

Intensity-dependent enhancements in strong-field ionization by quantum lightD. Habibović¹ and D. B. Milošević^{1,2,*}¹*Faculty of Science, University of Sarajevo, Zmaja od Bosne 35, 71000 Sarajevo, Bosnia and Herzegovina*²*Academy of Sciences and Arts of Bosnia and Herzegovina, Bistrik 7, 71000 Sarajevo, Bosnia and Herzegovina*

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We study strong-field ionization by quantum lights with emphasis on high-order above-threshold ionization and the intensity-dependent enhancements in the photoelectron spectra. We find that the length of the plateau in the photoelectron energy spectrum generated by such quantum lights can be extended by an order of magnitude in comparison with that generated by the classical coherent laser light and that within this plateau resonantly enhanced groups of sharp peaks appear at energies which are integer multiple of the photon energy. We relate the observed intensity-dependent enhancements to the channel closing effect. Our results are particularly interesting in the context of the recent interplay and merging of quantum optics with strong-field physics and attoscience.

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Introduction. In strong-field physics [1–16] and attoscience [17–25] the electrons are driven by the laser field, which is treated as classical, i.e., the influence of the quantum nature of light [26–29] is disregarded. The most interesting strong-field processes, such as high-order harmonic generation (HHG) [30,31], above-threshold ionization [32], and high-order above-threshold ionization (HATI) [33], were analyzed using semiclassical theories in which the laser field is classical and the interaction with the matter in ionization and recombination (HHG) or rescattering (HATI) processes is quantum mechanical. Furthermore, since the driving laser field is strong, the strong-field approximation (SFA) (for direct ionization it was introduced in [34–37], while for HHG and HATI see [38] and [39,40], respectively) appeared to be very useful for quantitative analysis and intuitive understanding of the strong-field processes [41].

Until relatively recently, all sources of light with high intensities and short pulse durations were classical laser fields, which are described by the Glauber coherent states in quantum optics [26–29]. However, nowadays quantum light sources with high intensities are experimentally accessible and numerous novel phenomena were discovered so that we are now witnessing an interplay of strong-field physics and attoscience with quantum optics [42–66]. For example, it was theoretically demonstrated that strong-field ionization is accompanied by the emission of radiation [50,67,68]. It was also predicted that quantum light influences the dynamical properties of the driven electron [69,70] with extension of the cutoff photoelectron energy [71,72].

In the present work we formulate the quantum strong-field ionization theory and explore how the quantum effects of light change the strong-field ionization and HATI. We consider how the differential ionization rate is changed if the classical coherent laser field is replaced with the bright squeezed vacuum (BSV), which is a quantum state of light (for the history of squeezed light generation see [73]; see also recent experiments in [52,60]). These processes have already been studied using various approaches [63,67,71,72]. In our theory the influence of the quantum nature of light is taken into account by integrating the product of the differential ionization probability and the corresponding quasiprobability distribution over the coherent-state electric-field amplitude.

We are particularly interested in the intensity-dependent enhancements in HATI. These resonantlike enhancements were discovered in experiments with inert gases [74,75]. The observed photoelectron spectrum consisted of narrow individual peaks whose energy positions do not shift with the change of the laser-field intensity. Subsequent experiments confirmed these results [76–78], even for molecular systems [79]. For more information about these enhancements, which are quantum mechanical in nature and cannot be explained classically, see the review articles in [2,13]. The intensity-dependent enhancements also appear for other strong-field processes such as HHG [80] and nonsequential double ionization. In general, studying such enhancements is important in the context of threshold anomalies in scattering theory and reactions [81–83]. For HATI the intensity-dependent enhancements fade away for few-cycle driving pulses [13]. These enhancements appear for the laser intensities such that the following resonance condition is fulfilled (in atomic system of units):

$$I_p + U_{p,c} = n_c \omega. \quad (1)$$

Here I_p is the ionization potential, $U_p = I/(4\omega^2)$ is the electron ponderomotive energy in the laser field of intensity $I = E_0^2$ (E_0 is the electric-field amplitude) and angular frequency ω , and $U_{p,c}$ is the critical value of the ponderomotive energy

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corresponding to the intensity I_c and the integer number n_c . The kinetic energy of an electron generated by ionization, after it has left the field, is $E_p = p^2/2 = n\omega - I_p - U_p$, $p = |\mathbf{p}|$, where $\mathbf{p}$ is the electron momentum and n is the number of absorbed photons. With the increase of the laser intensity, at some intensity I_c , absorption of n_c photons no longer suffices for ionization, and $n_c + 1$ photons are required. In this case we say that the n_c -photon channel has closed. Since this intensity is exactly given by Eq. (1), it is clear that the above-mentioned enhancements are related to the channel closing. The kinetic energy of electrons liberated by the field having this intensity is equal to an integer multiple of the photon energy. We will follow the interpretation of this effect as explained in [82]. We want to explore how the quantum light influences this effect.

Theory. We consider an atom exposed to a quantum field which is described by a superposition of coherent states $|\alpha\rangle$ and $|\beta\rangle$ in accordance with the generalized Glauber P representation $P(\alpha, \beta^*)$ in quantum optics [28,29,84,85]. The density matrix of our system is

$$\rho(t) = \int d^2\alpha d^2\beta P(\alpha, \beta^*) \rho_{\alpha\beta}(t) / \langle \beta | \alpha \rangle. \quad (2)$$

At the initial time t_0 the states of the electromagnetic field and the atom are independent so that $\rho_{\alpha\beta}(t_0) = |\alpha\rangle\langle\alpha| \psi_0(t_0) \langle\psi_0(t_0)|\langle\beta|$, where the atomic ground state is $|\psi_0(t_0)\rangle = |\psi_0\rangle e^{iI_p t_0}$, with ψ_0 the eigenstate of the atomic Hamiltonian $H_A = \hat{\mathbf{p}}^2/2 + V(\mathbf{r})$, $H_A \psi_0 = -I_p \psi_0$. The Hamiltonian of our system in the interaction picture is $\hat{H}(t) = H_A + \mathbf{r} \cdot \hat{\mathbf{E}}(t)$, where the electron-field interaction in the dipole approximation is described by the operator $\mathbf{r} \cdot \hat{\mathbf{E}}(t)$, with $\hat{\mathbf{E}}(t) = i\epsilon \hat{\mathbf{e}}(ae^{-i\omega t} - a^\dagger e^{i\omega t})$, $\epsilon = \sqrt{2\pi\omega/\mathcal{V}}$, and $\mathcal{V}$ the quantization volume. For simplicity we consider a single mode of angular frequency ω with the photon creation (annihilation) operator $a^\dagger$ (a), and linear polarization with the unit vector $\hat{\mathbf{e}}$.

We introduce the displacement operator $D(\alpha) = e^{\alpha a^\dagger - \alpha^* a}$, which connects the coherent state $|\alpha\rangle$ and the vacuum state $|0\rangle$, $|\alpha\rangle = D(\alpha)|0\rangle$, and which satisfies the relations $D^\dagger(\alpha)aD(\alpha) = a + \alpha$ and $D^\dagger(\alpha)a^\dagger D(\alpha) = a^\dagger + \alpha^*$. With this operator we perform a unitary transformation that shifts the initial coherent state to the vacuum state so that the Schrödinger equation which corresponds to the Hamiltonian $\hat{H}$ is transformed to [50,63]

$$i \frac{\partial |\Psi_\alpha(t)\rangle}{\partial t} = \hat{H}_\alpha(t) |\Psi_\alpha(t)\rangle, \quad (3)$$

$$\hat{H}_\alpha(t) = H_A + \mathbf{r} \cdot [\mathbf{E}_\alpha(t) + \hat{\mathbf{E}}(t)],$$

where $\mathbf{E}_\alpha(t) = \langle \alpha | \hat{\mathbf{E}}(t) | \alpha \rangle = (E_{\alpha 0}^* e^{i\omega t} - E_{\alpha 0} e^{-i\omega t}) \hat{\mathbf{e}} / (2i) = (\text{Re } E_{\alpha 0} \sin \omega t - \text{Im } E_{\alpha 0} \cos \omega t) \hat{\mathbf{e}}$, with the amplitude $E_{\alpha 0} = 2\epsilon\alpha$, is the classical part of the electric-field vector. Introducing the evolution operators $\hat{U}_\alpha(t, t_0)$ and $U_\alpha(t, t_0)$ which correspond to the Hamiltonian $\hat{H}_\alpha(t)$ and the semiclassical Hamiltonian $H_\alpha(t) = H_A + \mathbf{r} \cdot \mathbf{E}_\alpha(t)$, respectively, we obtain $|\Psi_\alpha(t)\rangle = D^\dagger(\alpha) \hat{U}_\alpha(t, t_0) |\alpha\rangle |\psi_0(t_0)\rangle$ and $|\Psi_\alpha(t_0)\rangle = |0\rangle |\psi_0(t_0)\rangle$ (we use the notation with the capital Ψ for states and the hat symbol for operators if they contain both the quantum field and atomic part; otherwise we use the notation with ψ and omit the hat). The matrix $\rho_{\alpha\beta}$, in terms of the states Ψ_α, β , takes the form $\rho_{\alpha\beta}(t) = D(\alpha) |\Psi_\alpha(t)\rangle \langle \Psi_\beta(t) | D^\dagger(\beta) =$

$\hat{U}_\alpha(t, t_0) |\alpha\rangle |\psi_0(t_0)\rangle \langle \psi_0(t_0) | \langle \beta | \hat{U}_\beta(t_0, t)$. The time evolution operators satisfy the relation $\hat{U}_\alpha(t, t_0) = U_\alpha(t, t_0) \mathcal{T} \exp[-i \int_{t_0}^t d\tau U_\alpha(t_0, \tau) \mathbf{r} \cdot \hat{\mathbf{E}}(\tau) U_\alpha(\tau, t_0)]$, with $\mathcal{T}$ the time-ordering operator. In the ionization experiment we do not measure the light field, so we trace it out and obtain the reduced density matrix $\text{Tr}_F \rho_{\alpha\beta}(t)$. If we add (or remove) one photon from a coherent state, the state hardly changes, so we can neglect the interaction $\mathbf{r} \cdot \hat{\mathbf{E}}(\tau)$ with the quantized field during the ionization process. In this case, the role of the quantum field is contained only in the quasiprobability distribution and we get $\text{Tr}_F \rho(t) = \int d^2\alpha d^2\beta P(\alpha, \beta^*) |\psi_\alpha(t)\rangle \langle \psi_\beta(t)|$. Therefore, the probability of observing the electron with the momentum $\mathbf{p}$ in the process in which the interaction with the quantum field $\hat{\mathbf{E}}(\tau)$ is neglected and the light field is traced out since it is not measured in the experiment is $W(\mathbf{p}) = p \lim_{t \rightarrow \infty} \text{Tr} [|\psi_{\mathbf{p}}(t)\rangle \langle \psi_{\mathbf{p}}(t) | \rho(t)] = p \lim_{t \rightarrow \infty} \langle \psi_{\mathbf{p}}(t) | \text{Tr}_F [\rho(t)] | \psi_{\mathbf{p}}(t) \rangle$, where $|\psi_{\mathbf{p}}(t)\rangle = |\mathbf{p}\rangle e^{-iE_{\mathbf{p}} t}$ is the state of the free electron having the momentum $\mathbf{p}$ and the energy $E_{\mathbf{p}}$. Defining the ionization probability amplitude by [2,5] $M_\alpha(\mathbf{p}) = \lim_{t \rightarrow \infty} \langle \psi_{\mathbf{p}}(t) | \psi_\alpha(t) \rangle = \lim_{t \rightarrow \infty, t_0 \rightarrow -\infty} \langle \psi_{\mathbf{p}}(t) | U_\alpha(t, t_0) | \psi_0(t_0) \rangle$, we obtain our final result for the differential ionization probability

$$W(\mathbf{p}) = p \int d^2\alpha d^2\beta P(\alpha, \beta^*) M_\alpha(\mathbf{p}) M_\beta^*(\mathbf{p}). \quad (4)$$

The generalized P representation can be expressed as [84,85]

$$P(\alpha, \beta^*) = \frac{e^{-|\alpha - \beta|^2/4}}{4\pi} Q(\gamma), \quad \gamma = (\alpha + \beta)/2, \quad (5)$$

where the Husimi-Kano Q function is $Q(\alpha) = 4\pi P(\alpha, \alpha^*)$ [29]. For $\epsilon \rightarrow 0$ ($\mathcal{V} = L^3 \rightarrow \infty$) and $\alpha \rightarrow \infty$ the field amplitude $E_{\alpha 0} = 2\epsilon\alpha$ remains constant. Then, using the relation $\lim_{\epsilon \rightarrow 0} e^{-|E_{\alpha 0} - E_{\beta 0}|^2/(16\epsilon^2)}/(16\pi\epsilon^2) = \delta^{(2)}(E_{\alpha 0} - E_{\beta 0})$, we get

$$P(\alpha, \beta^*) d^2\alpha d^2\beta = Q(\alpha) \delta^{(2)}(E_{\alpha 0} - E_{\beta 0}) d^2\alpha d^2\beta. \quad (6)$$

Introducing this into Eq. (4), we finally obtain

$$W(\mathbf{p}) = p \int d^2\alpha Q(\alpha) |M_\alpha(\mathbf{p})|^2. \quad (7)$$

Therefore, the differential ionization probability for ionization by arbitrary quantum light is expressed as an integral over the coherent field amplitude $E_{\alpha 0} = 2\epsilon\alpha$ with the subintegral function given by the product of the Q function and the differential ionization probability calculated for the coherent field with the amplitude $E_{\alpha 0}$. Note that this result is in accordance with that of Refs. [63,72], but is different from that of Ref. [71], where the subintegral function contains the probability amplitude. We calculate the differential ionization probability $p|M_\alpha(\mathbf{p})|^2$ for the coherent field using our improved SFA, which includes both the direct and rescattered electrons [82]. It should be mentioned that Eq. (7) is a general result and that this probability can be calculated using other methods [86], for example, the Coulomb quantum-orbit strong-field approximation [15,87] or the solutions of the time-dependent Schrödinger equation [88,89].

The Husimi-Kano Q function for a different type (state) of light is given in [28,29] (see also, for example, the Supplemental Material in [47,49]). For the coherent state it reduces

to the δ function since $d^2\alpha Q_{\text{coh}}(\alpha) = d^2E_{\alpha 0}\delta^{(2)}(E_0 - E_{\alpha 0})$. For the BSV light we have $|E_0|\sqrt{2\pi}d^2\alpha Q_{\text{BSV}}(\alpha) \rightarrow d(\text{Re } E_{\alpha 0})d(\text{Im } E_{\alpha 0})\delta(\text{Re } E_{\alpha 0})\exp[-(\text{Im } E_{\alpha 0})^2/(2|E_0|^2)]$, where the field intensity is $|E_0|^2 = 8\pi\bar{n}_{\text{ph}}/\mathcal{V}$, with $\bar{n}_{\text{ph}}$ the average number of photons in the state. The δ function cancels the integral over $\text{Re } E_{\alpha 0}$, while the integral over $\text{Im } E_{\alpha 0}$ can be simply denoted by the integral over the new real variable: $\int_{-\infty}^{\infty} d(\text{Im } E_{\alpha 0}) \rightarrow 2 \int_0^{\infty} dE_{\alpha 0}$, $E_{\alpha 0} = |E_{\alpha 0}|$.

For a $T = 2\pi/\omega$ periodic field the ionization probability amplitude takes the form [3]

$$M_{\alpha}(\mathbf{p}) = -2\pi i \sum_n \delta(E_{\mathbf{p}} + U_{p\alpha} + I_p - n\omega) T_{p\alpha}(n), \quad (8)$$

where $U_{p\alpha} = |E_{\alpha 0}|^2/(4\omega^2)$ is the ponderomotive energy which corresponds to the field with the amplitude $E_{\alpha 0}$, n is the number of absorbed photons, the δ function expresses the energy conservation, and the T -matrix element $T_{p\alpha}(n)$ is calculated using the improved strong-field approximation [2,3,82]. The differential ionization rate (probability per unit time) is then

$$w(\mathbf{p}) = W(\mathbf{p})/\tau = 2\pi p \sum_n \int d^2\alpha Q(\alpha) \times \delta(E_{\mathbf{p}} + U_{p\alpha} + I_p - n\omega) |T_{p\alpha}(n)|^2, \quad (9)$$

where we used the relation $2\pi\delta(E_{\mathbf{p}} + U_{p\alpha} + I_p - n'\omega) = \lim_{\tau \rightarrow \infty} \int_{-\tau/2}^{\tau/2} dt e^{i(E_{\mathbf{p}} + U_{p\alpha} + I_p - n'\omega)t} = \tau$ for $n'\omega = E_{\mathbf{p}} + U_{p\alpha} + I_p$.

For a coherent state the δ function $\delta^{(2)}(E_0 - E_{\alpha 0})$ cancels the integral over $d^2E_{\alpha 0}$ and we obtain

$$w_{\text{coh}}(\mathbf{p}) = 2\pi p \sum_n \delta(E_{\mathbf{p}} + U_{p0} + I_p - n\omega) |T_{p\alpha}(n)|^2, \quad (10)$$

so only the discrete values of energy $E_{\mathbf{p}} = n\omega - U_{p0} - I_p > 0$, $U_{p0} = |E_0|^2/(4\omega^2)$, are allowed and n is an integer (see also Ref. [90]). For the BSV state we use the δ function $\delta(g(E_{\alpha 0}))$, with $g(E_{\alpha 0}) = (E_{\alpha 0}^2 - E_{\alpha 0,0}^2)/(4\omega^2)$ and $E_{\alpha 0,0} = 2\omega\sqrt{n\omega - E_{\mathbf{p}} - I_p}$, to cancel the one-dimensional integral over $dE_{\alpha 0}$. We get

$$w_{\text{BSV}}(\mathbf{p}) = 2\pi p \sum_{n \geq n_0} \frac{4\omega^2 Q_{\text{BSV}}(E_{\alpha 0})}{E_{\alpha 0}} |T_{p\alpha}(n)|_{E_{\alpha 0}=E_{\alpha 0,0}}^2, \quad (11)$$

where $Q_{\text{BSV}}(E_{\alpha 0}) = \exp[-E_{\alpha 0}^2/(2|E_0|^2)]/(|E_0|\sqrt{2\pi})$ [28,29,47,49]. For a fixed value of $E_{\mathbf{p}}$ the summation is over $n \geq n_0 = \text{int}[(E_{\mathbf{p}} + I_p)/\omega] + 1$.

Numerical results. We first illustrate our theoretical results using the example of the Ne atom having the p ground state and the ionization potential $I_p = 21.56$ eV. The employed laser field is linearly polarized with a wavelength of 800 nm and intensity $E_0^2 = 10^{14}$ W/cm². This corresponds to the coherent light, while the BSV quantum light can reach much higher intensities $E_{\alpha 0}^2$ (the intensity E_0^2 is an average over the BSV distribution). For high intensities $E_{\alpha 0}^2$ the ground-state depletion effect becomes important since all atoms in the laser focus are already ionized, so they do not contribute to the process. The depletion effect can be taken into account by multiplying the factor $|T_{p\alpha}(n)|^2$ in Eq. (11) with the survival

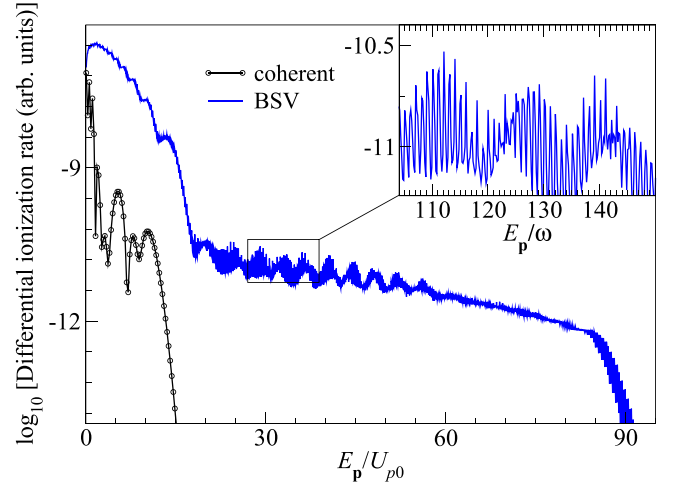


FIG. 1. Differential ionization rate for the Ne atom exposed to the classical laser coherent light (black curve with circles) and the quantum BSV light (blue curve; this curve with the abscissa in units of $E_{\mathbf{p}}/\omega$ is also shown in the inset) as a function of the photoelectron energy in units of the ponderomotive energy U_{p0} . The laser wavelength is 800 nm and the intensity is $E_0^2 = 10^{14}$ W/cm².

probability (see Sec. 2.4 in [91]). Since the survival probability depends on the experimental characteristics of the quantum light (such as pulse duration and shape) we will use a simpler method in which we put an upper limit on the quantum light intensity, i.e., we assume that $E_{\alpha 0}^2 < I_{\text{sat}}$, where the saturation intensity is determined by the barrier-suppression ionization model [92], which gives $I_{\text{BSI}}(10^9 \text{ W/cm}^2) = 4[I_p(\text{eV})]^4$. For the Ne atom we obtain $I_{\text{sat}} = I_{\text{BSI}} = 8.64 \times 10^{14}$ W/cm². The experimental value of the saturation intensity from [93] for a 620-nm 100-fs laser pulse is 8.3×10^{14} W/cm² (see also Table 1 in [94]). The cutoff of our spectrum is at $10U_{p\alpha}^{\text{sat}}$, $U_{p\alpha}^{\text{sat}} = I_{\text{sat}}/(4\omega^2)$. We have also found that the positions of the intensity-dependent enhancements we are exploring do not depend on I_{sat} . Therefore, we decided to use in this work the simpler model with I_{sat} [95].

In Fig. 1 we present the differential ionization rate for emission in the polarization direction of an electron with the energy $E_{\mathbf{p}}$, expressed in units of the ponderomotive energy $U_{p0} = E_0^2/(4\omega^2)$. For the coherent light the cutoff of the spectrum is at the expected value $10U_{p0}$. However, for the BSV quantum light the spectrum is extended to much higher energy values. More precisely, the cutoff is at $10U_{p0}I_{\text{sat}}/E_0^2 = 86.4U_{p0}$ [96]. The direct part of the spectrum, characterized by the four order of magnitude higher rate than that of the rescattered electrons, is also extended to much higher values than those of the coherent light (the coherent light cutoff is at $2U_{p0}$).

Particularly interesting are groups of enhanced peaks which appear in the rescattering plateau region. These sharp peaks appear at the energies $E_{\mathbf{p}} = k\omega$, with k an integer, and are particularly pronounced for $20U_{p0} < E_{\mathbf{p}} < 58U_{p0}$. They are caused by the channel closing mechanism, which was explained in the Introduction. The intensity-dependent enhancements are quantum-mechanical effects which cannot be explained classically. They appear at the resonant intensities for which the condition $I_p + U_{p\alpha} = n_c\omega$, with n_c an integer,

is fulfilled. Various explanations of this effect were offered (see references in [82]), but we mention the following two. If the electron drift momentum at the rescattering time is small, a large number of quantum orbits interfere constructively at the channel closings, leading to the resonant enhancement. The second interpretation, complementary to the first one, is based on the threshold anomalies which appear in the time-dependent scattering theory and reactions [81]. In [82] we expressed the rescattering part of the T -matrix element as a sum over the integer number n' and showed that it can be written as $T_{p\alpha}^{(1)}(n) \approx R_{p\alpha}(n) + A_p(n, n'_c) \sqrt{n'_c - n_c}$, where from all values of n' we explicitly separated the term $n' = n'_c$, with n'_c the integer nearest n_c . If we consider $T_{p\alpha}^{(1)}(n)$ as an analytical function of the parameter n_c we see that the nonanalyticity (and the threshold anomaly) appears for $n_c = n'_c$. It is shown in [82] that the derivatives of $|T_{p\alpha}^{(1)}(n)|^2$ with respect to n_c around the n'_c channel closing exhibit an upward cusp (this is the so-called type-II enhancement) or a rounded downward step (type-I enhancement). The difference of the quantum BSV light from the classical coherent light is that much higher intensities $E_{\alpha 0}^2$ can be achieved. Since it can be $E_{\alpha 0}^2 \gg E_0^2$, many more channel closings are available. For example, for the parameters of Fig. 1 for quantum light we have $n_c = 14, 15, \dots, 46, 47$, while for the classical coherent light this number is much lower and limited by $\text{int}[(I_p + U_{p0})/\omega] = 17$. This is manifested by the appearance of a much larger number of intensity-enhanced groups of peaks in Fig. 1 for quantum light.

In order to better illustrate this, in Fig. 2 we present (in false colors with a logarithmic color bar on the right) the quantity $2\pi p Q_{\text{BSV}}(E_{\alpha 0}) |T_{p\alpha}(n)|^2$, which appears in Eq. (9), in the plane with the abscissa $(I_p + U_{p\alpha})/\omega$ and the ordinate E_p/U_{p0} . We see that many channel closing enhancements appear exactly at or slightly below the integer values $n_c = \text{int}[(I_p + U_{p\alpha})/\omega]$. The sharp type-II enhancements appear exactly at n_c , while the broader type-I enhancements appear slightly before n_c .

In Fig. 3 we present the photoelectron momentum distribution, i.e., the differential ionization rate presented in false colors in the $p_x p_y$ plane, for the same laser parameter as in Fig. 2. We see that the cutoffs, for both direct electrons [Fig. 3(a)] and rescattered electrons [Fig. 3(b)], for the BSV quantum light extend to much higher energies than those of the classical coherent laser light (which are $p_{x,\text{max}}^{\text{dir}} = \sqrt{4U_{p0}} \approx 0.937$ a.u. and $p_{x,\text{max}}^{\text{res}} = \sqrt{20U_{p0}} = 2.096$ a.u.). In the rescattered photoelectron momentum distribution one can notice typical off-axis low-energy structures in the form of a fork [97].

Finally, in order to show that our conclusions are valid for other atomic and laser-field parameters, in Fig. 4 we present the results for the Xe atom ($I_p = 12.13$ eV) and for a laser wavelength of 1600 nm and intensity $E_0^2 = 10^{13}$ W/cm². We see that, in spite of the ten times lower laser intensity, the results shown in Fig. 4 are qualitatively similar to that of Fig. 1. In addition to the results obtained using the model with I_{sat} (blue BSV curve), we present the results obtained using a more precise model, which takes into account the contributions for intensities above saturation by multiplying $|T_{p\alpha}(n)|^2$ in Eq. (11) with the atom survival probability which depends on the pulse length [91] (for the pulse length of 25

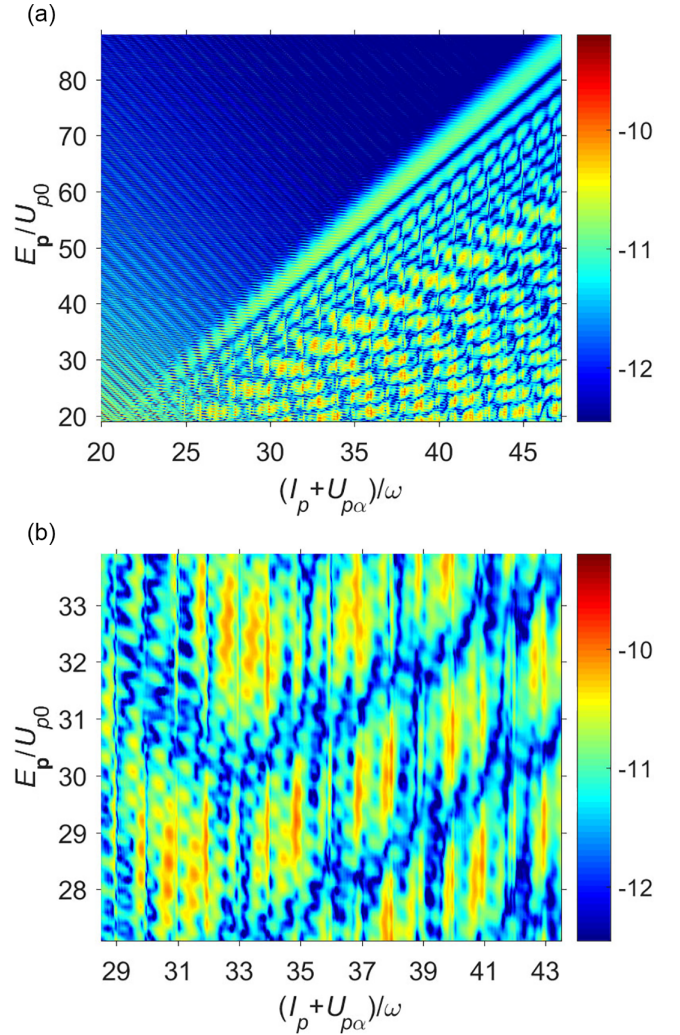


FIG. 2. Differential ionization rate multiplied by the Q function for the BSV light presented using the logarithmic false color scale (color bar on the right) as a function of the parameters $(I_p + U_{p\alpha})/\omega$ (abscissa) and E_p/U_{p0} (ordinate) for the same parameters as in Fig. 1. Shown in (b) is an enlarged part of (a).

fs the results are presented by the BSV-S red line, where S stands for survival; the inset shows the survival probability as a function of the intensity for two different pulse lengths).

Summary and outlook. In order to consider HATI by the quantum BSV light, we presented this light in the form of a superposition of coherent states. In this way we expressed the corresponding differential ionization probability as an integral over the coherent state amplitude $E_{\alpha 0}$ of a product of the Husimi-Kano Q function and the differential ionization probability for coherent light with the intensity $E_{\alpha 0}^2$. Since we could easily calculate the Q function and the differential ionization probability for laser light (for example, using our improved strong-field approximation), we were able to calculate the differential ionization probability for an arbitrary quantum light. For a periodic field, we expressed the ionization amplitude by Eq. (8) and used the δ function to cancel the integral over the amplitude $E_{\alpha 0}$. In this way the calculation was simplified and reduced to a sum over the number of exchanged photons of

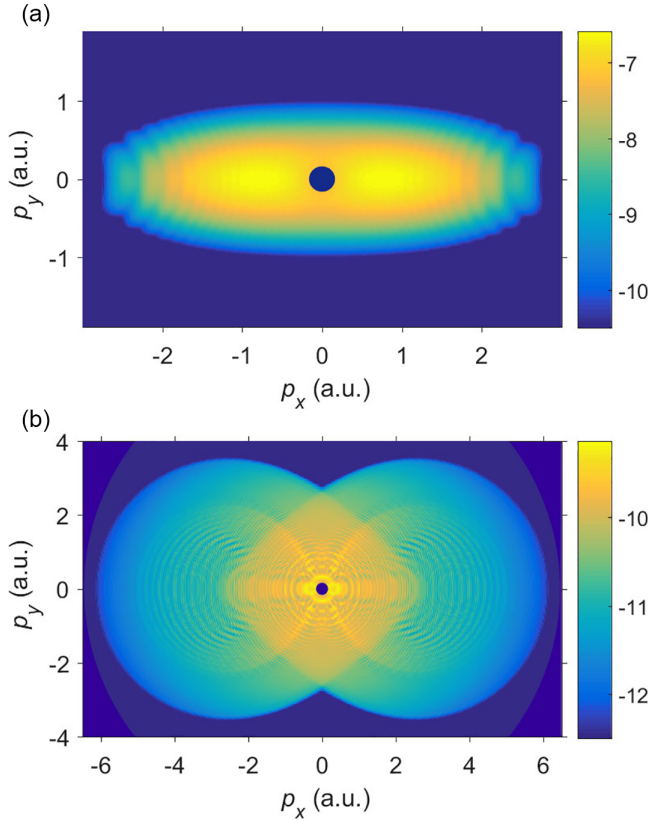


FIG. 3. Photoelectron momentum distributions for ionization of the Ne atom by the BSV quantum light with the same field parameters as in Figs. 1 and 2: (a) direct electrons and (b) rescattered electrons.

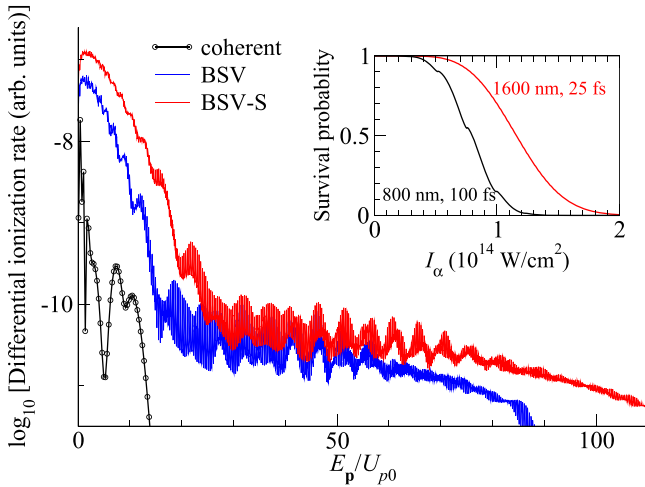


FIG. 4. Differential ionization rate for the Xe atom exposed to the laser field with a wavelength of 1600 nm and intensity $E_0^2 = 10^{13}$ W/cm² as a function of the photoelectron energy in units of the ponderomotive energy U_{p0} . The results are obtained for classical laser coherent light (black curve with circles) and the quantum BSV light [blue curve shows the model with the saturation intensity $I_{\text{sat}} = 8.66 \times 10^{13}$ W/cm² and red BSV-S curve the results obtained calculating the atom survival probability for each intensity using the method from [91] for a pulse length of 25 fs (in the inset the survival probability as a function of the intensity is shown also for a pulse length of 100 fs and a wavelength of 800 nm)].

the product of the Q function and the differential ionization rate for the coherent laser light.

The BSV quantum light, with the average intensity E_0^2 equal to the intensity of classical coherent laser light, can reach much higher intensities $E_{\alpha 0}^2$. Therefore, many more photons can be absorbed and the photoelectrons can reach much higher energies. More precisely, the cutoff of the spectrum appears at the energy above $10U_{p0}I_{\text{sat}}/E_0^2$, where the saturation intensity I_{sat} can be much higher than the laser intensity E_0^2 . Since the intensity $E_{\alpha 0}^2$ is high, the channel closing condition (1) is satisfied for many more values of n_c , i.e., many more channel closings contribute to the process (in our example it is $n_c = 17$ for coherent light, while for the quantum light the whole interval from $n_c = 14$ to $n_c = 47$ contributes to the process). Therefore, we have many more groups of enhanced peaks separated by the photon energies (see the interval $20U_{p0} < E_p < 58U_{p0}$ in Fig. 1). We have also shown that the type-I (type-II) intensity-dependent enhancements appear for integer values of n_c in the form of rounded downward steps (upward cusps) (compare Fig. 2). These enhancements are related to the threshold anomalies of the time-dependent scattering theory and reactions.

In conclusion, we have formulated a theory of HATI by quantum light and have shown that the length of the plateau in the photoelectron energy spectrum generated by such light can be extended by an order of magnitude in comparison with that generated by the classical coherent laser light. We also showed that within this plateau, resonantly enhanced groups of sharp peaks appear at the energy $E_p = k\omega$, with k an integer. We explained these enhancements using the channel closing effect. Our results are important in the light of recent applications of quantum-optics technologies to strong-field physics and attoscience. For example, very recently in the experiment in [98] the photoemission of electrons from a metal needle tip driven by quantum light was observed. The analysis presented in [98] shows that the BSV-driven electrons behave as if driven by an ensemble of coherent states of light. These results are in agreement with our findings. The difference is that for strong-field ionization of the metal needle tip the saturation arises from nonlinear emission dynamics, while in our case the saturation occurs due to the atomic depletion effect. Another difference is that optical field enhancement at the tip apex enhances the intensity in the focus (by a factor 3.4 in the experiment in [98]). As concerns future experiments, it will be interesting to explore channel closing effects for tailored fields [99]. Recent studies have demonstrated that all four types of threshold anomalies [81] appear for HATI by an elliptically polarized field [100]. It will also be important to explore how these enhancements manifest in molecular systems, where additional degrees of freedom may lead to qualitatively new features. Finally, future experiments with quantum light sources should provide a realistic avenue for testing our predictions and exploring their implications for quantum-enhanced attosecond science.

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- [96] This cutoff is sharp since we cut the integral over the quantum field strength E_{a0} at a fixed value determined by I_{sat} . (The survival probability is suddenly set to zero for $I > I_{\text{sat}}$. On the other hand, for the red line in Fig. 4, the results are obtained using a different method, which takes into account the atom survival probability for $I > I_{\text{sat}}$ and leads to a smooth decrease of the rate.)
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