

Parametric instability of ultracold Bose gases with long-range interaction and quantum fluctuations trapped in optical lattices

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(Received 25 March 2025; accepted 29 September 2025; published 3 November 2025)

We investigate the dynamical instabilities of a one-dimensional ultracold Bose-Bose mixture with long-range dipole-dipole interactions, trapped in deep optical lattices and subject to periodically varying contact interaction. The effect of beyond-mean-field corrections due to quantum fluctuations is considered. In the tight-binding regime, we employ Wannier functions to derive a discrete nonlinear Schrödinger equation that captures the dynamics of the system. Using linear stability analysis coupled to multiple scale expansion, we systematically study the modulational and parametric instabilities of the system in the miscible regime, identifying the conditions under which these instabilities emerge. The corresponding instability domains are found and examined. The roles of long-range interactions and quantum fluctuations are highlighted, demonstrating their significant impact on the stability and dynamics of the lattice system. Our analytical predictions are validated through direct numerical simulations, which confirm the instability diagrams through the effective onset of instabilities and reveal the intricate interplay between interaction strength, quantum fluctuation, and the long-range nature of the dipole-dipole forces. This work provides insights into the control and manipulation of ultracold quantum gases in optical lattices, which have potential applications in quantum simulation and condensed-matter physics.

DOI: [10.1103/15qz-q9vm](https://doi.org/10.1103/15qz-q9vm)

I. INTRODUCTION

Ultracold quantum gases trapped in optical lattices have emerged as a versatile platform for exploring fundamental phenomena in quantum many-body systems, including superfluidity, quantum phase transitions, and nonlinear dynamics [1–3]. The high degree of controllability in these systems, achieved through tunable lattice depths, interatomic interactions, and external fields, has enabled the quantum simulation of complex condensed-matter systems and the study of novel quantum states [4–7]. The broad range of phenomena exhibited by these systems includes parametric and modulational instabilities, which have garnered significant attention due to their role in the breakdown of coherent states and the emergence of complex dynamical behavior [8–10]. These instabilities are particularly relevant in the context of Bose-Einstein condensates (BECs), where nonlinear

interactions and lattice periodicity can lead to the amplification of small perturbations, driving the system into unstable regimes [11,12]. In the context of media described by the nonlinear Schrödinger equation and its variants, parametric instability refers to a phenomenon where small perturbations grow exponentially in a system at parametric resonance due to periodic modulation of system parameters [13,14]. Such a phenomenon is of particular interest as it can lead to the emergence of novel quantum states and dynamical patterns [15].

Recent advances in the manipulation of ultracold atoms with long-range interactions, such as those arising from dipole-dipole forces of dipolar atoms and molecules or Rydberg atoms, have opened new avenues for exploring the interplay between nonlocal interactions and lattice effects [16–20]. Unlike short-range contact interactions, long-range interactions introduce additional complexity into the system, leading to novel instability mechanisms and dynamical features [21,22]. In such a context, however, the effect of long-range interactions on parametric instabilities remains less understood. Furthermore, the inclusion of quantum fluctuations, often neglected in mean-field descriptions, has been shown to play a crucial role in stabilizing quantum gases, particularly in regimes where nonlinear effects dominate [23,24]. In regimes of anomalous quantum correction of the condensate, the energy correction may even change from attractive to repulsive, causing collapse at high densities [25]. These

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destabilizing effects are canceled upon interplay in presence of dipolar instability [26]. For instance, zero-point quantum fluctuations, which are known to stabilize attractive Bose gases against collapse, induce quantum droplets (QDs). These fluctuations are well described theoretically by beyond-mean-field (BMF) corrections. In one [27], two [28], and three [23] dimensions, respectively, the BMF correction in the energy density is proportional to $n^{3/2}$ (attractive), $n^2 \ln(n)$, and $n^{5/2}$ (repulsive), where n is the condensate density, which plays an important role in stabilizing QDs. Detailed expressions of these corrections are given in Appendix A. In the specific case of three-dimensional systems, the beyond-mean-field correction is of Lee-Huang-Yang (LHY) type [29]. In extreme regimes of strong correlations, entanglement and even quantum quenches, a quite accurate description of the outcome of experiments on dipolar systems would resort to more elaborate methods of quantum many-body dynamics, including multiconfigurational time-dependent Hartree for indistinguishable particles [30,31]. However, these methods are quite complex to implement and require significantly increased computational costs. The Gross-Pitaevskii (GP) equation offers a cheaper but powerful and acceptably accurate alternative. Then, we base our model on the GP theory, which happens to be an efficient workhorse for most ultracold atom problems. Moreover, the above extreme regimes are beyond the scope of the present work. In a Gross-Pitaevskii equation, known to well describe the dynamics of quantum gases at ultralow temperatures, the LHY term represents a higher-order nonlinear repulsive interaction, which eliminates the collapse of the gas induced by the mean-field attractive force [23,32,33].

In this work, we intend to investigate the parametric and modulational instabilities of binary ultracold quantum gases with long-range dipole-dipole interactions and quantum fluctuations in deep optical lattices. We expect the inclusion of quantum fluctuations in our model to allow us to explore regimes where mean-field theory breaks down and provide a more comprehensive understanding of the instability mechanisms in ultracold quantum gases. The organization of the work is as follows: In Sec. II, starting from an extended Gross-Pitaevskii model that incorporates long-range interactions and beyond-mean-field corrections, we derive a discrete nonlinear Schrödinger equation using Wannier functions, which accurately describe the tight-binding regime of deep optical lattices. In Sec. III, we then perform analytical investigations based on linear stability methods to identify the conditions under which parametric and modulational instabilities arise. That allows us to emphasize the role of long-range interactions in modifying the stability thresholds and dynamical behavior. In Sec. IV, our analytical predictions are validated through direct numerical simulations, which reveal the intricate interplay between lattice depth, interaction strength, and the nonlocal nature of dipole-dipole forces. The work ends in Sec. V with a conclusion along with a summary of our findings.

II. MODEL

We intend to target two-component bosonic systems, where quantum fluctuations are significantly enhanced by low

dimensionality and dipolar interactions. We will establish a general theoretical framework that can describe a nonidentical dipolar Bose-Bose mixture under periodic driving. Then, we will examine in details the simpler case of identical components, which can serve as a natural first benchmark. This allows us not only to demonstrate the consistency of our model but also to lay the groundwork for immediate extensions. For this, consider a binary mixture of Bose-condensed gases with dipole-dipole interaction close to the absolute zero. The system is trapped in an optical lattice (OL) given by the (external) potential

$$V_{\text{ext}} = V_{0x} \sin^2(kx) + V_{0y} \sin^2(ky) + V_{0z} \sin^2(kz), \quad (1)$$

where $k = 2\pi/\lambda$, with λ being the laser wavelength, i.e., the lattice spacing is $\lambda/2$. The depths V_{0i} of the lattice in the three directions $i = x, y, z$ are determined by the intensity of the corresponding pair of laser beams, which is easily tunable in quantum-gas experiments. In this section, we start with the continuum description of the dipolar bosonic system in an optical lattice. Then we provide the corresponding discrete or latticelike version of our model and justify the validity of the model.

A. Evolution equation in continuous form

1. Full evolution beyond mean field

The mean-field evolution of the two-component BEC system is described by the macroscopic wave functions $\Psi_\nu(\mathbf{r}, t)$, with $\nu = 1, 2$. These condensate wave functions obey the following nonlocal Gross-Pitaevskii equations [34,35]:

$$i\hbar \frac{\partial \Psi_\nu}{\partial t} = \left[-\frac{\hbar^2 \nabla^2}{2m_\nu} + V_{\text{ext}}(\mathbf{r}) + \sum_{\nu'=1}^2 \mathcal{G}_{\nu\nu'} |\Psi_{\nu'}(\mathbf{r}, t)|^2 + \sum_{\nu'=1}^2 \int \mathcal{U}_{\nu\nu'}(\mathbf{r} - \mathbf{r}') |\Psi_{\nu'}(\mathbf{r}', t)|^2 d^3\mathbf{r}' \right] \Psi_\nu. \quad (2)$$

When all dipoles are polarized along the z axis, the potential or kernel of the dipole-dipole interaction (DDI) can be expressed in the form [20,36]

$$\mathcal{U}_{\nu\nu'}(\mathbf{r} - \mathbf{r}') = C_{\nu\nu'} \frac{1 - 3 \cos^2 \Theta_{\nu\nu'}}{|\mathbf{r} - \mathbf{r}'|^3}, \quad (3)$$

where $\Theta_{\nu\nu'}$ is the angle between the relative position vector $\mathbf{r} - \mathbf{r}'$ of the particles and the polarization axis z . The prefactor $C_{\nu\nu'}$ is the coupling constant, which depends on the dipole moment μ_D as well as the permeability μ_0 and the permittivity ε_0 of vacuum. For simplicity, one may assume equal atomic masses for the two components, like in the typical dysprosium-dysprosium mixture [37]. This assumption is of high experimental relevance due to the increased cost of manipulating simultaneously different ultracold dipolar samples like erbium and dysprosium and a lower cost of manipulating dipolar spin mixtures [38]. Even for dysprosium-erbium mixtures, the assumption is still a quite reasonable approximation [39]. Then, the short-ranged s -wave contact interaction coupling constant of the equal-mass system reads $\mathcal{G}_{\nu\nu'} = 4\pi\hbar^2 a_{\nu\nu'}/m$, where $a_{\nu\nu'}$ is the s -wave scattering length. The ratio between the strengths of the dipolar interaction (3) to the

contact interactions, denoted ϵ_{dd} , represents an important parameter in the dynamics of the system. To improve the model, one may go beyond mean-field approximation by including an LHY contribution due to quantum fluctuations. Beyond-mean-field corrections turn out to be much more relevant in weakly interacting dipolar condensates than in their nondipolar counterparts [25]. This inclusion is also more relevant for our model since our target is low-dimensional systems, where quantum fluctuations even play more intriguing roles. Then, in the dilute regime as well satisfied in ultracold atomic experiments, a good description of a two-component dipolar quantum gas beyond mean field can be given by the extended Gross-Pitaevskii equation (EGPE) that follows [40–42]:

$$i\hbar \frac{\partial \Psi_v}{\partial t} = \left[-\frac{\hbar^2 \nabla^2}{2m} + V_{\text{ext}}(\mathbf{r}) + \sum_{v'=1}^2 \mathcal{G}_{vv'} |\Psi_{v'}(\mathbf{r}, t)|^2 + \sum_{v'=1}^2 \int \mathcal{U}_{vv'}(\mathbf{r} - \mathbf{r}') |\Psi_{v'}(\mathbf{r}', t)|^2 d^3 \mathbf{r}' + \Delta \mu_{\text{LHY}}^{(v)} \right] \times \Psi_v(\mathbf{r}, t), \quad (4)$$

where the LHY correction to the chemical potential resulting from quantum fluctuations is given by $\Delta \mu_{\text{LHY}}^{(v)} = \partial \mathcal{E}_{\text{LHY}} / \partial n_v$, with $\mathcal{E}_{\text{LHY}}$ being the LHY energy density correction and n_v the constant density of the v th component of the mixture. The density dependence in dipolar condensates of the chemical potential happens to be dependent on ϵ_{dd} [25]. For a homogenous system, i.e., $V_{\text{ext}}(\mathbf{r}) = 0$, the energy density correction reads $\mathcal{E}_{\text{LHY}}^{\text{hom}} = -4\chi(n_1, n_2)$, with [37]

$$\chi(n_1, n_2) = -\frac{1}{2} \int \frac{d^D k}{(2\pi)^D} \sum_{\lambda=\pm} \frac{[\zeta_\lambda(\mathbf{k}) - E(k)]^3}{4\zeta_\lambda(\mathbf{k})E(k)}, \quad (5)$$

in dimension D , with $E(k) = \hbar^2 k^2 / (2m)$ being the Bogoliubov modes of the mixture. We have $\zeta_\lambda(\mathbf{k}) = \sqrt{E(k)[E(k) + U_\lambda(\theta_k)]}$, where θ_k is the angle between vector $\mathbf{k}$ and the dipole moments or polarization direction. The squared sound velocity terms are given by [37,39,41,42]

$$U_\pm = \sum_{v=1}^2 \vartheta_{vv} n_v \pm \sqrt{(\vartheta_{11} n_1 - \vartheta_{22} n_2)^2 + 4\vartheta_{12}^2 n_1 n_2}, \quad (6)$$

where $\vartheta_{vv'} = \mathcal{G}_{vv'} + \mathcal{G}_{vv'}^{\text{dd}}(3 \cos^2 \theta_k - 1)$, with the dipole-dipole interaction strength being $\mathcal{G}_{vv'}^{\text{dd}} = \mu_0 \mu_v \mu_{v'} / 3 = 4\pi C_{vv'} / 3 \equiv 4\pi \hbar^2 a_{vv'}^{\text{dd}} / m$. For an inhomogeneous system, we may only include the real part of the quantum fluctuations and thus set the quantum fluctuation energy density to be $\mathcal{E}_{\text{LHY}} = \text{Re}\{\mathcal{E}_{\text{LHY}}^{\text{hom}}[n_1(\mathbf{r}), n_2(\mathbf{r})]\}$ in the local density approximation [41].

2. One-dimensional setting with time-varying interaction strength

Low-dimensional quantum gas systems have recently gained more interest because they happen to be quite challenging due to the enhanced impact of quantum fluctuations on static and dynamic properties [25]. In particular, and even though there is no Bose-Einstein condensation in one dimension in a strict sense, it has become clear that the excitation spectrum of weakly interacting one-dimensional ultracold gases can be well predicted assuming a conden-

sate [28]. In that context, and to simplify our calculations, we consider a one-dimensional optical lattice described by $V_{\text{ext}}(x) = V_0 \sin^2(kx)$, where the energy barrier V_0 between adjacent sites is usually expressed in units of the recoil energy $E_{\text{rec}} = \hbar^2 k^2 / (2m)$. Due to strong optical confinement in the y and z directions, the transverse dynamics are frozen, and the condensate wave function factorizes into transverse and longitudinal parts. After integrating over the transverse modes, we obtain an effective one-dimensional (1D) equation, where the dipole-dipole interaction and quantum corrections are projected onto the axial dimension. Then the evolution of the system is ruled by the equation

$$i\hbar \frac{\partial \Psi_v}{\partial t} = \left[-\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} + V_{\text{ext}}(x) + \sum_{v'=1}^2 \mathcal{G}_{vv'} |\Psi_{v'}(x, t)|^2 + \sum_{v'=1}^2 \int \mathcal{U}_{vv'}(x - x') |\Psi_{v'}(x', t)|^2 dx' + \frac{\partial \mathcal{E}_{\text{BMF}}^{\text{1D}}}{\partial n_v} \right] \times \Psi_v(x, t), \quad (7)$$

where the BMF energy density correction is given by

$$\mathcal{E}_{\text{BMF}}^{\text{1D}} = \frac{m^{1/2}}{3\pi\sqrt{2}\hbar} \sum_{\lambda} U_{\lambda}^{3/2}, \quad (8)$$

as shown in Appendix B. Furthermore, we consider a symmetric system, which can consist of equal-mass (or homonuclear) bosonic species with same density and intraspecies scattering length, i.e., the repulsion strengths $\mathcal{G}_{11} = \mathcal{G}_{22} \equiv \mathcal{G}$ and thus $\vartheta_{11} = \vartheta_{22} = \vartheta$ [27,28]. The ratio between the strengths of the dipolar and contact interactions is given by $\epsilon_{dd} = \mu_0 \mu_D^2 / (3\mathcal{G})$. Using the Fourier components of the DDI kernel as given in Ref. [42], we expect a standard procedure similar to the usual practice for nondipolar gas [28] to lead, in the GP equation, to a QF term in the form $\eta_{\text{QF}} n^{1/2}$. The full EGPE calculations are analytically and numerically intensive, especially due to the nonlocalities. For simplification purpose, one may rely on the same-shape approximation (SSA) [39,41], where the mixture is balanced, with both components being miscible. Then, assuming $\Psi_1 = \Psi_2 \equiv \Psi / \sqrt{2}$, similarly to the results in Refs. [27,32,43,44] for nondipolar gases, the beyond-mean-field dynamics of the two-component dipolar BEC system is described by the following extended Gross-Pitaevskii equation:

$$i\hbar \frac{\partial \Psi}{\partial t} = \left[-\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} + \delta \mathcal{G} |\Psi|^2 + \eta_{\text{QF}} |\Psi| + V_{\text{ext}}(x) + \int \mathcal{U}(x - x') |\Psi(x', t)|^2 dx' \right] \Psi, \quad (9)$$

where parameters $\mathcal{G}$ and $\delta \mathcal{G} \equiv \mathcal{G} + \mathcal{G}_{12}$ are positive, $\mathcal{U} = 2\mathcal{U}_{11} = 2\mathcal{U}_{22}$, and $\eta_{\text{QF}} \propto \vartheta^{3/2}$. Within the SSA, the EGPE reduces to an effective single-component system. However, the rich nature of interspecies interactions, obeying $\mathcal{G}_{12}^2 < \mathcal{G}_{11}\mathcal{G}_{22}$, can still lead to more interesting dynamics compared to a standard single-component problem. For instance, the model generalizes nondipolar self-bound mixtures, which are necessarily miscible, self-bound dipolar mixtures in miscible ground-state phase without population imbalance. In particular, the system can explore parts of ground-state phase

diagrams like the one for Dy-Dy mixtures presented in Ref. [37].

For the sake of generality, one can make the interaction strength to be time dependent, i.e., $\mathcal{G} \equiv \mathcal{G}(t)$; that is, experimentally achieved by inducing Feshbach resonances [20,45], which are highly convenient tools in ultracold atomic systems for changing the scattering length. Time-dependent scattering lengths are incredibly important because they provide a master knob for controlling the fundamental interactions between atoms in a quantum gas, allowing us to engineer and probe a vast range of quantum phenomena, including BEC-BCS crossover, interaction ramping towards unitarity, and Floquet engineering, that would otherwise be inaccessible [46]. In particular, one can make a periodic driving of the interaction strength, which may allow us, for instance, to achieve parametric resonances.

B. Discrete evolution equation

The lattice description of quantum gas dynamics is known to be quite appropriate for quantum simulations. It is derived under specific experimental conditions. For deep OLs, the interwell barriers of the periodic structure are high enough, i.e., $V_0 \gg \mu$, where μ is the chemical potential. Then the wells, which are centered at the minima of the laser potential V_{ext} located at points $x_n = n\lambda/2$, with n integer, can be assumed to be isolated enough from each other. Such a system can be studied using the tight-binding model, which substitutes each wave function by an approximate set of wave functions $\psi_n(t)$ at different lattice sites n . The wave functions are separated into a complex dynamical part ψ_n and real spatial part Φ_n through the nonlinear tight-binding ansatz [3,47,48],

$$\Psi(x, t) = \sum_n \psi_n(t) \Phi_n(x - x_n; N_n(t)), \quad (10)$$

where $N_n(t) \equiv |\psi_n|^2$ is the number of atoms in the n th site at time t . The total number of atoms of each species in the lattice, $N \equiv \sum_n |\psi_n|^2$, ensures the normalization. The Wannier functions Φ_n , localized around the center of the n th well, are normalized to 1 and quasiorthogonal, i.e., $\int \Phi_n \Phi_{n\pm 1} dx \simeq 0$ and $\int |\Phi_n|^2 dx = 1$. Due to deep OLs, only tunnelling to nearest neighbors are important, and terms proportional to the integrals $\int \Phi_n^3 \Phi_{n\pm 1} dx$ and $\int \Phi_n^2 \Phi_{n\pm 1}^2 dx$ are neglected for the sake of simplicity. In the highly deep optical lattice regime, Wannier functions $\Phi_n(x)$ are reliably approximated by Gaussians centered at each lattice site [49]. This approximation is used to compute the overlap integrals for tunneling, nonlinearity, dipole-dipole interaction, and beyond-mean-field terms in the discrete nonlinear Schrödinger (DNLS) equation (and not to approximate the time-dependent dynamics). Then the DNLS model for the set of macroscopic wave functions $\psi_n(t)$ can be derived by multiplying the equation on the left by Φ_n^* and integrating over space. We obtain the following set of effective one-dimensional DNLS equations:

$$i\hbar \frac{\partial \psi_n}{\partial t} = -J(\psi_{n+1} + \psi_{n-1}) + U_c(t)|\psi_n|^2 \psi_n + \mathcal{E}_n \psi_n + U_d(|\psi_{n+1}|^2 + |\psi_{n-1}|^2) \psi_n - U_q(t)|\psi_n| \psi_n. \quad (11)$$

As simplification hypothesis to approximate Wannier's functions, we assumed a fixed average number N_0 of atoms per

site, and kept only the zeroth-order terms, such that $\Phi_n(x - x_n, N_n) \rightarrow \Phi_n(x - x_n, N_0)$. The hopping parameter J (i.e., the tunneling rate between two neighboring lattice sites), the cubic nonlinear coefficient U_c , the effective strengths of DDI U_d , and quantum fluctuation U_q are expressed through overlap integrals of the Wannier functions as follows:

$$\begin{aligned} J &\simeq - \int dx \left[\frac{\hbar^2}{2m} \frac{\partial \Phi_n^*}{\partial x} \frac{\partial \Phi_{n\pm 1}}{\partial x} + \Phi_n^* V_{\text{ext}}(x) \Phi_{n\pm 1} \right], \\ U_c(t) &\simeq \delta \mathcal{G}(t) \int dx |\Phi_n|^4, \quad \delta \mathcal{G}(t) \equiv \mathcal{G}_{12} + \mathcal{G}(t), \\ \mathcal{E}_n &\simeq \int dx \Phi_n^* \left[-\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} + V_{\text{ext}}(x) \right] \Phi_n, \\ U_d &\simeq \int dx dx' |\Phi_n^*(x)|^2 C_{dd} \frac{1 - 3 \cos^2 \Theta}{|x - x'|^3} |\Phi_{n\pm 1}(x')|^2, \\ U_q(t) &\simeq - \int dx \eta_{\text{QF}} \vartheta(t)^{3/2} |\Phi_n|^3. \end{aligned} \quad (12)$$

We recall that it is the orientation of dipoles, which determines the sign of U_d . For dipolar particles in a head-to-tail configuration, we have $\Theta = 0$, i.e., $U_d < 0$, and the dipole-dipole interaction is attractive. Meanwhile, for dipoles sitting side by side, we have $\Theta = \pi/2$, i.e., $U_d > 0$, and the dipole-dipole interaction is repulsive. The interaction vanishes at the magic angle $\Theta = \arccos(1/\sqrt{3}) \approx 54.7^\circ$. When the species have a quite strong dipolar moment, the DDI term in Eq. (11) may even contain next-nearest-neighbor components and beyond, as shown in Ref. [49]. The explicit expressions of parameters J , U_c , U_d , and U_q are not important in the present work, but they may be found if one has an approximate expression of the ground-state wave function. To further simplify Eq. (11), let us introduce the rotational frame change $\psi_n \rightarrow \psi_n \exp(-i\mathcal{E}_n t/\hbar)$ and $t \rightarrow tJ$, and work in units where $\hbar = 1$. Then, the DNLS equation that rules the system reads

$$\begin{aligned} i \frac{\partial \psi_n}{\partial t} &= -(\psi_{n+1} + \psi_{n-1}) + g(t)|\psi_n|^2 \psi_n \\ &\quad + \gamma(|\psi_{n+1}|^2 + |\psi_{n-1}|^2) \psi_n - \eta(t)|\psi_n| \psi_n, \end{aligned} \quad (13)$$

where $g(t) = U_c(t)/J$, $\gamma = U_d/J$, and $\eta(t) = U_q(t)/J$. For a system under Feshbach resonance management, one may set

$$\mathcal{G}(t) = \mathcal{G}_c + \mathcal{G}_1 \sin(\omega t), \quad (14)$$

and then $\vartheta(t) = \vartheta_c + \mathcal{G}_1 \sin(\omega t)$, where $\vartheta_c = \mathcal{G}_c + \mathcal{G}^{\text{dd}}(3 \cos^2 \theta_k - 1)$. Consequently, to the first order in ϵ , the rescaled two-body interaction strength and LHY coefficients may have the form $g(t) \equiv g_0[1 + \epsilon \sin(\omega t)]$ and $\eta(t) \equiv \eta_0[1 + \epsilon \sigma \sin(\omega t)]$, where

$$\begin{aligned} g_0 &= (\mathcal{G}_{12} + \mathcal{G}_c) \int dx |\Phi_n|^4, \quad \epsilon = \frac{\mathcal{G}_1}{\mathcal{G}_{12} + \mathcal{G}_c}, \\ \eta_0 &= - \int dx \eta_{\text{QF}} \vartheta_c^{3/2} |\Phi_n|^3, \quad \sigma \simeq \frac{3}{2} \left(1 + \frac{\mathcal{G}_{12}}{\vartheta_c} \right), \end{aligned} \quad (15)$$

with $\mathcal{G}_c \gg \mathcal{G}_1$. Without loss of generality, we restrict ourself to the particular case when the dimensionless parameter $\sigma = 1$, which is achieved for $\mathcal{G}_{12} \simeq -\vartheta_c/3$. This is actually not a shortcoming of the model because any arbitrary values of $\mathcal{G}_{12}$

can be experimentally realized. Furthermore, the results for any σ can be straightforwardly deduced following the steps of the current analysis and applying an analytical rescaling. As σ only multiplies the modulation amplitude of the quantum fluctuation term, no additional simulation is required as long as the product $\epsilon\sigma$ remains small compared to 1. For larger values of σ , the modulation is no more a perturbation, and a new analytical procedure would be required.

C. Validity of our model

Ultracold atoms and molecules with dipolar interactions have proven to be accurate quantum simulators in realizing (Bose)-Hubbard models in one and two dimensions [30]. These simulators unravel how such interactions impact the physics of important phases of matter, including quasicrystals and Tonks-Girardeau states. In regimes of strong dipole-dipole interactions and high densities, however, the continuum system fails to recreate the desired lattice model, raising the question of validity of quantum simulation of lattice models in such regimes. The validity of our model and analysis is mostly restricted to regimes of mild dipole-dipole interactions and low densities. Strickly speaking, preparing a true 1D system is a theoretical exercise. All experimental systems are actually quasi-1D. A true 1D system would be reached with infinite confinement, which is not attainable in the laboratory. However, they can be so strongly confined that they behave as if they were true 1D systems, nearly freezing completely the transverse degrees of freedom and achieving fully 1D kinematics. For accessing a quasi-1D regime, as described in Refs. [25,27,50], the transverse confinement in a 2D or 3D gas is made sufficiently tight, $\hbar\omega_{\perp} \gg \mu$, which demands $|1 - \epsilon_{\text{dd}}| \ll 1/(2n_{\text{1D}}a_s)$, with a_s being the s -wave scattering length common to both components of the gas mixture. With such a lower chemical potential, the condensate remains kinematically 1D such that its wave function splits into radial and longitudinal parts [25]. The condensate should remain dilute enough, with $n_{\text{1D}}a_s \ll 1$, to avoid transverse excitations. In such a case, requiring $\epsilon_{\text{dd}} \leq 1$ may yield the formation of droplets. In addition, dipole-dipole interactions in confined directions should be kept negligible.

III. ANALYTICAL INVESTIGATIONS

While modulational and parametric instabilities are characterized by an exponential growth in the wave amplitude, the latter occurs at best in a modulationally stable system under the effect of periodic driving. Then, we will first study the conditions under which modulated plane waves become unstable in the system ruled by Eq. (13) under a time-independent particle-particle interaction strength. Next, we will consider time-periodic modulation of that interaction strength and examine parametric instability in the system.

A. Modulational instability

In order to examine the instability of modulated plane wave under perturbation, we use the linear stability approach, which directly allows getting the instability criteria through a linearization process.

1. Linear stability analysis

The stationary carrier wave solution of the DNLS Eq. (13) is a plane wave of the form

$$\psi_n(t) = u_0 e^{i(qn + 2t \cos q - \int_0^t [g(s)|u_0|^2 - \eta(s)|u_0| + 2\Delta_1] ds)}, \quad (16)$$

where $\Delta_1 \equiv \gamma|u_0|^2$. To examine the modulational instability (MI) of the BEC system via the linear stability analysis, we introduce a perturbed ansatz by replacing u_0 in Eq. (16) by $u_0(1 + \delta\phi_n(t))$, where $\delta\phi_n(t)$ is a small complex perturbation, i.e., $|\delta\phi_n| \ll 1$. Substituting the perturbed plane wave into the DNLS Eq. (13), and neglecting terms of order higher than 1 in $\delta\phi_n$ and its complex conjugate $\delta\phi_n^*$, we find that the evolution of the perturbation obeys the linearized equation:

$$\begin{aligned} i \frac{\partial(\delta\phi_n)}{\partial t} = & -(\delta\phi_{n-1} e^{-iq} + \delta\phi_{n+1} e^{iq} - 2 \cos q \delta\phi_n) \\ & + \Delta(t)(\delta\phi_n^* + \delta\phi_n) \\ & + \Delta_1(\delta\phi_{n+1}^* + \delta\phi_{n-1}^* + \delta\phi_{n+1} + \delta\phi_{n-1}), \end{aligned} \quad (17)$$

where $\Delta(t) \equiv g(t)|u_0|^2 - \eta(t)|u_0|/2$, which can be readily rewritten as $\Delta(t) = \Delta_0[1 + \epsilon \sin(\omega t)]$, with $\Delta_0 \equiv g_0|u_0|^2 - \eta_0|u_0|/2$. Let us look for plane-wave perturbations of the form

$$\delta\phi_n(t) = \alpha(t)e^{ikn} + \beta(t)^*e^{-ikn}, \quad (18)$$

where k is the perturbation wave number and $\alpha(t)$ and $\beta(t)$ are time-dependent complex fields, which are weak compared to 1. Replacing the ansatz (18) into Eq. (17) and linearizing, we obtain the following pair of first-order coupled equations:

$$\begin{aligned} i \frac{d\alpha}{dt} &= (B + C)\alpha + \Gamma(\alpha + \beta), \\ i \frac{d\beta}{dt} &= (B - C)\beta - \Gamma(\alpha + \beta), \end{aligned} \quad (19)$$

where $B = 2 \sin q \sin k$, $C = 2 \cos q (1 - \cos k)$, and $\Gamma(t) = \Delta(t) + 2\Delta_1 \cos k$, which can be rewritten as $\Gamma(t) = \Gamma_0 + \epsilon \Delta_0 \sin(\omega t)$, with $\Gamma_0 \equiv \Delta_0 + 2\Delta_1 \cos k$ being the effective nonlinearity strength of the system. The above two equations can be combined to form a second-order equation as follows:

$$\begin{aligned} \frac{d^2\alpha}{dt^2} = & \left[2iB - \frac{d(\ln \Gamma)}{dt} \right] \frac{d\alpha}{dt} \\ & + \left[C^2 - B^2 + 2C\Gamma - i(C + B) \frac{d(\ln \Gamma)}{dt} \right] \alpha = 0. \end{aligned} \quad (20)$$

In the static case when $\epsilon = 0$, Eq. (20) corresponds to a second-order equation with constant coefficients, whose solution $\alpha(t) \propto \exp(iB \pm \sqrt{-(C^2 + 2C\Gamma_0)}t)$, which yields the MI condition

$$C(C + 2\Gamma_0) \equiv \Omega^2 < 0. \quad (21)$$

In that case, the local growth rate of instability corresponding to unstable modes is given by $\xi = \text{Im}(\Omega)$. Obviously the MI condition (21) is in accordance with the results of the BECs with two-body interatomic interaction [13]. Indeed, in the absence of both quantum fluctuations (QFs) and dipole-dipole

interaction, i.e., $\eta_0 = \gamma = 0$, we have $\Gamma_0 = g_0|u_0|^2$ and retrieve the MI criteria

$$\cos^2 q (1 - \cos k) + g_0|u_0|^2 \cos q < 0, \quad (22)$$

for plane waves of background wave number q in a standard BEC with contact two-body interaction embedded in a deep OL [48] and perturbed with wave numbers k . It is worth mentioning that the instability criterion (21) is symmetric in the (q, k) domain, since it is invariant under the change $q \rightarrow -q$ and $k \rightarrow -k$. Therefore it is enough to only focus on positive q and k . Of all wave numbers that satisfy the instability condition (21), there exists particular ones that yield the largest instability growth rate in the system. In safe parameter domains, these critical wave numbers are given by

$$k_{\text{cr}} = \begin{cases} \pm \arccos\left(\frac{\Delta_0 - 2\Delta_1 + 2\cos q}{-4\Delta_1 + 2\cos q}\right), & q \in]-\frac{\pi}{2}, \frac{\pi}{2}[\\ \pi, & q \in]\frac{\pi}{2}, \frac{3\pi}{2}[\end{cases} \quad (23)$$

where k and q are defined modulo 2π . The corresponding maximum instability growth rate, $\xi_{\text{max}} \equiv \xi(k = k_{\text{cr}})$, is given by

$$\xi_{\text{max}} = \text{Re}\left(\left[\frac{(\Delta_0 + 2\Delta_1)^2 \cos q}{-2\Delta_1 + \cos q}\right]^{1/2} + 2\sqrt{2}[-\cos q(\Delta_0 - 2\Delta_1 + 2\cos q)]^{1/2}\right). \quad (24)$$

Compared to the gain ξ per wave number k , the maximum gain ξ_{max} has the advantage to better reproduce the results of computer experiments with single-wavelength or noisy perturbations and even laboratory experiments.

2. Instability domains

From the obtained instability criteria, the strength of quantum fluctuations may allow us to distinguish two main regimes: the BEC and quantum droplet regimes, given by $\eta_0 < \eta_{0c}$ and $\eta_0 > \eta_{0c}$, respectively, with $\eta_{0c} = 2g_0|u_0|$. The left (BEC) and right (QDs) sides of Fig. 1, separated by a yellow vertical line, depict these regimes. They were obtained by looking at the behavior of the term $\Gamma_0/(1 - \cos k)$ deductible from the condition (21). The whole set of subdomains presents a 180° rotation symmetry around the origin $(\eta_0 - \eta_{0c}, \gamma) = (0, 0)$. The subdomains are demarcated by well-known straight lines. From top to bottom, the three lines with negative slope are given by $\gamma = -\frac{1}{4|u_0|}(\eta_0 - \eta_{0c}) + \frac{1}{|u_0|^2}$, $\gamma = -\frac{1}{4|u_0|}(\eta_0 - \eta_{0c})$, and $\gamma = -\frac{1}{4|u_0|}(\eta_0 - \eta_{0c}) - \frac{1}{|u_0|^2}$. The line with positive slope is given by $\gamma = \frac{1}{4|u_0|}(\eta_0 - \eta_{0c})$. Each side contains four subdomains in which the system behaves differently. Considering the wave numbers, one may discriminate the intervals of interest $q \in]-\pi/2, \pi/2[$ (long wavelengths) and $q \in]\pi/2, 3\pi/2[$ (short wavelengths) mod $[2\pi]$. (i) In subdomains N, for any $q \in]-\pi/2, \pi/2[$, there exists no value of the perturbation wave number k , which can yield the instability in the system. Modes with $q \in]\pi/2, 3\pi/2[$ are expected to be unstable in subdomain N1 for only some values of k , and unstable in subdomain N2 for all values k . (ii) In subdomains P, all modes with $q \in]\pi/2, 3\pi/2[$ are stable for any perturbation wave number k . Meanwhile, modes with $q \in]-\pi/2, \pi/2[$ are expected to be unstable in subdomain

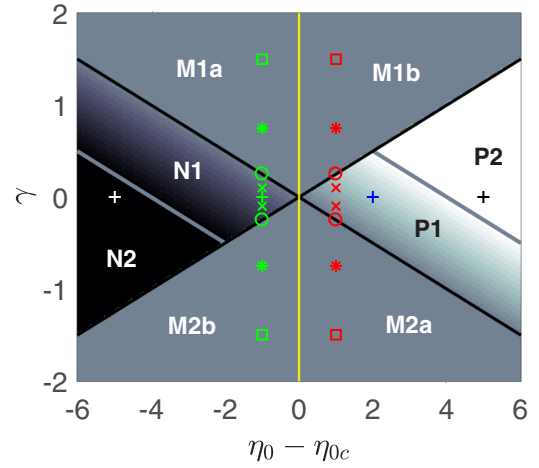


FIG. 1. Parameter subdomains that determine the stability of the system in the (η_0, γ) plane. We have $u_0 = 1.0$ and $g_0 = 1.0$, which implies $\eta_{0c} = 2$. Markers depict some selected points where the structure of gain profiles will be examined. The green markers are taken in the BEC regime ($\eta_0 < \eta_{0c}$) while the red ones lie in the QDs regime ($\eta_0 > \eta_{0c}$). The yellow vertical line ($\eta_0 = \eta_{0c}$) separates these two regimes.

P1 for only some values of k , and unstable in subdomain P2 for all values k . (iii) In subdomains M, i.e., M_{1a} , M_{1b} , M_{2a} , and M_{2b} , for any given value of q , they will always exist at least one value of k such that the mode (q, k) is stable, and at least one value of k such that the mode is unstable.

3. Instability gain profiles

To display the instability gain profiles corresponding to the above instability domains, we first consider the system in the absence of dipole-dipole interaction, i.e., $\gamma = 0$. The MI gain profiles in the (q, k) plane are shown in Fig. 2, where Figs. 2(a1) and 2(a2) correspond to the BEC side while Figs. 2(b1) and 2(b2) correspond to the QDs side. We observe different behaviors in the QDs and BEC regimes. In the QDs regime, the unstable bandwidths are located in the long-wavelengths interval $q \in]-\pi/2, \pi/2[$. Conversely, in the BEC regime, they lie within the short-wavelengths interval $q \in]\pi/2, 3\pi/2[$. A disk-shaped stability window exists at the center of each unstable region. Furthermore, in the BEC regime, as the density increases, the instability gain increases while the domain of unstable wave numbers in the (q, k) plane broadens. In the QDs regime, the system behaves differently, as the unstable bandwidths become narrower and the MI gain smaller when the density increases. It is worth mentioning that away from the critical value η_{0c} , the QF strength enhances the instability in both regimes by enlarging both the bandwidths and growth rate of instability. For strengths η_0 such that $|\eta_0 - \eta_{0c}| \leq 4$, as portrayed in Figs. 2(a3) and 2(b3), the stability window closes and all modes (q, k) in the respective regimes and q interval become unstable. Besides, unconditional MI suppression, for all perturbation wave numbers k , is expected for long and short wavelengths, respectively, in the BEC and QDs regimes, as seen through the zero gain areas in Fig. 2.

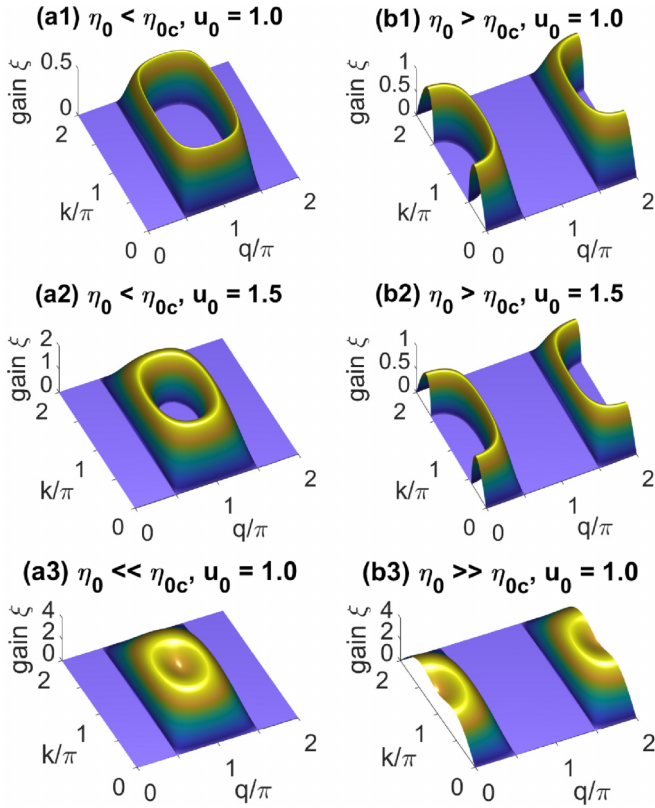


FIG. 2. Effect of wave amplitude $|u_0|$ on the modulational instability gain profiles analytically predicted for (a1) $\eta_0 = 1, u_0 = 1.0$; (b1) $\eta_0 = 4, u_0 = 1.0$; (a2) $\eta_0 = 1, u_0 = 1.5$; (b2) $\eta_0 = 4, u_0 = 1.5$; (a3) $\eta_0 = -3, u_0 = 1.0$; (b3) $\eta_0 = 7, u_0 = 1.0$. The other parameters are $\gamma = 0, N = 2048, g_0 = 1.0$. For $\eta_0 = 2g_0|u_0| \equiv \eta_{0c}$, the gain is zero at all (q, k) points. We have $\eta_{0c} = 2$ and 3 for $u_0 = 1.0$ and 1.5 , respectively. The left panels (a1), (a2), and (a3) correspond to the BEC side while the right ones, (b1), (b2), and (b3), correspond to the QDs side. Panels (a3), (a1), (b1), and (b3) correspond to white, green, blue, and black plus (+) markers, respectively, depicted in Fig. 1.

Let us now consider the system in the presence of dipole-dipole interaction. For a fixed $\gamma \neq 0$, the above described behaviors of the system at $\gamma = 0$ still apply to pure BEC and QDs regimes defined by $\eta_0 < \eta_{0c-}$ and $\eta_0 > \eta_{0c+}$, respectively, where $\eta_{0c\pm} = \eta_{0c} \mp 4\gamma|u_0|$. For strengths of $\eta_0 \in [\eta_{0c-}, \eta_{0c+}]$, however, the system lies in a mixed regime where its stability is conditional and determined by the specific values of wave numbers. For different values of γ , however, there can be more possible scenarios as displayed in Figs. 3 and 4, on the BEC and QDs sides, respectively, and for both attractive ($\gamma < 0$) and repulsive ($\gamma > 0$) dipole-dipole interaction. For weak strengths of the DDI, i.e., $|\gamma| \leq 0.25 \equiv |\eta_{0c} - \eta_0|/(4|u_0|)$, corresponding to subdomains N1 and P1 of Fig. 1, there is absence of MI in the long-wavelengths interval $q \in [-\pi/2, \pi/2]$ and short-wavelengths interval $q \in [\pi/2, 3\pi/2]$ on the BEC and QDs sides, respectively. For stronger strengths of the DDI, i.e., $|\gamma| \geq 0.25$, corresponding to subdomains M of Fig. 1, the maximum value of the gain is higher compared to the previous case, and the absence or presence of MI at both long and short wavelengths depends only the selected pair (q, k) . This increase of maximum growth rate

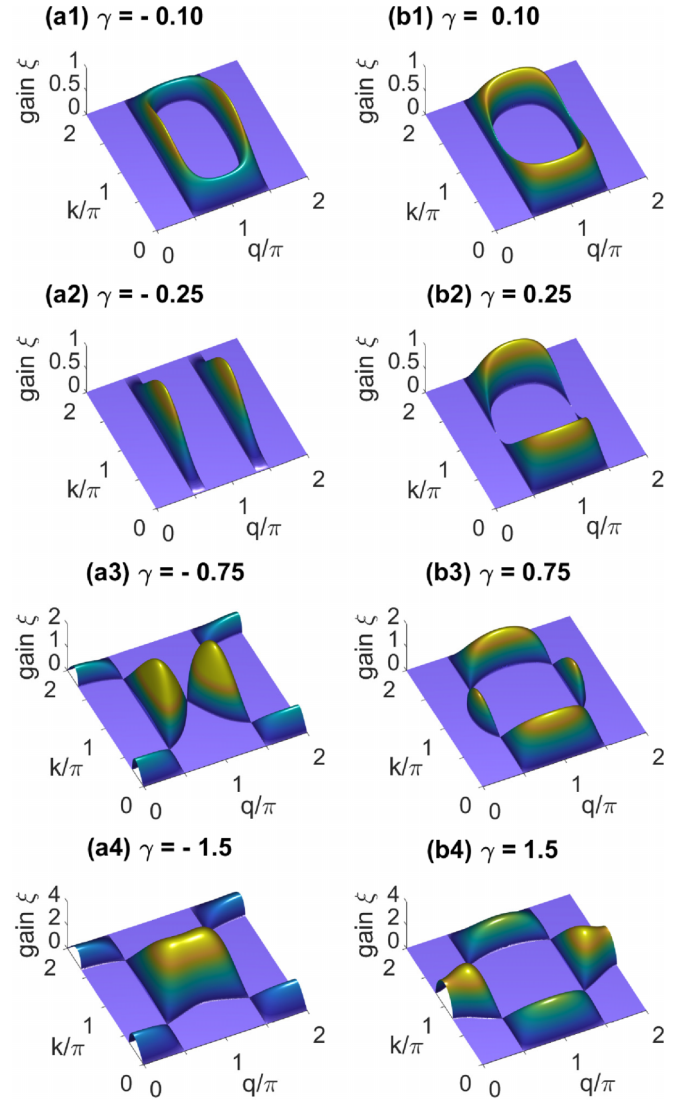


FIG. 3. Effect of QFs strength γ on the modulational instability gain profiles analytically predicted for (a) $\gamma < 0$ (attractive DDI), (b) $\gamma > 0$ (repulsive DDI) on the BEC side ($\eta_0 < \eta_{0c}$). We used $\eta_0 = 1, N = 2048, u_0 = 1.0, g_0 = 1.0$. The case $\gamma = 0$ was already presented in Fig. 2(a1) with same parameters. Panels (a1) and (b1), (a2) and (b2), (a3) and (b3), and (a4) and (b4) correspond to green cross, circle, asterisk, and square markers, respectively, depicted in Fig. 1.

as well as the ubiquity of unstable modes at all wavelengths infer a possible enhancement of the instability at large DDI strengths.

These rich results are obtained for $\epsilon = 0$, i.e., in the absence of any temporal modulation in the parameters of the system. A natural question that emerges is how such a stability would be impacted by the modulation of contact interaction. We explore this question in the following subsection.

B. Parametric instability

Similarly to modulational instability, we consider an instability, referred to as *parametric*, characterized by an exponential growth in the wave amplitude, happening in the

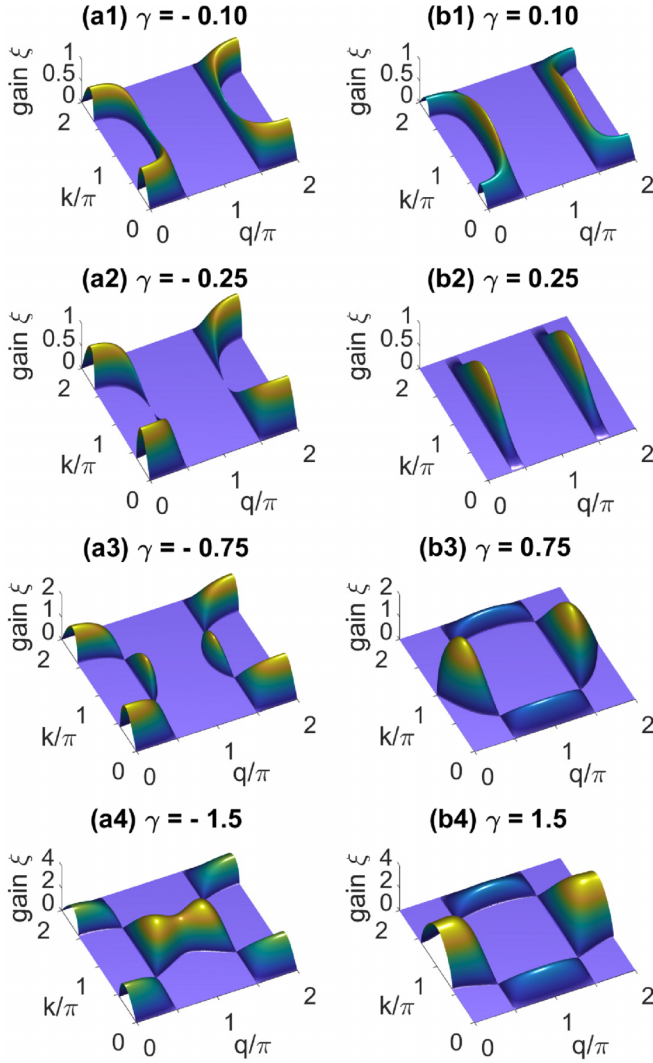


FIG. 4. Effect of QFs strength γ on the modulational instability gain profiles analytically predicted for (a) $\gamma < 0$ (attractive DDI), (b) $\gamma > 0$ (repulsive DDI) on the QDs side ($\eta_0 > \eta_{0c}$). We used $\eta_0 = 3$, $N = 2048$, $u_0 = 1.0$, $g_0 = 1.0$. The case $\gamma = 0$ was already presented in Fig. 2(a1), with same parameters. Panels (a1) and (b1), (a2) and (b2), (a3) and (b3), and (a4) and (b4) correspond to red cross, circle, asterisk and square markers, respectively, depicted in Fig. 1.

context of periodically driven interactions. Up to order $\mathcal{O}(\epsilon)$, the above equation (20) can be recast as

$$\ddot{\alpha} = \left[2iB - \epsilon \omega \frac{\Delta_0}{\Gamma_0} \cos(\omega t) \right] \dot{\alpha} + \left(C^2 - B^2 + 2C\Gamma_0 + \epsilon \left[2C\Delta_0 \sin(\omega t) - i\omega(C+B) \frac{\Delta_0}{\Gamma_0} \cos(\omega t) \right] \right) \alpha = 0, \quad (25)$$

where the dots stand for the differentiation with respect to time. Because (25) is an ordinary differential equation with periodic coefficients, it is natural to examine whether such a temporal modulation may induce parametric instabilities in the system.

1. Instability criteria and domains: Multiple-scale analysis

To study parametric instabilities, we consider a *modulationally stable* system, i.e., $\Omega^2 > 0$, and introduce a regular perturbation expansion of the perturbation amplitude α as a series in ϵ as follows:

$$\alpha(t) = \alpha_0(t) + \epsilon \alpha_1(t) + \mathcal{O}(\epsilon^2). \quad (26)$$

The next step consists in using the ansatz (26) to solve Eq. (25) at different orders. It follows that $\alpha_0 = e^{iB}(c_1 e^{i\Omega t} + c_2 e^{-i\Omega t})$, where $c_{1,2}$ are constants. Secular behaviors in α may arise if the denominator of the first-order term α_1 turns to zero or equivalently if at least one right-hand side term of the α_1 equation picks up the natural frequency of the system, i.e., $\omega \pm \Omega = \pm \Omega$. In such a case, we expect the amplitude to diverge. That happens when $\Omega = \omega/2 \equiv \Omega_0$, i.e., the driving frequency is twice the natural frequency, and thus $C = -\Gamma_0 + \sqrt{\Gamma_0^2 + \omega^2/4} \equiv C_0$, which corresponds to

$$q = \pm \arccos(C_0/C_m) \equiv \pm q_0. \quad (27)$$

The parameter $C_m = 2(1 - \cos k)$ denotes the maximum value of C over the q domain. The condition (27) holds only for driving frequencies $\omega \leq 2\sqrt{C_m(C_m + 2\Gamma_0)}$, and when the fixed background wave parameters u_0 , k and system's parameters g_0 , γ , η_0 satisfy the relation $\Gamma_0 \geq -C_m/2$. To explore the dynamics near the first parametric resonance value, we implement a multiple-scale expansion by letting $\Omega = \Omega_0 + \Omega_1 \epsilon + \mathcal{O}(\epsilon^2)$, with multiple scale ansatz

$$\alpha(t) = \alpha_0(t, t_1) + \epsilon \alpha_1(t, t_1) + \dots, \quad (28)$$

where $t_1 = \epsilon t$. Since Ω is a function of q , consequently, it follows the expansion $q = q_0 + q_1 \epsilon + \mathcal{O}(\epsilon^2)$. Performing the multiple-scale expansion yields

$$q_1 = \pm \frac{-\Gamma_0 + \sqrt{\Gamma_0^2 + \frac{\omega^2}{4}}}{4\sqrt{\Gamma_0^2 + \frac{\omega^2}{4}}(1 - \cos k) \sin q_0}. \quad (29)$$

Substituting (29) into the expansion of q , we identify the instability domain boundaries in the (q, ϵ) domain to be

$$\epsilon = \pm \frac{2(C_0 + \Gamma_0)\sqrt{C_m^2 - C_0^2}}{C_0}(q - q_0). \quad (30)$$

These instability boundaries may allow us to locate and draw the parametric instability bands within the modulationally stable parts of the parameter domains.

In Fig. 5, we display canals of parametric instability in the (q, γ) plane for three values of the QFs parameter η_0 . The walls of the canals are given by Eq. (30). For wave numbers within the canals, parametric instability is expected to occur, while for wave numbers in the neighborhood of the canals, we expect parametric stability. It can be seen that the canals, which are not straight, narrow for increasing background wave numbers q and for increasing γ and η_0 . Each of these canals actually lies in the modulationally stable part of the parameter domain, as shown in Fig. 6. Depending on the choice of parameters, the instability windows, which represents a cut at specific ϵ and γ in Fig. 5, can go from tiny in Fig. 6(a) to broader and well visible enough in Fig. 6(d).

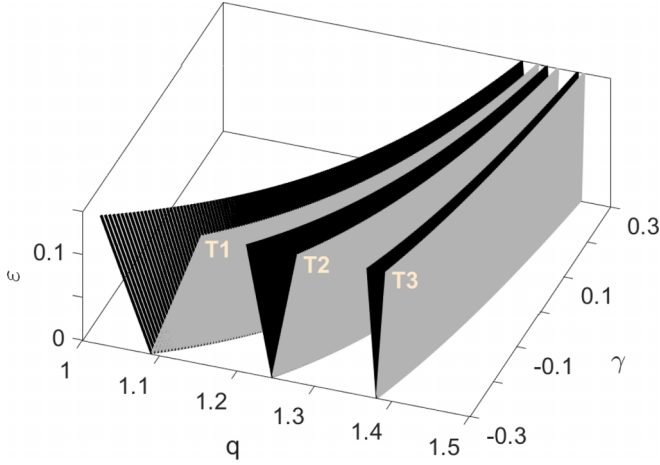


FIG. 5. Parametric instability canals in the (q, γ) plane at wave numbers around $q = q_0$ for (T1) $\eta_0 = -0.5$, (T2) $\eta_0 = 0$, and (T3) $\eta_0 = 1.0$. We used $k = 0.25\pi$, $\omega = 1.0$, $g_0 = 1$, and $u_0 = 1$.

2. Parametric instability suppression

Relation (30) can be helpful also for searching the suppression of parametric instability, or at least its avoidance in the system. The idea of instability suppression is to make the bandwidth as narrow as possible or even close it. The analytically predicted value of the bandwidth of parametric instability is

$$\Delta q_{\text{anal}} \equiv 2q_1\epsilon = \frac{C_0\epsilon}{(C_0 + \Gamma_0)\sqrt{C_m^2 - C_0^2}}. \quad (31)$$

In the context of periodic driving, C_0 cannot be canceled. However, we can make the ratio (31) small by selecting

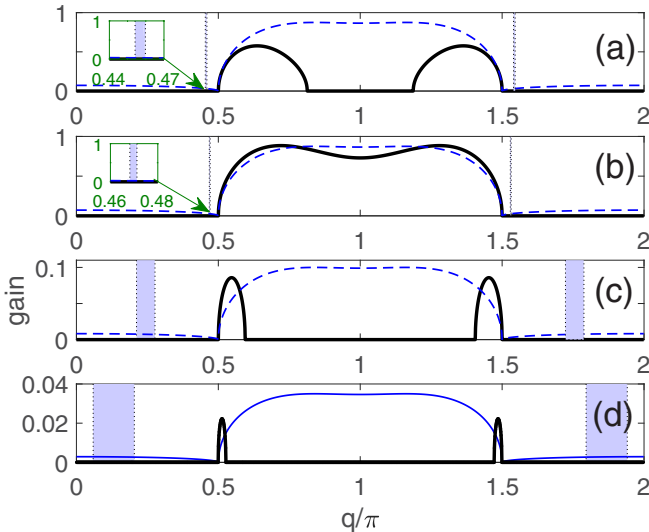


FIG. 6. Locations of parametric instability windows within the modulationally stable regions in the q domain, when (a) $k = 0.40\pi$, $\gamma = -1.5$; (b) $k = 0.40\pi$, $\gamma = -1.0$; (c) $k = 0.25\pi$, $\gamma = -1.0$; and (d) $k = 0.235\pi$, $\gamma = -1.0$. The other parameters are $\epsilon = 0.10$, $\eta_0 = -1$, $\omega = 1.0$, $g_0 = 1$, and $u_0 = 1.0$. Solid black lines depict the instability gain ξ for a specific k . Blue dashed lines represent the maximum gain ξ_{max}/ℓ , where the scaling factor $\ell = 4, 4, 35$, and 100 for (a), (b), (c), and (d), respectively.

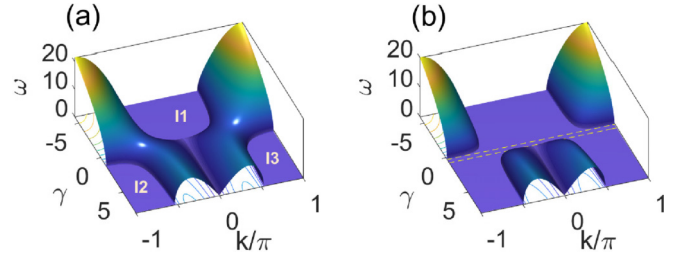


FIG. 7. Separatrix sheets, which cover parametrically unstable modes in tunnels below it and keeps parametrically stable ones above it. We used (a) $g_0 = 1.0$ and (b) $g_0 = -1.0$, with $\eta_0 = 1.0$ and $u_0 = 1.0$. There are three stability islands I1, I2, and I3 and two full instability tunnels in panel (a). The instability tunnels are constricted in panel (b), creating an isthmus of stability depicted by yellow dashed lines and four dead-end tunnels.

large effective nonlinearity strengths Γ_0 and carefully sizing the system to filter permitted perturbation wave numbers. As already shown in the section above, the instability windows can be made broader or narrower. Besides, we can completely suppress parametric instability via an appropriate choice of parameters, which we use Fig. 7 to illustrate. The separatrix sheet, defined by the critical frequency

$$\omega_c \equiv \text{Re}[2\sqrt{C_m(C_m + 2\Gamma_0)}], \quad (32)$$

in the (k, γ) plane, forms in Fig. 7(a) two tunnels inside which parametrically unstable modes can be picked. These modes satisfy $\omega < \omega_c$. Modes picked anywhere outside the instability tunnels, i.e., $\omega > \omega_c$ are expected to be parametrically stable. Among the latter are, for instance, high-frequency modes with repulsive DDI, which lie well above the sheet. Another way for suppressing parametric instability consists in choosing the parameters such that $\Gamma_0 < -C_m/2$. In that case, we may select the DDI strength $\gamma \in [-\frac{\eta_0 - \eta_{0c}}{4|u_0|} + \frac{1}{|u_0|^2}, \frac{\eta_0 - \eta_{0c}}{4|u_0|}]$ under the requirement $\eta_0 > \eta_{0c} + 2/|u_0|$, which happens to correspond to modes in subdomain P_2 of Fig. 1. Concrete examples of these scenarios can be attained with $u_0 = 1$, $g_0 = 1$, and $\eta_0 = 5 > 2/|u_0| + \eta_{0c} = 4$, i.e., repulsive contact interaction and strong quantum fluctuations, where $\gamma \in [\frac{1}{4}, \frac{3}{4}]$. A more realistic example with weaker quantum fluctuations but attractive contact interaction is gotten when $u_0 = 1$, $g_0 = -1$, and $\eta_0 = 1 > 2/|u_0| + \eta_{0c} = 0$, where $\gamma \in [\frac{1}{4}, \frac{3}{4}]$, as portrayed in Fig. 7(b). Infinitely many examples of these kinds can be envisioned. In all these cases, there is a constriction around $\gamma = 1/(2|u_0|)$ in the two instability tunnels from Fig. 7(a), which opens a stability isthmus of width $\Delta\gamma = \frac{\eta_0 - \eta_{0c}}{2|u_0|} + \frac{1}{|u_0|^2}$ for any perturbation wave number k as shown in Fig. 7(b). The stability isthmus both creates a k -independent stable zone and connects the three stability islands from Fig. 7(a). Hence, one may suppress parametric instability at some frequencies and DDI. In contexts where we cannot completely suppress parametric instability, we can still either lessen it by shrinking the unstable bandwidth or avoid it by operating the system with parameters away from the bandwidth.

IV. NUMERICAL EXPERIMENTS

The analytical results obtained in the sections above are based on linear instability analysis method. Such a technique,

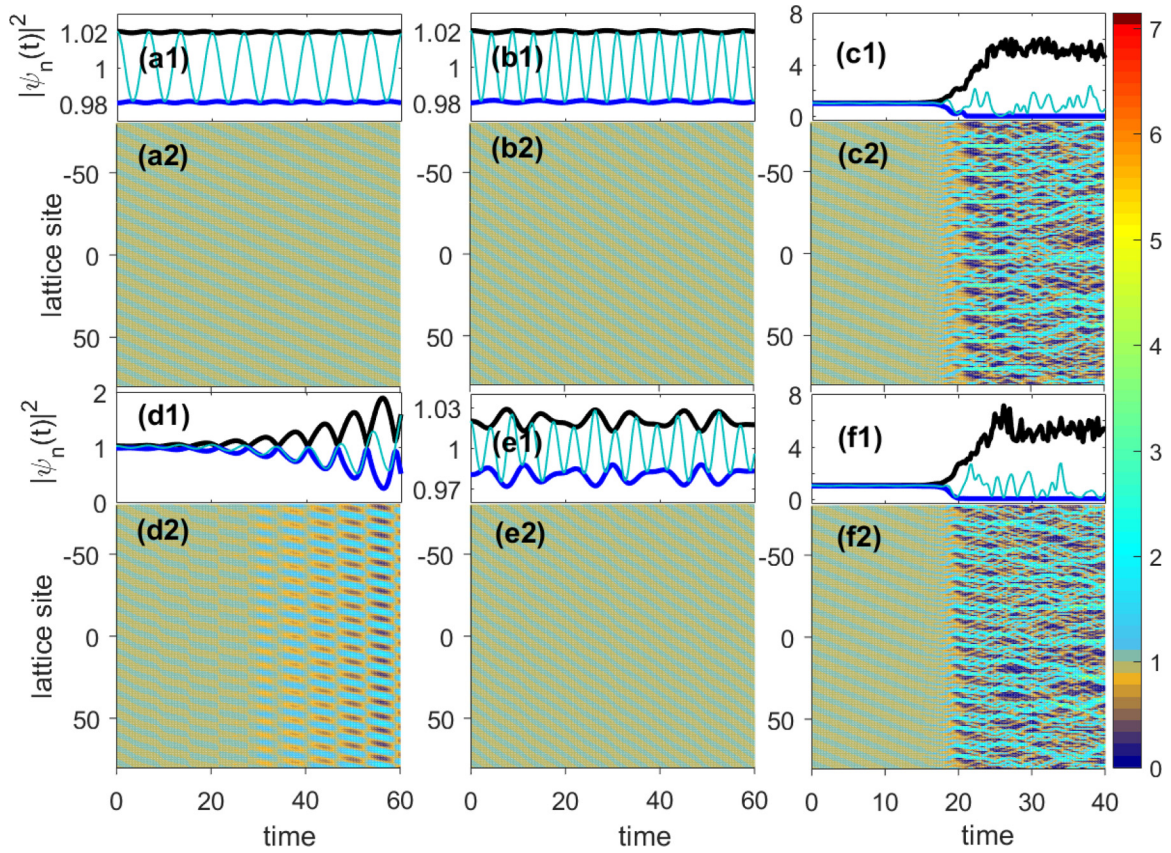


FIG. 8. Spatiotemporal dynamics of the system for parametrically undriven (top two rows) and driven (bottom two rows) systems, i.e., $\epsilon = 0.0$ and $\epsilon = 0.10$, respectively. The first and third rows present the temporal evolution of the maximum density (black line), the minimum density (blue line), and the density at the central well (cyan line) of the lattice. The second and fourth rows are 3D contour plots of the density in space and time. We used $q = 0.1006\pi$ (left column), $q = 0.3008\pi$ (middle column), and $q = 0.5137\pi$ (right column). The other parameters are $k = 0.2354\pi$, $\gamma = -1.0$, $\eta_0 = -1.0$, $u_0 = 1.0$, and $g_0 = 1.0$.

which is based on linearization, can well predict the linear phase of the instability as well as the instability phase diagram. However, it cannot tell what happens when the non-linear phase of the instability sets in. To validate these results and gain more insight into the dynamics of the system, we rely on direct numerical simulations, which are the focus in this section.

A. Setup and space-time dynamics

The model being investigated numerically is the equation of motion (13), and we use a fourth-order Runge-Kutta scheme. We choose the fixed parameters of the system to be $\omega = 1.0$, $g_0 = 1.0$, and a number of sites $N = 2048$. The time step is $\Delta t = 0.005$, and the initial condition is a modulated wave of the form

$$\psi_n(0) = u_0 e^{iqn} (1 + \varepsilon e^{ikn}). \quad (33)$$

It should be emphasized that the smallness parameter of perturbation $\varepsilon = 0.01$ is not connected to the driving strength ϵ . The only fixed parameter of the initial state is $u_0 = 1.0$, and we vary the wave numbers q and k . Periodic boundary conditions are imposed to the chain of sites, leading to a quantization of the wave numbers, i.e., $q = 2\pi n_1/N$ and $k = 2\pi n_2/N$, where n_1 and n_2 are integers between 0 and N . The

solution algorithm is built upon the fourth-order Rung-Kutta scheme.

With this setup, we proceed with direct numerical simulations of Eq. (13) using the initial condition (33). There are many features one can harness from the simulations. Here we focus on the evolution of the atomic density, $|\psi_n(t)|^2$, in space n and time t . Figure 8 displays in its top two rows an illustrative spatiotemporal dynamics of the system in its static regime, i.e., $\epsilon = 0.0$ for a mode picked in the subdomain M_{2b} (BEC side) of Fig. 1. In this case, there is no periodic driving, such that an amplitude growth in the system is attributable only to modulation instability. In Figs. 1(a) and 1(b), for which the system is predicted to be modulationally stable (MS), the amplitude undergoes only smooth and small oscillations around its average value. In Fig. 1(c), the system predicted to be modulationally unstable (MU) effectively undergo an exponential growth of its amplitude [see Fig. 1(c1)], and localized excitations are generated [see Fig. 1(c2)]. The bottom two rows of Fig. 8 portray the spatiotemporal dynamics of a parametrically driven system, i.e., $\epsilon = 0.10$. In this case, periodic driving leads to an amplitude growth in the system in a parametrically unstable (PU) regime, as depicted in Fig. 1(d). In a parametrically stable (PS) regime as shown in Fig. 1(e), it yields only a limited excitation, which does not change the stability of the system. These modes obviously

correspond to a parameter domain where modulation instability cannot manifest itself. In the case where the system is MU, parametric driving cannot help but to keep the system nearly unstable modulationally, as shown in Fig. 1(f). This is why it is of no use to investigate parametric instability in an MU parameter domain, as we assumed in the analytical calculations. These numerical findings results well corroborate our analytical predictions. It is worth noting that a comparison between Figs. 1(d1) and 1(f1) may reveal one striking feature of parametric instability. While modulation-dominated instability is characterized by a smooth or pure exponential growth of the maximum amplitude [see Fig. 1(f1)], the maximum amplitude in a parametric-dominated instability is modulated in time with frequency ω during the exponential growth [see Fig. 1(d1)]. Indeed, these modulations have a period $T = 6.214 \pm \Delta t$, with $\Delta t = 0.005$ the time step, corresponding to a period $\omega \simeq 1.01$, which is nothing but the driving frequency.

B. Numerical instability growth rate and parametric amplification

In our analytical investigations, we derived expressions for the gains ξ and $\xi_{\max}$, as given in Eqs. (21) and (24). In this section, we are concerned by their numerical counterparts, as well as amplitude growth induced by parametric driving.

1. Gain estimation method

From the linear stability ansatz and the initial condition (33), the density of any MS mode is expected to oscillate between a global maximum $\max_{n,t}(|\psi_n(t)|^2) = |u_0|^2(1 + 2\epsilon)$ and a global minimum $\min_{n,t}(|\psi_n(t)|^2) = |u_0|^2(1 - 2\epsilon)$. In the nonlinear stage of MI, we expect the maximum squared wave amplitude $\max_n(|\psi_n|^2)$ to exponentially grow well beyond $|u_0|^2(1 + 2\epsilon)$. A quite interesting method to get the instability growth rate consists in recording the time τ_s when a given value ρ_s of the maximum amplitude over space is reached during the early exponential growth phase [51,52]. Then the numerically computed instability gain can be defined as the inverse of the time when $\max_x(|\psi|^2)$ crosses the fixed value ρ_s , i.e.,

$$\xi_{\text{num}} \propto \frac{1}{\tau_s}, \quad \tau_s = t|_{\max_n(|\psi_n|^2) = \rho_s}. \quad (34)$$

In this work, we choose to detect the wave when the perturbation strength is five times higher than initially, which corresponds to a 10% growth of the wave density, i.e., $\rho_s = 1.10|u_0|^2$. Any other close choice would lead to similar results, at least qualitatively.

2. Numerical gain profiles

The numerical gain (34) allows us to successfully identify all unstable modes. An example is shown in Fig. 9 for modes picked in the subdomain M_{2b} of Fig. 1. In Fig. 1(a), the green marks depict a mode of the system in absence of driving, when the zone $q =]\pi/2, 3\pi/2[$ contains MU modes, while the zone $q =]0, \pi/2[\cup]3\pi/2, 2\pi[$ contain MS modes characterized by a zero gain. In the presence of driving, while the MU zone is kept, instability windows, characterized by two small bumps of nonzero gains, appear in the MS zones. This evidence endorses the analytical results displayed in Fig. 1(b), where gray

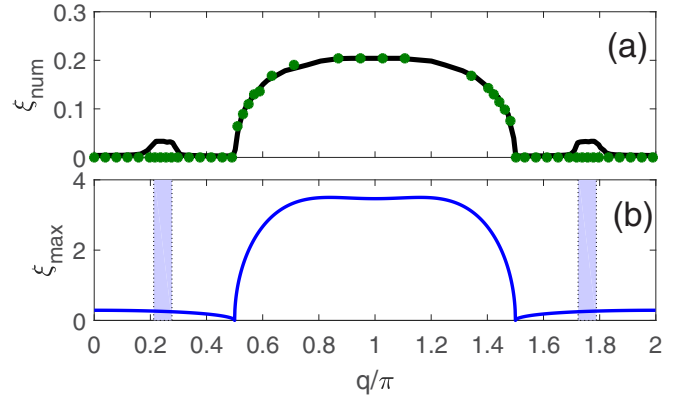


FIG. 9. Numerical vs analytical instability growth rate. Panel (a) shows the numerical instability growth rate for a parametrically driven system and $\epsilon = 0.10$ (solid black line) and $\epsilon = 0.0$ (green markers). We used $\omega = 1.0$, $\gamma = -1.0$, $k = 0.25\pi$, $u_0 = 1.0$, $\eta_0 = -1.0$, $g_0 = 1.0$, and $t_{\max} = 70$. The bottom panel (b) provides, for comparison purpose, the corresponding analytically predicted growth rate $\xi_{\max}$, which was already shown in Fig. 6(c).

bands represent PU windows. The agreement between the analytical predictions and the numerical results looks perfect.

3. Parametric amplification

We can think of parametric amplification as a phenomenon whereby a parametric nonlinearity and a pump wave are used to amplify a signal [53]. In the context of our DNLS equation, parametric nonlinearity and the pump wave are both provided by periodic driving of the two-particle contact interaction. Due to the driving, for PU modes and for background wave numbers q around q_0 , the system enters a parametric resonance regime, and the wave amplitude begins to undergo a modulated exponential growth in time. To study this parametric amplification, we recorded the amplitude after a given runtime for various parameters values. As shown in Figs. 10(a) and 10(b) for two values of the DDI strength,

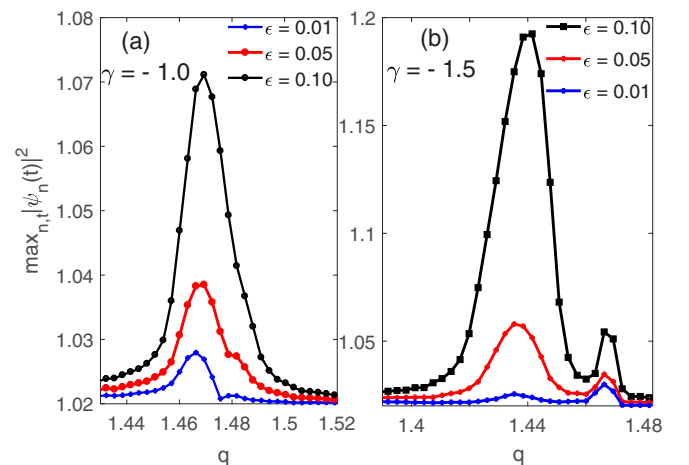


FIG. 10. Parametric resonances at wave numbers around $q = q_0$ for various values of $\epsilon = 0.01, 0.05, 0.10$, when (a) $\gamma = -1.0$ and (b) $\gamma = -1.5$. The other parameters are $k = 0.4\pi$, $\omega = 1.0$, $u_0 = 1.0$, $\eta_0 = -1.0$, $t_{\max} = 50$.

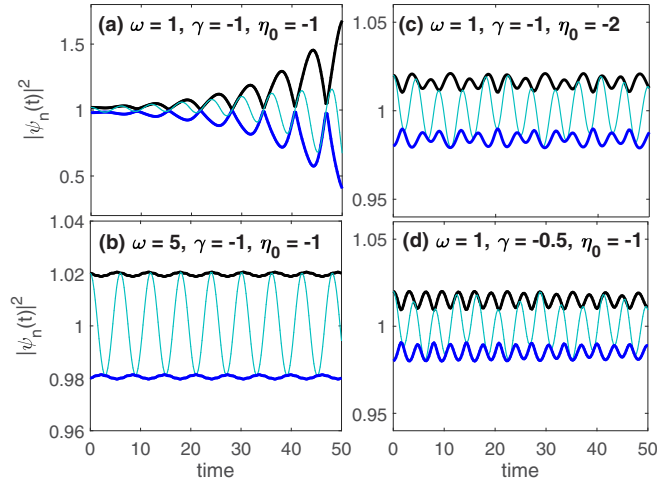


FIG. 11. Suppression of parametric instability in the system. Parametric amplification (a) is suppressed using fast driving (b), stronger QFs (c), and weaker DDI (d). We used $q = 0.1315\pi$, $k = 0.2354\pi$, $u_0 = 1.0$, and $g_0 = 1.0$.

a resonance peak is always reached around $q = q_0$ as predicted in Eq. (27). The peaks heighten and broaden as the driving strength ϵ is increased. In the case of quite strong DDI as portrayed in Fig. 10(b), we observe an additional small peak, which was not predicted by our analytical result. The additional peak heightens with increasing driving strengths. From the full-width at half maximum (FWHM) of the wave amplitude around resonance, one may check the validity of our linearization. When $\gamma = -1$ like in Fig. 10(a), the analytical bandwidth of parametric instability (31) yields $\Delta q_{\text{anal}} \simeq 0.0003\pi$, 0.0014π , and 0.0030π for $\epsilon = 0.01$, 0.05 , and 0.10 , respectively. When $\gamma = -1.5$ like in Fig. 10(b), it yields $\Delta q_{\text{anal}} \simeq 0.0005\pi$, 0.0029π , and 0.0057π for $\epsilon = 0.01$, 0.05 , and 0.10 , respectively. The numerical results based on the FWHM for $\gamma = -1$ yield $\Delta q_{\text{num}} \simeq 0.0046\pi$, 0.0059π , and 0.0066π for $\epsilon = 0.01$, 0.05 , and 0.10 , respectively. We realize that the numerically computed bandwidths are broader than the ones predicted via linearization, and do not vary linearly as per our predictions. However, both the analytical and numerical results agree on the increase of the unstable bandwidth with the driving strength ϵ .

C. Parametric instability suppression

In the quest to achieve suppression of parametric instability in the periodically driven system, we propose three possible scenarios. A prime method consists in periodically driving at large frequencies. As shown in Fig. 11, parametric instability, which exists at $\omega = 1$, in the mode in Fig. 11(a), disappears for $\omega = 5$, as portrayed in Fig. 11(b). This result, sometimes referred to as nonlinearity management [54], is well known in Floquet engineering, where averaging over rapid modulations of the nonlinearity creates an effective time-independent nonlinearity. Another method of parametric instability suppression may consist in controlling quantum fluctuations in the system. Changing the QFs strength from $\eta_0 = -1$ to -2 as presented in Fig. 11(c) stabilizes the system against parametric amplification. The third method for suppressing parametric

instability amounts to controlling the strength of dipole-dipole interaction. For instance, changing it from $\gamma = -1$ to -0.5 as in Fig. 11(d) helps the system avoid parametric instability. Our results highlight the significant impact of frequency, quantum fluctuations and long-range interactions on the stability and dynamics of the system under periodic driving. This offers insights into the control and manipulation of quantum gases in optical lattices.

V. CONCLUSION AND OUTLOOK

In this work, we examined the dynamical instabilities of a one-dimensional ultracold Bose-Bose mixture with long-range dipole-dipole interactions, trapped in optical lattices and subject to periodically varying contact interaction, including effects of beyond-mean-field corrections due to quantum fluctuations. In the deep OL wells regime, we derived a discrete nonlinear Schrödinger equation that rules the dynamics of the system. Then using the linear stability analysis and multiple scale analysis, we revealed the stability diagrams in parameter spaces, including windows and canals of instability. Direct numerical calculations were conducted to endorse the analytical findings. The roles of long-range interactions and quantum fluctuations were explored. We found that quantum fluctuations significantly attenuate the instability of the system. Our results also highlight the influence of dipole-dipole interactions on the parametric instability. We demonstrate that the anisotropy (which controls the sign) and long-range nature of these interactions can either suppress or enhance instability thresholds. In particular, repulsive dipole-dipole interaction lessens the instability. We showed that under some conditions, it is even possible to achieve a complete suppression of parametric instability. The interplay between the contact interaction strength, the quantum fluctuations, and the dipolar interaction reveals new instability regimes that are otherwise absent in systems with purely short-range interactions. This study not only advances our understanding of parametric instabilities in ultracold quantum gases but also provides a framework for exploring the role of long-range interactions in other quantum systems. The insights gained from this work could have implications for the design of quantum simulators and the realization of exotic quantum phases in ultracold atomic systems in optical lattices. Since the present study used the same-shape approximation, the logical and immediate extension of this work is to investigate driven dipolar mixtures where the components have different masses, scattering lengths, or dipolar strengths—a scenario of significant experimental interest with atoms like dysprosium and erbium. In that general case, the spin and density modes split, and their interplay can lead to entirely new, component-specific instabilities that are inaccessible to an effective single-component theory. By deriving the full two-component formalism here, we provided a direct path to those future results. These investigations may be based, for instance, on the single-mode approximation, the immiscible/imbalanced approximation or the full extended Gross-Pitaevskii equation for a miscible-immiscible problem with nonsymmetric components, which would certainly yield a richer dynamics.

ACKNOWLEDGMENTS

E.W. thanks the Rhineland Palatinate Technical University of Kaiserslautern-Landau for support with high performance computing resources from the RHRZ (*Regionales Hochschulinformationssystem Kaiserslautern-Landau*).

DATA AVAILABILITY

The data that support the findings of this article are not publicly available. The data are available from the authors upon reasonable request.

APPENDIX A: BEYOND MEAN-FIELD ENERGY CORRECTIONS

The mathematical treatment of ultracold quantum gas systems is commonly given in the mean-field regime using the Bogoliubov theory. While such a description has proven useful to explain a broad range of phenomena observed in these systems, it fails to predict, for instance, the formation of quantum droplets of dipolar [20] and nondipolar atomic mixtures [55]. This occurs in a regime of attractive interactions, where mean-field predicts a collapse. The observed stability leading to droplets is due to quantum fluctuations, which are described by the next-order term beyond mean-field approximation. In three-dimensional systems, these fluctuations are described by the LHY correction to the ground-state energy [29]. For a homogenous (untrapped) nondipolar gas, the beyond mean-field energy density reads [29,56]

$$\mathcal{E}_{\text{BMF}}^{\text{3D}} = \frac{gn^2}{2} \left[1 + \frac{128}{15\sqrt{\pi}} (na_s^3)^{1/2} + \dots \right], \quad (\text{A1})$$

where $\mathcal{E}_{\text{BMF}}^{\text{3D}} \equiv E_{\text{BMF}}^{\text{3D}}/V$, n is the 3D atomic density, a_s the s -wave scattering length, and $g = 4\pi\hbar^2 a_s/m$ is the strength of two-body interactions, with m being the mass of an atom. The underbars indicate the single-component nature of the energy (density). We readily see that the correction is proportional to the diluteness parameter $\sqrt{na_s^3}$ in 3D. In a homogenous 2D setting, the energy density with BMF correction was first derived by Schick [57] and coworkers. It reads [58]

$$\mathcal{E}_{\text{BMF}}^{\text{2D}} = \frac{g_{2D}n^2}{2} \left[1 + \frac{1}{\ln(1/na_s^2)} + \dots \right], \quad (\text{A2})$$

where $g_{2D} = \frac{4\pi\hbar^2}{m|\ln(na_s^2)|}$ and n is the 2D atomic density. In 1D, the BMF correction was derived within the Lieb-Liniger model to be in the following form [59]:

$$\mathcal{E}_{\text{BMF}}^{\text{1D}} = \frac{g_{1D}n^2}{2} \left(1 - \frac{4}{3\pi} \sqrt{\gamma_l} + \dots \right), \quad (\text{A3})$$

where n is the 1D atomic density. While the coefficient γ_l was initially evaluated only numerically [59], its expression was explicitly evaluated later, in the context of dimensional crossover and quasi-one-dimensional quantum gases induced by confinement. It leads to $g_{1D} = 2\hbar^2/(ma_{1D})$ and [60]

$$\gamma_l = \frac{2}{|na_{1D}|} = \frac{mg_{1D}}{\hbar^2 n}, \quad a_{1D} = \frac{\ell_\perp^2}{2\pi a_s} \left(1 - C_l \frac{a_s}{\ell_\perp} \right), \quad (\text{A4})$$

where the constant C_l determines the shift to the mean-field energy due to the confinement, and $\ell_\perp$ is a characteristic length, which can be a box potential length [60] or the transverse harmonic oscillator length [61,62]. Hence, neither of Lee, Huang, and Yang ever calculated beyond-mean-field terms for any dimension other than three [27]. Then, strictly speaking, BMF terms are supposed to be LHY ones in 3D [29], Schick ones in 2D [57], and Lieb-Liniger ones in 1D [59]. However, bearing in mind the idea of dimensional crossover [63] in quantum gas systems, it has become customary to loosely refer to all BMF terms, which describe quantum fluctuations, as Lee-Huang-Yang terms or corrections, irrespective of the dimensionality; see for instance Refs. [64,65]. Similar expressions were recently derived for single-component and binary dipolar quantum gases in 3D [37,39].

APPENDIX B: DERIVATION OF THE EXTENDED GPE FOR TWO-COMPONENT DIPOLAR GASES IN 1D

To write Eq. (4) in one dimension, let us start by reconsidering the energy term (5) that determines the LHY contribution through the Hugenholtz-Pines formalism [66]. For three-dimensional settings as examined in Refs. [37,39], where $\mathbf{k} = (k, \theta_k, \varphi_k)$ in polar spherical coordinates, we have $\int \frac{d^D k}{(2\pi)^D} \rightarrow \int_0^{2\pi} d\varphi_k \int_0^\pi d\theta_k \int_0^\infty k^2 \sin \theta_k dk$, with $D = 3$, leading to

$$\begin{aligned} \mathcal{E}_{\text{LHY}}^{\text{h}} &= -4\chi(n_1, n_2) = -4 \left(-\frac{1}{2} \sum_\lambda 2\pi \int_0^\pi d\theta_k \int_0^\infty \frac{dk}{(2\pi)^3} k^2 \sin \theta_k \frac{[\zeta_\lambda(\mathbf{k}) - E(k)]^3}{4\zeta_\lambda(\mathbf{k})E(k)} \right) \\ &= \frac{1}{8\pi^2} \sum_\lambda \int_0^\pi \sin \theta_k d\theta_k \int_0^\infty \frac{2mE}{\hbar^2} \sqrt{\frac{m}{2E\hbar^2}} \frac{(\sqrt{E(E+U_\lambda)} - E)^3}{E\sqrt{E(E+U_\lambda)}} dE \\ &= \frac{1}{8\pi^2} \frac{m^{3/2}\sqrt{2}}{\hbar^3} \sum_\lambda \int_0^\pi \sin \theta_k d\theta_k \int_0^\infty \sqrt{E} \frac{(\sqrt{E+U_\lambda} - \sqrt{E})^3}{\sqrt{E+U_\lambda}} dE \\ &= \frac{1}{8\pi^2} \frac{m^{3/2}\sqrt{2}}{\hbar^3} \int_0^\pi \sin \theta_k d\theta_k \left(\frac{4}{15} \sum_\lambda U_\lambda^{5/2} \right) = \frac{m^{3/2}\sqrt{2}}{30\pi^2\hbar^3} \int_0^\pi \sin \theta_k d\theta_k \sum_\lambda U_\lambda^{5/2} \\ &= \frac{m^{3/2}\sqrt{2}}{15\pi^2\hbar^3} \int_0^{\pi/2} \sin \theta_k d\theta_k \sum_\lambda U_\lambda^{5/2}. \end{aligned} \quad (\text{B1})$$

To get to the last step of these equations, we used the fact that the integrand contains only terms in $\sin \theta_k$ and $\cos^2 \theta_k$ (inside U_λ), which makes it symmetric around $\theta_k = \pi/2$. The contribution from imaginary parts has been neglected. It is clear that this result is equivalent to similar expressions in Refs. [37,39,40,42].

Now, let us consider the one-dimensional problem. The system can be experimentally realized by using a single pair of lasers to produce a 1D optical lattice and then imposing large trapping frequencies in two directions to confine the atoms in one direction. For a one-dimensional system as examined in our work, we have $\theta_k = \pi/2$ and $\int \frac{d^D k}{(2\pi)^D} \rightarrow \int_{-\infty}^{\infty} \frac{dk}{2\pi}$, with $D = 1$, leading to

$$\begin{aligned} \mathcal{E}_{\text{BMF}}^{\text{h,1D}} &= -4 \left(-\frac{1}{2} \sum_{\lambda} \int_{-\infty}^{\infty} \frac{dk}{2\pi} \frac{[\zeta_{\lambda}(\mathbf{k}) - E(k)]^3}{4\zeta_{\lambda}(\mathbf{k})E(k)} \right) \\ &= \frac{1}{2\pi} \sum_{\lambda} \int_0^{\infty} dk \frac{[\zeta_{\lambda}(k) - E(k)]^3}{\zeta_{\lambda}(k)E(k)} \\ &= \frac{1}{2\pi} \sum_{\lambda} \int_0^{\infty} \sqrt{\frac{m}{2E\hbar^2}} \frac{(\sqrt{E(E+U_{\lambda}}) - E)^3}{E\sqrt{E(E+U_{\lambda})}} dE \\ &= \frac{1}{2\pi} \frac{m^{1/2}}{\sqrt{2}\hbar} \sum_{\lambda} \int_0^{\infty} \frac{(\sqrt{E+U_{\lambda}} - \sqrt{E})^3}{\sqrt{E}\sqrt{E+U_{\lambda}}} dE \\ &= \frac{1}{2\pi} \frac{m^{1/2}}{\sqrt{2}\hbar} \sum_{\lambda} \left(\frac{2}{3} U_{\lambda}^{3/2} \right) = \frac{m^{1/2}}{3\pi\sqrt{2}\hbar} \sum_{\lambda} U_{\lambda}^{3/2} \\ &= \frac{1}{3\pi\sqrt{2}} \left(\frac{m}{\hbar^2} \right)^{1/2} \sum_{\lambda} U_{\lambda}^{3/2}. \end{aligned} \quad (\text{B2})$$

The expression above of the 1D analog of the LHY correction can be rewritten in the form

$$\mathcal{E}_{\text{BMF}}^{\text{h,1D}} = \frac{m^{1/2}}{3\pi\sqrt{2}\hbar} (n_1 \vartheta_{11})^{3/2} f\left(1, \frac{\vartheta_{12}^2}{\vartheta_{11}\vartheta_{22}}, \frac{n_2 \vartheta_{22}}{n_1 \vartheta_{11}}\right), \quad (\text{B3})$$

where $f(1, x, y) = \sum_{\pm} (1 + y \pm \sqrt{(1-y)^2 + 4xy})^{3/2}$. In the balanced mixture limit [39], one considers equal intraspecies interactions and particle numbers in each component, i.e., $\vartheta_{11} = \vartheta_{22}$, $n_1 = n_2 = n/2$, and $\psi_1 = \psi_2 = \psi/\sqrt{2}$. Then, when both components are miscible and have the same shape, the EGPE can be reduced to an effective single-component problem, with BMF energy density as

$$\mathcal{E}_{\text{BMF}}^{\text{h,1D}} = \frac{m^{1/2}}{3\pi\sqrt{2}\hbar} h(r_{\vartheta}) \vartheta_{11}^{3/2} n^{3/2}, \quad (\text{B4})$$

where $h(x) = (1-x)^{3/2} + (1+x)^{3/2}$ and $r_{\vartheta} = \vartheta_{12}/\vartheta_{11}$. Hence, we have $\mathcal{E}_{\text{BMF}}^{\text{h,1D}} \propto n^{3/2}$. Taking the derivative of the energy density with respect to n leads to a beyond mean-field correction to the chemical potential $\Delta\mu_{\text{BMF}}^{\text{1D}} \propto n^{1/2}$, as used in Eq. (9), with coefficient

$$\eta_{\text{QF}} = \frac{m^{1/2}}{2\pi\sqrt{2}\hbar} h(r_{\vartheta}) \vartheta_{11}^{3/2}. \quad (\text{B5})$$

In the limit of vanishing dipole-dipole interactions, one readily obtains Petrov and Astrakharchik's result [28]. It is clearly seen that contributions for 1D and quasi-1D limits are mathematically different in the context of dipolar droplets as concluded in Ref. [67], being proportional to $n^{1/2}$ and $n^{3/2}$, respectively.

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