

Geometric and resource-theoretic characterization of nonstabilizerness in quantum algorithms

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While there is strong evidence for advantages of quantum over classical computation, the repertoire of computational primitives with proven or conjectured quantum advantage remains limited. A big challenge of quantum algorithmic design is a still incomplete understanding of the sources of quantum computational power. Advancing toward systematic quantum advantage calls for a better understanding of the efficient use of nonclassical resources like nonstabilizer states. We present an approach to track nonclassical contributions in the form of nonstabilizerness across various algorithms by pairing resource theory of nonstabilizer entropies with the geometry of quantum state evolution, and introduce permutation agnostic distance measures that reveal and quantify nonstabilizer effects previously hidden by a subset of Clifford operations. We find different efficiency in the use of nonstabilizerness for structured and unstructured variational approaches and show that greater freedom for classical optimization in quantum-classical methods increases unnecessary nonstabilizer consumption. Our results open alternative means of analyzing the efficient utilization of quantum resources and contribute toward the targeted construction of algorithmic quantum advantage.

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I. INTRODUCTION

Contrary to the general discussion of quantum versus classical computing that often treats these as separate computational models, quantum computing (QC) can also be seen as an extension to the classical computational model that adds alternative primitives and resources. Quantum computations can (and for many suggested approaches also do, particularly for any variational Ansatz) contain classical parts [1–3], which shifts the question of separating the two models to a more nuanced approach of identifying inherently quantum parts in computations. While possible speedups over purely classical approaches must obviously originate from quantum parts of a computation, not every quantum subcomputation in proposed algorithms necessarily needs to positively contribute to overall solution finding. Identifying reasons for and structure of quantum speedups is a crucial question to improve the understanding of chances and limitations of quantum computation. In this paper, we address this question from a unique point of view by using geometrical distance arguments within a solution space.

Several measures for quantumness have been established; the amount of entanglement that manifests in computations is a prime candidate. It has been studied extensively from the early times of quantum computing [4] into the present, drawing substantial interest during the last few decades [5–8]. Entanglement is a distinct, perhaps the most nonclassical aspect of quantum mechanics, and is considered one of the fundamental resources of QC [9]. However, its effect on computational power is not easy to characterize from a computer science point of view. It is generally acknowledged and understood that entanglement plays a fundamental role in many quantum algorithms and protocols, at least when using the appropriate amount [10–14]. Trying to pinpoint exactly where and how such nonclassical advantage is exploited necessitates more fine-grained insights. In particular, it is well-known that not all forms of entanglement are equal (or equally useful) [15]. Even maximally entangled states like the seminal Greenberger-Horne-Zeilinger (GHZ) state can be prepared by Clifford circuits; it is known that these can be efficiently simulated by a classical computer [16]. States within the orbit of the Clifford group are called stabilizer states (STABs). Conversely, states outside of STABs are referred to as nonstabilizer state (examples of non-STAB entangled states include W states with three or more qubits) [17]. Circuits required for their preparation are believed to be classically hard to simulate [16,18–20].

Stabilizer Rényi entropies (SREs) have been recently introduced to entropically measure nonstabilizerness, also referred to as magic, of quantum states [21]. We adopt SRE as a measure of intermediate states to locate how and where

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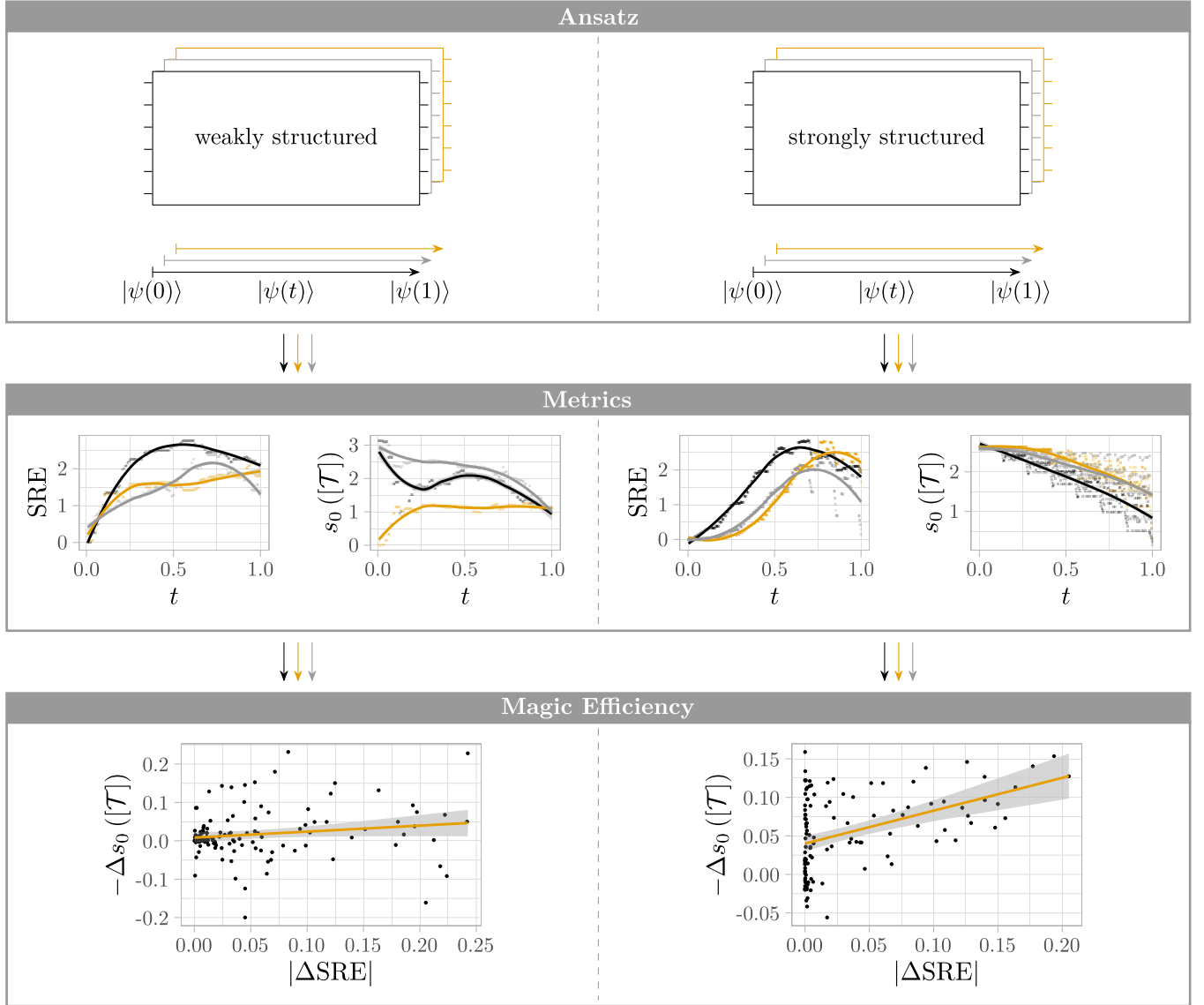


FIG. 1. Overview of our main methodology and contributions. In this paper, we compare the state evolution $|\psi(t)\rangle$ (as $0 \mapsto 1$) throughout the circuit of a weakly and a strongly structured Ansatz. Along this state evolution, we compute two metrics, the first being the stabilizer Rényi entropy (SRE) and the second a geodesic distance from the target space invariant under qubit permutations. As can be seen in the middle pane, both metrics behave much more uniformly in the strongly structured state evolution compared to the weakly structured one. In a final step (bottom panel), we compare the magic consumption $|\Delta \text{SRE}|$ with the resulting geometric approach to the target $-\Delta s_0([\mathcal{T}])$. Doing so, we can show that the strongly structured state evolution has a higher magic efficiency than the weakly structured one.

nonclassical effects appear during the execution of contemporary quantum algorithms.

The structure of the paper is as follows: In Sec. II, we review the history and significance of nonstabilizer resource theory and measures, and particular stabilizer Rényi entropies. This is followed by Sec. III, where we provide an introduction into the relevant definitions and characteristics of stabilizer Rényi entropies. We then present a geometric perspective in Sec. IV and show how to calculate geodesic distances to target spaces by means of taking the expectation value of a special problem Hamiltonian. We then provide intuition for qubit permutation invariant structures throughout the state evolution, from which we derive an extension of our tools to work

under invariance of the qubit order and show how this enables us to reveal previously overlooked nonstabilizer effects. After that, we put our theoretic framework to use in Sec. V to analyze the differences of intermediate nonstabilizer consumption in structured and unstructured state evolutions. By combining the quantum resource theoretic SRE measures with a geometric perspective, we are able to qualify the efficiency of nonstabilizer consumption. We observed a significantly higher efficiency for the structured evolution than for the unstructured case. We conclude and discuss the potential of combining resource theoretic tools with geometric perspectives in Secs. VI and VII. For a better overview our main methodology and contributions are summarized in Fig. 1.

II. RELATED WORK

Given the significance of nonstabilizer effects in quantum computing, it is no surprise that understanding their properties and effects has been considered in numerous contexts. In [18], Gottesman presented the stabilizer formalism in his Ph.D. thesis, laying the foundations of quantum error correction protocols. This formalism already covers many quantum specific phenomena like Greenberger-Horne-Zeilinger (GHZ) entanglement. The seminal Gottesman-Knill theorem [16], presented shortly after, showed a quantum-classical separation by stating that every stabilizer circuit can be efficiently simulated by a classical computer. A stabilizer circuit is restricted to using gates from the Clifford group. An interesting conflict arises as even stabilizer circuits are able to harness some quantum effects, yet they can be efficiently simulated. This means the threshold to achieve quantum speedups must be somewhere behind the obvious first line drawn between classical and nonclassical physics. For Clifford circuits, universality can be recovered by an injection process [19] where magic states—nonstabilizer ancillas—serve as consumable resources restoring universality by interfering with the Clifford part of the circuit. Consequently, reaching a quantum advantage must be affected by magic state consumption. This motivated the recent development of a resource theory for nonstabilizerness [22], producing a diverse set of nonstabilizerness measures including stabilizer rank [23], stabilizer fidelity [24], or stabilizer nullity [20]. Following up on these results, more abstract characterizations of measures like nonstabilizer monotones have been defined [25]. The most relevant for our work are SREs, as introduced in [21], and it has been hypothesized that tracking magic state consumption during temporal evolution [26] might be a measure for algorithmic quantumness. Stabilizer Rényi entropies are also known to be monotones for nonstabilizerness resource theory [27]. While most measures of nonstabilizerness are hard to compute, SRE can be efficiently determined for low entanglement systems and can be represented as matrix product states [28]. Further, SREs can also be determined empirically through measurements [29].

III. NONSTABILIZERNESS

Before defining a measure for nonstabilizerness, it seems pertinent to take a brief moment to discuss the term nonstabilizerness. We already mentioned that the STAB is given by the orbit of the Clifford group (recall that the orbit of an element x in a group G is given by $G(x) := \{gx \in G : g \in G\}$). The Clifford group $\mathcal{C}_n = \{V \in \mathcal{U}_{2^n} : V\mathcal{P}_n V^\dagger = \mathcal{P}_n\}$ is the normalizer of the Pauli group $\mathcal{P}_n = \langle X, Y, Z \rangle^n$, where X, Y, Z are the Pauli operators and n denotes the number of qubits. In the following, we drop suffix n if the number of qubits can be deduced from the context or to describe systems of arbitrary (but finite) size.

Definition 1 (see Ref. [21]). The α -stabilizer Rényi entropy is defined as

$$\text{SRE}_\alpha(|\psi\rangle) = \frac{1}{1-\alpha} \ln \sum_{P \in \mathcal{P}_n / \langle \pm \mathbb{I}_n \rangle} \Xi_P^\alpha(|\psi\rangle) - \ln 2^n \quad (1)$$

$$\Xi_P(|\psi\rangle) = \frac{1}{2^n} \langle \psi | P | \psi \rangle^2. \quad (2)$$

Definition 1 may require some explanation to establish an intuitive understanding. Let us start from Eq. (2). Note that $\Xi_P(|\psi\rangle) \leq 1$ and $\sum_{P \in \mathcal{P}_n / \langle \pm \mathbb{I}_n \rangle} \Xi_P(|\psi\rangle) = 1$. Thus, $\{\Xi_P\}_{P \in \mathcal{P}_n / \langle \pm \mathbb{I}_n \rangle}$ induces a probability distribution on a state $|\psi\rangle$. Here we also immediately see why we used the factor group of $\mathcal{P}_n$, ignoring the scalar unit factors ± 1 and $\pm i$. Due to the applied square in Eq. (2), they would only double up in the sum, not contributing any valuable information to the distribution. This also explains the normalization factor of 2^{-n} corresponding to $|\mathcal{P}_n / \langle \pm \mathbb{I}_n \rangle| = 2^n$ in contrast to the perhaps expected $4^n = |\mathcal{P}_n|$. We also see, with $\{\Xi_P\}$ being a probability distribution, Eq. (1) simply defines a family of Rényi entropies offset by $\ln 2^n$.

To demonstrate and provide intuition about how stabilizer Rényi entropies work, it is worth looking into their main characteristics to understand how SREs characterize stabilizer states. For this, we will revisit the property which is most important for our work, namely, that non-STAB states are characterized by a nonzero SRE.

Let $|\psi\rangle \in \text{STAB}$ be some not further specified stabilizer state. Then $|\psi\rangle$ is in the Clifford orbit of $|0\rangle$, meaning that there exists a $U \in \mathcal{C}$ such that $U|0\rangle = |\psi\rangle$. Due to the Clifford group normalizing the Pauli group, we have $U^\dagger P_j U = P_j$ for $P_i, P_j \in \mathcal{P}_n / \langle \pm \mathbb{I}_n \rangle$. In fact, $\mathcal{C}$ is isomorphic to the group of permutations in a sense that $U^\dagger P_j U : P_j \mapsto P_{\pi(j)}$. Therefore, $\{\Xi_{P_j}(|\psi\rangle)\} = \{\Xi_{P_{\pi(j)}}(|0\rangle)\}$ and, consequently, $\sum_{P \in \mathcal{P} / \langle \pm \mathbb{I} \rangle} \Xi_P^\alpha(|\psi\rangle) = \sum_{P \in \mathcal{P} / \langle \pm \mathbb{I} \rangle} \Xi_P^\alpha(|0\rangle)$. Note that $\langle a | Z | a \rangle_{a=0,1} = \pm 1$, $\langle a | \mathbb{I} | a \rangle_{a=0,1} = 1$, and $\langle a | X, Y | a \rangle_{a=0,1} = 0$, which leads to the conclusion that

$$\langle 0 | P | 0 \rangle = \begin{cases} 0 & \text{if } \exists i : \sigma_i \in \{X, Y\} \\ 1 & \text{otherwise} \end{cases} \quad (3)$$

for all $P = \sigma_i \otimes \dots \otimes \sigma_n \in \mathcal{P}_n / \langle \pm \mathbb{I}_n \rangle$. There are 2^n many $P \in \mathcal{P}_n / \langle \pm \mathbb{I}_n \rangle$ such that $\langle 0 | P | 0 \rangle = 1$. As a result, $\sum_{P \in \mathcal{P}_n / \langle \pm \mathbb{I}_n \rangle} \Xi_P^\alpha(|\psi\rangle) = 2^n 2^{-n\alpha} = 2^{n(1-\alpha)}$. Here is where the offset of $\ln(2^n)$ in Eq. (1) comes into play, as $\text{SRE}_\alpha(|\psi\rangle) = (1-\alpha)^{-1} \ln 2^{n(1-\alpha)} - \ln 2^n = \ln 2^n - \ln 2^n$. We conclude that $\text{SRE}_\alpha(|\psi\rangle) = 0$ for all $|\psi\rangle \in \text{STAB}$.

Now we will investigate the other direction, for which we will use an alternative characterization of stabilizer states, which is that $|\psi\rangle$ is in STAB if and only if there exists a subset $S \subset \mathcal{P}_n$ such that $|S| = 2^n$ and $A|\psi\rangle = |\psi\rangle$ for all $A \in S$. Now let us assume $\text{SRE}_\alpha(|\psi\rangle) = 0$ for some arbitrary state $|\psi\rangle$. Written out, that gives us $(1-\alpha)^{-1} \ln \sum_{P \in \mathcal{P} / \langle \pm \mathbb{I} \rangle} \Xi_P^\alpha(|\psi\rangle) - \ln 2^n = 0$ or rewritten $\ln \sum_{P \in \mathcal{P} / \langle \pm \mathbb{I} \rangle} \Xi_P^\alpha(|\psi\rangle) = \ln 2^{n(1-\alpha)}$. Thus, $\ln \sum_{P \in \mathcal{P} / \langle \pm \mathbb{I} \rangle} 2^{-n\alpha} \langle \psi | P | \psi \rangle^{2\alpha} = 2^{n(1-\alpha)}$. From this, we can derive a condition on $|\psi\rangle$ for $\text{SRE}_\alpha(|\psi\rangle)$ to equal to 0:

$$f(\alpha) = 2^n, \quad (4)$$

where $f(\alpha) = a_1(\alpha) + \dots + a_{2^n}(\alpha)$ with $a_i = \langle \psi | P_i | \psi \rangle^{2\alpha}$ and $P_i \in \mathcal{P}_n / \langle \pm \mathbb{I}_n \rangle$. Due to f being a constant function, we have $\frac{d}{d\alpha} f = 0$. Additionally, we know that all $a_i \geq 0$ and therefore $\frac{d}{d\alpha} \langle \psi | P | \psi \rangle^{2\alpha} = 0$ for all $P \in \mathcal{P}_n / \langle \pm \mathbb{I}_n \rangle$. From $\frac{d}{d\alpha} \langle \psi | P | \psi \rangle^{2\alpha} = 2 \langle \psi | P | \psi \rangle^{2\alpha} \ln \langle \psi | P | \psi \rangle = 0$, we conclude that $\langle \psi | P | \psi \rangle \in \{0, 1\}$ for all $P \in \mathcal{P}_n / \langle \pm \mathbb{I}_n \rangle$. Note that $\langle \psi | P | \psi \rangle = 1$ only if $P|\psi\rangle = |\psi\rangle$ and $f(\alpha) = 2^n$. Thus, there exists a subset $S \subset \mathcal{P}_n / \langle \pm \mathbb{I}_n \rangle$ such that $|S| = 2^n$ and $A|\psi\rangle = |\psi\rangle$ for all $A \in S$; showing that $|\psi\rangle \in \text{STAB}$. We arrive at the

following key characteristic of the SRE measure proven in its introductory paper [21].

Theorem 1 (originally proven in [21]). A state $|\psi\rangle$ is in STAB if and only if $\text{SRE}_\alpha(|\psi\rangle) = 0$.

We refer readers to the original paper for the original proof and a more in-depth discussion [21]. Also note the following observation, which makes SRE such an interesting measure for identifying nonclassical computation steps in quantum circuits.

Corollary 1. Stabilizer Rényi entropies are invariant under Clifford operations.

Proof. This follows directly from the isomorphism between the Clifford group and permutations. Let $|\psi\rangle$ be an arbitrary state and $U \in \mathcal{C}$, then we get that $\{\Xi_{P_j}(U|\psi)\} = \{\Xi_{P_{\pi(j)}}(|\psi\rangle)\}$ and thus $\text{SRE}_\alpha(U|\psi) = \text{SRE}_\alpha(|\psi\rangle)$. ■

IV. GEOMETRIC PERSPECTIVE

If we take a random Haar sampled state $|\psi\rangle \sim \text{Haar}$. Then the expected SRE is $\mathbb{E}_{|\psi\rangle \sim \text{Haar}}(\text{SRE}_\alpha(|\psi\rangle)) \in \mathcal{O}(n)$ for all $\alpha \geq 2$, with overwhelming probability [30]. Additionally, SREs are linear upper bounded by $\text{SRE}_\alpha(|\psi\rangle) \leq \ln(2^n) \in \mathcal{O}(n)$ [21] (technically, more precise, stabilizer entropy scales linearly with concentration bounds). This means, although intermediate states with $\text{SRE} \geq 0$ are linked with and even necessary for quantum advantage, their occurrence is nothing special and has to be expected. Consequently, this raises the question if observed nonstabilizerness contributes to the computation or if it is merely a by-product of a suboptimal choice of unitary propagators. To make this notion more precise, we need to clarify the meaning of contributing to the computation mean: Every computational process can be interpreted as a state evolution that drives a specific initial state $|\psi_0\rangle$ to a target state $|\psi_T\rangle$ that encodes a problem solution (or a superposition thereof). Geometrically speaking, such a state evolution resembles a rotation of the state vector. The whole circuit represents one singular unitary, which in turn corresponds to a direct rotation from the initial to the final state around the rotational axis defined by said unitary. This only applies from the most top-level view and discard the actual realization of the circuit's unitary given by a concrete partitioning into quantum gates and their correct sequencing. The circuit's gate level realization induces a path of the resulting state evolution, which is most likely diverging from the shortest path at some point. In geometric terms, the shortest, most direct path of this state evolution would be characterized by the geodesic from the initial state to the target. In [31], Anandan and Aharonov presented exactly this geometric perspective in conjunction with the concept of geodesic efficiency $\mu_{\text{gd}} = s_0/s$ of a state evolution where s_0 is the geodesic distance and s the actual distance traveled. If we want to specify the distance to a specific state $|\phi\rangle$, we write $s_0(|\phi\rangle)$ and $s(|\phi\rangle)$ and if the initial state is not clear from this context, we write $s(|\psi\rangle, |\phi\rangle)$ and $s_0(|\psi\rangle, |\phi\rangle)$.

A. Problem Hamiltonian

Usually, there is more than one unique solution to a computational problem, for instance, all binary variable assignments satisfying a propositional satisfiability (SAT) problem. This

adds variability to the geometric perspective discussed above. Instead of rotating our initial state to a specific target state $|\psi_T\rangle$, a quantum algorithm has the freedom to reach any state within the target space, which is the subspace of $\mathcal{H}^{\otimes n}$ that contains all superpositions of quantum states encoding problem solutions. Thus, the geodesic distance s_0 from above needs to be reinterpreted to be the shortest geodesic distance to one of the states in the target space. In the following, we will address this by first defining the target space based on an indicator function of problem solutions and a two-level problem Hamiltonian projecting on said target space. In contrast to Hamiltonians usually found in combinatorial optimization, such two-level Hamiltonians encode solution spaces of decision problems, where the solutions are not weighted and the task is to find any valid solution. We then show that the expected value of this problem Hamiltonian corresponds to the scaled inverse geodesic distance from the initial state to the target space.

Definition 2. Let $c : \mathbb{F}_2^n \rightarrow \mathbb{F}_2$ be the solution verifier of a problem with a finite set of classical solutions $T = \{t \in \mathbb{F}_2^n : c(t) = 1\}$. We then extend $c(\cdot)$ to quantum states $|\psi\rangle = \sum_{b \in \mathbb{F}_2^n} \alpha_b |b\rangle$ as a linear function on the basis states $\{|b\rangle\}_{b \in \mathbb{F}_2^n}$:

$$c(|\psi\rangle) = \sum_{b \in \mathbb{F}_2^n} |\alpha_b|^2 c(b). \quad (5)$$

Further, we define a quantum target space:

$$\mathcal{T} = \{|t\rangle : c(|t\rangle) = 1\} \subset \mathcal{H}^{\otimes n}. \quad (6)$$

Remark 1. Note that $\mathcal{T}$ is indeed a complete subspace of $\mathcal{H}^{\otimes n}$, spanned by $\{|b\rangle : b \in \mathbb{F}_2^n, c(b) = 1\}$. Thus, $\mathcal{T}$ has a dimension of $|T|$.

Definition 3. Based on $c(|\psi\rangle)$, we define a two-level problem Hamiltonian H_c by the condition that

$$\langle H_c \rangle = c(|\psi\rangle). \quad (7)$$

H_c is a projector onto $\mathcal{T}$ that can explicitly defined by $H_c = \sum_{t \in T} |t\rangle\langle t|$.

Theorem 2. Given a target space $\mathcal{T}$ and the corresponding problem Hamiltonian H_c according to Definition 1, we have

$$s_0(\mathcal{T}) := \min_{|t\rangle \in \mathcal{T}} s_0(|t\rangle) = 2 \arccos(\langle H_c \rangle). \quad (8)$$

Proof. Let $\mathcal{B}_\mathcal{T} = \{|t\rangle : t \in T\}$ be the basis of $\mathcal{T}$. We then expand $\mathcal{B}_\mathcal{T}$ with $\mathcal{B}_{\mathcal{T}^\perp} = \{|b\rangle : b \in \mathbb{F}_2^n \setminus T\}$ such that $\mathcal{B}_\mathcal{T} \cup \mathcal{B}_{\mathcal{T}^\perp}$ forms a basis of $\mathcal{H}^{\otimes n}$. Given that expanded basis, we can write every state $|\psi\rangle \in \mathcal{H}^{\otimes n}$ as $|\psi\rangle = \sum_{i=1}^{|T|} \tau_i |t_i\rangle + \sum_{i=1}^{n-|T|} \beta_i |b_i\rangle$, with $\sum_{i=1}^{|T|} |\tau_i|^2 + \sum_{i=1}^{n-|T|} |\beta_i|^2 = 1$, and for all $|t\rangle \in \mathcal{T}$ we have $\sum_{i=1}^{|T|} |\tau_i|^2 = 1$. Now, let H_c be a problem Hamiltonian as defined in Definition 3, then $H_c = \sum_{|t\rangle \in \mathcal{B}_\mathcal{T}} |t\rangle\langle t|$ and

$$0 \leq \langle H_c \rangle = \sum_{i=1}^{|T|} |\tau_i|^2 \leq 1.$$

Now, let us take an arbitrary state $|\psi\rangle$, then $\langle H_c \rangle = \langle \psi | H_c | \psi \rangle$ is exactly the overlap between $|\psi\rangle$ and its projection onto the target space $H_c |\psi\rangle$, which satisfies

$$\langle H_c \rangle = \max_{|t\rangle \in \mathcal{T}} |\langle \psi | t \rangle|.$$

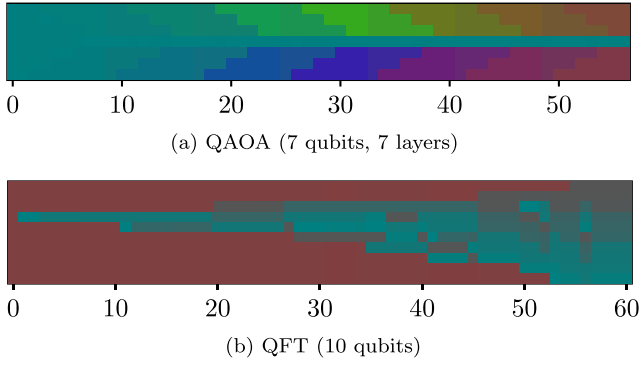


FIG. 2. Evolution of the color representation of the state in quantum circuits. Every vertical slice at $x = i$ represents the color spectrum of the state after the i th gate. The reduced one qubit density matrices are mapped to a hue-saturation-value color with $\text{hsv}(\langle P_0 \rangle, \langle P_+ \rangle, \langle P_{+i} \rangle)$. Within a vertical slice, they are sorted according to the hsv tuple. This reveals stark differences in the evolution of qubit permutation invariant structures throughout the state evolutions driven by QAOA and QFT circuits.

Now we use that $s_0(|t\rangle) = 2 \arccos |\langle \psi | t \rangle|$ [32]. Now, due to the monotonicity of arccos in $[0, 1]$, we can pull out the max from $2 \arccos \langle H_C \rangle$ to end up at Eq. (8). ■

B. Permutations

In a typical quantum circuit, qubits are sequentially numbered. This numbering implies an unsubstantiated sense of order of qubits in the quantum circuit and state vector picture. Indeed, it is actually completely arbitrary and nothing more of a naming convention. Qubits q_i and q_j could also be remapped $q_{\sigma(i)}$ and $q_{\sigma(j)}$ for some permutation $\sigma \in S_n$. Yet, geometrically the state vectors of $|0010\rangle$ and $|0001\rangle$ are orthogonal. In this section, we will extend the geometric distance measure to address this dilemma. From a computational standpoint, looking at a concrete problem instance, there is no order of variables by naming, there are relations between variables.¹ Variables are then being mapped to qubits and intervariable connections usually result in qubit interactions, i.e., multi-qubit gates. Those connections could better be represented as a graph, which in turn is isomorphic under vertex permutations. On this graph, we assign to each vertex a color based on properties of its linked qubit. By keeping track of the necessary permutations, we can, at each intermediate time, determine the permutation order of qubits based on their assigned color.

One could thus visualize a quantum state evolution as the change of a color spectrum through time. Solving a classical problem, we are basically interested in the measurement probabilities of all qubits and the resulting bitstring, hopefully encoding a possible solution to the problem. We therefore exemplarily map $\langle P_0 \rangle$, $\langle P_+ \rangle$, and $\langle P_{+i} \rangle$ to the hue, saturation, and value component of an HSV color. Here $\langle P_0 \rangle$, $\langle P_+ \rangle$, and $\langle P_{+i} \rangle$ are the probabilities of the reduced mixed system of said qubit

being in the states $|0\rangle$, $|+\rangle$, and $|+i\rangle$. Now the qubits can be ordered according to their hue. Figure 2 demonstrates how this representation, which is qubit permutation invariant, still reveals highly specific structures of quantum state evolutions {e.g. for QAOA [Fig. 2(a)] vs for QFT [Fig. 2(b)]}. As we will show below, the ordering does not alter nonstabilizerness, as it can be performed by an efficient Clifford circuit. Therefore, it can be ignored regarding our analysis of nonstabilizerness resource consumption. Additionally, introducing, at the worst case, one ordering and reordering before and after each computational step does also not change the complexity theoretic characterization of the circuit, as it, given the presumption of a polynomial sized initial circuit, only adds a polynomial amount of permutation circuits which themselves also only have a polynomial complexity. Therefore, questions regarding the link between nonstabilizer consumption and quantum advantages can be investigated with frameworks factoring out permutational degrees of freedom.

Our color spectrum representation of states served as a visual intuition for qubit permutation invariant similarities between states. As an intermediate step, we will now reformulate this idea in a more mathematical way before defining a rigorous permutation invariant distance measure. Consider $\text{hsv}(\langle P_0 \rangle, \langle P_+ \rangle, \langle P_{+i} \rangle)$ as a point in a color space, then the color spectrum state representation is a set of points in this space, where each point corresponds to the vector of Pauli expectations $\langle P_0 \rangle, \langle P_+ \rangle, \langle P_{+i} \rangle$ for a specific qubit of the state. Given two states, we get two clouds of n points in the color space. The similarity of both clouds can be computed by matching points of both clouds in pairs. The pairwise matching cost will be the Euclidean distance between Pauli expectation vectors of both paired states. Finally, we get the similarity measure by taking the mean pairwise matching cost of the minimal matching. Informally speaking, we calculated how far the points of both clouds drifted apart on average. Mathematically, we define a function $f : \mathbb{N} \times \mathcal{H}^{\otimes n} \rightarrow \mathbb{R}^3$ such that

$$f(j, |\psi\rangle) = (\text{tr}(\rho_j P_0), \text{tr}(\rho_j P_+), \text{tr}(\rho_j P_{+i})), \quad (9)$$

with ρ_j being the reduced density matrix of the j th qubit. Using f , we now define a state map $C : \mathcal{H}^{\otimes n} \rightarrow \mathbb{R}^{n \times 3}$:

$$C(|\psi\rangle) = [f(i, |\psi\rangle)]_{i=1}^n. \quad (10)$$

On this state representation, we calculate a distance $d_{\text{MMC}}(|\psi_1\rangle, |\psi_2\rangle)$ which is the mean of the minimal bipartite row matching between $C(|\psi_1\rangle)$ and $C(|\psi_2\rangle)$ with matching costs $\alpha_{j,l} = \|f(l, |\psi_2\rangle) - f(j, |\psi_1\rangle)\|$.

Figure 3 shows how this neatly bridges between the color spectrum intuition and the rigorous permutation invariant distance we will define now. As a first step, we define a permutation operator capturing the notions discussed above.

Definition 4. Given a permutation $\sigma \in S_n$ and $|\mathbf{b}\rangle \in \mathcal{B}^n$, where $|\mathbf{b}\rangle = |b_1\rangle \otimes \cdots \otimes |b_n\rangle$ with $|b_i\rangle \in \{|0\rangle, |1\rangle\}$, then we define $\hat{\sigma} \in U(\mathcal{H}^{\otimes n})$ as

$$\hat{\sigma} |\mathbf{b}\rangle = |b_{\sigma(1)}\rangle \otimes \cdots \otimes |b_{\sigma(n)}\rangle \quad (11)$$

and

$$\hat{\sigma} \sum_{b \in \mathcal{B}^n} \alpha_b |\mathbf{b}\rangle = \sum_{b \in \mathcal{B}^n} \alpha_b \hat{\sigma} |\mathbf{b}\rangle. \quad (12)$$

¹Which then could enforce an order not necessarily present in the naming of variables.

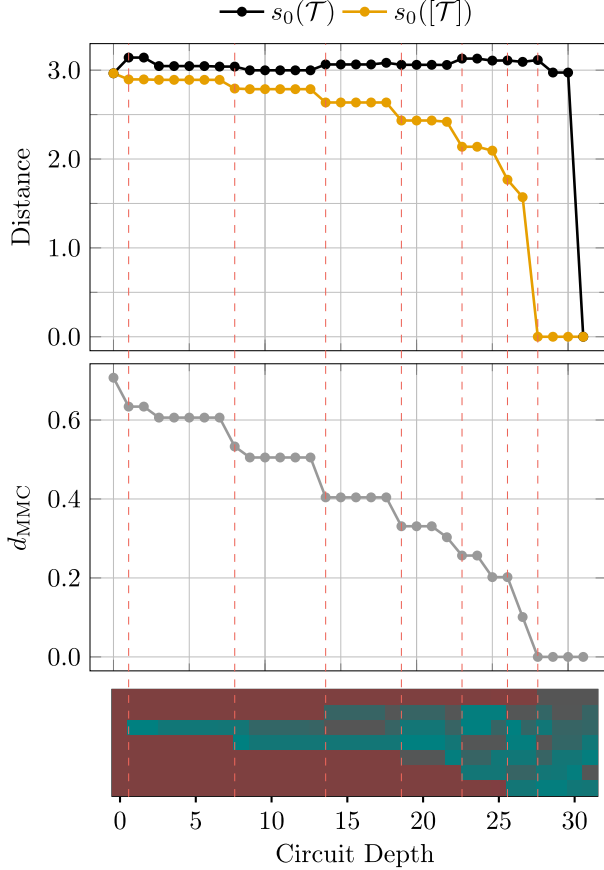


FIG. 3. Minimal geodesic distance for increasing circuit depths. Non-Clifford computational progress can be observed prior to the final qubit order reversal when all target space permutations $s_0([T])$ (ochre) are considered. As such effects are not visible in the direct distance to the target space $s_0(\mathcal{T})$ (black) that neglects permutations, this demonstrates how potential nonstabilizer effects can be masked by non-Clifford-agnostic measures (lines are used to guide the eye and have no significance). The permutation invariant distance $s_0([T])$ is also much more inline with the structural changes observed in the color spectrum representation of the state evolution under the QFT circuit. See Fig. 2 for a detailed description of the color spectrum representation. Also, the mean minimal matching cost distance (d_{MMC}) to the final circuit state aligns with both $s_0([T])$ and the structural changes of the color spectrum representation.

Note that in Definition 4 the inverse operator $\hat{\sigma}^\dagger \in U(\mathcal{H}^{\otimes n})$ corresponds to the inverse permutation $\sigma^{-1} \in S_n$. Next, we have to show that the SRE measure is invariant under such permutation operators.

Theorem 3. Let $\hat{\sigma}$ be a permutation operator as defined in Definition 4, then

$$\text{SRE}_\alpha(\hat{\sigma}|\psi\rangle) = \text{SRE}_\alpha(|\psi\rangle).$$

Proof. Every permutation $\sigma \in S_n$ can be decomposed into a sequence of 2-cycles, which can be realized by a single SWAP gate. Thus, $\hat{\sigma} \in \mathcal{H}^{\otimes n}$ can be realized by a sequence of SWAP gates, which are Clifford operations. Since, SRE_α is invariant under Clifford operations, it also is for all permutation operators constructed as defined in Definition 4. ■

Now, after we have formalized the idea of invariance under permutation on the operational side, we will do the same for the objects of interest. We do this by subsuming all states equal under permutation into equivalence classes and then extend this to the target space itself.

Definition 5. We define an equivalence relation $|\psi_l\rangle \sim |\psi_r\rangle$ which is satisfied if and only if there exists a permutation operator $\hat{\sigma}$ as defined in Definition 4, such that $|\psi_r\rangle = \hat{\sigma}|\psi_l\rangle$. Then

$$[|\psi\rangle] = \{|\psi'\rangle : \forall \hat{\sigma}\} \quad (13)$$

is the corresponding equivalence class of $|\psi\rangle$ under $\sim$ and further $\text{SRE}_\alpha([|\psi\rangle]) = \text{SRE}_\alpha(|\psi\rangle)$. Let $\mathcal{T}$ be a subspace of $\mathcal{H}^{\otimes n}$; we then extend this notion by defining

$$[\mathcal{T}] = \bigcup_{|t\rangle \in \mathcal{T}} [|t\rangle]. \quad (14)$$

From Theorem 3, it also immediately follows that $\text{SRE}_\alpha(|\psi\rangle) = \text{SRE}_\alpha([|\psi\rangle])$. For the geodesic distance, we need to extend the definition to equivalence classes.

Definition 6. Let $[|\phi\rangle]$ be an equivalence class of states; then

$$s_0([|\phi\rangle]) = \min_{|\phi'\rangle \in [|\phi\rangle]} s_0(|\phi'\rangle). \quad (15)$$

For $[\mathcal{T}]$, we extend s_0 in a similar fashion to

$$s_0([\mathcal{T}]) = \min_{|t\rangle \in \mathcal{T}} s_0(|t\rangle). \quad (16)$$

Determining the distance $[\mathcal{T}]$ requires tracing all permutations of all possible solution states, which can be a bit tricky. Luckily, we can show that the distance from $|\psi\rangle$ to $\mathcal{T}$ is equal to the distance from $[|\psi\rangle]$ to $\mathcal{T}$.

Theorem 4. Given a target space permutation equivalence class $[\mathcal{T}]$ as defined in Definition 4, it holds that

$$s_0([\mathcal{T}]) = s_0(|\psi\rangle, [\mathcal{T}]) = s_0([|\psi\rangle], \mathcal{T}). \quad (17)$$

Proof. By definition, we have that $s_0([|\psi\rangle], \mathcal{T}) = \min_{|\psi'\rangle \in [|\psi\rangle]} s_0(|\psi'\rangle, \mathcal{T})$, which equals $\min_{\hat{\sigma}} s_0(\hat{\sigma}|\psi\rangle, \mathcal{T}) = \min_{\hat{\sigma}} 2 \arccos\langle\psi|\hat{\sigma}^\dagger H_c \hat{\sigma}|\psi\rangle$. As we are minimizing over the whole group of all permutation operators, we can also minimize over all complex conjugate operators instead: $\min_{\hat{\sigma}^\dagger} 2 \arccos\langle\psi|\hat{\sigma} H_c \hat{\sigma}^\dagger|\psi\rangle$. By the canonical definition of H_c , we have $\hat{\sigma} H_c \hat{\sigma}^\dagger = \sum_{t \in \mathcal{T}} \hat{\sigma}|t\rangle\langle t|\hat{\sigma}^\dagger$. Recall that $\{|t\rangle : t \in \mathcal{T}\}$ is the basis of the corresponding quantum target space $\mathcal{T}$. This means, by applying $\hat{\sigma} H_c \hat{\sigma}^\dagger$, we are performing a basis transformation on the target space, measuring the expected probability of $|\psi\rangle$ being in the permuted target space. By minimizing over all permutations, we get $s_0(|\psi\rangle, [\mathcal{T}])$, thus, in conclusion, $s_0(|\psi\rangle, [\mathcal{T}]) = s_0([|\psi\rangle], \mathcal{T})$. ■

The quantum Fourier transform (QFT) is a good example to demonstrate this effect, and additionally shows how to calculate the distance of the closest target space permutation. As of today, the QFT is regarded as *the* seminal primitive contributing to quantum advantage, finding application in a wide range of quantum algorithms reaching proven quantum speedups [33,34], thus making it an interesting quantum primitive to study [35]. If one intends to use a distance measure to quantify computational progress, it must be applicable to analyze the QFT as well. The interesting parts of the QFT

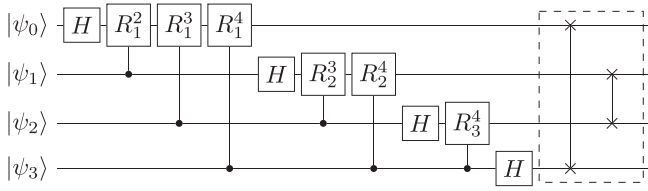


FIG. 4. QFT circuit with four qubits. The dashed box marks the qubit order inversion block of SWAP gates. Nonstabilizer computations take place before this block, but their computational influence on the geodesic distance is masked by the final qubit reordering.

circuit take part before the qubits are reordered in a final step (marked section in Fig. 4). This is problematic when looking at distance measures based on state to space overlaps like the geodesic distance. The block of SWAP gates implementing the reordering is entirely Clifford, yet looking at the geodesic instance $s_0(\mathcal{T})$ one could be under the impression that all the computational progress takes place in this section of the circuit. This cannot be the case, as the QFT algorithm enables exponential speed-ups. Therefore, valuable computational progress has to be made before taking possible target space permutations into account to reveal such effects (see Fig. 3).

V. NUMERICAL SIMULATIONS

General state evolution algorithms usually are quite high level from an algorithmic standpoint. The logical structure of problem instances usually is encoded in a Hamiltonian either driving the state evolution like in quantum annealing and its gate based counterparts [e.g., quantum approximate optimization algorithm (QAOA)] or serving as a cost function representation expressing the solution quality, which then can be used to optimize free parameters of a quantum circuit. In both cases, the problem structure is quite removed from the description of the algorithmic dynamics. This divide between descriptive dynamics and problem structures introduces a high level of abstraction masking the actual dynamics.

A. Problem structure in state evolution Ansätze

When choosing an Ansatz, one can decide between different levels of expressiveness. While a more expressive and thus more general Ansatz seems to be the obvious choice, they usually need a high number of free parameters. Alternatively, information about the problem can be used to construct more specific aAnsätze. An example for a general Ansatz would be a hardware efficient variational quantum eigensolver (VQE). In the case of QAOA on the other hand, a problem Hamiltonian is driving the unitary evolution of problem layers alternating with mixing layers. Here, the induced problem structure restricts the state evolution of the Ansatz but therefore guides the less variational parameters needed.

In this work, we will use the expressibility [36,37] of an Ansatz to quantify that circuit induces structure on the resulting state evolution. Following [37], we define the expressibility of a circuit by the Kullback-Leibler divergence (D_{KL}) between the fidelity distributions of pairwise sampled

states from the Ansatz's parametrized output distribution and Haar randomly sampled state pairs.

Definition 7. Let $C(\theta)$ be a parametrized Ansatz, with an angle parametrization $\theta \in [0, 2)^k$ for some $k \in \mathbb{N}$. Given two random states $|\psi\rangle$ and $|\phi\rangle$, the fidelity $F = |\langle\psi|\phi\rangle|^2$ is a random variable. Then, $P_C(F)$ is the probability distribution of fidelities $F(\theta, \varphi) = |\langle\psi(\theta)|\psi(\varphi)\rangle|^2$ with $|\psi(\theta)\rangle = C(\theta)|0\rangle$, $|\psi(\varphi)\rangle = C(\varphi)|0\rangle$ for angle parametrizations θ, φ sampled uniformly at random. Further, $P_{\text{Haar}}(F)$ is the probability distribution of fidelities $F = |\langle\psi|\phi\rangle|^2$ with $|\psi\rangle$ and $|\phi\rangle$ randomly sampled from the Haar distribution. Then, we define the expressibility of C by

$$\text{Expr}(C) = D_{KL}(P_C(F) || P_{\text{Haar}}(F)).$$

If an Ansatz C has an expressibility $\text{Expr}(C) \approx 0$, then its output distribution $\{C(\theta)|0\rangle\}_\theta$ with regard to a random angle parametrization θ is close to the Haar distribution. Otherwise, the structure of C induces a bias to the output state distribution forcing it to diverge from the Haar distribution. In the picture of state evolutions, we consider $|0\rangle \mapsto C(\theta)|0\rangle$ to be either a weakly structured state evolution if $\text{Expr}C \approx 0$ or a strongly structured state evolution if $\text{Expr}C \gg 0$.

Like [37], we will estimate $\text{Expr}(C)$ by uniformly sampling pairs of parametrizations $\theta, \varphi \in [0, 2)^k$ and computing the corresponding fidelity $F(\theta, \varphi) = |\langle\psi(\theta)|\psi(\varphi)\rangle|^2$, where $|\psi(\cdot)\rangle = C(\cdot)|0\rangle$. In this picture, F is a random variable and we can estimate $P_C(F)$ by the histogram of the sampled $F(\theta, \varphi)$ realizations of F . The histogram of the F distribution for random Haar pairs can be analytically computed [36]. This gives us the estimator:

$$\hat{\text{Expr}}(C) = D_{KL}(\hat{P}_C(F) || P_{\text{Haar}}(F)). \quad (18)$$

We will now show two exemplary Ansätze, one being weakly and the other strongly structured according to our expressibility-based argument.

B. Weakly structured state evolution Ansatz

A highly expressible (i.e., weakly structured) state evolution Ansatz requires a generic circuit template leaving maximal flexibility to be adjusted later in an optimization step minimizing a cost function which is minimal if the final state is in the target space $\mathcal{T}$. The initial circuit Ansatz is the same for all problems and problem instances, otherwise problem or instance information would potentially induce circuit structures. Concrete instance or problem specific structure only gets introduced during the cost function optimization process and is contained in the final optimized circuit parametrization $\theta \in [0, 2\pi)$. These considerations motivated us to investigate as an exemplary Ansatz a hardware efficient VQE. We chose a layered architecture where one layer exists of a stack of $R_y(\theta_i^y)$ gates applied to each qubit i followed by a similar stack of $R_z(\theta_i^z)$ gates and a ladder of cnot gates to provide entanglement. For a full circuit for the layer structure, see Fig. 5.

Figure 6 shows the estimated expressibility $\hat{\text{Expr}}$ of the hardware efficient VQE Ansatz with different numbers of layers p as described above. To compute $\hat{\text{Expr}}$, we sampled 1000 random fidelities $F(\theta, \varphi)$ and estimated P_C and P_{Haar} using histograms with 100 bins of uniform bin width. One can

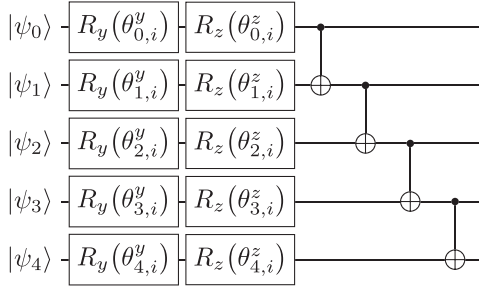


FIG. 5. The i th layer of the hardware efficient Ansatz used for unstructured state evolution.

observe that the expressibility of this Ansatz converges close to 0, meaning that it does describe a weakly structured state evolution.

C. Strongly structured state evolution Ansatz

In contrast to weakly structured state evolution techniques, in the structured case the Ansatz already gets infused with instance structures. One can show that problem structures extrapolated from common instance structures are sufficient to successfully approximate key parameters like the expected target space overlap of such structured state evolution methods [38]. This shows that the structural infusion significantly impacts the Ansatz even before instance specific cost function optimization techniques are applied. As a representative for strongly structured state evolution, we chose a standard QAOA Ansatz where the driving problem Hamiltonian is the problem Hamiltonian H_c Krüger defined above. This equals the construction by Krüger and Maurer [38].

For the sake of completeness, we will shortly recap the construction presented in [38]. Additionally, as it was not required in the original paper, we will also provide a gate

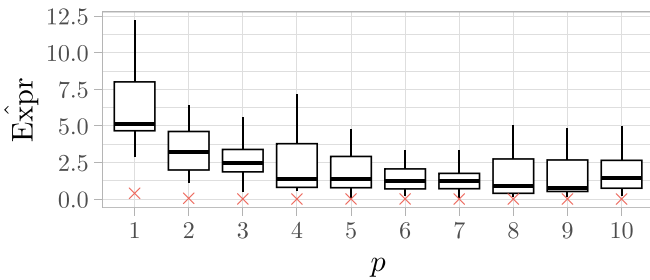


FIG. 6. Expressibility of a strongly structured Ansatz (QAOA) and a weakly structured Ansatz (VQE) with various layer depths p . In the strongly structured case, the Ansatz is constructed with instance information. The distribution of expressibilities for 20 random instances per p is shown as box plots. The weakly structured VQE Ansatz is instance independent and therefore is depicted as a single data point. The weakly structured Ansatz achieves close to full expressibility [i.e., $\text{Expr}(0)$] while the strongly structured one converges above $\text{Expr}(0)$.

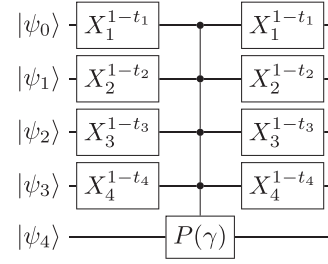


FIG. 7. The decomposition of the $P(t, \gamma)$ gate.

level decomposition of the mixer and problem unitaries. The i th layer of the described QAOA Ansatz is given by

$$U_L(\beta_i, \gamma_i) = e^{-i\beta_i H_x} e^{-i\gamma_i H_c}, \quad (19)$$

where $H_x = \sum_{j=1}^n \sigma_j^x$. Here σ_j^x is the Pauli-X operator on the j th qubit. The resulting mixing layer $e^{-i\beta_i H_x}$ can thus be decomposed into single qubit rotating gates,

$$e^{-i\beta_i H_x} = \prod_{j=1}^n R_x^j(\beta_i),$$

where R_x^j is an x rotation applied to the j th qubit. To decompose $e^{-i\gamma_i H_c}$, we need to understand its effect. Being a projector onto the target space $\mathcal{T}$, the constraint Hamiltonian is diagonal $H_c = \text{diag}(c(x_0), c(x_1), \dots, c(x_{2^n-1}))$ with $x_i \in \mathbb{F}_2^n$ being the binary representation of i and $c(x_i) \in \mathbb{F}_2$, the constraint function which is only 1 if x_i is a valid solution. Thus, $e^{-i\gamma_i H_c}$ is a phase gate which maps $|t\rangle \mapsto e^{-i\gamma_i} |t\rangle$ iff $t \in \mathcal{T}$. To decompose $e^{-i\gamma_i H_c}$ we construct a subcircuit $U_C(\gamma) = \prod_{t \in \mathcal{T}} P(t, \gamma)$ where $P(t, \gamma)|x\rangle = e^{-i\gamma \mathbf{1}_{x=t}} |x\rangle$, with $\mathbf{1}_{x=t} \in \mathbb{F}_2$ being the indicator function which equals 1 iff $x = t$. Further, $P(t, \gamma)$ can be implemented by applying a X gate to each qubit $i \leq n-1$ if the i th bit t_i in t equals 0. After that, a $n-1$ multiqubit controlled phase gate $C_{n-1}P(\gamma)$ is applied to the n th qubit. Finally, the X gates are being reversed, resulting in

$$P(t, \gamma) = \prod_{i=1}^{n-1} X_i^{1-t_i} C_{n-1} P(\gamma) \prod_{i=1}^{n-1} X_i^{1-t_i}. \quad (20)$$

Note that the X gates on the n th qubit can be omitted as the phase gate only applies to the 1 state of the qubit. Figure 7 shows the construction of the $P(t, \gamma)$ gate. In summary, the p -layer QAOA Ansatz used in our simulations is constructed as follows:

$$U_L(\beta_p, \gamma_p) \cdots U_L(\beta_2, \gamma_2) U_L(\beta_1, \gamma_1). \quad (21)$$

U_L as described by Eq. (19) with its decomposition as described above.

Just as in the weakly structured case, we utilize the expressibility to argue that this Ansatz is strongly structured. The first argument we make is that its expressibility depends on the specific problem instance. This shows that instance information is used to induce structure into the circuit. Further, Fig. 6 shows that even with more layers the median expressibility converges at a higher value than the weakly structured Ansatz. As the box plots show, it also converges significantly above 0 ($\langle \text{Expr} \rangle \gg 0$). Therefore, we argue that the Ansatz represents strongly structured state evolution Ansatzes.

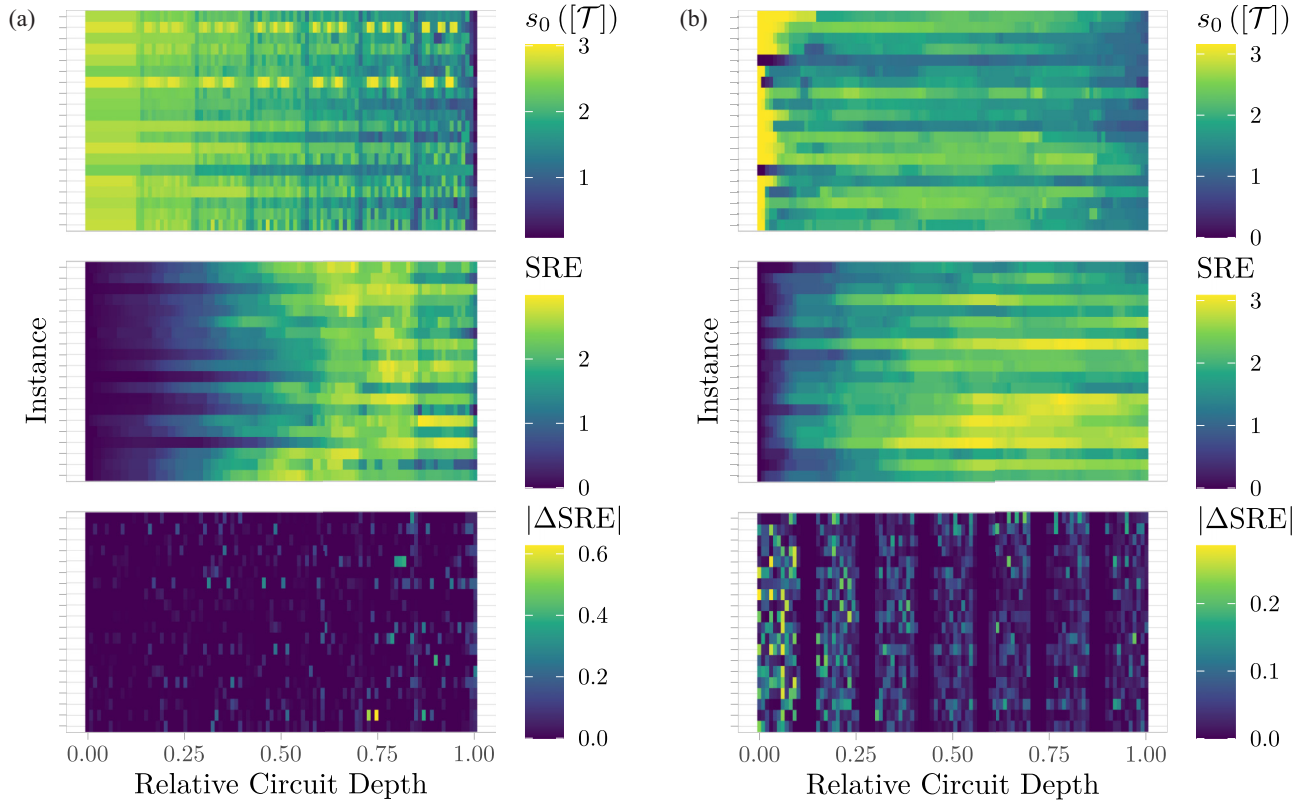


FIG. 8. Comparison of intermediate geodesic distances $s_0([\mathcal{T}])$, nonstabilizerness SRE and nonstabilizer consumption $|\Delta\text{SRE}|$ between unstructured and structured state evolution. As the circuits vary in gate count, the depicted values are linearly interpolated to 100 steps to achieve uniform value distributions over the relative circuit depth across all circuits. (a) Properties of intermediate states (x axis) evolving under a weakly structured Ansatz for different instances (y axis). Top: The state approaches the target space with notable erraticity, as evidenced by irregular fluctuations in the geodesic distance from $[\mathcal{T}]$. While nonstabilizerness (middle) appears to evolve comparatively smoothly, the variation in resource consumption, as quantified by $|\Delta\text{SRE}|$ (bottom), continues to display a markedly irregular behavior. (b) Properties of intermediate states (x axis) evolving under a strongly structured Ansatz for different instances (y axis). Top: The state approaches the target space $[\mathcal{T}]$ smoothly. The accumulation of nonstabilizerness (middle) follows a structured trajectory and reaches its apex at about 75% relative circuit depth, after which it gradually diminishes. This behavior is mirrored in the patterns of nonstabilizer resource consumption (bottom).

D. Problem description

We now want to showcase our methods introduced above to reveal actual differences in the evolution of structured and unstructured state evolution techniques. As an exemplary problem, we chose the seminal NP complete problem of boolean SAT, more precisely the problem of finding a satisfying variable assignment of a 3-CNF boolean formula $F : \mathbb{F}_2^n \rightarrow \mathbb{F}_2$. Let us define a problem Hamiltonian satisfying Definition 3. We start by defining the classical solution space T where $t = t_1 t_2 \cdots t_n \in T$ iff $c(t) := F(t) = 1$. Then target space shall be defined as $\mathcal{T} = \{\bigotimes_{i=1}^n |t_i\rangle : t_1 t_2 \cdots t_n \in T\}$. Note that F is a 3-CNF boolean formula, therefore $F = \prod_{i=1}^m f_i$ with $f_i : \mathbb{F}_2^3 \rightarrow \mathbb{F}_2$ are disjunctions. This means every f_i has one unique unsatisfying assignment $\bar{t}_i$. Now it is easy to see that the Hamiltonian $H_c := \mathbb{1} - \prod_{i=1}^m |t_i\rangle\langle t_i|$ satisfies Eq. (7). See Ref. [38] for more details.

E. Setup of simulations

For each Ansatz, we solved 20 SAT instances with the circuits spanning $n = 7$ qubits and $p = 7$ layers. Every instance was randomly sampled with a clause to variable ratio of $|C|/|V| = 3$, which generates SAT instances that are con-

strained enough to be at the start of the easy to hard phase transition. At the same time those instances are still not too hard to solve such that we can expect the state evolutions to get fairly close to the target space, assuring that we witness a state space traversal traveling a significant part of the distance necessary to successfully solve the problem. For our simulations we used the QuTiP library [39].

F. Results

Comparing the state evolution of strongly structured and weakly structured circuits, we notice that the former apart from local fluctuations generally approaches the target space $[\mathcal{T}]$ in a more direct path, smoothly reducing the geodesic distance. In contrast, the weakly structured evolution seems to more erratically move through the state space, witnessed by bigger deviations away from the target space while passing through the circuit. The differences become apparent when comparing the top plots of Figs. 8(a) and 8(b). Although only exemplary, we also refer to the metric plots of Fig. 1, where the aforementioned difference becomes especially apparent. We now further analyze how both state evolutions approached the target space on a step by step basis. For this, we calcu-

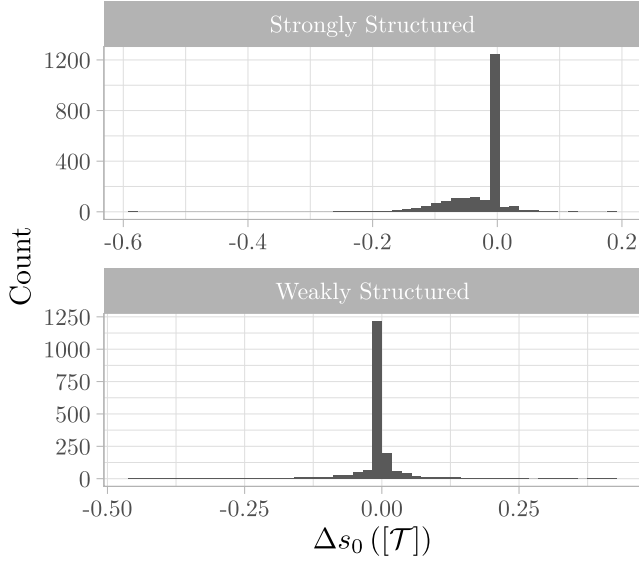


FIG. 9. Distribution of increments and decrements in distance to the target space ($x \sim \pm \Delta s_0([\mathcal{T})$) of computational steps with magic consumption $|\Delta \text{SRE}| > 1 \times 10^{-5}$ and Δs_0 outliers (z score > 3) removed. Top: Heavy skew toward decrements is observed for the structured Ansatz, and most distance changes are negative. Bottom: The distribution of distance changes is approximately centered around zero for the unstructured Ansatz, discounting a small number of outliers on the negative side.

late the delta of s_0 before and after each step. Our interest is focused on nonclassical computation steps. Therefore, we filter out all steps where no magic consumption took place. To account for numerical errors we filter for states where $|\Delta \text{SRE}| > \epsilon$, with $\epsilon = 1 \times 10^{-5}$. Additionally, we discard outliers with a z score > 3 . The z score is defined by

$$z(x) = \frac{|x - \mu|}{\sigma}, \quad (22)$$

where μ is the distribution mean and σ the standard deviation. Figure 9 shows that the distribution of the $\Delta s_0([\mathcal{T})$ is symmetrically centered around zero in the weakly structured case. For the strongly structured evolution, on the other hand, we observe that the distribution of Δs_0 values is skewed toward the regime below zero (Table I shows more detailed numbers). This indicates that speaking on a per-step basis, the strongly

TABLE I. 25% (Q1), 50% (Q2), and 75% (Q3) quartiles of the $\Delta s_0([\mathcal{T})$ distributions for strongly structured and weakly structured state evolution. Outliers with z score > 3 were discarded. The last two columns depict fractions of steps that decrease ($\Delta s_0 < -\epsilon$) or increase ($\Delta s_0 > \epsilon$) target distance, where $\epsilon = 1 \times 10^{-5}$. On the whole, the distribution of Δs_0 for strongly structured state evolutions exhibits a pronounced skewness toward negative values, in marked contrast to the weakly structured scenario, where it is mostly symmetrically centered about zero.

Structuredness	Q1	Q2	Q3	$\Delta s_0 < -\epsilon$	$\Delta s_0 > \epsilon$
Strongly	-0.0383	0.0000	0.0000	35.4%	5.99%
Weakly	-0.0001	0.0000	0.0000	25.6%	23.1%

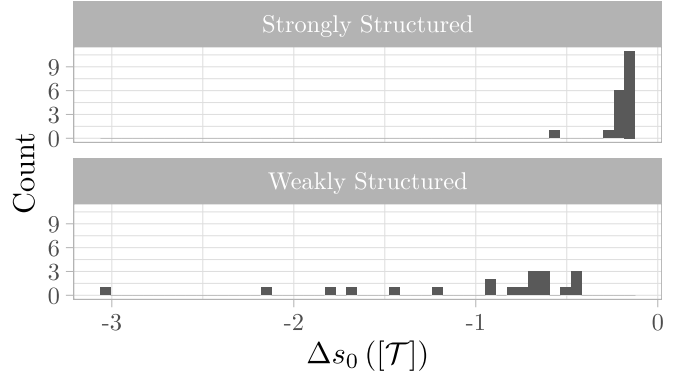


FIG. 10. Distribution of negative $\Delta s_0([\mathcal{T}) < -\epsilon$ outliers (z score > 3) discarded in Fig. 9 and Table I.

structured Ansatz more efficiently approaches the target, exhibiting a higher magic efficiency. For the weakly structured Ansatz, the majority of steps seem to move toward or away from the target with equal probability, de-facto canceling each other out on the macroscopic level. That being said, negative Δs_0 outliers (see Fig. 10) of bigger value seem to suggest that the weakly structured approach is able to reach further in larger individual steps, reaching the target faster if utilized efficiently.

Another important aspect of efficiency is the consumption of nonstabilizerness. As already mentioned above, SRE is invariant under Clifford gates. In conclusion, a change in the SRE of the intermediate state being evolved indicates the use of a non-Clifford operation. Therefore, we will use the absolute step SRE difference $|\Delta \text{SRE}|$ as an indicator of nonstabilizerness consumption, which is inherently linked to costly operations. Comparing Figs. 8(a) and 8(b) (bottom), one sees that the weakly structured Ansatz has a significantly denser distribution of magic consuming evolution steps compared to the strongly structured Ansatz, whereas the latter shows a higher peak magic consumption throughout its evolution steps. This begs the question whether this shows a higher magic efficiency of the structured state evolution, especially taking the more uniform target approach of them compared to weakly structured evolutions. To explore this question, we will compare each geometric step $\Delta s_0([\mathcal{T})$ with its corresponding magic consumption $|\Delta \text{SRE}|$. Figure 9 suggests a bimodal distribution off $\Delta s_0([\mathcal{T})$ values differentiating between two cases: actual geometric steps ($\Delta s_0 > \epsilon$) and computational steps where the state neither decreased nor increased its geometric distance to the target $\mathcal{T}$ ($|\Delta s_0| < \epsilon$), both up to a precision of ϵ . During the strongly structured state evolution, only 23.5% of the computational steps had no effect on the geometric distance to $\mathcal{T}$ whereas during the weakly structured case, 42.2% of the steps had no effect on s_0 . Looking further into it, we see that the distance invariant computational steps on average consumed more magic in the weakly structured state evolution compared to the strongly structured one (see Fig. 11). For the second mode of computational steps with $|\Delta s_0([\mathcal{T})| > \epsilon$, there is a clear positive correlation between stepwise geodesic distance reductions to the target space and magic consumption for the strongly structured state evolution. In contrast to that observation, there is no such correlation for the weakly structured case, as

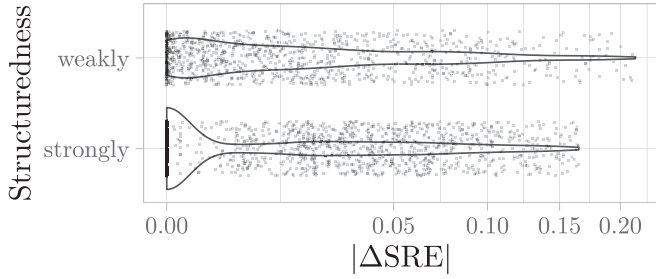


FIG. 11. The distribution of magic consumption $|\Delta\text{SRE}|$ of computational steps that had no effect on the geometric target distance $s_0([\mathcal{T}])$. Outliers with z score >3 were discarded. The $|\Delta\text{SRE}|$ axis is square root scaled.

shown in Fig. 12. This further substantiates the hypothesis that the strongly structured Ansatz utilizes nonstabilizer resources more efficiently.

VI. DISCUSSION

We have extended the concept of geodesic distance measures in state evolutions targeting a specific state to evolutions targeting a more complex target space $\mathcal{T}$. We showed how the geodesic distance to $\mathcal{T}$ can be derived from the expected value of a Hamiltonian satisfying Eq. (7). This Hamiltonian-based definition fits well into widely used frameworks of Hamiltonian cost function encodings. It also further allows for establishing empirical measurement-based setups that integrating nicely with existing toolkits of quantum computing practitioners.

We further provided a qubit order agnostic version of the geodesic framework by introducing equivalence classes of states that are equal under permutation. Considering the quantum Fourier transform as a use case, we demonstrated how our approach can cut through Clifford layers, and thus unveil previously hidden computational progress in the circuit. We then applied the developed methods to comparatively analyze of strongly structured and weakly structured state evolutions.

The different distributions of geodesic distance changes in addition with a less dense distribution of magic consuming evolution steps suggest a higher geodesic and magic efficiency for the strongly structured evolution.

Our analysis demonstrates a bimodal distribution of geodesic step sizes for weakly and strongly structured state evolution. One mode captures all gates that have no effect on the geometric target distance. Here magic consumption can be considered unproductive. The weakly structured state evolution consumes more magic in these steps compared to the strongly structured one. The second mode captures all gates that influence the geometric target distance, either positively or negatively. By combining resource theoretic stabilizer Rényi entropy and geometric geodesic distance measures, we were able to show that for these gates the strongly structured Ansatz is significantly more efficient in the consumption of nonstabilizer resources than the weakly structured Ansatz. On a methodical level, this demonstrates the potential of our combination of methodologies. In summary, the structured evolution is less wasteful in magic consumption with gates that do not affect the geometric target distance and

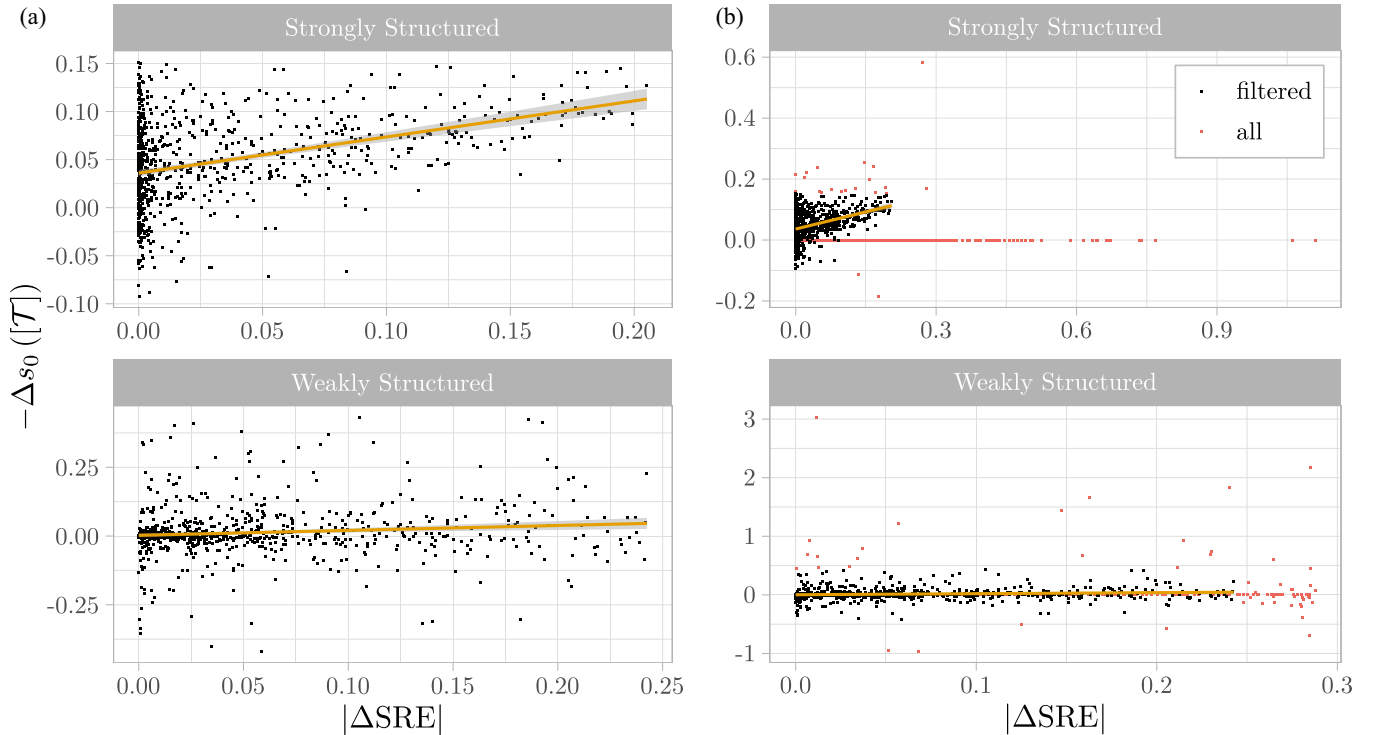


FIG. 12. Correlation of magic consumption $|\Delta\text{SRE}|$ and geometric target distance variation $\Delta s_0([\mathcal{T}])$ for all steps with $\Delta s_0 > \epsilon$: Strongly structured state evolution (top) shows clear correlation between magic consumption $|\Delta\text{SRE}|$ and steps reducing the geodesic distance to the target space $-\Delta s_0([\mathcal{T}])$. In contrast, we cannot observe a similar correlation for the weakly structured Ansatz (bottom). Outliers with z score >3 in $|\Delta\text{SRE}|$ or $\Delta s_0([\mathcal{T}])$ were discarded. (a) A close up of the filtered data points. (b) The filtered data points in the context of all data points.

additionally shows a higher magic efficiency when actually geometrically moving the state toward the target. Overall, the structured state evolution is significantly more magic efficient.

It is also interesting to note that the magic consumption of the structured state evolution we investigated seems to match the observations of [40] quite well, where magic buildup peaked roughly in the middle of the evolution before it got reduced while closing in on the target state. This is interesting, as in [40] the authors investigate QAOA in the context of combinatorial optimization with low degeneracy in the ground state (i.e., target space in our terms). The authors observed how the state has to pass, in their terms, a magic barrier to reach the nondegenerated (stabilizer) state. We argue that in a case where the initial and target state are both low in magic, this shape is necessary to achieve quantum speedups. This becomes clear under our perspective, where indeed not magic in itself is the quantum resource associated with quantum computations (and thus speedups) but rather the magic consumption $|\Delta\text{SRE}|$. Thus, the magic barrier observed in [40] can be seen as the accumulated magic consumption during the state evolution as both steps increasing and decreasing magic are inherently nonclassical and can both be efficient computational steps. In fact, evolving a stabilizer initial state to a stabilizer target state through a flat magic landscape would hint at a primarily classic computation not touching any quantum potential. In a variational scheme, this would mean moving the hard computational effort to the classical optimization loop.

In contrast to [40], our construction captures a decision problem that quite naturally has structured solution spaces leading to entangled nonstabilizer high magic targets spaces. Just think of valid solutions of SAT formulas which have inherent dependencies between single bit flips cascading to other bits. Or, in a more pathological case, we could also force our structured Ansatz to prepare the $|w\rangle$ state, which has magic of $\text{SRE}(|w\rangle) = 3 \ln_2(n) - \ln_2(7n - 6)$ [41]. Despite this stark difference, we also witnessed a magic-barrier-like effect overreaching the final magic. This raises the question if such a magic barrier shape is a sign of efficient quantum resource consumption in a circuit. Further, one could analyze how the distribution between magic consumption during the state evolution and the final state magic differs whether the quantum speedup is routed more in the state evolution itself or the sampling from a high magic state.

Another interesting aspect probably worth a follow-up investigation is the role of the parameter space. Intuition would suggest that the encoded problem structure of strongly structured state evolution allows for a reduced number of well-placed parametrized operators. In the more general weakly structured case, a higher density of parameterized gates is necessary to cover different problems and instances. On a per instance level, perhaps only an unknown subset of gates would suffice, allowing the evolution to unproductively meander through the state space as the optimizer in a way turning the wrong knobs. Our framework could be utilized, for instance, to investigate the importance of individual gates in weakly structured Ansätze to come up with more Ansätze for certain computational problems.

Recently, magic measures were proposed that capture non-local nonstabilizerness of bipartite systems [42,43]. It would

be interesting to deploy our framework with nonlocal magic measures. However, the impact of disregarding local magic to computational progress needs to be studied in more detail to infer sound conclusions. In general, our framework is flexible enough to facilitate arbitrary quantum state measures, including entanglement measures.

VII. CONCLUSION AND PERSPECTIVE

We believe our methodology opens alternative means of analyzing nonstabilizer effects and the efficient utilization of nonstabilizer resources in quantum circuits. A nuanced understanding of such effects seems crucial for advancing the systematic development of quantum algorithms, particularly with regard to realizing quantum speedups in a well-principled manner. Furthermore, we anticipate that our results will become increasingly pertinent as the field transitions into the era of early fault-tolerant quantum computing: In such regimes, nonstabilizer operations pose significantly greater challenges for error correction compared to stabilizer operations. Consequently, the use of this resource must be optimized, and we believe that our analytical framework offers a valuable instrument in progressing toward this objective.

We showed how permutation agnostic distance measures can reveal internal nonstabilizer effects previously hidden by a subset of Clifford operations. Our construction based on permutation operators $\hat{\sigma}$ could be extended to accept general Clifford operators in the sense that two states $|\psi_1\rangle \sim |\psi_2\rangle$ iff there exists a $U \in \mathcal{C}$ such that $U|\psi_1\rangle = |\psi_2\rangle$. Although Clifford circuits are classically simulable, finding minimal Clifford decompositions is not always efficient. Thus, such an extension would need to impose some complexity theoretic bounds on U to avoid grouping states that can only be reached by overly powerful oracles.

We showed that the combination of resource theoretic and geometric tools offers a mean to qualify resource consumptions by efficiency. We see a potential to embed quantum resource theoretic measures like stabilizer Rényi entropies into a proper differential geometric framework. This is a second promising avenue for improvement that would allow us to analyze resources consumed by state evolutions following different paths over the projective state manifold.

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DATA AVAILABILITY

A reproduction package [44] that contains all code required to run the simulations in this paper is available at [45]. All simulations are done on premise with seeded random number generators. No cloud services were used. Thus, given the availability of the used software packages, our simulations are fully reproducible.

- [1] K. Bharti, A. Cervera-Lierta, T. H. Kyaw, T. Haug, S. Alperin-Lea, A. Anand, M. Degroote, H. Heimonen, J. S. Kottmann, T. Menke, W.-K. Mok, S. Sim, L.-C. Kwek, and A. Aspuru-Guzik, Noisy intermediate-scale quantum algorithms, *Rev. Mod. Phys.* **94**, 015004 (2022).
- [2] S. Thelen, H. Safi, and W. Mauerer, Approximating under the influence of quantum noise and compute power, in *2024 IEEE International Conference on Quantum Computing and Engineering (QCE), Montreal, QC, Canada* (IEEE, Piscataway, NJ, 2024), pp. 15–20.
- [3] M. Franz, T. Winker, S. Groppe, and W. Mauerer, Hype or heuristic? Quantum reinforcement learning for join order optimization, in *2024 IEEE International Conference on Quantum Computing and Engineering (QCE), Montreal, QC, Canada* (IEEE, Piscataway, NJ, 2024), pp. 15–20.
- [4] R. Jozsa, Entanglement and quantum computation, [arXiv:quant-ph/9707034](https://arxiv.org/abs/quant-ph/9707034).
- [5] J. Preskill, Quantum computing and the entanglement frontier, [arXiv:1203.5813](https://arxiv.org/abs/1203.5813).
- [6] R. Jozsa and N. Linden, On the role of entanglement in quantum-computational speed-up, *Proc. R. Soc. Lond. Ser. A* **459**, 2011 (2003).
- [7] A. Datta and G. Vidal, Role of entanglement and correlations in mixed-state quantum computation, *Phys. Rev. A* **75**, 042310 (2007).
- [8] W. Mauerer, On colours, keys, and correlations: Multi-mode parametric downconversion in the photon number basis, Ph.D. thesis, Friedrich-Alexander-Universität Erlangen-Nürnberg, 2009.
- [9] A. Khrennikov, Roots of quantum computing supremacy: Superposition, entanglement, or complementarity? *Eur. Phys. J.: Spec. Top.* **230**, 1053 (2021).
- [10] T. Rohe, D. Schuman, J. Nüßlein, L. Sünkel, J. Stein, and C. Linnhoff-Popien, The questionable influence of entanglement in quantum optimization algorithms, in *2024 IEEE International Conference on Quantum Computing and Engineering (QCE), Montreal, QC, Canada* (IEEE, Piscataway, NJ, 2024) pp. 1497–1503.
- [11] A. C. Nakhil, T. Quella, and M. Usman, Calibrating the role of entanglement in variational quantum circuits, *Phys. Rev. A* **109**, 032413 (2024).
- [12] P. Díez-Valle, D. Porras, and J. J. García-Ripoll, Quantum variational optimization: The role of entanglement and problem hardness, *Phys. Rev. A* **104**, 062426 (2021).
- [13] Andreas J. C. Woitzik, P. K. Barkoutsos, F. Wudarski, A. Buchleitner, and I. Tavernelli, Entanglement production and convergence properties of the variational quantum eigensolver, *Phys. Rev. A* **102**, 042402 (2020).
- [14] D. Gross, S. T. Flammia, and J. Eisert, Most quantum states are too entangled to be useful as computational resources, *Phys. Rev. Lett.* **102**, 190501 (2009).
- [15] J. Odavić, M. Viscardi, and A. Hamma, Stabilizer entropy in non-integrable quantum evolutions, *Phys. Rev. B* **112**, 104301 (2025).
- [16] D. Gottesman, The Heisenberg representation of quantum computers, in *22nd International Colloquium on Group Theoretical Methods in Physics* (International Press, Hobart, Tasmania, Australia, 1998), pp. 32–43.
- [17] H. J. García, I. L. Markov, and A. W. Cross, On the geometry of stabilizer states, *Quantum Inf. Comput.* **14**, 683 (2014).
- [18] D. Gottesman, *Stabilizer Codes and Quantum Error Correction* (California Institute of Technology, 1997).
- [19] S. Bravyi and A. Kitaev, Universal quantum computation with ideal Clifford gates and noisy ancillas, *Phys. Rev. A* **71**, 022316 (2005).
- [20] M. Beverland, E. Campbell, M. Howard, and V. Kliuchnikov, Lower bounds on the non-Clifford resources for quantum computations, *Quantum Sci. Technol.* **5**, 035009 (2020).
- [21] L. Leone, Salvatore F. E. Oliviero, and A. Hamma, Stabilizer Rényi entropy, *Phys. Rev. Lett.* **128**, 050402 (2022).
- [22] E. Chitambar and G. Gour, Quantum resource theories, *Rev. Mod. Phys.* **91**, 025001 (2019).
- [23] S. Bravyi, G. Smith, and J. A. Smolin, Trading classical and quantum computational resources, *Phys. Rev. X* **6**, 021043 (2016).
- [24] S. Bravyi, D. Browne, P. Calpin, E. Campbell, D. Gosset, and M. Howard, Simulation of quantum circuits by low-rank stabilizer decompositions, *Quantum* **3**, 181 (2019).
- [25] T. Haug and L. Piroli, Stabilizer entropies and nonstabilizerness monotones, *Quantum* **7**, 1092 (2023).
- [26] T. Krüger and W. Mauerer, Quantum dark magic: Efficiency of intermediate non-stabilizerness, *2025 IEEE International Conference on Quantum Computing and Engineering (QCE), Albuquerque, NM* (IEEE, Piscataway, NJ, 2025).
- [27] L. Leone and L. Bittel, Stabilizer entropies are monotones for magic-state resource theory, *Phys. Rev. A* **110**, L040403 (2024).
- [28] Salvatore F. E. Oliviero, L. Leone, and A. Hamma, Magic-state resource theory for the ground state of the transverse-field Ising model, *Phys. Rev. A* **106**, 042426 (2022).
- [29] S. F. E. Oliviero, L. Leone, A. Hamma, and S. Lloyd, Measuring magic on a quantum processor, *npj Quantum Inf.* **8**, 148 (2022).
- [30] A. Gu, L. Leone, S. Ghosh, J. Eisert, S. F. Yelin, and Y. Quek, Pseudomagic quantum states, *Phys. Rev. Lett.* **132**, 210602 (2024).
- [31] J. Anandan and Y. Aharonov, Geometry of quantum evolution, *Phys. Rev. Lett.* **65**, 1697 (1990).
- [32] C. Cafaro, E. Clements, and A. Alanazi, Aspects of complexity in quantum evolutions on the Bloch sphere, *Eur. Phys. J. Plus* **140**, 349 (2025).
- [33] A. Y. Kitaev, Quantum measurements and the Abelian stabilizer problem, [arXiv:quant-ph/9511026](https://arxiv.org/abs/quant-ph/9511026).
- [34] P. W. Shor, Polynomial-time algorithms for prime factorization and discrete logarithms on a quantum computer, *SIAM J. Comput.* **26**, 1484 (1997).
- [35] N. Linden and R. de Wolf, Average-case verification of the quantum Fourier transform enables worst-case phase estimation, *Quantum* **6**, 872 (2022).
- [36] G. Ilário Correr, I. Medina, P. C. Azado, A. Drinko, and D. O. Soares-Pinto, Characterizing randomness in parameterized quantum circuits through expressibility and average entanglement, *Quantum Sci. Technol.* **10**, 015008 (2025).
- [37] S. Sim, P. D. Johnson, and A. Aspuru-Guzik, Expressibility and entangling capability of parameterized quantum circuits for hybrid quantum-classical algorithms, *Adv. Quantum Technol.* **2**, 1900070 (2019).
- [38] T. Krüger and W. Mauerer, Out of the loop: Structural approximation of optimization landscapes and non-iterative quantum optimization, *Quantum* **9**, 1903 (2025).

- [39] N. Lambert, E. Giguère, P. Menczel, B. Li, P. Hopf, G. Suárez, M. Gali, J. Lishman, R. Gadhvi, R. Agarwal, A. Galicia, N. Shammah, P. Nation, J. R. Johansson, S. Ahmed, S. Cross, A. Pitchford, and F. Nori, QuTip 5: The quantum toolbox in Python, *Phys. Rep.* **1153**, 1 (2026).
- [40] C. Capecci, G. C. Santra, A. Bottarelli, E. Tirrito, and P. Hauke, Role of nonstabilizerness in quantum optimization, *arXiv:2505.17185*.
- [41] J. Odavić, T. Haug, G. Torre, A. Hamma, F. Franchini, and S. M. Giampaolo, Complexity of frustration: A new source of non-local non-stabilizerness, *SciPost Phys.* **15**, 131 (2023).
- [42] D. A. Korbany, M. J. Gullans, and L. Piroli, Long-range nonstabilizerness and phases of matter, *Phys. Rev. Lett.* **135**, 160404 (2025).
- [43] D. Qian and J. Wang, Quantum nonlocal nonstabilizerness, *Phys. Rev. A* **111**, 052443 (2025).
- [44] W. Maurer and S. Scherzinger, 1-2-3 reproducibility for quantum software experiments, in *2022 IEEE International Conference on Software Analysis, Evolution and Reengineering (SANER), Honolulu, HI* (IEEE, Piscataway, NJ, 2022).
- [45] W. M. Tom Krueger, Reproduction package, https://github.com/lfld/quantum_dark_magic, doi: 10.5281/zenodo.21811049.