

Many-electron characterizations of higher-charge superconductors

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The theoretical understanding of conventional superconductivity as the phonon-assisted formation and condensation of two-electron Cooper pairs is a significant triumph in condensed matter physics. Here, we propose many-electron characterizations of higher-charge superconductivity with Cooper pairs consisting of more than two electrons, by implementing translation symmetrization on parent pair-density-wave-ordered states. In particular, we demonstrate many-electron constructions with vanishing charge- $2e$ sectors, but with sharp signatures in charge- $4e$ or charge- $6e$ expectation values instead. Such characterizations are consistent with previous phenomenology of vestigial order and Ginzburg-Landau theory. Furthermore, we demonstrate that at the microscopic level, momentum conservation alone may be vital and sufficient for the robust emergence of higher-charge superconductivity. Our study thus offers a general and microscopic route to understand and characterize higher-charge superconductivity, including nontrivial experimental signatures such as fractional magnetic flux and period in interferometry, as well as localized Cooper pairs at lattice topological defects.

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Introduction. Superconductors naturally exhibit zero electrical resistance and the Meissner effect below a critical temperature [1–3]. Discovered by Onnes in 1911, the origin of superconductivity remained a microscopic mystery, despite the phenomenological success of its Ginzburg-Landau theory [4], until the BCS theory was proposed in 1957, which explains the dissipationless current through the formation and condensation of Cooper pairs—two electrons bound together via lattice-mediated attractive interactions [5,6]. Such Cooper pairs, when generalized to finite momenta, give rise to pair density waves (PDWs), whose superconducting order parameter breaks translation symmetries and varies periodically in space [7–16]. Signatures of such exotic phases have been established in recent experiments [17–21], including those involving the layered kagome material CsV_3Sb_5 [19,20].

Higher-charge superconductors can emerge as vestigial orders of an incommensurate PDW, which may be (partially) melted by disorder, thermal fluctuations, quantum fluctuations, or quasicrystal structures [9,22–26]. Subsequently, the broken continuous translation symmetry is restored, and only the $U(1)$ gauge symmetry (charge conservation) breaking is retained. In a Ginzburg-Landau theory framework, these exotic superconducting phases reflect higher orders of the PDW order parameter fields, characterized by multiplets of Cooper pairs, such as with charges of $4e$ or $6e$

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[9,22–24,27–32]. The corresponding fractional magnetic flux quanta, aside from alternatives such as multicomponent or time-reversal-symmetry-breaking superconductors [33–36], have been potentially observed in quantum-interference experiments [37]. While theoretical mechanisms have been explored via the Hubbard models [38–41], vestigial fluctuations [42–45], and geometric frustration [46–48], a comprehensive, microscopic many-electron characterization remains lacking, limiting our description and understanding of these fascinating quantum phenomena and correlated systems.

Candidate wavefunctions play a crucial role in understanding exotic quantum matters, often bypassing the need for exact ground-state solutions of intractable microscopic Hamiltonians. A landmark example is the Laughlin wavefunctions, which successfully characterize fractional quantum Hall liquids and encode the fractional statistics of topological order [49–51], profoundly guiding subsequent quantum-transport and shot-noise experiments [52,53]. In the realm of quantum spin liquids, Gutzwiller-projected wavefunctions derived from slave-particle approaches offer a powerful variational ansatz [54–57], supporting an effective field theory for universal behaviors such as long-range entanglement and emergent gauge structures, and successfully predicting the gapless spinon Fermi surfaces later observed in organic materials [58,59]. Overall, these wavefunctions serve not only as trial states for variational studies [60–68] but also as conceptual bridges linking microscopic models to experimental observables of emergent quantum phases. Furthermore, the restoration of translation symmetry via projection is a well-established operation for recovering well-defined quantum numbers in variational studies across quantum chemistry [69], nuclear physics [70,71], and condensed matter physics [72,73].

In this work, we propose and study candidate many-electron quantum states of higher-charge superconductors, based on translation-symmetrizing parent PDW states. For the simplest scenario of a charge- $4e$ superconducting state rooted from a stripe PDW in two dimensions (2D), the resulting states display vanishing single-Cooper-pair sectors, yet finite expectation values over certain charge- $4e$ Cooper pairs, e.g., $\langle c_{\mathbf{k},\uparrow}^\dagger c_{-\mathbf{k}+\mathbf{Q},\downarrow}^\dagger c_{\mathbf{k}',\uparrow}^\dagger c_{-\mathbf{k}'-\mathbf{Q},\downarrow}^\dagger \rangle$, a composite of two Cooper pairs with momentum $\pm\mathbf{Q}$, respectively. Generalizations to even higher-charge states, such as charge- $6e$, and to higher dimensions, are also straightforward. The formalism is consistent with the physical picture of vortex binding between the PDW order parameter fields in previous vestigial order and classical Ginzburg-Landau theory studies. Moreover, while point-group symmetries often play a vital role in stabilizing the coexisting parent PDW states, we demonstrate that the underlying kinematics for the subsequent emergence of higher-charge superconductivity is fundamentally driven by momentum conservation. We further elaborate on this general mechanism in a perturbation theory.

Furthermore, we examine higher-charge superconductors with respect to an interferometry geometry and observe apparent periodicity in the threading magnetic fluxes associated with the proposed higher-charge Cooper pairs, which differs significantly from that of the parent charge- $2e$ PDW. Interestingly, we find that such higher-charge superconductors interact nontrivially with certain topological defects appearing in the form of edge dislocations [74] of the underlying lattices, with Burgers vectors compatible with the parent PDW wave vectors, allowing localized single Cooper pairs to re-emerge. Also, such charge- $4e$ superconductivity naturally offers nontrivial pairing symmetry and exchange statistics resembling those of a topological $p + ip$ superconductor [75–80], yet without the necessity of broken time-reversal symmetry. These discoveries offer distinctive experimental signatures and characterizing probes for higher-charge superconductivity and its parent PDW phases.

Many-electron states for higher-charge superconductors. For simplicity, we first start with a tight-binding model with a (mean-field) stripe PDW in 2D as our parent Hamiltonian:

$$\hat{H}_{\text{PDW}} = -t \sum_{\langle \mathbf{r}\mathbf{r}' \rangle, s} (c_{\mathbf{r}',s}^\dagger c_{\mathbf{r},s} + c_{\mathbf{r},s}^\dagger c_{\mathbf{r}',s}) - \mu \sum_{\mathbf{r},s} c_{\mathbf{r},s}^\dagger c_{\mathbf{r},s} + V \sum_{\mathbf{r}} \cos(Qx) (c_{\mathbf{r},\uparrow}^\dagger c_{\mathbf{r},\downarrow}^\dagger + \text{H.c.}), \quad (1)$$

where $s = \uparrow, \downarrow$ labels the electron spin and $\mathbf{r} = (x, y)$ labels the sites on an $L \times L$ square lattice under periodic boundary conditions (PBCs), and we set $L = 200$ unless noted otherwise; $t = 1$ is the hopping amplitude as well as our unit of energy; and V describes the strength of a stripe

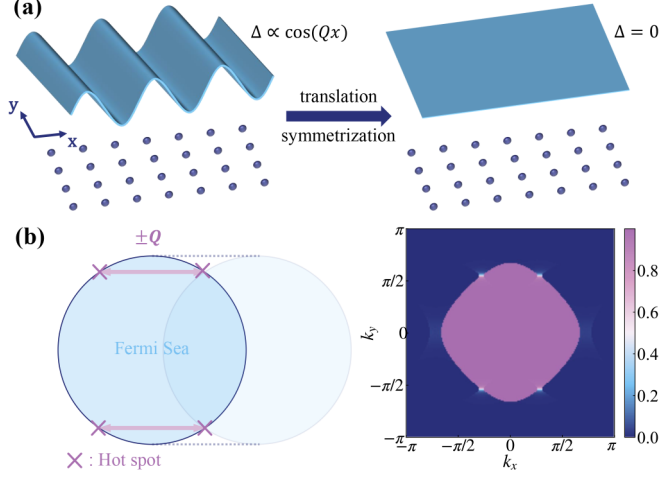


FIG. 1. (a) Illustration of the model in Eq. (1) on a two-dimensional lattice (blue dots). In contrast to the parent state with a stripe PDW (left), the many-electron state following translation symmetrization exhibits vanishing $2e$ Cooper-pair density Δ . (b) A schematic diagram illustration of the Fermi sea and the $\pm\mathbf{Q}$ momentum transfers induced due to the PDW. The hot spots (purple crosses) are located at the intersections of the original and shifted Fermi surfaces. Correspondingly, the momentum-space electron density $\langle c_{\mathbf{k},s}^\dagger c_{\mathbf{k},s} \rangle$ of the model in Eq. (1) clearly reveals its Fermi sea and the hot spots. Here, $V = 0.1$, $Q = 1.76$, and $\mu = -1.0$.

PDW with wave vector $\mathbf{Q} = Q\hat{x}$ along the x direction [see illustration in Fig. 1(a)]. Ideally, our choice of Q should be incommensurate; however, such Q would obstruct the PBC and later the translation symmetrization in numerical models, where we are compelled to employ $Q = 2\pi q/p$ with a large $p \sim O(L)$, whose distinction with an incommensurate Q becomes negligible for a sufficiently large L . Through a particle-hole transformation $a_{\mathbf{r},\downarrow} = c_{\mathbf{r},\downarrow}^\dagger$ and diagonalization of $\hat{H}_{\text{PDW}}$, we obtain the ground state $|\Psi\rangle = \prod_{\epsilon_n < 0} \sum_{\mathbf{r}} [\psi_n(\mathbf{r}, \uparrow) c_{\mathbf{r},\uparrow}^\dagger + \psi_n(\mathbf{r}, \downarrow) a_{\mathbf{r},\downarrow}^\dagger] |\emptyset\rangle$, where $|\emptyset\rangle$ is the vacuum state of $c_\uparrow$ and $a_\downarrow$ (full state of $c_\downarrow$). On top of the electron Fermi sea dictated by the dispersion $\epsilon_{\mathbf{k}} = -2t(\cos k_x + \cos k_y) - \mu$, the PDW introduces coupling between $c_{\mathbf{k},\uparrow}^\dagger$ and $c_{-\mathbf{k} \pm \mathbf{Q},\downarrow}^\dagger$ ($a_{\mathbf{k} \mp \mathbf{Q},\downarrow}$), introducing a momentum transfer of $\pm\mathbf{Q}$, respectively; such pairing is most active around selected hot spots on the otherwise sharp Fermi surface, as shown in Fig. 1(b).

Next, we introduce a translation symmetrization P_T , where $P_T = 1 + T_x + T_x^2 + \dots + T_x^{L-1}$ and T_x is the translation operator of one lattice spacing along the $\hat{x}$ direction. Such a P_T is sufficient for restoring the translation symmetry broken by the stripe PDW in Eq. (1) and straightforwardly generalizable in more complex scenarios, some of which we will discuss later and in Ref. [81]. The subsequent projected wavefunction, $P_T|\Psi\rangle = T_x(P_T|\Psi\rangle)$, is translation invariant, just as the vestigial order after the (partial) melting of the initial PDW. Interestingly, unlike the parent PDW state $|\Psi\rangle$ [Fig. 2(b) [82]], the Cooper-pair expectation values, $\langle c_{\mathbf{r},\uparrow}^\dagger c_{\mathbf{r}',\downarrow}^\dagger \rangle$, vanish universally for arbitrary $\mathbf{r}$ and $\mathbf{r}'$, showing that the single $2e$ Cooper-pair sector is fully absent in $P_T|\Psi\rangle$ [see Fig. 2(c)]. Further, we can fully suppress the charge- $2e$ defining feature, the long-range pair-pair correlations, by introducing a finite coherence length to the parent PDWs (see Ref. [81] for details). Instead, the minimum charge difference is $4e$ —two Cooper pairs, as shown by the finite expectation value of $\langle c_{\mathbf{k},\uparrow}^\dagger c_{-\mathbf{k}+\mathbf{Q},\downarrow}^\dagger c_{\mathbf{k}',\uparrow}^\dagger c_{-\mathbf{k}'-\mathbf{Q},\downarrow}^\dagger \rangle$ in Fig. 2(a). We detail our method for evaluating such expectation values in the Supplemental Material [81]. Such a contrast between two-electron and four-electron terms already indicates that $P_T|\Psi\rangle$ deviates significantly from a simple Hartree-Fock state, whose correlations always follow straightforward factorizations of Wick's theorem.

The Gutzwiller projection of spinful electrons offers quantum spin-liquid wavefunctions for variational studies and universal characterizations [55,60–68,83–86], by retaining the physically

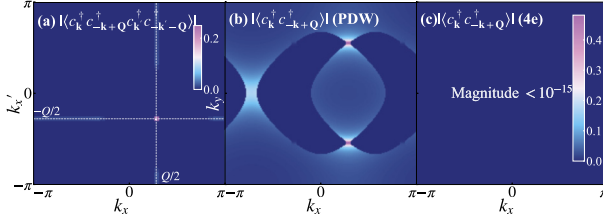


FIG. 2. (a) Heat map of the four-point correlation functions $|\langle c_{\mathbf{k}}^{\dagger} c_{-\mathbf{k}+\mathbf{Q}}^{\dagger} c_{\mathbf{k}'}^{\dagger} c_{-\mathbf{k}'-\mathbf{Q}}^{\dagger} \rangle|$ of the charge- $4e$ superconducting state $P_T|\Psi\rangle$ shows a sharp peak at $(k_x, k'_x) = (Q/2, -Q/2)$, which corresponds to the hot spots on the Fermi surface in Fig. 1(b); we have fixed $k_y = k'_y = 1.71$ for simpler display. (b) While the charge- $2e$ Cooper-pair density $\Delta(\mathbf{k}) = \langle c_{\mathbf{k}}^{\dagger} c_{-\mathbf{k}+\mathbf{Q}}^{\dagger} \rangle$ flourishes in the parent PDW state $|\Psi\rangle$, (c) it is completely suppressed in the higher-charge superconducting state $P_T|\Psi\rangle$.

relevant spinon degrees of freedom while removing the chargon degrees of freedom. Effectively, the projection binds the spin-up electrons and spin-down holes as an auxiliary (gauge) field. Likewise, a stripe PDW consists of two order parameter fields $\Delta_{\mathbf{Q}}(\mathbf{r}), \Delta_{-\mathbf{Q}}(\mathbf{r}) \in \mathbb{C}$, corresponding to its $\pm\mathbf{Q}$ components, whose antibinding $\Delta_{\mathbf{Q}}^*(\mathbf{r})\Delta_{-\mathbf{Q}}(\mathbf{r})$ and binding $\Delta_{\mathbf{Q}}(\mathbf{r})\Delta_{-\mathbf{Q}}(\mathbf{r})$ are associated with the CDW and the charge- $4e$ superconductivity, breaking translation symmetry and charge conservation, respectively—The translation symmetrization projects out the former and retains the latter degrees of freedom. Phenomenologically, the projection P_T effectively integrates over the relative spatial phases of the PDW order parameters. This process fundamentally corresponds to the proliferation of phase fluctuations—dual to vortices—that melts the single-Cooper-pair order while binding the two order parameter fields, consistent with previous Ginzburg-Landau theories [9,22–24,27,28,32]. Although a local “vortex density” operator is not directly definable within this purely electronic many-body wavefunction, this vortex-proliferation picture can be indirectly visualized through topological lattice defects, as we will demonstrate later.

Intuitively, we may understand the above unusual properties of $P_T|\Psi\rangle$ through the perturbation theory for relatively small V . The zeroth-order state $|\Psi_0\rangle$, the electron Fermi sea of the first line in Eq. (1), is fully translation symmetric with a zero total momentum, thus remains intact upon P_T . Following first-order perturbation, the contributions to the state, $|\Psi_1\rangle \propto V \sum_{\mathbf{k}} (c_{\mathbf{k},\uparrow}^{\dagger} c_{-\mathbf{k}+\mathbf{Q},\downarrow}^{\dagger} + \text{H.c.}) |\Psi_0\rangle$, introduce charge- $2e$ Cooper pairs, yet alter the total momentum by $\pm\mathbf{Q}$, and subsequently, become projected out by P_T . It is not until the second order that a nontrivial contribution $|\Psi_2\rangle \propto V^2 \sum_{\mathbf{k},\mathbf{k}'} c_{\mathbf{k},\uparrow}^{\dagger} c_{-\mathbf{k}+\mathbf{Q},\downarrow}^{\dagger} \cdot c_{\mathbf{k}',\uparrow}^{\dagger} c_{-\mathbf{k}'-\mathbf{Q},\downarrow}^{\dagger} |\Psi_0\rangle$ emerges. This contribution corresponds to sequential scatterings from the PDW with opposing, and thus mutually canceling, momentum transfers, enabling the state to conserve its total momentum and survive the projection P_T . Indeed, our numerical results indicate that $P_T|\Psi\rangle$'s expectation values of charge- $4e$ Cooper pairs are most significant at momenta $\pm\mathbf{Q}$: $\langle c_{\mathbf{k},\uparrow}^{\dagger} c_{-\mathbf{k}+\mathbf{Q},\downarrow}^{\dagger} c_{\mathbf{k}',\uparrow}^{\dagger} c_{-\mathbf{k}'-\mathbf{Q},\downarrow}^{\dagger} \rangle$, with amplitudes quadratic in V [see Fig. 3(g)].

Importantly, while previous Ginzburg-Landau theories often incorporate point-group symmetries—inversion symmetry for charge- $4e$ superconductivity [9,22,24,27,28] and C_3 rotation symmetry for charge- $6e$ superconductivity [23,32]—to stabilize and ensure the degeneracy of coexisting PDW components, our quantum many-electron states explicitly demonstrate that momentum conservation alone is sufficient to drive the subsequent emergence of higher-charge condensates. For instance, we may generalize $P_T|\Psi\rangle$ to cases without inversion symmetry, e.g., either a nonreciprocal PDW $V_{\mathbf{Q}} \sum_{\mathbf{k}} c_{\mathbf{k},\uparrow}^{\dagger} c_{-\mathbf{k}+\mathbf{Q},\downarrow}^{\dagger} + V_{\mathbf{Q}'} \sum_{\mathbf{k}} c_{\mathbf{k},\uparrow}^{\dagger} c_{-\mathbf{k}+\mathbf{Q}',\downarrow}^{\dagger} + \text{H.c.}$ ($\mathbf{Q}' = -\mathbf{Q}$), or more fundamentally, an asymmetric Fermi sea $|\Psi_0\rangle$. Note that in the latter case, the translation symmetrization should be adjusted as $P_T = 1 + T_x e^{i\mathbf{K}} + T_x^2 e^{i2\mathbf{K}} + \dots + T_x^{L-1} e^{i(L-1)\mathbf{K}}$ for a Fermi sea with a finite total momentum $\mathbf{K}$; for simplicity, here we consider $\mathbf{K} = K\hat{x}$. Consistent with the aforementioned suppression of charge- $2e$ sectors, Figs. 3(a) and 3(b) demonstrate that a finite charge- $4e$ signal is clearly evident in both scenarios. Further, the charge- $4e$ amplitude at $\langle c_{\mathbf{k},\uparrow}^{\dagger} c_{-\mathbf{k}+\mathbf{Q},\downarrow}^{\dagger} c_{\mathbf{k}',\uparrow}^{\dagger} c_{-\mathbf{k}'-\mathbf{Q},\downarrow}^{\dagger} \rangle$, which we dub the charge- $4e$ order parameter V_{4e} and summarize in Fig. 3(g), shows a clear

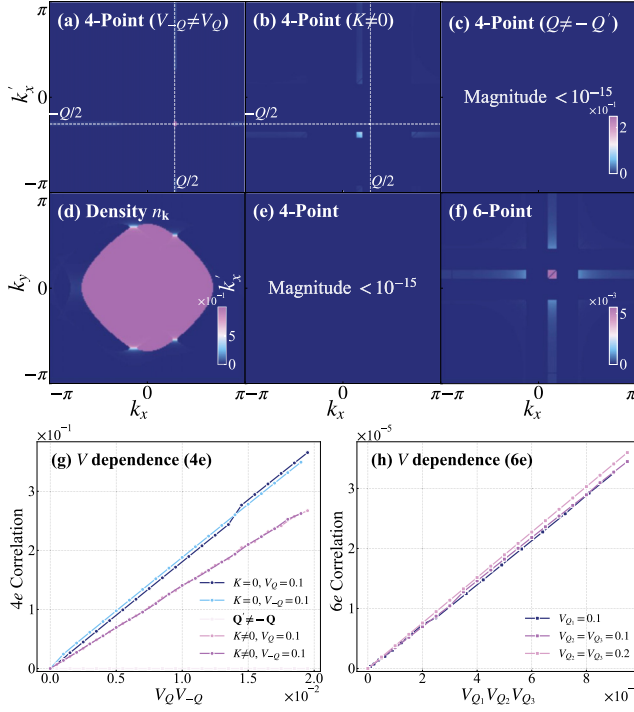


FIG. 3. The correlations $|\langle c_{\mathbf{k}}^\dagger c_{-\mathbf{k}+\mathbf{Q}}^\dagger c_{\mathbf{k}'}^\dagger c_{-\mathbf{k}'-\mathbf{Q}}^\dagger \rangle|$ of the $P_T|\Psi\rangle$ states with (a) asymmetric pairing amplitude $V_Q \neq V_{-\mathbf{Q}}$, (b) asymmetric Fermi sea with nonzero total momentum $\mathbf{K}$, and (c) nonreciprocal PDW momentum $\mathbf{Q}' \neq -\mathbf{Q}$ suggest that the charge-4e superconductivity may exist as long as momentum conserves despite broken inversion symmetry. (d) The electron density $\langle c_{\mathbf{k},s}^\dagger c_{\mathbf{k},s} \rangle$ of a charge-6e superconductor displays hot spots on its Fermi surface. The corresponding $P_T|\Psi\rangle$ exhibits vanishing charge-2e and (e) charge-4e expectation values, yet (f) a sharp signal in charge-6e correlations (we have fixed $k_y = k'_y = k''_y = 2.0$). (g) The charge-4e order parameter V_{4e} scales proportionally to both V_Q and $V_{-\mathbf{Q}}$; (h) similarly, the charge-6e order parameter scales trilinearly to each of the three parent PDWs' amplitudes. Here, we set $L = 1000$.

linear dependence on both V_Q and $V_{-\mathbf{Q}}$: $V_{4e} \propto V_Q V_{-\mathbf{Q}}$, consistent with our perturbative analysis and irrespective of broken inversion symmetry. On the contrary, all charge-4e expectation values $\langle c_{\mathbf{r},\uparrow}^\dagger c_{\mathbf{r}',\downarrow}^\dagger c_{\mathbf{r}'',\uparrow}^\dagger c_{\mathbf{r}''',\downarrow}^\dagger \rangle$ vanish in $P_T|\Psi\rangle$ in the absence of momentum conservation, e.g., the momentum components $\mathbf{Q}' \neq -\mathbf{Q}$ within the PDW, as we summarize in Figs. 3(c) and 3(g). These results and conclusions also extend to relatively large values of $V_{\pm\mathbf{Q}}$, as we discuss further in the Supplemental Material [81].

Further, we can generalize such a construction to higher-charge superconductivity, e.g., charge-6e Cooper pairs with three PDW momentum components, $\mathbf{Q}_1$, $\mathbf{Q}_2$, and $\mathbf{Q}_3$, which conserves total momentum $\mathbf{Q}_1 + \mathbf{Q}_2 + \mathbf{Q}_3 = 0$. For instance, we employ the following PDW term on top of the Fermi sea (first line) in Eq. (1):

$$H_{\text{PDW}} = \sum_{\mathbf{r}} V(x) c_{\mathbf{r},\uparrow}^\dagger c_{\mathbf{r},\downarrow}^\dagger + \text{H.c.},$$

$$V(x) = V_Q e^{iQx} + V_{-\mathbf{Q}/2} e^{-iQx/2}. \quad (2)$$

Following translation symmetrization, the resulting state $P_T|\Psi\rangle$ witnesses vanishing charge-2e and charge-4e correlations, yet a clear charge-6e pairing signature ($\langle c_{\mathbf{k},\uparrow}^\dagger c_{-\mathbf{k}+\mathbf{Q},\downarrow}^\dagger c_{\mathbf{k}',\uparrow}^\dagger c_{-\mathbf{k}'-\mathbf{Q}/2,\downarrow}^\dagger c_{\mathbf{k}'',\uparrow}^\dagger c_{-\mathbf{k}''-\mathbf{Q}/2,\downarrow}^\dagger \rangle$), which scales as V^3 , or proportionally to the amplitude

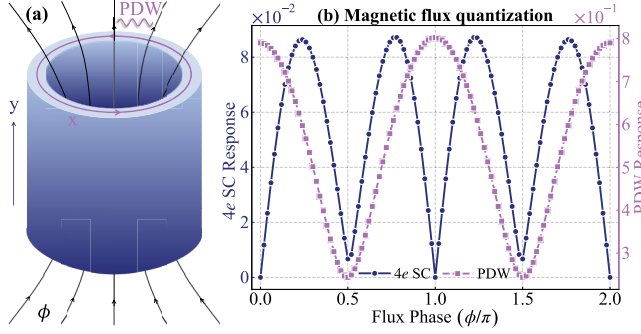


FIG. 4. (a) An illustration of our interferometer setup $P_T|\Psi(\phi)\rangle$: translation symmetrizing the parent PDW model in Eq. (1) with a PDW in the x direction and a magnetic flux ϕ in the y direction. (b) Unlike the parent PDW system $|\Psi(\phi)\rangle$ with a π periodicity in ϕ corresponding to conventional charge- $2e$ Cooper pairs, the susceptibility $\mathcal{L}(\phi)$ in Eq. (3) of $P_T|\Psi(\phi)\rangle$ exhibits an unmistakable charge- $4e$ signature: $\pi/2$ period with respect to the threaded magnetic flux ϕ . Here, our system size is $L = 1000$.

of each of the participating PDWs. These results are presented in Figs. 3(d)–3(f) and 3(h). We note that potential signatures on sequential or coexisting charge- $4e$ and charge- $6e$ orderings observed in CsV_3Sb_5 materials [37] are consistent with the previously hosting PDW patterns [19,20], which conserve the total momentum under both charge- $4e$ and charge- $6e$ mechanisms. Generalizations to higher dimensions or other PDW scenarios, such as the checkerboard order, are straightforward, albeit a corresponding translation symmetrization P_T may need to be introduced.

Phenomena of higher-charge superconductivity. Our microscopic many-electron models of higher-charge superconductors have opened opportunities to understand and discover their properties. For instance, we may probe their exotic magnetic-flux quantization by measuring the periodicity of their physical properties in an interferometer geometry. In particular, we may introduce to the parent PDW model in Eq. (1) a magnetic flux along the y direction through a phase ϕ across the boundary along the x direction; see our setup geometry in Fig. 4(a). We neglect the Zeeman effect on the electron spins and the magnetic flux vortices for simplicity.

After the translation symmetrization, fully compatible with the magnetic flux, we evaluate the susceptibility of the charge- $4e$ Cooper pair:

$$\mathcal{L}(\phi) = \frac{\partial \langle c_{\mathbf{k},\uparrow}^\dagger c_{-\mathbf{k}+\mathbf{Q},\downarrow}^\dagger c_{\mathbf{k}',\uparrow}^\dagger c_{-\mathbf{k}'-\mathbf{Q},\downarrow} \rangle |_{P_T|\Psi(\phi)\rangle}}{\partial V}, \quad (3)$$

for the subsequent many-electron state $P_T|\Psi(\phi)\rangle$, which we summarize in Fig. 4(b). Indeed, $\mathcal{L}(\phi)$ exhibits clear, distinctive $\pi/2$ periodicity in magnetic fluxes and offers strong evidence for its charge- $4e$ fundamentals. We attribute the slight deviations from a perfect $4e$ -period to residue contributions of the electrons, especially the remaining Fermi surface, and the finite-size effects; please refer to the Supplemental Material for additional results and generalizations [81]. This is in sharp contrast with its parent PDW state, where the π magnetic-flux period plays a dominant role in the pairing susceptibility contributed by charge- $2e$ Cooper pairs.

In addition, our formalism may reveal features of higher-charge superconducting states: While the charge- $2e$ Cooper-pair channel is initially gapped out, localized Cooper pairs may re-emerge at certain topological defects, as long as their Burgers vector $\mathbf{B}$ is compatible, i.e., parallel or antiparallel, with the parent PDW wave vector $\mathbf{Q}$. Intuitively, a topological defect may serve as a source or drain of momentum along its Burgers vector [81], thus assisting charge- $2e$ Cooper pairs originating from single PDW scatterings to conserve momentum and exist locally. We note that a topological defect and its Burgers vector—essentially a topological invariant—remains robust upon perturbations; likewise, such localized Cooper pairs also remain relatively stable. When and

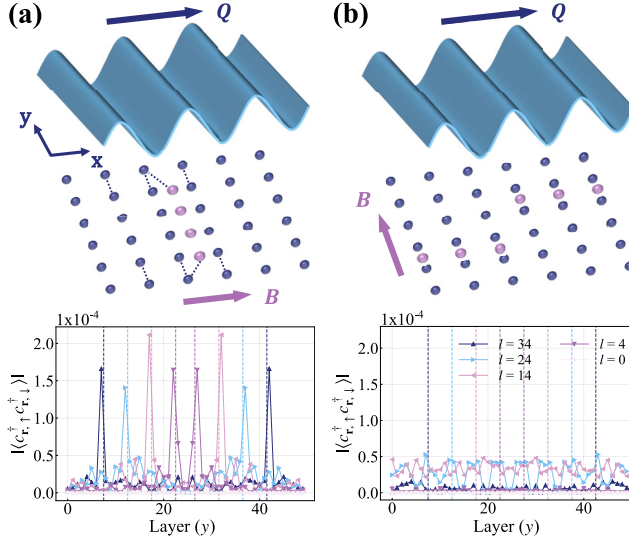


FIG. 5. (a) Charge- $2e$ Cooper pairs re-emerge locally at topological defects of the underlying lattice when the Burgers vector $\mathbf{B}$ aligns with the parent PDW wave vector $\pm\mathbf{Q}$, yet (b) no such modes are visible when $\mathbf{B}$ and $\mathbf{Q}$ are vertical. The upper panels show schematic illustrations of the lattices with a pair of topological defects and the PDWs before translation symmetrization P_T . The dashed lines illustrate the local hopping connectivity, which becomes 1-to-2 on the edge-dislocation sites. The lower panels are the charge- $2e$ expectation values $\langle c_{\mathbf{r},\uparrow}^\dagger c_{\mathbf{r},\downarrow}^\dagger \rangle$ for such $P_T|\Psi\rangle$ with various distances l between the two topological defects. Here, we have $L = 50$.

only when two topological defects with opposing Burgers vectors encounter and annihilate, their respective local charge- $2e$ Cooper pairs with $\pm\mathbf{Q}$ momentum merge into the charge- $4e$ Cooper pair condensate.

To demonstrate such localized charge- $2e$ Cooper pairs, we introduce a pair of edge dislocations with opposing Burgers vectors $\pm\mathbf{B}$ into the 2D lattice hosting the parent Hamiltonian in Eq. (1) (see Fig. 5, upper panels). In particular, we insert l additional lattice sites in the bulk, thus separating the two edge dislocations by a distance l . Except in the vicinity of the topological defects, the system remains effectively translation invariant, and we can generalize our translation symmetrization P_T accordingly; see further details in Ref. [81].

As a result, lower panels of Fig. 5 display the charge- $2e$ Cooper pair density $\langle c_{\mathbf{r},\uparrow}^\dagger c_{\mathbf{r},\downarrow}^\dagger \rangle$ for the many-electron state $P_T|\Psi\rangle$. When the Burgers vector $\mathbf{B}$ is perpendicular to the PDW wave vector $\mathbf{Q}$, charge- $2e$ Cooper pairs remain fully absent. In contrast, when $\mathbf{B}$ is parallel to $\mathbf{Q}$, clear charge- $2e$ pairing signals emerge at the dislocations, while everywhere else in the lattice remains oblivious to the topological defects and minimally charge- $4e$. From the perspective of phase fluctuations, such a topological defect breaks translation symmetry and acts as a local “vacuum” for the proliferating vortices, allowing the charge- $2e$ Cooper pairs to re-emerge nearby. As we gradually decrease the separation l between the two dislocations, the localized charge- $2e$ Cooper pairs approach and eventually annihilate each other. During this process, the pinned vortices resaturate the former defect cores, causing the local charge- $2e$ signals to vanish and fully restoring the translation-symmetric, vestigial charge- $4e$ superconducting state without topological defects. Similar conclusions also hold for open boundaries of higher-charge superconductors, which, when normal to $\mathbf{Q}$, may transfer momentum and thus host localized charge- $2e$ Cooper pairs, as we further discuss in Ref. [81]. These unusual localized-Cooper-pair phenomena are fully consistent with our theoretical expectations and offer smoking-gun signatures for identifying charge- $4e$ superconductivity as well as parent PDW in candidate materials, e.g., CsV_3Sb_5 , via scanning tunneling spectroscopy experiments.

Discussion. We have established many-electron characterizations of higher-charge superconductivity by translation-symmetrizing parent PDW systems, consistent with the Ginzburg-Landau theory of vestigial orders. In particular, with simple stripe PDWs, we clearly demonstrate charge- $4e$ and charge- $6e$ superconductivity, with vanishing charge- $2e$ Cooper-pair sectors as well as long-range pair-pair correlations [81]. Interestingly, momentum conservation, rather than stricter point-group symmetries, is in principle sufficient for their establishment and thus significantly broadens their potential material platforms.

Beyond serving as a conceptual framework, our many-electron formalism provides concrete predictions for experimental observables. The obtained higher-charge superconductors exhibit consistent magnetic flux quantization and periodicity in their interferometry. We also show that charge- $2e$ Cooper pairs may re-emerge at topological defects of the underlying lattice, if their Burgers vector and the parent PDW wave vector are compatible. In addition, as a potential manifestation under specific settings (e.g., complex hopping amplitudes), the spin-up (or spin-down) electrons of the charge- $4e$ superconductors can exhibit topological exchange statistics that resemble a nodeless $p + ip$ -wave superconductor, even while the overall system preserves macroscopic time-reversal symmetry. This feature thus offers an arena for nontrivial topological phenomena and applications, as we discuss further in the Supplemental Material [81]. Further theoretical insights from the perspective of topological quantum field theory are detailed in Refs. [87,88]. Overall, our formalism offers opportunities to understand and identify these exotic quantum matters.

Finally, generalizations of the current translation-symmetrization formalism to higher dimensions, other density-wave scenarios, and alternative parent numerical ansatzes, such as matrix product states and quantum Monte Carlo methods, are straightforward. Interestingly, applications to spin density waves and charge density waves may offer a perspective and method for the physics of intertwined orders [89–92] in strongly correlated systems [93,94]. There, in addition to the universal behaviors of the superconducting phases that we have mainly focused on, details such as the parent density wave’s pattern, strength, and wave vector may impact their overall energetic competitiveness. Besides, as in the topological theories of gapped quantum spin liquids, it will be crucial to investigate the physics of such Cooper-pair-binding vortices, based on particular Fermi seas entirely gapped out by the parent density waves and thus clean from residual contributions.

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Data availability. The data that support the findings of this article are available from the authors upon reasonable request.

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