATI).

The most important and most extensively investigated strong-laser-field-induced processes are ATI [6] (see also the review articles [7,8]) and high-order harmonic generation (HHG [9,10], see the review articles [11,12]). ATI is distinguished by the emitted electron having absorbed more laser photons than is necessary for ionization. On the other hand, in HHG high-energy photons are emitted, the resulting radiation

is coherent, and the process is described by the three-step model [13,14]. In the first step, the electron is virtually liberated from the atom, in the second step the laser field drives the electron first away from and subsequently back to the parent ion, and in the third step the electron recombines with the ion and one high-energy photon is emitted. In the spirit of the current paper, HHG can also be envisioned as the process where the electron subject to the laser field transitions from the initial ground state back into the same final ground state and emits an additional photon along the way. For ATI, the final state is in the continuum, and again a photon may be emitted at any stage in between.

In the theoretical description of ATI and HHG, the laser field has usually been treated as a classical electromagnetic field. However, more recently, the consequences of having a quantum field both for the driving field and, for HHG, for the emitted radiation have become of increasing interest and been leading to strong-field physics merging with quantum optics (see, for example, Refs. [15–21]). Recently, one of the present authors formulated a theory of photon emission during ATI [1]. The adopted approach was similar to the one used previously for laser-assisted x-ray–atom scattering [22,23] and for HHG [24,25], where the emitted photon field is quantized. Using the *S*-matrix formalism and the strong-field approximation (SFA) we obtained an expression for the rate of simultaneous emission of an ATI electron and a spontaneous photon, and analyzed it for various laser-field parameters. The rate for spontaneous emission of a photon with given momentum and polarization is obtained by integrating over the rates for all momenta of the final electron. This result was obtained in the lowest-order approximation [1]. Very recently, it was shown [26] that the first-order correction to our result gives a com-

parable and, in some parameter range, higher emission rate. This correction allows for the emission of a photon while the electron is already in the continuum. In the present paper we apply our formalism to analyze this correction to the results presented in Ref. [1].

The paper is organized as follows. In Sec. II, we present the zeroth-order SFA and the correction, which includes the continuum dynamics of the electron. Section III is devoted to numerical results and their discussion, and our conclusions are summarized in Sec. IV. Atomic units are used throughout the paper.

II. THEORY

A. Definitions and zeroth-order approximation

The S -matrix element for the laser-induced transition from the initial bound atomic state $|\psi_b(t)\rangle = e^{iH_p t}|\psi_b\rangle$ to the final state $e^{-iE_p t}|\mathbf{p}\rangle$ of the free electron having the momentum $\mathbf{p}$, with the emission of a photon having the frequency ω_k and the polarization vector $\hat{\mathbf{e}}_k$, is [1]

$$S_{\mathbf{p}}(\omega_k, \hat{\mathbf{e}}_k) = -ic_k \int_{-\infty}^{\infty} dt e^{i\omega_k t} \hat{\mathbf{e}}_k^* \cdot \mathbf{d}_{\mathbf{p}}(t). \quad (1)$$

In deriving this result, we supposed that the laser field is described classically, while the emitted photon mode is quantized, with $c_k = -i(2\pi\omega_k/V)^{1/2}$ and V the quantization volume. The ionization potential is denoted by I_p , $E_p = p^2/2$ is the final electron energy ($p = |\mathbf{p}|$), the time-dependent dipole matrix element is

$$\mathbf{d}_{\mathbf{p}}(t) = \langle \Phi_{\mathbf{p}}^{(-)}(t) | \mathbf{r} | \Phi_b^{(+)}(t) \rangle, \quad (2)$$

with the exact laser-dressed initial ($\Phi_b^{(+)}$) and final ($\Phi_{\mathbf{p}}^{(-)}$) states, and the process is induced by the laser field with the electric-field vector $\mathbf{E}(t) = -d\mathbf{A}(t)/dt$.

We introduce the Green's operator $G^{(+)}(t, t')$ which corresponds to the Hamiltonian $H(t) = -\nabla^2/2 + V_{\text{at}}(\mathbf{r}) + V_{\text{le}}(t)$, with $\nabla \equiv \partial/\partial\mathbf{r}$, $V_{\text{at}}(\mathbf{r})$ the atomic interaction, and $V_{\text{le}}(t) = \mathbf{r} \cdot \mathbf{E}(t)$ the interaction of the laser field with the electron (in length gauge and dipole approximation). It satisfies the Lippmann-Schwinger equation

$$G^{(+)}(t, t') = G_{\text{le}}(t, t') + \int d\tau G^{(+)}(t, \tau) V_{\text{at}}(\mathbf{r}) G_{\text{le}}(\tau, t'), \quad (3)$$

where the Volkov Green's operator

$$G_{\text{le}}(t, t') = -i\theta(t - t') \int d\mathbf{q} |\chi_{\mathbf{q}}(t)\rangle \langle \chi_{\mathbf{q}}(t')| \quad (4)$$

corresponds to the Hamiltonian $H_{\text{le}}(t) = -\nabla^2/2 + V_{\text{le}}(t)$, with the Volkov state

$$|\chi_{\mathbf{p}}(t)\rangle = |\mathbf{p} + \mathbf{A}(t)\rangle e^{-iS_{\mathbf{p}}(t)}, \quad dS_{\mathbf{p}}(t)/dt = [\mathbf{p} + \mathbf{A}(t)]^2/2. \quad (5)$$

The exact initial laser-dressed state $|\Phi_b^{(+)}(t)\rangle$ satisfies the Lippmann-Schwinger equation [27]

$$|\Phi_b^{(+)}(t)\rangle = |\psi_b(t)\rangle + \int dt' G^{(+)}(t, t') V_{\text{le}}(t') |\psi_b(t')\rangle, \quad (6)$$

so that

$$\mathbf{d}_{\mathbf{p}}(t) = \mathbf{d}_{\text{pb}}(t) + \mathbf{d}_{\text{pc}}(t), \quad (7)$$

where the time-dependent dipole matrix element which describes direct emission of a photon during transition from the bound state to continuum is

$$\mathbf{d}_{\text{pb}}(t) = \langle \Phi_{\mathbf{p}}^{(-)}(t) | \mathbf{r} | \psi_b(t) \rangle, \quad (8)$$

while the matrix element

$$\mathbf{d}_{\text{pc}}(t) = \langle \Phi_{\mathbf{p}}^{(-)}(t) | \mathbf{r} \int dt' G^{(+)}(t, t') V_{\text{le}}(t') |\psi_b(t')\rangle \quad (9)$$

includes the continuum dynamics.

Using the relation $\langle \Phi_{\mathbf{p}}^{(-)}(\tau) | = i \lim_{t \rightarrow \infty} e^{iE_p t} \langle \mathbf{p} | G^{(+)}(t, \tau)$ and Eq. (3) we can write

$$\langle \Phi_{\mathbf{p}}^{(-)}(t) | = \langle \chi_{\mathbf{p}}(t) | + \int d\tau \langle \chi_{\mathbf{p}}(\tau) | V_{\text{at}}(\mathbf{r}) G_{\text{le}}(\tau, t) + \dots \quad (10)$$

The lowest-order term of this expansion gives

$$\mathbf{d}_{\text{pb}}^{(0)}(t) = \langle \chi_{\mathbf{p}}(t) | \mathbf{r} | \psi_b(t) \rangle \equiv \mathbf{d}_{\mathbf{p}}^{(0)}(t), \quad (11)$$

which is the time-dependent dipole considered in Ref. [1], and

$$\mathbf{d}_{\text{pc}}^{(0)}(t) = \langle \chi_{\mathbf{p}}(t) | \mathbf{r} \int dt' G_{\text{le}}(t, t') V_{\text{le}}(t') |\psi_b(t')\rangle \equiv \mathbf{d}_{\mathbf{p}}^{(1)}(t), \quad (12)$$

which we will consider in the present paper. Here, in accordance with Eq. (3), we have also approximated $G^{(+)}(t, t')$ with $G_{\text{le}}(t, t')$. The lowest-order term in Eq. (10) yields the well-known SFA [28]. We consider only the leading order of the SFA, both for the direct emission of a photon during transition from the bound state to the continuum [Eq. (11)] and for the continuum dynamics term (12). Using the next-order term of this expansion we can derive the so-called improved SFA, which includes rescattering off the atomic potential $V_{\text{at}}(\mathbf{r})$ and describes the high-order ATI process [28]. The improved SFA or rescattering correction is usually a few orders of magnitude lower than the leading-order SFA result. However, it may be important for the long-range Coulomb interaction and for long laser wavelengths (for example, for the so-called low-energy structure [29,30]). It should be mentioned that there are corrections to the standard SFA, such as the Coulomb-Volkov SFA (see, for example, Refs. [31,32]) and atom-Volkov SFA [33], which use the Coulomb-Volkov or atom-Volkov Green's operator instead of the Volkov Green's operator. We will not consider these approximations in the present paper.

In Ref. [1], the lowest-order term $\mathbf{d}_{\mathbf{p}}^{(0)}(t)$ was considered, and, supposing that the fundamental frequency of the laser field is $\omega = 2\pi/T$, for a T -periodic field the T -matrix vector

$$\mathbf{T}_{\mathbf{p}}^{(0)}(n) = \int_0^T \frac{dt}{T} \mathcal{T}_{\mathbf{p}}^{(0)}(t) e^{in\omega t} \quad (13)$$

was introduced, where the T -periodic function $\mathcal{T}_{\mathbf{p}}^{(0)}(t) = \sum_{n=-\infty}^{\infty} \mathbf{T}_{\mathbf{p}}^{(0)}(n) e^{-in\omega t}$ is given by

$$\mathcal{T}_{\mathbf{p}}^{(0)}(t) = \langle \mathbf{p} + \mathbf{A}(t) | \mathbf{r} | \psi_b \rangle e^{i[\mathbf{p} \cdot \mathbf{A}(t) + U_1(t)]}. \quad (14)$$

Here $d\mathbf{A}(t)/dt = \mathbf{A}(t)$ and $U_1(t) = \int^t dt'' \mathbf{A}^2(t'')/2 - U_p t$, with $U_p = \int_0^T dt \mathbf{A}^2(t)/(2T)$ the ponderomotive energy. We

finally get [1]

$$S_{\mathbf{p}}^{(0)}(\omega_k, \hat{\mathbf{e}}_{\mathbf{k}}) = -2\pi i c_k \sum_n \delta(E - n\omega) \hat{\mathbf{e}}_{\mathbf{k}}^* \cdot \mathbf{T}_{\mathbf{p}}^{(0)}(n), \quad (15)$$

where n photons are absorbed from the laser field and the condition of energy conservation is

$$n\omega = E \equiv \omega_k + E_p + I_p + U_p. \quad (16)$$

Using this result we obtain the rate for emission of a photon with the frequency ω_k during ionization from the initial bound state to the final continuum state with the electron momentum $\mathbf{p}$ [1]:

$$w_{\mathbf{p}}^{(0)}(\omega_k, \hat{\mathbf{e}}_{\mathbf{k}}) = \frac{\omega_k^3}{2\pi c^3} |\hat{\mathbf{e}}_{\mathbf{k}}^* \cdot \mathbf{T}_{\mathbf{p}}^{(0)}(n)|^2. \quad (17)$$

The total rate for emission of a photon with specified energy and polarization, regardless of the electron, is

$$W(\omega_k, \hat{\mathbf{e}}_{\mathbf{k}}) = \int d\mathbf{p} w_{\mathbf{p}}(\omega_k, \hat{\mathbf{e}}_{\mathbf{k}}). \quad (18)$$

B. Continuum dynamics in lowest order

In the present paper we consider the electron's continuum dynamics in the intermediate state before and after the photon emission. In the SFA and to lowest order the corresponding time-dependent dipole is given by Eq. (12). Introducing (4) into (12) we get

$$\mathbf{d}_{\mathbf{p}}^{(1)}(t) = -i \int d\mathbf{q} \langle \chi_{\mathbf{p}}(t) | \mathbf{r} | \chi_{\mathbf{q}}(t) \rangle \int_{-\infty}^t dt' \mathbf{E}(t') \cdot \mathbf{d}_{\mathbf{q}}^{(0)}(t'). \quad (19)$$

From Eqs. (1) and (19), with $\int_{-\infty}^{\infty} dt \int_{-\infty}^t dt' = \int_{-\infty}^{\infty} dt' \int_{t'}^{\infty} dt$, we obtain

$$S_{\mathbf{p}}^{(1)}(\omega_k, \hat{\mathbf{e}}_{\mathbf{k}}) = (-i)^2 c_k \int_{-\infty}^{\infty} dt' \int_{t'}^{\infty} dt e^{i\omega_k t} \hat{\mathbf{e}}_{\mathbf{k}}^* \cdot \int d\mathbf{q} \langle \chi_{\mathbf{p}}(t) | \mathbf{r} | \chi_{\mathbf{q}}(t) \rangle \mathbf{E}(t') \cdot \mathbf{d}_{\mathbf{q}}^{(0)}(t'). \quad (20)$$

For an infinitely extended T -periodic laser field, utilizing $\int_{-\infty}^{\infty} dt' f(t') = \sum_{m=-\infty}^{\infty} \int_0^T dt' f(t' + mT)$, $S_{\mathbf{q}}(t' + mT) = S_{\mathbf{q}}(t') + (E_q + U_p)mT$, $\mathbf{d}_{\mathbf{q}}^{(0)}(t' + mT) = \mathbf{d}_{\mathbf{q}}^{(0)}(t') e^{i(E_q + U_p + I_p)mT}$, and substituting $t'' = t - mT$, this becomes

$$S_{\mathbf{p}}^{(1)}(\omega_k, \hat{\mathbf{e}}_{\mathbf{k}}) = (-i)^2 c_k \sum_m \int_0^T dt' \int_{t'}^{\infty} dt'' e^{i\omega_k(t'' + mT)} \hat{\mathbf{e}}_{\mathbf{k}}^* \cdot \int d\mathbf{q} \langle \mathbf{p} | \mathbf{r} | \mathbf{q} \rangle \mathbf{E}(t') \cdot \mathbf{d}_{\mathbf{q}}^{(0)}(t') \times e^{i[S_{\mathbf{p}}(t'') - S_{\mathbf{q}}(t'') + (E_p + U_p + I_p)mT]}. \quad (21)$$

Using the relation $\sum_m e^{imET} = \omega \sum_m \delta(E - m\omega)$, we obtain

$$S_{\mathbf{p}}^{(1)}(\omega_k, \hat{\mathbf{e}}_{\mathbf{k}}) = -2\pi i c_k \sum_n \delta(E - n\omega) \hat{\mathbf{e}}_{\mathbf{k}}^* \cdot \mathbf{T}_{\mathbf{p}}^{(1)}(n), \quad (22)$$

where (with $t' \rightarrow t$)

$$\mathbf{T}_{\mathbf{p}}^{(1)}(n) = -i \int d\mathbf{q} \langle \mathbf{p} | \mathbf{r} | \mathbf{q} \rangle \int_0^T \frac{dt}{T} \mathbf{E}(t) \cdot \mathbf{d}_{\mathbf{q}}^{(0)}(t) \times \int_t^{\infty} dt'' e^{i[\omega_k t'' + S_{\mathbf{p}}(t'') - S_{\mathbf{q}}(t'')]} \quad (23)$$

This is the analog of Eq. (15) with Eq. (16). The integral over the intermediate electron momentum is solved with the help of

$$\int d\mathbf{q} \langle \mathbf{p} | \mathbf{r} | \mathbf{q} \rangle f(\mathbf{q}) = [-idf(\mathbf{q})/d\mathbf{q}]_{\mathbf{q}=\mathbf{p}}, \quad (24)$$

which yields (with $t'' \rightarrow t'$)

$$\mathbf{T}_{\mathbf{p}}^{(1)}(n) = (-i)^2 \int_0^T \frac{dt}{T} e^{i[S_{\mathbf{p}}(t) + I_p t]} \int_t^{\infty} dt' g_{\mathbf{p}}(t, t') e^{i\omega_k t'}, \quad (25)$$

where

$$g_{\mathbf{p}}(t, t') = i \frac{\partial S_{\mathbf{p}}(t) - \partial S_{\mathbf{p}}(t')}{\partial \mathbf{p}} m(\mathbf{p}, t) + \frac{\partial m(\mathbf{p}, t)}{\partial \mathbf{p}}, \quad (26)$$

with $\partial S_{\mathbf{p}}(t)/\partial \mathbf{p} = \mathbf{p}t + \boldsymbol{\alpha}(t)$ and

$$m(\mathbf{p}, t) \equiv \langle \mathbf{p} + \mathbf{A}(t) | V_{\text{ic}}(t) | \psi_b \rangle. \quad (27)$$

After the substitution $\tau = t' - t$ we finally obtain

$$\mathbf{T}_{\mathbf{p}}^{(1)}(n) = -i \int_0^T \frac{dt}{T} e^{i[S_{\mathbf{p}}(t) + I_p t]} \int_0^{\infty} d\tau e^{i\omega_k(t+\tau)} \times \left[\boldsymbol{\alpha}(t) - \boldsymbol{\alpha}(t+\tau) - \mathbf{p}\tau - i \frac{\partial}{\partial \mathbf{p}} \right] m(\mathbf{p}, t). \quad (28)$$

The physical interpretation of this result is the following. Due to the action of the laser field the electron is liberated with the momentum $\mathbf{p}$ at the time t . Thereafter, it propagates in the field and at the time $t + \tau$ the photon with the energy ω_k is emitted. Subsequently, the electron continues to propagate in the field. In total, n photons are absorbed from the laser field, and the energy-conservation condition is $n\omega = \omega_k + E_p + I_p + U_p$.

In Appendix A we show that the T -matrix vector (28) can be rewritten as the sum of three elements

$$\begin{aligned} \mathbf{T}_{\mathbf{p}}^{(1,1)}(n) &= -i \frac{\partial T_{\mathbf{p}}^{\text{SFA}}(n)}{\partial \mathbf{p}} \frac{1}{\omega_k}, \\ \mathbf{T}_{\mathbf{p}}^{(1,2)}(n) &= - \sum_{s,r,j} T_{\mathbf{p}}^{\text{SFA}}(n + sr) \frac{\alpha_{srj} \hat{\mathbf{e}}_j}{\omega_k + sr\omega + i\varepsilon}, \\ \mathbf{T}_{\mathbf{p}}^{(1,3)}(n) &= -iT_{\mathbf{p}}^{\text{SFA}}(n) \mathbf{p} / \omega_k^2, \end{aligned} \quad (29)$$

with the SFA ionization T -matrix element (A1).

The process described by the continuum correction to the direct T -matrix element is schematically depicted in Fig. 1. The three terms of the total matrix element have very different properties and will be extensively analyzed in the following section for various laser fields. In particular, for $s = -1$ and $\varepsilon \rightarrow 0$ the second term in (29) is resonant for $\omega_k = r\omega$. This manifests itself as a sharp peak in the photon emission spectrum for $\omega_k = r\omega$. Under realistic experimental conditions the resonance peak has a particular width and shape, depending on the laser and atomic parameters. Even for our infinitely

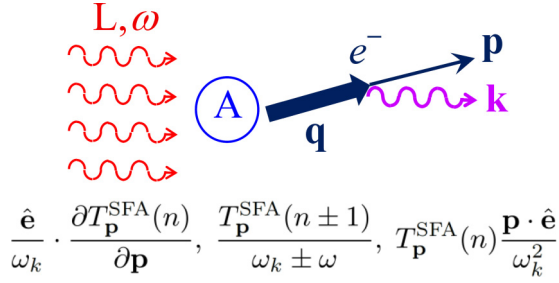


FIG. 1. Schematic of photon emission during above-threshold ionization with the electron dynamics in the intermediate state included. The laser field L of frequency ω releases an electron (e^-) with the intermediate momentum $\mathbf{q}$ and the final momentum $\mathbf{p}$ from the atom (A) accompanied by emission of a photon with wave vector $\mathbf{k}$. The three components of the T -matrix elements (29) with $\hat{\mathbf{e}}_k = \hat{\mathbf{e}}$ are presented.

long periodic laser pulse with fixed frequency $r\omega$, the resonance has a finite (natural) width. This can be shown taking into account that the atomic state has a finite lifetime $1/\Gamma$ (Γ is its width) so that the ionization potential I_p should be replaced by $I_p - \Delta + i\Gamma/2$. The real shift Δ is usually small and can be neglected but the term $i\Gamma/2$ is important and replaces the term $i\varepsilon$ in Eq. (29) [cf. Eq. (16) where I_p appears].

The third term in Eq. (29) diverges for $\omega_k \rightarrow 0$. Such behavior is well known from, for example, Coulomb scattering accompanied by the emission of a photon and was dubbed the “infrared catastrophe” for soft-photon emission [34]. The problem of this infrared divergence is purely academic [34]. Namely, in reality, the emission of a very soft photon is indistinguishable from the case where no such photon is emitted, that is, the rate (17) cannot be observed in the limit where $\omega_k \rightarrow 0$. An observable quantity, adapted to the properties of the detector in question, would be $\int_{\omega_{k,\min}}^{\omega_{k,\max}} d\omega_k w_{\mathbf{p}}(\omega_k, \hat{\mathbf{e}}_k)$ from Eq. (17), which is finite. If and only if one insists on letting $\omega_{k,\min} = 0$, the infrared catastrophe does become manifest. In this case, radiative corrections to the electron propagator need be included in order to obtain a finite result [35]. The procedure is analogous to the cancellation of infrared divergencies, for example, in Coulomb or Compton scattering. The problem of infrared divergencies is elaborated in Appendix C.

C. Low-frequency approximation

The result (29) can be simplified if we suppose that $\omega \ll \omega_k$, i.e., in the low-frequency approximation (LFA). Expanding $\alpha(t) - \alpha(t + \tau) = -\mathbf{A}(t)\tau + \mathbf{E}(t)\tau^2/2 + \dots = -\mathbf{A}(t)\tau[1 + o(\omega\tau)]$, where we used the relations $\alpha'(t) = \mathbf{A}(t)$ and $\alpha''(t) = -\mathbf{E}(t)$, we get

$$\begin{aligned} T_{\mathbf{p}}^{(1)}(n) &\approx -i \int_0^\infty d\tau e^{i\omega_k \tau} \int_0^T \frac{dt}{T} e^{i[\mathbf{p} \cdot \alpha(t) + U_1(t) + n\omega t]} \\ &\quad \times \left\{ -[\mathbf{p} + \mathbf{A}(t)]\tau - i \frac{\partial}{\partial \mathbf{p}} \right\} m(\mathbf{p}, t) \\ &= \int_0^T \frac{dt}{T} e^{i[\mathbf{p} \cdot \alpha(t) + U_1(t) + n\omega t]} \\ &\quad \times \left\{ [\mathbf{p} + \mathbf{A}(t)] i \frac{\partial}{\partial \omega_k} - i \frac{\partial}{\partial \mathbf{p}} \right\} \frac{m(\mathbf{p}, t)}{\omega_k + i\varepsilon}. \end{aligned} \quad (30)$$

Here we made use of (A7) and $(-i\partial/\partial\omega_k)e^{i\omega_k\tau} = \tau e^{i\omega_k\tau}$. Our final result in the LFA is

$$\begin{aligned} T_{\mathbf{p}}^{(1,\text{LFA})}(n) &= -\frac{i}{\omega_k} \int_0^T \frac{dt}{T} e^{i[\mathbf{p} \cdot \alpha(t) + U_1(t) + n\omega t]} \\ &\quad \times \left[\frac{\mathbf{p} + \mathbf{A}(t)}{\omega_k} + \frac{\partial}{\partial \mathbf{p}} \right] m(\mathbf{p}, t). \end{aligned} \quad (31)$$

III. NUMERICAL RESULTS

A. Linearly polarized field

In this subsection we present our numerical results for the linearly polarized bichromatic field

$$\mathbf{E}(t) = E(t)\hat{\mathbf{e}} = [E_1 \sin \omega t + E_2 \sin(2\omega t + \phi)]\hat{\mathbf{e}}, \quad (32)$$

with the unit polarization vector $\hat{\mathbf{e}}$, component intensities $I_1 = E_1^2$ and $I_2 = E_2^2$, and the relative phase ϕ . We have $A(t) = A_1 \cos \omega t + A_2 \cos(2\omega t + \phi)$, $U_p = U_{p1} + U_{p2}$, $U_{pj} = A_j^2/4$, $A_j = E_j/(j\omega)$, $j = 1, 2$, $U_1(t) = \{2U_{p1} \sin(2\omega t) + U_{p2} \sin(4\omega t + 2\phi) + 8\sqrt{U_{p1}U_{p2}}[\sin(\omega t + \phi) + \sin(3\omega t + \phi)/3]\}/(4\omega)$, $\alpha(t) = \sum_{s=\pm 1, r=1,2} \alpha_{sr} e^{isr\omega t}$, $\alpha_{s1} = sA_1/(2i\omega)$, $\alpha_{s2} = sA_2 e^{i\phi}/(4i\omega)$. The total rate $W(\omega_k, \hat{\mathbf{e}}_k) = \int d\mathbf{p} w_{\mathbf{p}}(\omega_k, \hat{\mathbf{e}}_k)$ for the field (32) can be simplified using Ref. [1]: $\int d\mathbf{p} \rightarrow 2\pi \sum_{n>n_0} p_n \int_{-1}^1 dx$, with $x = \cos \theta_p = \hat{\mathbf{e}} \cdot \mathbf{p}/p$ and $n_0 = [(\omega_k + I_p + U_p)/\omega]$. We choose $\hat{\mathbf{e}}_k = \hat{\mathbf{e}}$.

Figure 2 shows the total rate for photon emission during ATI of argon atoms (ionization potential $I_p = 15.76$ eV) by a linearly polarized monochromatic laser field with the intensity 3×10^{14} W/cm² and wavelengths of either 800 or 1300 nm. Separately, the rates are displayed that correspond to the T -matrix vectors (13) and (29) and their coherent sums: $W^{(0)}(\omega_k, \hat{\mathbf{e}})$ (black solid line), $W^{(1,1)}$ (red dotted line), $W^{(1,2)}$ (green dashed line), $W^{(1,3)}$ (blue long-dashed line), $W^{(1)}$ (magenta dot-dashed line), $W^{(0+1)}$ (cyan long dot-dashed line), and $W^{(1,\text{LFA})}$ (orange solid line). The continuum-correction contributions $W^{(1,1)}$ and $W^{(1,2)}$ are dominant, but they interfere destructively, except in the resonant region at $\omega_k \approx \omega$ where $W^{(1,2)}$ prevails. The contribution $W^{(1,3)}$ is dominant for very low energies (it is proportional to $\omega_k^3/\omega_k^4 = 1/\omega_k$). Interestingly, for not too low photon energies ω_k the lowest-order contribution $W^{(0)}$ is comparable with the coherent sum of the contributions $W^{(1)}$. The rate obtained as the coherent sum of all contributions is higher than its zeroth-order and continuum-continuum contributions, but lower than the individual rates $W^{(1,1)}(\omega_k, \hat{\mathbf{e}})$ and $W^{(1,2)}(\omega_k, \hat{\mathbf{e}})$, which destructively interfere. While the direct (lowest-order) photon-emission T -matrix element (13) includes the interaction with the laser field only via the Volkov state, the correction (29) contains the SFA matrix element, which is proportional to the laser electric-field amplitude. We also display the result due to application of the LFA. It agrees very well with the exact result for $\omega_k \gg \omega$. It does not separately single out the terms $W^{(1,1)}$ and $W^{(1,2)}$, which destructively interfere.

Comparison of Figs. 2(a) and 2(b) shows that the results do not very much depend on the wavelength of the driving laser. Especially, the destructive interference between the contributions $W^{(1,1)}$ and $W^{(1,2)}$ is universal, and the LFA works very well regardless of the wavelength. For 800 nm

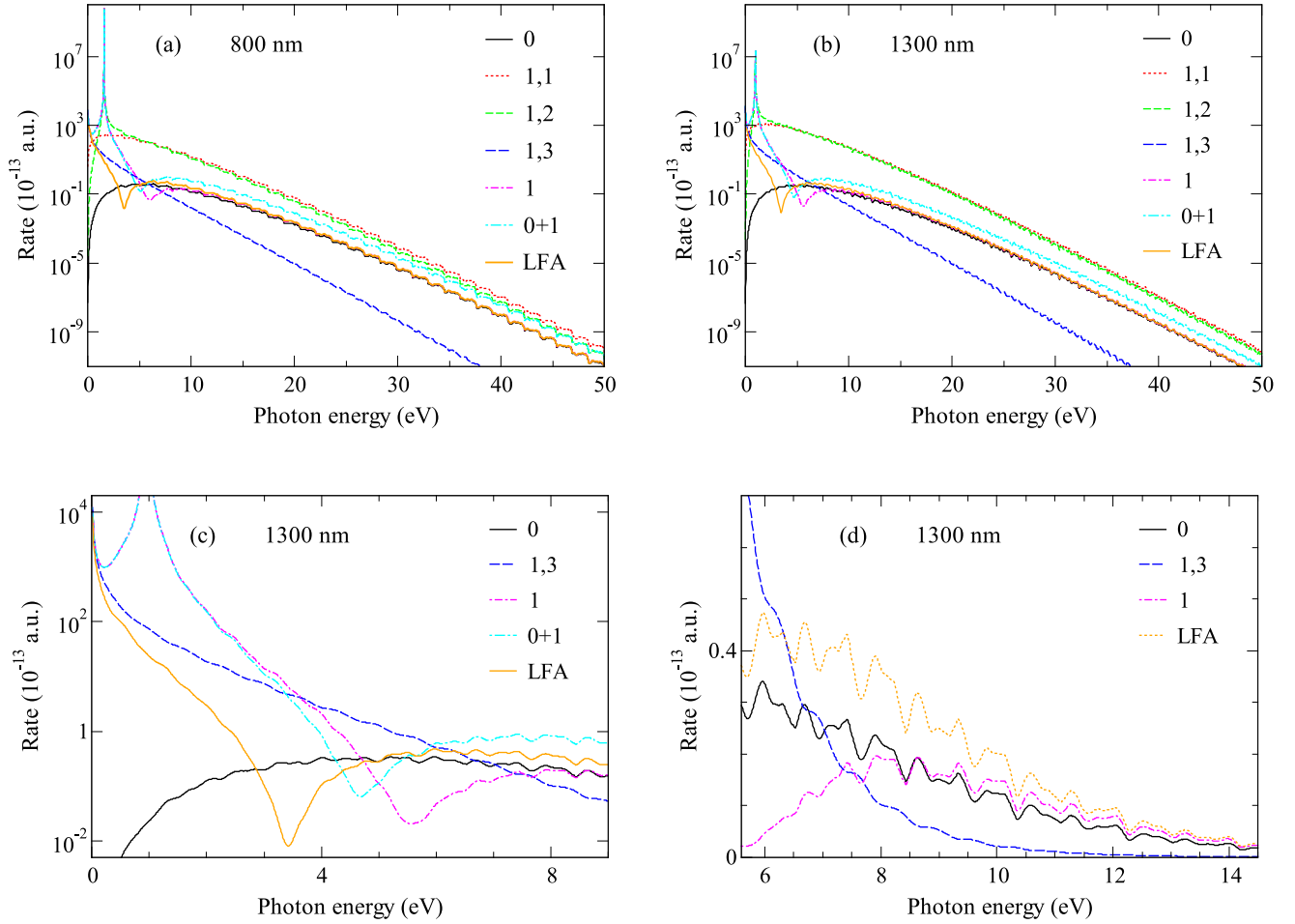


FIG. 2. The total rate $W(\omega_k, \hat{\mathbf{e}})$ as a function of the energy ω_k of the emitted photon for emission during ATI of argon atoms by a linearly polarized monochromatic laser field with the intensity $3 \times 10^{14} \text{ W/cm}^2$ and wavelength (a) 800 nm and (b) 1300 nm. Panel (c) is an enlargement of panel (b) for low photon energies, while panel (d) represents the corresponding rate on a linear scale. In all panels, the partial contributions and their coherent sums are presented separately, as denoted in the legend.

and energies $\omega_k \gg \omega$, the various contributions as well as their coherent sum exhibit a periodic structure with a period $\Delta\omega_k < \omega$. For 1300 nm, Fig. 2(c) enlarges the region of low photon energy where the interference is not yet effective. In contrast to the lowest-order contribution, the continuum correction is infrared divergent, which is due to the contribution $W^{(1,3)}$. Figure 2(d) shows, for low photon energies, the rate on a linear scale, which allows for a quantitative comparison of the various contributions. In particular, the aforementioned characteristic oscillations of the rate are similar for all contributions presented. These oscillations are caused by the behavior of the action, $\mathbf{p} \cdot \boldsymbol{\alpha}(t) + U_1(t) + n\omega t$, in the exponent of the SFA T -matrix element in Eq. (A1), which is the same for all contributions to the rate.

In Fig. 3 we present the results for the linearly polarized bichromatic laser field (32) and the argon atom. The fundamental wavelength is 1030 nm, the component intensities are equal, $I_1 = E_1^2 = I_2 = E_2^2 = 1.5 \times 10^{14} \text{ W/cm}^2$, the relative phase is (a) $\phi = 0^\circ$ and (b) $\phi = 90^\circ$, and we exhibit the total rate. As for the monochromatic case, the most remarkable result is that for energies below 6 eV the total rate, presented on a linear scale, is completely dominated by the

continuum contribution. The zeroth-order contribution is essential for energies in the region 6–10.8 eV, where the continuum contribution exhibits a minimum. For higher energies the two contributions are comparable and their sum exhibits a maximum near 10 eV. This shows that both the zeroth-order and the continuum contributions have to be taken into account and that their amplitudes interfere constructively. From the inset, where the total rates are shown on a logarithmic scale, we see that the (1,1) and (1,2) contributions are dominant for energies higher than 0.7 eV. However, as in the monochromatic case, these contributions interfere destructively. The contribution (1,3) is dominant for lower energies. It is responsible for the infrared divergence and, probably, for strong terahertz-radiation emission [36–38]. The results are practically independent of the relative phase ϕ between the field components.

So far we presented our results for the linearly polarized laser field (32) and for photon emission in the laser-field polarization direction, $\hat{\mathbf{e}}_k = \hat{\mathbf{e}}$. In this case the total rate, which is obtained solving the three-dimensional integral over the photoelectron momentum $\mathbf{p}$, is calculated summing over the number of absorbed laser photons and integrating over the variable $x = \cos\theta_p$, where θ_p is the polar angle in spherical

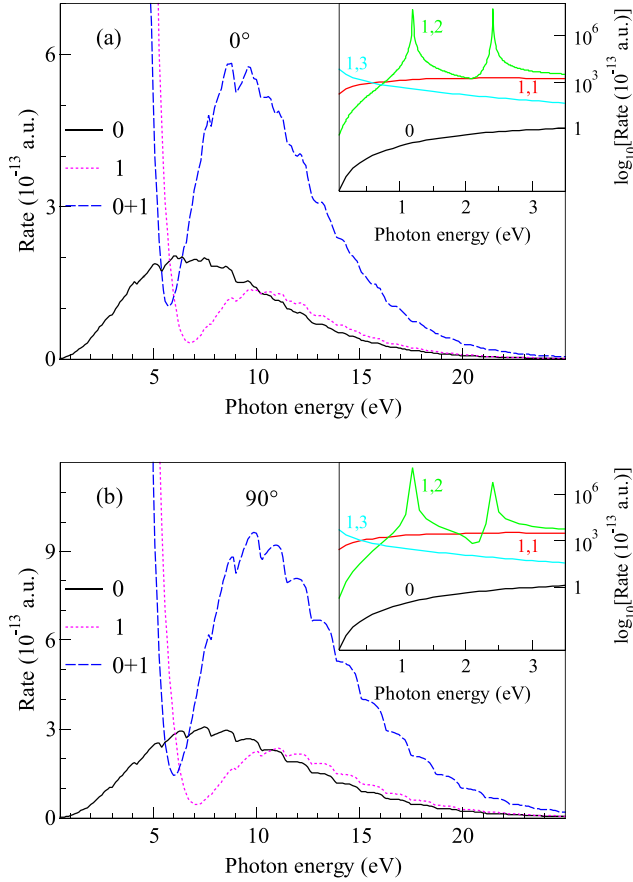


FIG. 3. Same as in Fig. 2, but for the linearly polarized bichromatic laser field (32) with equal components intensities $I_1 = I_2 = 1.5 \times 10^{14} \text{ W/cm}^2$, the fundamental wavelength 1030 nm, and the relative phase (a) $\phi = 0^\circ$ and (b) $\phi = 90^\circ$. The zeroth-order contribution (black solid line), the continuum correction (magenta dotted line), and their coherent sum (blue dashed line) are shown on a linear scale. The inset shows the contributions (1,1), (1,2), and (1,3), as denoted, on a logarithmic scale.

coordinates (the integral over the azimuthal angle φ_p gives the factor 2π). For arbitrary polarization of the emitted photon, the integral over $d\varphi_p$ should also be solved numerically. We now calculate the total rate for this more general case. For $\hat{\mathbf{e}} = \hat{\mathbf{e}}_k = \hat{\mathbf{e}}_x$ we obtain the same results as those presented in Figs. 2 and 3. For emission of photons with the polarization $\hat{\mathbf{e}}_k = \hat{\mathbf{e}}_z$ we get a much lower total rate. For example, for the parameters of Fig. 3 and $\phi = 0$, the rate for $\hat{\mathbf{e}}_k = \hat{\mathbf{e}}_z$ is lower than for $\hat{\mathbf{e}}_k = \hat{\mathbf{e}}_x$ by factors of 15.9 and 4480 for $W^{(0)}(\omega_k, \hat{\mathbf{e}}_k)$ and $W^{(1,1)}(\omega_k, \hat{\mathbf{e}}_k)$, respectively.

B. Circularly polarized field

We consider the circularly polarized monochromatic laser field

$$\mathbf{E}(t) = \frac{E_0}{\sqrt{2}}(\sin \omega t \hat{\mathbf{e}}_x - \cos \omega t \hat{\mathbf{e}}_y) \quad (33)$$

and present the results in Fig. 4. For photon emission in the $\hat{\mathbf{e}}_x$ direction the rates $W^{(1,j)}(\omega_k, \hat{\mathbf{e}}_x)$, $j = 1, 2$, are higher than the zeroth-order rate and the rate (1,2) exhibits a resonant

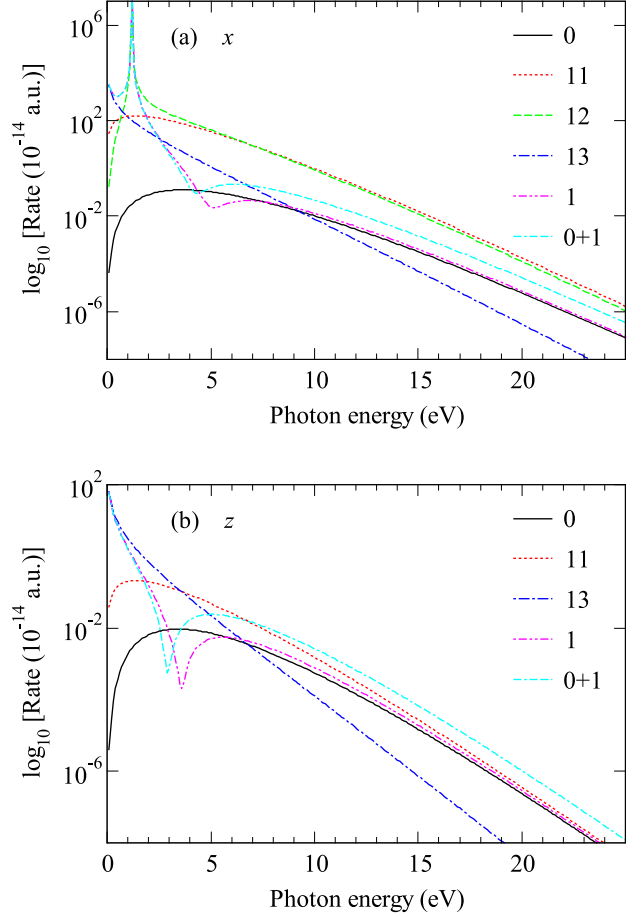


FIG. 4. Same as in Fig. 2, but for the circularly polarized monochromatic laser field (33) with intensity $3 \times 10^{14} \text{ W/cm}^2$ and wavelength 1030 nm. The zeroth-order contribution (black solid line), and the contributions (1,1) (red dotted line), (1,2) (green dashed line), and (1,3) (blue dot-dashed line) as well as their sum (magenta double-dot-dashed line) are shown for photon polarization in the direction of (a) the x axis and (b) the z axis.

peak at $\omega_k = \omega$. The rate (1,3) is dominant at very low photon energies ω_k where it is responsible for the infrared divergence. As in the linearly polarized case, the continuum T -matrix contributions interfere destructively, so the zeroth-order and continuum rates are comparable, except in the low-energy region where the continuum term is dominant. Near 5 eV the rate $W^{(1)}(\omega_k, \hat{\mathbf{e}}_x)$ exhibits a broad minimum and in consequence the zeroth-order term is dominant. The rate obtained using the coherent sum of all contributions is higher than the zeroth-order rate. Hence, again, including the continuum correction is imperative.

For emission of a photon with the polarization in the $\hat{\mathbf{e}}_z$ direction, the (resonant) rate $W^{(1,2)}(\omega_k, \hat{\mathbf{e}}_z)$ is zero, $W^{(1,1)}$ is dominant above 3.6 eV, while $W^{(1,3)}$ is dominant below 3.6 eV. However, the destructive interference of the (1,1) and (1,3) contributions gives a minimum around 3.6 eV, where the zeroth-order term is dominant.

The total rate for emission in the $\hat{\mathbf{e}}_y$ direction is the same as in the $\hat{\mathbf{e}}_x$ direction, which is compatible with circular polarization of the emitted radiation.

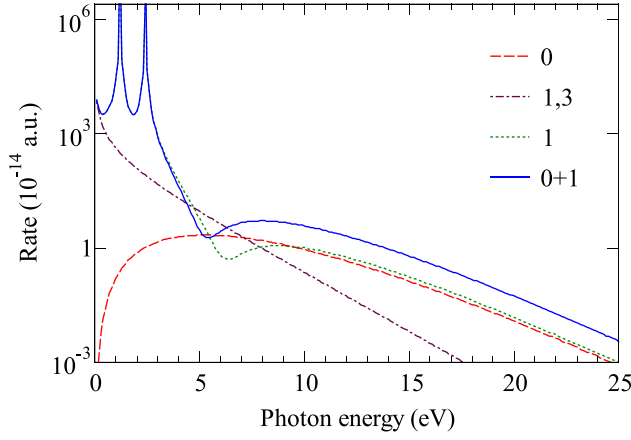


FIG. 5. Same as in Fig. 2 but for the bicircular field (34), with the wavelength 1030 nm and the component intensities $I_1 = I_2 = 1.5 \times 10^{14}$ W/cm². The partial contributions to the rate are presented separately as denoted in the legend.

C. Bicircular field

A bicircular field $\mathbf{E}(t) = \mathbf{E}_1(t) + \mathbf{E}_2(t)$,

$$\begin{aligned} \mathbf{E}_1(t) &= \frac{E_1}{\sqrt{2}} [\hat{\mathbf{e}}_x \sin(\omega t) - \hat{\mathbf{e}}_y \cos(\omega t)], \\ \mathbf{E}_2(t) &= \frac{E_2}{\sqrt{2}} [-\hat{\mathbf{e}}_x \cos(2\omega t) + \hat{\mathbf{e}}_y \sin(2\omega t)], \end{aligned} \quad (34)$$

consists of two coplanar counterrotating circularly polarized components, which we take in the xy plane. In Fig. 5 we exhibit the total rate $W(\omega_k, \hat{\mathbf{e}}_x)$. For $\hat{\mathbf{e}}_k = \hat{\mathbf{e}}_y$ the results are the same, compatible with circular polarization of the emitted photons (the rate for emission of photons with $\hat{\mathbf{e}}_k = \hat{\mathbf{e}}_z$ is much lower [1]). For energies below 5 eV the continuum-continuum contribution is dominant, while for energies from 5 to 9 eV the zeroth-order contribution is stronger. For higher energies these two contributions are comparable and interfere constructively. Resonant peaks due to the (1,2) contribution appear at $\omega_k = \omega$ and 2ω . The contribution (1,3) is responsible for the low-energy divergence which behaves as $1/\omega_k$. As already observed for linear and circular polarization of the driving field, for low energies the continuum contribution dominates the zeroth order by many orders of magnitude.

D. Orthogonal two-color field

An orthogonal two-color (OTC) field is defined by

$$\mathbf{E}(t) = E_1 \hat{\mathbf{e}}_x \sin(\omega t) + E_2 \hat{\mathbf{e}}_y \sin(2\omega t + \phi), \quad (35)$$

where ϕ is the relative phase between the two components. The total rate is highest for the relative phase $\phi = 90^\circ$ and for $\hat{\mathbf{e}}_k = \hat{\mathbf{e}}_x$ and $\hat{\mathbf{e}}_y$ [1]. In Fig. 6 we show our results for the OTC field with the relative phase $\phi = 90^\circ$. For the x component [Fig. 6(a)] the zeroth-order term is dominant in the region from 5 to 9 eV; for all other energies the continuum term prevails. The contributions (1,1) and (1,2) interfere destructively. The contribution (1,3) is dominant for very low energies, while the resonant peak at $\omega_k = \omega$ is due to the (1,2) rate. For the y component [Fig. 6(b)] the results are similar, except that the resonant peak appears for $\omega_k = 2\omega$. This is expected

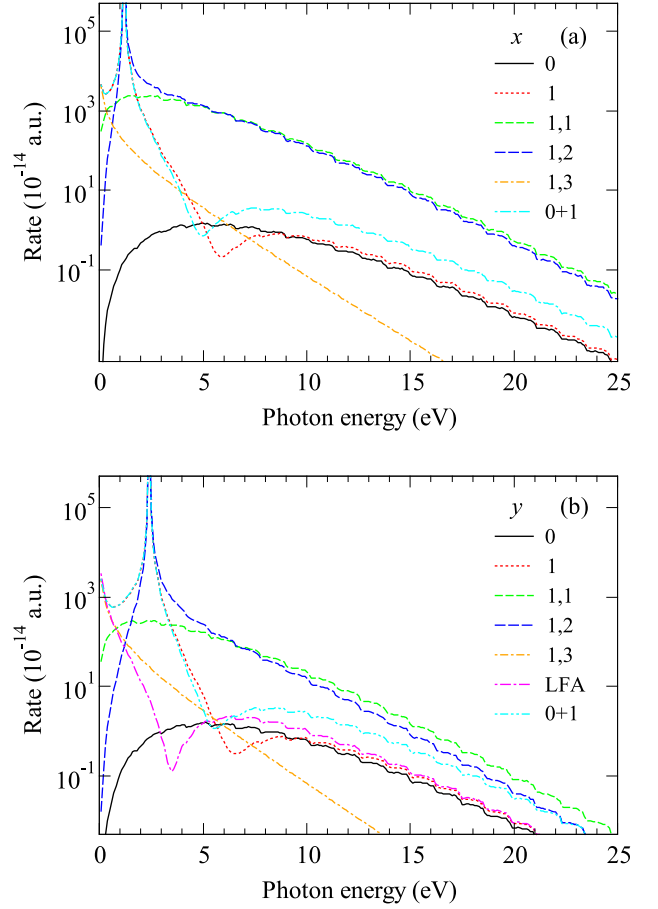


FIG. 6. Same as in Fig. 2 but for an OTC field, Eq. (35), with the wavelength 1030 nm, the component intensities $I_1 = I_2 = 1.5 \times 10^{14}$ W/cm², and the relative phase $\phi = 90^\circ$, for (a) $\hat{\mathbf{e}}_k = \hat{\mathbf{e}}_x$ and (b) $\hat{\mathbf{e}}_k = \hat{\mathbf{e}}_y$. Various partial contributions are shown, as denoted in the legends.

since the y component of the OTC field oscillates with the frequency 2ω . In Fig. 6(b) we also present the LFA result in order to show that for higher photon energies ($\omega_k > 10$ eV) it agrees well with the continuum-continuum result. This confirms the strong destructive interference of the (1,1) and (1,2) contributions for higher photon energies.

IV. CONCLUSIONS

We considered the emission of a photon during the process of above-threshold ionization. A previous paper studied the lowest-order contribution where the photon is emitted while the electron transits to the continuum. In this paper, we investigated the correction to this result where the photon is emitted while the electron is propagating in the continuum. We employed various laser fields to drive the process: a linearly polarized monochromatic field without or with its second harmonic, a monochromatic circularly polarized field, a counterrotating $(\omega, 2\omega)$ bicircular field, and an orthogonally polarized $(\omega, 2\omega)$ two-color field. The total rates for the emission of a photon with specified momentum and polarization during the ATI process, with the momentum of the electron summed over, are very similar for all these different

driving fields. Most importantly, it turns out that the lowest-order rates are very significantly modified by the continuum-continuum corrections for all but some intermediate photon energies. For all laser fields that we considered, the photon emission rate is proportional to ω_k^{-1} in the limit of vanishing photon energy ω_k . This behavior is well known from scattering processes where the initial and final states of the electron are in the continuum [4,5]. Here, however, the electron is initially bound. This infrared divergence only occurs in the continuum-continuum correction; the lowest-order rate is finite when ω_k goes to zero. The explanation of how this divergence cancels with virtual soft-photon emission in ATI is presented in Appendix C.

We decomposed the continuum-continuum photon emission amplitude $\mathbf{T}_p^{(1)}(n)$ into three terms, $\mathbf{T}_p^{(1)}(n) = \sum_{j=1,2,3} \mathbf{T}_p^{(1,j)}(n)$. The second term $\mathbf{T}_p^{(1,2)}(n)$ is resonant when the photon energy ω_k agrees with the energy of a driving-field photon. The third term $\mathbf{T}_p^{(1,3)}(n)$ leads to an infrared-divergent emission rate. For photon energies ω_k larger than about 10 eV (for our parameters) the two terms $\mathbf{T}_p^{(1,1)}(n)$ and $\mathbf{T}_p^{(1,2)}(n)$, which are individually very large, destructively interfere almost completely. This would suggest to combine them into one term. However, for smaller energies ω_k , they behave very differently. We also investigated the low-frequency approximation, which is expected to hold for $\omega \ll \omega_k$. The LFA combines all three terms, works very well for large ω_k , and incorporates the aforementioned destructive interference.

Klaiber *et al.* [26] considered the first-order correction to the lowest-order result for spontaneous radiation using a different approach. The ionization dynamics were also treated using the SFA but, contrary to our quantum approach, the classical Maxwell equations were employed to calculate the intensity of the emitted radiation using the obtained current density of the liberated electron. Their numerical results show that for low energies the first-order contribution to the emitted radiation is much larger than the zeroth order. However, their results do not exhibit the resonant contribution or the ω_k^{-1} behavior for $\omega_k \rightarrow 0$ (see Appendix B for our results obtained using the velocity gauge for interaction with the emitted-photon field as in Ref. [26]). For high energies the functional shape of their zeroth-order and first-order results is similar to ours, but the two contributions interfere destructively, contrary to our findings.

The third term of our continuum-dynamics correction exhibits an infrared divergence since, according to Eqs. (17) and (29), the corresponding total rate is $W^{(1,3)}(\omega_k, \hat{\mathbf{e}}) = (2\pi c^3 \omega_k)^{-1} \int d\mathbf{p} (\hat{\mathbf{e}} \cdot \mathbf{p})^2 |\mathbf{T}_p^{\text{SFA}}(n)|^2 \propto 1/\omega_k$. However, any detector has a finite resolution and therefore cannot distinguish whether or not one or more sufficiently soft photons were spontaneously emitted during ATI. Bloch and Nordsieck [4] showed long ago that, if one considers the physically measurable sum of the probabilities of all soft-photon processes, then all infrared divergencies cancel and one obtains a finite result. As far as we know, the problem of infrared divergences for ATI has not been considered yet. A detailed treatment is beyond the scope of our present paper. For a general approach to radiative corrections, see Chap. 6 of Ref. [5]. For a more recent treatment of the infrared divergencies in plane-wave

backgrounds, see Ref. [41]. In Appendix C we shortly explain how the infrared divergences for virtual and real soft-photon emission in ATI cancel.

In an earlier paper [1] by one of the present authors about the zeroth-order process without the continuum correction, the magnitude of the photon emission rate for given momentum was compared with high-order harmonic emission of an analogous photon and found to be higher by orders of magnitude. But it was emphasized that for the process considered in Ref. [1] the contributions of different atoms will add incoherently, in contrast to HHG. This conclusion is not modified by the continuum correction considered here.

As mentioned before [1], in early HHG experiments [39] and plasma fluorescence experiments [40] low-energy continuum radiation was observed as a background, but the effect has not been explored in detail. Our present results show that photon emission during ATI is enhanced by the continuum dynamics the contribution of which may be dominant over the direct photon emission considered in Ref. [1]. This should be explored in future experiments. For very low emitted photon energies the third term in Eq. (29) is dominant and this may be important in the context of THz generation [2,3,36–38].

The significance of the continuum-continuum correction raises the question of the importance of more than one photon being emitted by the same electron. Moreover, the process of high-order ATI, where the liberated electron rescatters off the binding potential, will also allow for the emission of an additional photon. High-order ATI is not necessarily a weak process, since it generates the low-energy structure for very low electron momentum [29,30], which is comparable with or even exceeds the contribution of the direct electrons.

We ought to mention that Coulomb effects, which are neglected in our SFA approach, may be important, in particular in the low-energy regime. Such effects are still an important area of investigation (see, for example, the review [42] and the recent reference [43]). However, in the absence of the Coulomb effects, for example, for above-threshold detachment of negative ions, the SFA is more than satisfactory.

Since our results are obtained using the leading order of the SFA expansion, we expect that they are quantitatively valid for strong laser fields, for long wavelengths, and whenever Coulomb corrections are less important. For other values of the parameters, we expect that our results qualitatively describe the process.

DATA AVAILABILITY

The data that support the findings of this article are not publicly available. The data are available from the authors upon reasonable request.

APPENDIX A: DERIVATION OF EQ. (29)

In this appendix we derive the result (29) starting from the T -matrix vector (28). For this purpose we introduce the SFA ionization T -matrix element

$$T_p^{\text{SFA}}(n) = \int_0^T dt m(\mathbf{p}, t) e^{i[\mathbf{p} \cdot \boldsymbol{\alpha}(t) + U_1(t) + not]}/T \quad (\text{A1})$$

and its derivative

$$\frac{\partial T_{\mathbf{p}}^{\text{SFA}}(n)}{\partial \mathbf{p}} = \int_0^T \frac{dt}{T} \left[\frac{\partial m(\mathbf{p}, t)}{\partial \mathbf{p}} + i\alpha(t)m(\mathbf{p}, t) \right] \times \exp\{i[\mathbf{p} \cdot \boldsymbol{\alpha}(t) + U_1(t) + n\omega t]\}. \quad (\text{A2})$$

For the hydrogenlike atom model [28] we have

$$m(\mathbf{p}, t) = \frac{C[\mathbf{p} + \mathbf{A}(t)] \cdot \mathbf{E}(t)}{\{[\mathbf{p} + \mathbf{A}(t)]^2 + 2I_p\}^3} \quad (\text{A3})$$

and

$$\frac{\partial m(\mathbf{p}, t)}{\partial \mathbf{p}} = \frac{C}{\{[\mathbf{p} + \mathbf{A}(t)]^2 + 2I_p\}^3} \times \left\{ \mathbf{E}(t) - \frac{6[\mathbf{p} + \mathbf{A}(t)] \cdot \mathbf{E}(t)[\mathbf{p} + \mathbf{A}(t)]}{[\mathbf{p} + \mathbf{A}(t)]^2 + 2I_p} \right\} \quad (\text{A4})$$

with $C = -i2^{7/2}(2I_p)^{5/4}/\pi$. We present results for an arbitrary R -color laser field polarized in the xy plane, with

$$\boldsymbol{\alpha}(t) = \sum_{s=\pm 1, r=1, \dots, R, j=x, y} \alpha_{srj} e^{isr\omega t} \hat{\mathbf{e}}_j. \quad (\text{A5})$$

We rewrite (28) as

$$\begin{aligned} \mathbf{T}_{\mathbf{p}}^{(1)}(n) = & i \int_0^\infty d\tau e^{i\omega_k \tau} \left(\mathbf{p}\tau + i \frac{\partial}{\partial \mathbf{p}} \right) T_{\mathbf{p}}^{\text{SFA}}(n) \\ & + i \sum_{s, r, j} \alpha_{srj} \hat{\mathbf{e}}_j \int_0^\infty d\tau e^{i(\omega_k + sr\omega)\tau} T_{\mathbf{p}}^{\text{SFA}}(n + sr). \end{aligned} \quad (\text{A6})$$

The integral over τ can be solved using the function (see Refs. [34] [Chap. II, Sec. 8, Eqs. (13) and (17)] and [44])

$$\zeta(x) = -i \int_0^\infty d\tau e^{i\tau x} = \frac{1}{x + i\varepsilon} = \frac{\mathcal{P}}{x} - i\pi \delta(x), \quad (\text{A7})$$

where $\varepsilon \rightarrow 0$ and the symbol $\mathcal{P}$ denotes the principal value of the integral. The result is

$$\begin{aligned} \mathbf{T}_{\mathbf{p}}^{(1)}(n) = & i \left(\mathbf{p} \frac{\partial}{\partial \omega_k} - \frac{\partial}{\partial \mathbf{p}} \right) \zeta(\omega_k) T_{\mathbf{p}}^{\text{SFA}}(n) \\ & - \sum_{s, r, j} \alpha_{srj} \hat{\mathbf{e}}_j \zeta(\omega_k + sr\omega) T_{\mathbf{p}}^{\text{SFA}}(n + sr). \end{aligned} \quad (\text{A8})$$

Since $\omega_k > 0$, we can write $\mathbf{T}_{\mathbf{p}}^{(1)}(n) = \sum_{j=1,2,3} \mathbf{T}_{\mathbf{p}}^{(1,j)}(n)$, where the three contributions are given by Eq. (29).

APPENDIX B: CONTINUUM CORRECTION FOR MIXED GAUGES FROM REF. [26]

In Ref. [26] the spontaneous emission of a photon was analyzed expressing the interaction with the quantized field in the velocity gauge, while the interaction with the laser field was described in the length gauge as in our paper [1]. Replacing our interaction operator $V_{\text{ph}}(t) = \mathbf{r} \cdot \mathbf{E}_{\text{ph}}(t)$ with the corresponding interaction in velocity gauge $V_{\text{ph}}(t) = \hat{\mathbf{p}} \cdot \mathbf{A}_{\text{ph}}(t)$ where $\hat{\mathbf{p}}$ is the momentum operator and

$$\mathbf{A}_{\text{ph}}(t) = \sum_{\mathbf{k}} b_{\mathbf{k}} (a_{\mathbf{k}}^\dagger e^{i\omega_k t} \hat{\mathbf{e}}_{\mathbf{k}}^* + a_{\mathbf{k}} e^{-i\omega_k t} \hat{\mathbf{e}}_{\mathbf{k}}), \quad (\text{B1})$$

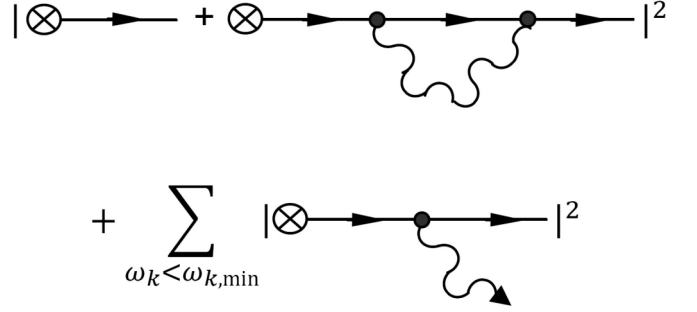


FIG. 7. Feynman diagrams for virtual and real soft-photon emission in ATI. The first term in the sum is the ATI process and its electron self-energy correction, while the second term is our “ATI + photon emission” process summed over all undetectable soft photons with energies $\omega_k < \omega_{k,\min}$.

with $b_k = [2\pi/(V\omega_k)]^{1/2}$ and $a_{\mathbf{k}}$ and $a_{\mathbf{k}}^\dagger$ the annihilation and creation operators of the emitted photon, we can repeat the derivation from Ref. [1], Eqs. (1)–(19), with $c_k \rightarrow b_k$ and $\mathbf{r} \rightarrow \hat{\mathbf{p}}$. The result is

$$w_{\mathbf{p}}^{(0)}(\omega_k, \hat{\mathbf{e}}_{\mathbf{k}}) = \frac{\omega_k}{2\pi c^3} |\hat{\mathbf{e}}_{\mathbf{k}}^* \cdot \mathbf{T}_{\mathbf{p}}^{(0)}(n)|^2, \quad (\text{B2})$$

with Eq. (13) and with Eq. (12) replaced by

$$\mathcal{T}_{\mathbf{p}}^{(0)}(t) = [\mathbf{p} + \mathbf{A}(t)] \langle \mathbf{p} + \mathbf{A}(t) | \psi_b \rangle e^{i[\mathbf{p} \cdot \boldsymbol{\alpha}(t) + U_1(t) + n\omega t]}. \quad (\text{B3})$$

Notice that in Eq. (B2) we have ω_k in the numerator while in Eq. (17) of Ref. [1] we had ω_k^3 .

In order to obtain the continuum correction in the mixed-gauges approach, we repeat our derivation (1)–(22) using the same replacement: $c_k \rightarrow b_k$ and $\mathbf{r} \rightarrow \hat{\mathbf{p}}$. Equation (23) should be replaced by $[\mathbf{p} + \mathbf{A}(t)]f(\mathbf{p})$, so that in Eq. (22) the integral over t'' reduces to $\int_t^\infty dt'' e^{i\omega_k t''} = e^{i\omega_k t} \int_0^\infty dt'' e^{i\omega_k t''} = i e^{i\omega_k t} / \omega_k$, where we used Eq. (A7) and $\delta(\omega_k) = 0$ since $\omega_k > 0$. Denoting

$$\mathbf{T}_{\mathbf{p}}^{(1)}(n) = \int_0^T dt [\mathbf{p} + \mathbf{A}(t)] m(\mathbf{p}, t) e^{i[\mathbf{p} \cdot \boldsymbol{\alpha}(t) + U_1(t) + n\omega t]} / T, \quad (\text{B4})$$

we get

$$w_{\mathbf{p}}^{(1)}(\omega_k, \hat{\mathbf{e}}_{\mathbf{k}}) = \frac{1}{2\pi c^3 \omega_k} |\hat{\mathbf{e}}_{\mathbf{k}}^* \cdot \mathbf{T}_{\mathbf{p}}^{(1)}(n)|^2, \quad (\text{B5})$$

which does exhibit the infrared divergence for $\omega_k \rightarrow 0$.

APPENDIX C: INFRARED DIVERGENCES FOR VIRTUAL AND REAL SOFT-PHOTON EMISSION IN ATI

In the ATI process the laser-dressed atom is ionized and one laser-dressed electron (described by a Volkov state) is emitted. In terms of Feynman diagrams this electron can be depicted by a straight line emerging from the atomic center. This line (i.e., the corresponding electron propagator) can be modified by the process in which a virtual photon is emitted and reabsorbed (see the upper part of Fig. 7). However, this higher-order contribution is divergent. In quantum field theory this divergence is cured using mass renormalization (the bare electronic mass m is replaced by the physical mass

$m_R = m + \delta m$, with δm the self energy). Another problem is that the renormalization constant is infrared divergent (in addition to the ultraviolet divergence). This infrared divergence cancels the infrared divergence found in our paper. Namely, in our continuum-continuum process a photon is emitted. This photon can be detected if its energy $\omega_k > \omega_{k,\min}$, where $\omega_{k,\min}$ is determined by the experimental resolution. The infrared divergent contribution for $\omega_k < \omega_{k,\min}$ cancels the higher-order term of the ATI process, as depicted in Fig. 7. Denoting by S the corresponding S -matrix-element contribution, we conclude that the meaning of the scheme presented in Fig. 7 is that for ATI with no detection of an emitted photon it is $|S_{\text{ATI}}^{(0)} + S_{\text{ATI}}^{(1,\text{IR divergent})}|^2 + \sum_{\omega_k < \omega_{k,\min}} |S_{\text{ATI},\omega_k}|^2 = |S_{\text{ATI}}^{(0)}|^2$, while for ATI with one emitted photon of energy $\omega_k > \omega_{k,\min}$ we have $|S_{\text{ATI},\omega_k}|^2$.

Therefore, the divergences of the higher-order ATI process with virtual photons cancel the infrared divergence of

our continuum-continuum process with soft-photon emission. The combined result does not depend on the renormalization parameters. For $\omega_k > \omega_{k,\min}$ our (1,3) continuum term really contributes to the process and should be observed in an experiment. The seminal paper in which the problem of infrared divergences was considered is Ref. [4]. More details can be found in the textbooks such as Refs. [5,34,35]. These conclusions are valid for arbitrary order of the perturbation theory as shown in Ref. [45] (see also the more recent Refs. [41,46]). As far as we know, the problem of quantum-electrodynamic divergences for ATI was not considered yet. This is beyond the scope of our paper. In this appendix we provided a physical explanation that the soft-photon emission, related to a term that appears in our continuum-dynamics correction to our ATI+photon emission process, is observable, and gave a prescription of how the problem of the infrared divergence can be solved.

-
- [1] D. B. Milošević, Quantum theory of photon emission during strong-laser-field-induced ionization, *Phys. Rev. A* **108**, 033110 (2023).
 - [2] X. C. Zhang, A. Shkurinov, and Y. Zhang, Extreme terahertz science, *Nat. Photon.* **11**, 16 (2017).
 - [3] P. Salén, M. Basini, S. Bonetti, J. Hebling, M. Krasilnikov, A. Y. Nikitin, G. Shamuilov, Z. Tibai, V. Zhaunerchyk, and V. Goryashko, Matter manipulation with extreme terahertz light: Progress in the enabling THz technology, *Phys. Rep.* **836-837**, 1 (2019).
 - [4] F. Bloch and A. Nordsieck, Note on the radiation field of the electron, *Phys. Rev.* **52**, 54 (1937).
 - [5] M. E. Peskin and D. V. Schroeder, *An Introduction to Quantum Field Theory* (Addison-Wesley, Reading, MA, 1995).
 - [6] P. Agostini, F. Fabre, G. Mainfray, G. Petite, and N. K. Rahman, Free-free transitions following six-photon ionization of xenon atoms, *Phys. Rev. Lett.* **42**, 1127 (1979).
 - [7] W. Becker, F. Grasbon, R. Kopold, D. B. Milošević, G. G. Paulus, and H. Walther, Above-threshold ionization: From classical features to quantum effects, *Adv. At Mol. Opt. Phys.* **48**, 35 (2002).
 - [8] P. Agostini and L. F. DiMauro, Atomic and molecular ionization dynamics in strong laser fields: From optical to x-rays, *Adv. At Mol. Opt. Phys.* **61**, 117 (2012).
 - [9] A. McPherson, G. Gibson, H. Jara, U. Johann, T. S. Luk, I. A. McIntyre, K. Boyer, and C. K. Rhodes, Studies of multiphoton production of vacuum-ultraviolet radiation in the rare gases, *J. Opt. Soc. Am. B* **4**, 595 (1987).
 - [10] M. Ferray, A. L' Huillier, X. F. Li, L. A. Lompre, G. Mainfray, and C. Manus, Multiple-harmonic conversion of 1064-nm radiation in rare gases, *J. Phys. B* **21**, L31 (1988).
 - [11] P. Salières, A. L' Huillier, P. Antoine, and M. Lewenstein, Study of the spatial and temporal coherence of high-order harmonics, *Adv. At Mol. Phys.* **41**, 83 (1999).
 - [12] M. C. Kohler, T. Pfeifer, K. Z. Hatsagortsyan, and C. H. Keitel, Frontiers of atomic high-harmonic generation, *Adv. At Mol. Opt. Phys.* **61**, 159 (2012).
 - [13] K. C. Kulander, K. J. Schafer, and J. L. Krause, Dynamics of short-pulse excitation, ionization and harmonic conversion, in *Super-Intense Laser-Atom Physics*, edited by B. Piraux, A. L'Huillier, and K. Rzażewski, NATO Advanced Science Institutes Series B: Physics (Plenum, New York, 1993), Vol. 316, p. 95.
 - [14] P. B. Corkum, Plasma perspective on strong field multiphoton ionization, *Phys. Rev. Lett.* **71**, 1994 (1993).
 - [15] M. Lewenstein, M. F. Ciappina, E. Pisanty, J. Rivera-Dean, P. Stammer, T. Lamprou, and P. Tzallas, Generation of optical Schrödinger cat states in intense laser-matter interactions, *Nat. Phys.* **17**, 1104 (2021).
 - [16] A. Gorlach, M. E. Tzur, M. Birk, M. Krüger, N. Rivera, O. Cohen, and I. Kaminer, High harmonic generation driven by quantum light, *Nat. Phys.* **19**, 1689 (2023).
 - [17] U. Bhattacharya, T. Lamprou, A. S. Maxwell, A. Ordonez, E. Pisanty, J. Rivera-Dean, P. Stammer, M. F. Ciappina, M. Lewenstein, and P. Tzallas, Strong-laser-field physics, non-classical light states and quantum information science, *Rep. Prog. Phys.* **86**, 094401 (2023).
 - [18] P. Stammer, J. Rivera-Dean, A. Maxwell, T. Lamprou, A. Ordóñez, M. F. Ciappina, P. Tzallas, and M. Lewenstein, Quantum electrodynamics of intense laser-matter interactions: A tool for quantum state engineering, *PRX Quantum* **4**, 010201 (2023).
 - [19] L. Cruz-Rodriguez, D. Dey, A. Freibert, and P. Stammer, Quantum phenomena in attosecond science, *Nat. Rev. Phys.* **6**, 691 (2024).
 - [20] D. Habibović and D. B. Milošević, Intensity-dependent enhancements in strong-field ionization by quantum light, *Phys. Rev. A* **112**, L051103 (2025).
 - [21] D. B. Milošević and W. Becker, Strong-field ionization by thermal light, *Ann. Phys.* **538**, e00620 (2026).
 - [22] D. B. Milošević and F. Ehlötzky, X-ray-atom scattering in the presence of a laser field, *Phys. Rev. A* **58**, 2319 (1998).
 - [23] D. B. Milošević and A. F. Starace, Static-electric-field-induced, high-energy plateau for scattered x-ray photons in laser-assisted, x-ray-atom scattering, *Phys. Rev. Lett.* **81**, 5097 (1998).
 - [24] W. Becker, A. Lohr, M. Kleber, and M. Lewenstein, A unified theory of high-harmonic generation: Application to polarization properties of the harmonics, *Phys. Rev. A* **56**, 645 (1997).

- [25] D. B. Milošević and F. Ehlotzky, Scattering and reaction processes in powerful laser fields, *Adv. At Mol. Opt. Phys.* **49**, 373 (2003).
- [26] M. Klaiber, K. Z. Hatsagortsyan, and C. H. Keitel, Role of quantum dynamics in coherent and incoherent radiation during tunneling ionization, *Phys. Rev. A* **113**, 033110 (2026).
- [27] In Eq. (10) in Ref. [1] V_{at} should be replaced by V_{le} . Also, three lines below Eq. (3) in Ref. [1] it should be $\omega_k = ck$. Finally, in the denominator of Eq. (23) in Ref. [1] it should be π^2 instead of π .
- [28] D. B. Milošević, G. G. Paulus, D. Bauer, and W. Becker, Above-threshold ionization by few-cycle pulses, *J. Phys. B* **39**, R203 (2006).
- [29] C. I. Blaga, F. Catoire, P. Colosimo, G. G. Paulus, H. G. Muller, P. Agostini, and L. F. DiMauro, Strong-field photoionization revisited, *Nat. Phys.* **5**, 335 (2009).
- [30] W. Quan, Z. Lin, M. Wu, H. Kang, H. Liu, X. Liu, J. Chen, J. Liu, X. T. He, S. G. Chen, H. Xiong, L. Guo, H. Xu, Y. Fu, Y. Cheng, and Z. Z. Xu, Classical aspects in above-threshold ionization with a midinfrared strong laser field, *Phys. Rev. Lett.* **103**, 093001 (2009).
- [31] D. B. Milošević and F. Ehlotzky, Coulomb and rescattering effects in above-threshold ionization, *Phys. Rev. A* **58**, 3124 (1998).
- [32] F. H. M. Faisal, Strong-field S -matrix theory with final-state Coulomb interaction in all orders, *Phys. Rev. A* **94**, 031401(R) (2016).
- [33] D. B. Milošević and W. Becker, Atom-Volkov strong-field approximation for above-threshold ionization, *Phys. Rev. A* **99**, 043411 (2019).
- [34] W. Heitler, *The Quantum Theory of Radiation*, 3rd ed. (Clarendon, Oxford, 1954).
- [35] J. D. Bjorken and S. D. Drell, *Relativistic Quantum Mechanics* (McGraw-Hill, New York, 1964).
- [36] K. Y. Kim, J. H. Glowina, A. J. Taylor, and G. Rodriguez, Terahertz emission from ultrafast ionizing air in symmetry-broken laser fields, *Opt. Express* **15**, 4577 (2007).
- [37] L.-L. Zhang, W.-M. Wang, D. Wu, R. Zhang, S.-J. Zhang, C.-L. Zhang, Y. Zhang, Z.-M. Sheng, and X.-C. Zhang, Observation of terahertz radiation via the two-color laser scheme with uncommon frequency ratios, *Phys. Rev. Lett.* **119**, 235001 (2017).
- [38] Y.-B. Wang, Z.-W. Ding, M. F. Ciappina, and X.-B. Bian, Broadband quantum terahertz wave generation in laser-induced plasmas, *Phys. Rev. A* **113**, L021501 (2026).
- [39] A. L' Huillier, K. J. Schafer, and K. C. Kulander, Theoretical aspects of intense field harmonic generation, *J. Phys. B* **24**, 3315 (1991).
- [40] E. Takahashi, V. Tosa, Y. Nabekawa, and K. Midorikawa, Experimental and theoretical analyses of a correlation between pump-pulse propagation and harmonic yield in a long-interaction medium, *Phys. Rev. A* **68**, 023808 (2003).
- [41] V. Dinu, T. Heinzl, and A. Ilderton, Infrared divergences in plane wave backgrounds, *Phys. Rev. D* **86**, 085037 (2012).
- [42] C. Figueira de Morisson Faria and A. S. Maxwell, It is all about phases: Ultrafast holographic photoelectron imaging, *Rep. Prog. Phys.* **83**, 034401 (2020).
- [43] S. Walker, A. Alqasem, A. Camacho Garibay, C. I. Blaga, A. S. Landsman, and L. F. DiMauro, Nonadiabatic strong-field photoionization revisited, *Phys. Rev. Lett.* **136**, 123203 (2026).
- [44] D. B. Milošević, Reexamination of the improved strong-field approximation: Low-energy structures in the above-threshold-ionization spectra for short-range potentials, *Phys. Rev. A* **88**, 023417 (2013).
- [45] S. Weinberg, Infrared photons and gravitons, *Phys. Rev.* **140**, B516 (1965).
- [46] S. Weinberg, Soft bremsstrahlung, *Phys. Rev. D* **99**, 076018 (2019).