

Quantum advantage from negativity of work quasiprobability distributions

Gianluca Francica 

Dipartimento di Fisica e Astronomia, [Università di Padova](#), via Marzolo 8, 35131 Padova, Italy



(Received 27 March 2025; accepted 5 August 2026; published 13 August 2026)

Quantum batteries can be charged by performing a work “instantaneously” in the limit of a large number of cells, achieving a so-called quantum advantage. In general, the work exhibits statistics that can be represented by a quasiprobability in the presence of initial quantum coherence in the energy basis. Here, we show that these two concepts of quantum thermodynamics, which apparently appear disconnected, can show a simple relation. Specifically, if a certain work distribution shows negativity asymptotically in the limit of a large number of cells and in a certain time interval, then we surely get a quantum advantage in the charging process. In particular, we prove this for a direct charging protocol performed with a class of charging Hamiltonian operators.

DOI: [10.1103/xwk6-qzzw](https://doi.org/10.1103/xwk6-qzzw)

I. INTRODUCTION

Quantum batteries [1–3] are thought of as quantum systems capable of storing energy. This energy can be used to perform useful work, e.g., through a cyclic change in some parameters of the quantum system [4]. The maximum extractable work is called ergotropy [5] and can be related to some quantum features of the initial state, e.g., quantum correlations [6–10] and quantum coherence [11–13]. This amount of work can also be extracted without generating entanglement during the process [14]. Furthermore, all operations that do not generate this useful work have recently been investigated [15]. Regarding battery charge, when the battery is formed by N cells (or copies) the duration time tends to zero as N increases, and thus a quantum advantage is achieved [16–18]. There are several proposals for the realization of quantum batteries, e.g., by using many-body interactions in spin systems [19], with a cavity assisted charging [20–26], in disordered chains [27], and fermionic Sachdev-Ye-Kitaev interactions [28–31], to name a few. In particular, rescaled Sachdev-Ye-Kitaev interactions [32,33], where the (correlated) disorder is chosen in order to asymptotically saturate the Bhatia-Davis bound [34], allow to achieve an optimal and extensive quantum advantage.

On the other hand, the work done exhibits statistics that satisfies fluctuation theorems [35,36] when it is taken into account with a two-point measurement scheme [37–40], for an incoherent initial state. If there is initial quantum coherence with respect to the energy basis, a no-go theorem [41] indicates that the statistics can be represented with some quasiprobability distribution [42–44], i.e., a normalized distribution that also takes negative [45,46] or complex [47] values. This can be seen as a consequence of the presence of quantum contextuality [48]. Concerning the form of the quasiprobability, it can be derived by starting from some axioms, e.g., the first law of

*Contact author: gianluca.francica@gmail.com

thermodynamics. For weak conditions, no unique quasiprobability distribution is selected, but we get a class of quasiprobability distributions [49,50]. A description in terms of quasiprobabilities allows one to perform an optimization with a utility function [51,52], i.e., based on higher moments of the work, and fluctuation theorems can also be formulated [53]. However, it is not clear what the role of quasiprobability of work in the charging quantum advantage is, where the duration time plays a key role. Here, we try to answer this question.

To do this, we start by introducing the charging process in Sec. II, which is usually known as the direct charging protocol [3], and a class of work quasiprobability distributions, identified by a parameter q , in Sec. III. For our purposes, we focus only on this charging protocol without going beyond the sudden quench setting or generic noise charging models, and we do not take into account alternative definitions of work. In particular, we give an intuitive simple calculation in Sec. III A. This suggests that there can be some relation between the negativity of some work distribution and the charging quantum advantage. This relation subsists for the work done in a certain time subinterval of the process and not for any time subinterval. Furthermore, although there is a quantum advantage, the $q = 0, 1$ quasiprobability distribution takes positive values, thus in this relation the work distribution is not negative in a representation-independent sense. Hence, we explain how the negativity of a certain work quasiprobability distribution in a certain time interval gives a sufficient condition to achieve quantum advantage in Sec. IV. In detail, this negativity survives in the limit of a large number of cells, suggesting that the process is strongly quantum. We further investigate this result with the help of a hybrid model in Sec. V. In the end, we summarize and discuss possible applications of our result in Sec. VI.

II. CHARGING PROCESS

In order to introduce the charging process, we focus on a quantum battery having N cells and the free Hamiltonian $H_0 = \sum_{i=1}^N h_i$, where h_i is the local Hamiltonian of the i th cell. In our process, the battery is initially prepared in the ground state of H_0 , which is $|E_0\rangle = |0\rangle^{\otimes N}$, with $|0\rangle$ such that $h_i|0\rangle = 0$ for all $i = 1, \dots, N$. Thus, the lowest energy of the Hamiltonian H_0 is $E_{\min}^0 = E_0 = 0$. For simplicity, we assume that the ground state is not degenerate. The charging process leads the system to the final state $|\psi(\tau)\rangle = U_{\tau,0}|\psi(0)\rangle$, where $U_{t,0}$ is the unitary operator of time evolution generated by the time-dependent Hamiltonian $H(t)$, with $t \in [0, \tau]$ such that $H(0) = H(\tau) = H_0$. For simplicity, let us focus on a time-dependent Hamiltonian such that $H(t) = H_0$ for $t = 0, \tau$ and $H(t) = H_1$ for $t \in (0, \tau)$. This situation can be realized by performing two sudden quenches at the initial ($t = 0$) and final ($t = \tau$) times. In this case, the unitary operator of time evolution is $U_{t,0} = e^{-iH_1 t}$ when $t \in (0, \tau)$. We consider a Hamiltonian H_1 such that $[H_1, H_0] \neq 0$ in order to achieve a nontrivial dynamics with a final state $|\psi(\tau)\rangle \propto |E_1\rangle$, where $|E_1\rangle$ is the highest excited state of H_0 , with energy $E_1 = E_{\max}^0 \sim N$, since we have $|E_1\rangle = |1\rangle^{\otimes N}$, where $|1\rangle$ is the eigenstate of h_i with maximum energy. For simplicity, we assume that the highest excited state is not degenerate. Thus, at the end of the process, we get a fully charged battery. In order to optimize this charging process, we want to minimize the duration time τ of the process. To do this, the choice of the charging Hamiltonian H_1 is typically constrained by some conditions, e.g., the gap $\Delta E_1 = E_{\max} - E_{\min}$ between the maximum and minimum eigenvalues of H_1 , which are $E_{\max}$ and $E_{\min}$, is not larger than the gap $\Delta E_0 = E_{\max}^0$ between the maximum and minimum eigenvalues of H_0 , which are $E_{\max}^0 \sim N$ and $E_{\min}^0 = 0$. If no interactions between cells are allowed (parallel charging), the charging Hamiltonian has the form $H_1 = \sum_i v_i$, where v_i is a local operator of the i th cell, and we get a minimum duration time τ that remains constant and nonzero as $N \rightarrow \infty$. On the other hand, if interactions are allowed, e.g., terms like $\sum_{i_1, i_2, \dots, i_m} \lambda_{i_1 i_2 \dots i_m} v_{i_1} \otimes v_{i_2} \otimes \dots \otimes v_{i_m}$ may be present in the Hamiltonian H_1 , we can outperform the parallel charging case, getting a duration time $\tau \rightarrow 0$ as $N \rightarrow \infty$, and thus we achieve a charging quantum advantage.

In order to give a sufficient scaling argument for a quantum advantage such that $\tau \rightarrow 0$ as $N \rightarrow \infty$, we note that the average work done in the process in general is $W = \langle H_0 \rangle_\tau - \langle H_0 \rangle_0$. In general, we define $\langle X \rangle_t = \text{Tr}[X \rho(t)]$ for any operator X and a density matrix $\rho(t)$. In our case, $t \in [0, \tau]$ and

$\rho(t)$ is the time-evolved pure state $\rho(t) = |\psi(t)\rangle\langle\psi(t)|$, with $|\psi(t)\rangle = U_{t,0}|\psi(0)\rangle$, so that $\langle X \rangle_t = \langle \psi(t) | X | \psi(t) \rangle$. In particular, we have $\langle H_0 \rangle_t = \sum_i \langle e^{iH_1 t} h_i e^{-iH_1 t} \rangle_0$, which can be expanded in a series of powers in t by using the Baker-Campbell-Hausdorff formula, i.e., $\langle H_0 \rangle_t = \sum_i \langle H_1 h_i H_1 \rangle_0 t^2 + \dots$, where we noted that $|\psi(0)\rangle = |E_0\rangle$ and $E_0 = E_{\min}^0 = 0$. Since $|\psi(\tau)\rangle \propto |E_1\rangle$ for our process, we get

$$\frac{E_1}{N} = \frac{1}{N} \sum_i \langle H_1 h_i H_1 \rangle_0 \tau^2 + \dots = \sum_k \sum_i \frac{w_{ik}}{N} \tau^k, \quad (1)$$

where we note that the series is convergent. We emphasize that before to perform the limit $N \rightarrow \infty$, we have to sum the infinite series, since the two limits are not interchangeable in general. To see how τ scales with N , we can use Eq. (1), which of course is satisfied for some duration time τ , i.e., τ is a solution of Eq. (1). Thus, if for some k the coefficient of the k th term is unbounded, i.e.,

$$\left| \sum_i \frac{w_{ik}}{N} \right| \rightarrow \infty, \quad (2)$$

as $N \rightarrow \infty$, then $\tau \rightarrow 0$ since the series in Eq. (1) converges to the sum $E_1/N \sim O(1)$, so that the series is bounded as $N \rightarrow \infty$. To understand this, we define the function $w(N)$ as the smallest function such that $|\sum_i w_{ik}/N|^{1/k} \leq w(N)$ for all k . We define the function $f(x) = \sum_k a_k x^k$, with $a_k = \sum_i w_{ik}/(N(w(N))^k)$, such that a_k tends to a finite constant (or zero) in the limit $N \rightarrow \infty$ for all k . Then, $\tau = x^*/w(N)$, where $x^* \sim O(1)$ is the solution $x = x^*$ of $f(x) = E_1/N$, so that $\tau \rightarrow 0$ if $w(N) \rightarrow \infty$. By noting that $|\sum_i w_{i2}| = \langle H_1 H_0 H_1 \rangle_0$, we get that if $\langle H_1 H_0 H_1 \rangle_0/N \rightarrow \infty$ as $N \rightarrow \infty$ then $\tau \rightarrow 0$. Thus, in this case, there is a quantum advantage.

III. QUASIPROBABILITY DISTRIBUTION

Having introduced our charging process, let us focus on the work done in a time interval $[t_1, t_2]$ for a generic out-of-equilibrium process generated by a time-dependent Hamiltonian $H(t)$. We are considering a time interval $[t_1, t_2]$ different from the one previously considered, which was $[0, \tau]$, since we will focus on $t_1 \in [0, t_2)$ and $t_2 \in (0, \tau]$ in our charging process, later in the paper. In general, the statistics of the work can be represented by some quasiprobability distribution, and for our purposes, we focus on

$$p_q(w) = \sum_{i,j,k} \text{Re} \langle E_i^{t_1} | \rho(t_1) | E_j^{t_1} \rangle \langle E_j^{t_1} | U_{t_2,t_1}^\dagger | E_k^{t_2} \rangle \langle E_k^{t_2} | U_{t_2,t_1} | E_i^{t_1} \rangle \delta(w - E_k^{t_2} + qE_i^{t_1} + (1-q)E_j^{t_1}), \quad (3)$$

where q is a real number, $\rho(t_1)$ is the density matrix at time $t = t_1$, and instantaneous eigenvalues E_i^t and eigenstates $|E_i^t\rangle$ are defined by the spectral decomposition $H(t) = \sum_i E_i^t |E_i^t\rangle\langle E_i^t|$. We recall that the time-evolution operator is obtained as $U_{t_2,t_1} = \mathcal{T} e^{-i \int_{t_1}^{t_2} H(t) dt}$, where $\mathcal{T}$ is the time ordering operator. Thus, the moments of the work are defined as $\langle w^n \rangle = \int w^n p_q(w) dw$. In particular, the first three moments read $\langle w \rangle = \langle (H^{(H)}(t_2) - H(t_1)) \rangle_{t_1}$, $\langle w^2 \rangle = \langle (H^{(H)}(t_2) - H(t_1))^2 \rangle_{t_1}$, and

$$\begin{aligned} \langle w^3 \rangle &= \langle (H^{(H)}(t_2) - H(t_1))^3 \rangle_{t_1} + \frac{1}{2} [[H(t_1) + H^{(H)}(t_2), [H(t_1), H^{(H)}(t_2)]]]_{t_1} \\ &\quad - 3q(1-q) \langle [H(t_1), [H(t_1), H^{(H)}(t_2)]] \rangle_{t_1}, \end{aligned} \quad (4)$$

where the average is calculated with respect to $\rho(t_1)$ as $\langle X \rangle_t = \text{Tr}\{X \rho(t)\}$ for any operator X and we define $H^{(H)}(t_2) = U_{t_2,t_1}^\dagger H(t_2) U_{t_2,t_1}$. We note that only the first two moments do not depend on the parameter q , whereas the higher moments, which will play a key role in our analysis, depend on the particular representation chosen, i.e., on the value of q . This quasiprobability distribution $p_q(w)$ is the only quasiprobability distribution that satisfies some conditions [50] (it reproduces the two-point measurement scheme for incoherent initial states and gives the values for $\langle w \rangle$ and $\langle w^2 \rangle$ of above) and helps us to understand what the work statistics in the quantum regime is. When the quasiprobability distribution $p_q(w)$ also takes negative values, we can get some advantage. As noted in Ref. [52], when there is negativity, the moments are not constrained by the Jensen's inequality. Then, given

a strictly increasing function $f(w)$ that is concave, i.e., such that $f'(w) > 0$ and $f''(w) \leq 0$, if $p_q(w) \geq 0$ for all w , from the Jensen's inequality we get the lower bound

$$\langle w \rangle \geq f^{-1}(\langle f(w) \rangle). \quad (5)$$

If $f(w) = aw + b$, with $a > 0$, this bound is trivially saturated, but for a function $f(w)$ such that $\min_w -f''(w)/f'(w) \geq \varepsilon$ for a given $\varepsilon > 0$, the bound cannot be saturated and if the bound is violated then there is negativity. We can refer to $f(w)$ as utility function, as usual. Thus, when $\langle w \rangle$ is negative for some work extraction processes, we can get a larger amount of work extracted $w_{ex} = -\langle w \rangle$ that goes beyond the bound given by the utility $f(w)$, provided that these processes show quasiprobability distributions of work also taking negative values. However, differently from the case discussed, it is not clear how the negativity of the quasiprobability can be related to the charging quantum advantage, which involves the duration time τ and the work moments seem to play no role.

In the end, we note that negativity can be measured with $\mathcal{N} = \int |p_q(w)|dw$, which is larger than 1, i.e., $\mathcal{N} > 1$, if and only if p_q takes also negative values [54]. We define the characteristic function $\chi_q(u) = \langle e^{iuw} \rangle$, such that the quasiprobability distribution is achieved by performing the Fourier transform $p_q(w) = \frac{1}{2\pi} \int e^{-i w u} \chi_q(u) du$. We get $\chi_q(u) = \frac{1}{2}(X_q(u) + X_{1-q}(u))$, where

$$X_q(u) = \text{Tr}\{e^{-iuqH(t_1)}U_{t_1,0}\rho(0)U_{t_1,0}^\dagger e^{-iu(1-q)H(t_1)}e^{iuH^{(H)}(t_2)}\}, \quad (6)$$

where we consider $\rho(t_1) = U_{t_1,0}\rho(0)U_{t_1,0}^\dagger$ and $t_1 \in [0, t_2]$, $t_2 \in (0, \tau]$. Thus, negativity can be inferred experimentally by measuring the characteristic function $\chi_q(u)$ through an interferometric measurement scheme (see, e.g., Ref. [49]). For instance, we can consider a detector that is a qubit in the initial state $\rho_D(0)$ with Hamiltonian $H_D = \omega_D|e\rangle\langle e|$. We consider $H_I = -\delta_e|e\rangle\langle e| + \delta_g|g\rangle\langle g|$ and $H'_I = -\delta'_e|e\rangle\langle e| + \delta'_g|g\rangle\langle g|$, where $|g\rangle$ is the ground state of the qubit and $|e\rangle$ is the excited state. Thus, if the total Hamiltonian is $H_{tot}(t) = H(t) + H_D + \delta(t - t_1)H(t_1) \otimes H_I + \delta(t - t_2)H(t_2) \otimes H'_I$, the coherence of the qubit at the final time τ reads

$$\langle e|\rho_D(\tau)|g\rangle = \langle e|\rho_D(0)|g\rangle e^{-i\omega_D\tau} \text{Tr}\{e^{i\delta_e H(t_1)}U_{t_1,0}\rho(0)U_{t_1,0}^\dagger e^{i\delta_g H(t_1)}e^{i(\delta'_e + \delta'_g)H^{(H)}(t_2)}\}, \quad (7)$$

from which we can determine $X_q(u)$.

A. Quantum advantage and work quasiprobability

To explain the intuition behind our main result, we focus on the charging process of Sec. II with $H_I = \lambda(|E_1\rangle\langle E_0| + |E_0\rangle\langle E_1|)$. In this case, the time-evolved state is $|\psi(t)\rangle = \cos(\omega t/2)|E_0\rangle - i\sin(\omega t/2)|E_1\rangle$, where $\omega = 2\lambda$. The duration time is $\tau = \pi/\omega$, so we get the final state $|\psi(\tau)\rangle = -i|E_1\rangle$ and the battery is fully charged. If $\lambda \rightarrow \infty$ as $N \rightarrow \infty$, then $\tau \rightarrow 0$ and there is a quantum advantage. Let us examine the statistics of the work done in a time interval $[t_1, t_2]$. For $t_1 = 0$, there is no coherence with respect to the energy basis at the initial time t_1 , then the work distribution is a probability distribution. For $t_1 \in (0, t_2)$ and $t_2 < \tau$, we get $H(t_1) = H(t_2) = H_I$ and since the time evolution is generated by H_I , we get a trivial situation with a probability distribution for the work. In contrast, the case $t_1 \in (0, \tau)$ and $t_2 = \tau$ appears to be more interesting. In particular, for $t_1 = \tau/2$ and $t_2 = \tau$, it is easy to see that the work quasiprobability distribution $p_q(w)$ reads

$$p_q(w) = \frac{1}{4} \sum_{i=\pm, k=0,1} \delta(w - E_k + E_i) + \delta p_q(w), \quad (8)$$

where $E_\pm$ are the only eigenvalues of H_I different from zero and $|E_\pm\rangle$ are the corresponding eigenstates. In detail, we define

$$\delta p_q(w) = \frac{1}{4} \sum_{k=0,1} (-1)^{k+1} (\delta(w - E_k + qE_+ + (1-q)E_-) + \delta(w - E_k + qE_- + (1-q)E_+)), \quad (9)$$

which, of course, also takes negative values. Then, the quasiprobability distribution $p_q(w)$ is a probability distribution for $q = 0, 1$, explicitly $p_0(w) = \frac{1}{2}\delta(w - E_1 + E_-) + \frac{1}{2}\delta(w - E_1 + E_+)$. In contrast, for $q \neq 0, 1$, $p_q(w)$ also takes negative values due to the term $\delta p_q(w)$. Thus, the charging quantum advantage appears to be related to the negativity of $p_q(w)$. However, negativity does not depend on the support of $p_q(w)$, while $\tau \sim 1/\lambda$ depends on the scaling of λ with N . Then, there is no relation between the magnitude of negativity and the degree of quantum advantage. In particular, the negativity achieved is not related to a violation of a Leggett-Garg inequality, which is always satisfied (see Appendix A).

IV. MAIN RESULT

In the following, we consider the charging process of Sec. II with $t_1 \in (0, \tau)$ and $t_2 = \tau$. We get the characteristic function $\chi_q(u) = \frac{1}{2}(X_q(u) + X_{1-q}(u))$, with

$$X_q(u) = \langle e^{-iu(1-q)H_1} e^{iuH_0} e^{-iuqH_1} \rangle_\tau, \quad (10)$$

which does not depend on the particular value of t_1 . Furthermore, the cumulant generating function is defined as $G_q(u) = \ln \chi_q(u)$ and the n th cumulant is $\kappa_n = (-i)^n G_q^{(n)}(0)$. For example, for $q = 1/2$, we get $\chi_{1/2}(u) = X_{1/2}(u)$. In particular, since $X_{1/2}(-u) = X_{1/2}^*(u)$, the Fourier transform of $X_{1/2}(u)$ takes real values, and thus $p_{1/2}(w)$ is a real-valued distribution. In contrast, for $q = 0$, we get $\chi_0(u) = X_0(u) = X_1(u) = e^{iuE_{\max}^0} \langle e^{-iuH_1} \rangle_\tau$. This means that the quasiprobability distribution for $q = 0$ is equal to the probability distribution of an observable. Then, the cumulants κ_n of the work for $q = 0$ that are higher than the first cumulant κ_1 are equal to the cumulants κ'_n of the Hamiltonian $-H_1$ coming from the generating function $G'(u) = \ln \langle e^{-iuH_1} \rangle_\tau$, which are $\kappa'_n = (-i)^n G'^{(n)}(0)$. In particular, the first cumulants are $\kappa'_1 = -\langle H_1 \rangle_\tau$, $\kappa'_2 = \langle H_1^2 \rangle_\tau - \langle H_1 \rangle_\tau^2$, and $\kappa'_3 = -\langle H_1^3 \rangle_\tau + 3\langle H_1^2 \rangle_\tau \langle H_1 \rangle_\tau - 2\langle H_1 \rangle_\tau^3$. Since the first two moments do not depend on q , we see that the first two cumulants κ_n are the same for any value of q . Thus, from the expression derived for $\chi_0(u)$, we get $\kappa_1 = E_{\max}^0 - \langle H_1 \rangle_\tau$ and $\kappa_2 = \kappa'_2$. In particular, the first cumulant κ_1 is the average work done in the interval $[t_1, t_2]$. In contrast, the third cumulant reads

$$\kappa_3 = \kappa'_3 - 6q(1-q)\langle H_1 \tilde{H}_0 H_1 \rangle_\tau, \quad (11)$$

where we defined the operation of inversion of the spectrum of a hermitian operator X as $\tilde{X} = X_{\max} - X$, where $X_{\max}$ is the maximum eigenvalue of X , so that $\tilde{H}_0 = E_{\max}^0 - H_0$. If the eigenvalues of $\tilde{H}_0$ are equal to the eigenvalues of H_0 , this operation can be realized through some unitary operator U_I that squares to the identity, i.e., $\tilde{H}_0 = U_I H_0 U_I$, with $U_I^2 = I$.

In general, we define $g_q(u)$ such that

$$X_q(u) = e^{iuE_{\max}^0} e^{N g_q(u)}. \quad (12)$$

The variance of the work distribution depends on $g_q''(0)$ and is nonzero if and only if $g_q''(0) \neq 0$. We note that $g_q'(0)$ is imaginary, $g_q''(0)$ can be complex, and $g_{1/2}'(0)$ is real, $g_{1/2}'(0) = -\kappa_2/N$. In particular, for $q = 1/2$, the cumulant generating function is $G_{1/2}(u) = iuE_{\max}^0 + N g_{1/2}(u)$ and $\kappa_n = (-i)^n N g_{1/2}^{(n)}(0)$ for $n \geq 2$. We define the quasiprobability distribution

$$\hat{p}_q(w) = \sqrt{N} p_q(\sqrt{N} w). \quad (13)$$

We aim to analyze the properties of the asymptotic formula of this distribution as $N \rightarrow \infty$. Its characteristic function is $\hat{\chi}_q(u) = \int e^{iuw} \hat{p}_q(w) dw = \chi_q(u/\sqrt{N})$. Thus, $\hat{p}_q(w)$ can be represented with a binned histogram, overcoming the delta peaks contained in the distribution. Since the length of the support of $\hat{p}_q(w)$ scales as $\sqrt{N}$, we can consider a number $M \sim \sqrt{N}$ of bins in the histogram. In particular, the histogram $\hat{p}_q^h(w)$ can be obtained from the characteristic function as explained in Appendix B. We proceed by giving a formal cumulant argument similar to the proof of the central limit theorem. Basically, as we will see, under certain circumstances the main nonzero contribution

for $q = 1/2$ comes from a neighbourhood of the mean value, as $N \rightarrow \infty$. If $|g_q^{(n)}(0)| < \infty$ as $N \rightarrow \infty$ for all n , then we get

$$X_q\left(\frac{u}{N^{\frac{1}{2}}}\right) \sim e^{iuN^{-\frac{1}{2}}E_{\max}^0 + N^{\frac{1}{2}}g'_q(0)u + \frac{g''_q(0)}{2}u^2 + O(N^{-\frac{1}{2}})}, \quad (14)$$

from which we note that the skewness is $O(N^{-\frac{1}{2}})$ and tends to zero as $N \rightarrow \infty$. By neglecting $O(N^{-\frac{1}{2}})$ terms, from Eq. (14) we get

$$\hat{p}_q(w) \sim \frac{1}{\sqrt{2\pi}} \operatorname{Re} \left(\frac{\exp\left(\frac{(\sqrt{N}w - E_{\max}^0 + iNg'_q(0))^2}{2Ng''_q(0)}\right)}{\sqrt{-g''_q(0)}} \right), \quad (15)$$

which for $q = 1/2$ reduces to the Gaussian

$$\hat{p}_{1/2}(w) \sim \frac{1}{\sqrt{2\pi}} \frac{\exp\left(\frac{(\sqrt{N}w - E_{\max}^0 + iNg'_{1/2}(0))^2}{2Ng''_{1/2}(0)}\right)}{\sqrt{-g''_{1/2}(0)}}. \quad (16)$$

The same result is derived by noting that the cumulants corresponding to the quasiprobability distribution $\hat{p}_q(w)$ are $\hat{\kappa}_n = \kappa_n/N^{\frac{n}{2}}$. If $|g_q^{(n)}(0)| < \infty$ as $N \rightarrow \infty$ for all n , then $\hat{\kappa}_n \lesssim O(N^{-\frac{1}{2}})$ for $n \geq 3$. Then, $\hat{\kappa}_n \rightarrow 0$ as $N \rightarrow \infty$ for $n \geq 3$ and the function $g_q(u/\sqrt{N})$, appearing in $X_q(u/\sqrt{N})$, can be replaced by a quadratic polynomial in u in the limit $N \rightarrow \infty$, and thus we achieve Eq. (15). We note that the rate at which $\hat{p}_{1/2}(w)$ approaches the asymptotic Gaussian in Eq. (16) is dictated by the largest error term. Then, the difference between $\hat{p}_{1/2}(w)$ and the asymptotic Gaussian in Eq. (16) tends to zero no slower than $1/\sqrt{N}$ as $N \rightarrow \infty$ for all w in the support. This matches the well-known Berry-Esseen theorem. Then, for values of w where the Gaussian tends to zero as $N \rightarrow \infty$, even if $\hat{p}_{1/2}(w)$ can take negative values, $\hat{p}_{1/2}(w)$ tends to zero no slower than $1/\sqrt{N}$ as $N \rightarrow \infty$. Thus, $\hat{p}_q(w)$ is positive for $q = 1/2$ as $N \rightarrow \infty$, i.e., $\hat{p}_{1/2}(w) \geq 0$ in the limit $N \rightarrow \infty$ for all w in the support, and also takes negative values for $q \neq 1/2$ if $g''_q(0)$ is not real. Therefore, for $q = 1/2$ if $\hat{p}_{1/2}(w) < 0$ for some w as $N \rightarrow \infty$, then $g''_{1/2}(0)$ is unbounded for some n , i.e., $|g''_{1/2}(0)| \rightarrow \infty$ as $N \rightarrow \infty$ for some n . Similarly, if $g''_q(0)$ is real, from Eq. (15) we get $\hat{p}_q(w) \sim \hat{p}_{1/2}(w)$. Thus, in this case, if $\hat{p}_q(w) < 0$ for some w as $N \rightarrow \infty$, then $g''_q(0)$ is unbounded for some n .

In order to proceed with the analysis, we define the class $\mathcal{C}$ of systems that are lattice models with local dimension d and total number of sites N . The Hamiltonian reads $H_1 = \sum_X v_X$, where $v_X = 0$ if $|X| > r$ for some r . We assume that there are k terms in the sum over X , with $k \leq N$. Moreover, we assume that $\|H_1\| \sim N$ and $\|v_X\| \sim \lambda$. In the end, we assume that the order of the interactions is $m \sim r^\nu$, where ν is the dimension of the lattice, i.e., we assume that for $|X| \leq r$ the nonzero v_X are

$$v_X = \lambda_{i_1 i_2 \dots i_m} v_{i_1}^1 \otimes v_{i_2}^2 \otimes \dots \otimes v_{i_m}^m, \quad (17)$$

with $m \sim r^\nu$, where $X = \{i_1, i_2, \dots, i_m\}$ and $\lambda_{i_1 i_2 \dots i_m} \sim \lambda$, and $[v_{i_k}^k, H_0] \neq 0$ for all local operators $v_{i_k}^k$ on the site i_k . For simplicity, we also assume that $\langle v_{i_k}^k \rangle_0 = \langle v_{i_k}^k \rangle_\tau = 0$ for all k and i_k . All these assumptions define the class $\mathcal{C}$.

We start to focus on systems with charging Hamiltonian $H_1 \in \mathcal{C}$ such that (1) κ'_3/N is bounded as $N \rightarrow \infty$. Then, from Eq. (11), we get the following sufficient condition to get a quantum advantage.

Lemma 1. For $H_1 \in \mathcal{C}$, when $|\kappa'_3|/N < \infty$ as $N \rightarrow \infty$, if $|\kappa_3|/N \rightarrow \infty$ as $N \rightarrow \infty$, then $\tau \rightarrow 0$.

Proof. If $|\kappa_3|/N \rightarrow \infty$ as $N \rightarrow \infty$, then, from Eq. (11), we get $\langle H_1 \tilde{H}_0 H_1 \rangle_\tau / N \rightarrow \infty$, since κ'_3/N is bounded. In particular, this can happen only for $q \neq 0, 1$. In the end, we note that the term $\langle H_1 \tilde{H}_0 H_1 \rangle_\tau$ in κ_3 , appearing for $q \neq 0, 1$, scales as $\langle H_1 H_0 H_1 \rangle_0$. To show this, we consider $H_1 \in \mathcal{C}$.

The charging Hamiltonian reads $H_1 = \sum_X v_X$. By noting that $h_i|\psi(0)\rangle = 0$, we get

$$\langle H_1 H_0 H_1 \rangle_0 = \sum_{i,X,X'|i \in X \cap X'} \langle v_X h_i v_{X'} \rangle_0. \quad (18)$$

Furthermore, since $\langle v_{i_k}^k \rangle_0 = 0$ for all k and i_k , we get

$$\langle H_1 H_0 H_1 \rangle_0 = \sum_{i,X|i \in X} \langle v_X h_i v_X \rangle_0. \quad (19)$$

Similarly, by noting that $\tilde{h}_i|\psi(\tau)\rangle = 0$, we get

$$\langle H_1 \tilde{H}_0 H_1 \rangle_\tau = \sum_{i,X,X'|i \in X \cap X'} \langle v_X \tilde{h}_i v_{X'} \rangle_\tau, \quad (20)$$

and, since $\langle v_{i_k}^k \rangle_\tau = 0$ for all k and i_k , we get

$$\langle H_1 \tilde{H}_0 H_1 \rangle_\tau = \sum_{i,X|i \in X} \langle v_X \tilde{h}_i v_X \rangle_\tau. \quad (21)$$

Then, since both sums in Eqs. (19) and (21) involve the same number of positive numbers, which scale as λ^2 , we get $\langle H_1 \tilde{H}_0 H_1 \rangle_\tau \sim \langle H_1 H_0 H_1 \rangle_0$. In particular, we get the equality $\langle H_1 \tilde{H}_0 H_1 \rangle_\tau = \langle H_1 H_0 H_1 \rangle_0$ if the inversion symmetry $U_I H_1 U_I = H_1$ is satisfied. Then, from the sufficient scaling argument of Sec. II, we get $\tau \rightarrow 0$. ■

Furthermore, when condition (1) is satisfied, for any Hamiltonian $H_1 \in \mathcal{C}$, for $q \neq 0, 1$, if $|g_q^{(n)}(0)| \rightarrow \infty$ as $N \rightarrow \infty$ for some n , then $|g_q'''(0)| \rightarrow \infty$.

Proof. We have to show that $|g_q'''(0)| < \infty$ for $N \rightarrow \infty$ implies $|g_q^{(n)}(0)| < \infty$ for $N \rightarrow \infty$ for all n . To show this, we consider $H_1 \in \mathcal{C}$. We get

$$N \sim \|H_1\| \leq \sum_X \|v_X\| \sim \lambda k. \quad (22)$$

Thus, $\lambda k \geq cN$ for some positive constant c . From Eq. (21), if the order of the interactions is $m \sim r^\nu$, we get

$$\langle H_1 \tilde{H}_0 H_1 \rangle_\tau = \sum_{i,X|i \in X} \langle v_X \tilde{h}_i v_X \rangle_\tau \sim \lambda^2 r^\nu k. \quad (23)$$

Then, if condition (1) holds, since $\kappa_3 = iN g_q'''(0)$, we get $|g_q'''(0)| \geq c' \lambda^2 r^\nu k / N$ for $q \neq 0, 1$, and thus $\lambda^2 r^\nu k / N < \infty$. Since $\lambda k \geq cN$, we get $|g_q'''(0)| \geq c'' r^\nu N / k$, then $r^\nu N / k < \infty$. If $N/k \rightarrow \infty$, $|g_q'''(0)| \rightarrow \infty$, then $1 \leq N/k < \infty$ and thus $r < \infty$. Then, v_X is local and the state

$$|\tilde{\psi}_q(u)\rangle = e^{-iu\tilde{H}_0} e^{-iuqH_1} |1\rangle^{\otimes N} \quad (24)$$

is short range entangled [55], thus can be expressed as a tensor product state with finite bond dimension. For $\nu = 1$, we get the matrix product state

$$|\tilde{\psi}_q(u)\rangle = \sum_{i_1, \dots, i_N} A_{i_1} \cdots A_{i_N} |i_1 \cdots i_N\rangle, \quad (25)$$

where A_{i_k} are $D_{k-1} \times D_k$ matrices with $D_k < \infty$ as $N \rightarrow \infty$. Similarly, the state

$$|\psi_{1-q}(-u)\rangle = e^{iu(1-q)H_1} |1\rangle^{\otimes N} \quad (26)$$

can be expressed as

$$|\psi_{1-q}(-u)\rangle = \sum_{i_1, \dots, i_N} B_{i_1} \cdots B_{i_N} |i_1 \cdots i_N\rangle, \quad (27)$$

where B_{i_k} are $D_{k-1} \times D_k$ matrices, with $D_k < \infty$ as $N \rightarrow \infty$. The overlap between these two states reads

$$\langle \psi_{1-q}(-u) | \tilde{\psi}_q(u) \rangle = \sum_{i_1, \dots, i_N} B_{i_N}^\dagger \cdots B_{i_1}^\dagger A_{i_1} \cdots A_{i_N}. \quad (28)$$

By performing a singular value decomposition, the matrix A_{i_k} can be expressed as $A_{i_k} = U_{i_k} \Lambda_{i_k} V_{i_k}^\dagger$, where U_{i_k} and V_{i_k} are unitary matrices and $\Lambda_{i_k} = \sum_j a_{i_k,j} |e_j\rangle\langle e_j|$, with $a_{i_k,j} \geq 0$, thus $A_{i_k} = \sum_j a_{i_k,j} |a'_{i_k,j}\rangle\langle a_{i_k,j}|$, with $|a'_{i_k,j}\rangle = U_{i_k} |e_j\rangle$ and $|a_{i_k,j}\rangle = V_{i_k} |e_j\rangle$. In particular, for $u = 0$, we get $A_{i_k} = \delta_{i_k,1} |e_1\rangle\langle e_1|$. Similarly, $B_{i_k} = \sum_j b_{i_k,j} |b'_{i_k,j}\rangle\langle b_{i_k,j}|$, then from Eq. (28) we get

$$\langle \psi_{1-q}(-u) | \tilde{\psi}_q(u) \rangle \sim \prod_{k=2}^N a_k b_k \langle a'_k | a_{k-1} \rangle \langle b_k | b'_{k-1} \rangle, \quad (29)$$

where a_k and b_k are certain optimal elements in the sets $\{a_{i_k,j}\}$ and $\{b_{i_k,j}\}$, e.g., we can have (but not necessarily) $a_k = \max_{i_k,j} a_{i_k,j}$ and $b_k = \max_{i_k,j} b_{i_k,j}$, such that $a_k = b_k = 1$ and $|a_k\rangle = |a_k\rangle = |b_k\rangle = |e_1\rangle$ for $u = 0$. Then, by noting that,

$$g_q(u) = \frac{1}{N} \ln \langle \psi_{1-q}(-u) | \tilde{\psi}_q(u) \rangle \quad (30)$$

we get $g_q(u) \sim \frac{1}{N} \sum_k g_{q,k}(u)$, where $g_{q,k}(0) = 0$, thus $g_q^{(n)}(0) \sim O(1)$ for all n . For $\nu > 1$, the proof is completely similar. ■

In contrast, when condition (1) is not satisfied, i.e., κ'_3/N is unbounded, for $H_1 \in \mathcal{C}$ we get $\tau \rightarrow 0$, and there is a quantum advantage. In general, for $H_1 \in \mathcal{C}$ we have the following lemma.

Lemma 2. For $H_1 \in \mathcal{C}$, if $|\kappa'_n|/N \rightarrow \infty$ as $N \rightarrow \infty$ for some $n \geq 2$, then $\tau \rightarrow 0$.

Proof. To show this, we consider $H_1 \in \mathcal{C}$. Given X such that $|X| \leq r$, the number of X' such that $|X'| \leq r$ and $X \cap X' \neq \emptyset$ is $r' < ar^\nu$ for some positive constant a , then

$$\kappa'_2 = \sum_{X, X' | X \cap X' \neq \emptyset} \langle v_X v_{X'} \rangle_\tau - \langle v_X \rangle_\tau \langle v_{X'} \rangle_\tau \quad (31)$$

$$\lesssim kr' \lambda^2, \quad (32)$$

$$\kappa'_3 = \sum_{X, X', X'' | X \cap X' \cap X'' \neq \emptyset} -\langle v_X v_{X'} v_{X''} \rangle_\tau + 3 \langle v_X v_{X'} \rangle_\tau \langle v_{X''} \rangle_\tau - 2 \langle v_X \rangle_\tau \langle v_{X'} \rangle_\tau \langle v_{X''} \rangle_\tau, \quad (33)$$

$$|\kappa'_3| \lesssim kr'^2 \lambda^3, \quad (34)$$

and so on. If $\|H_1\| \sim N$, then $\lambda \gtrsim N/k$, thus for $n \geq 2$, $|\kappa'_n|/N \lesssim (r'\lambda)^{n-1} k \lambda / N$, with $k \lambda / N \gtrsim 1$. Hence, if $|\kappa'_n|/N \rightarrow \infty$ as $N \rightarrow \infty$ for some $n \geq 2$, then we can have (1) $k \lambda / N < \infty$ and $r' \lambda \rightarrow \infty$, (2) $k \lambda / N \rightarrow \infty$ and $r' \lambda > 0$, or (3) $k \lambda / N \rightarrow \infty$ and $r' \lambda \rightarrow 0$. Similar to Eq. (23), if the order of the interactions is $m \sim r^\nu$, we have

$$\langle H_1 H_0 H_1 \rangle_0 = \sum_{i, X | i \in X} \langle v_X h_i v_X \rangle_0 \sim \lambda^2 r^\nu k, \quad (35)$$

thus $\langle H_1 H_0 H_1 \rangle_0 / N \sim r^\nu \lambda^2 k / N > c' r' \lambda^2 k / N$, so that if $|\kappa'_n|/N \rightarrow \infty$ for some $n \geq 2$, then $\langle H_1 H_0 H_1 \rangle_0 / N \rightarrow \infty$ [if (1) or (2) holds the proof is straightforward, if (3) holds it is enough to note that $r' \lambda \geq (r' \lambda)^{n-1}$] and thus from the sufficient scaling argument of Sec. II we get $\tau \rightarrow 0$. ■

Thus, we deduce our main result, which is the following theorem.

Theorem 1. For the charging process of Sec. II, for $t_1 \in (0, \tau)$ and $t_2 = \tau$, and for any charging Hamiltonian $H_1 \in \mathcal{C}$, if for $q = 1/2$ the asymptotic rescaled quasiprobability distribution of the work w done in the time interval $[t_1, t_2]$ also takes negative values, i.e., $\hat{p}_{1/2}(w) < 0$ for some w as $N \rightarrow \infty$, then $\tau \rightarrow 0$ as $N \rightarrow \infty$, and thus there is a quantum advantage in the charging process.

Furthermore, for $q \neq 1/2$, if $g_q''(0)$ is real and $\hat{p}_q(w) < 0$ for some w as $N \rightarrow \infty$, then $\tau \rightarrow 0$ as $N \rightarrow \infty$.

Proof. For $H_1 \in \mathcal{C}$, if condition (1) is not satisfied, from Lemma 2 we get $\tau \rightarrow 0$. In contrast, if condition (1) is satisfied, then, if $\hat{p}_{1/2}(w) < 0$ for some w as $N \rightarrow \infty$, $|g_{1/2}''(0)| \rightarrow \infty$ as $N \rightarrow \infty$. Thus, $|\kappa_3|/N \rightarrow \infty$ and from Lemma 1 we get $\tau \rightarrow 0$. For $q \neq 1/2$, if $g_q''(0)$ is real, the proof is analogous. ■

Basically, the theorem guarantees that, in some circumstances, if $\hat{p}_{1/2}$ shows negativity that does not vanish in the limit of a large number of cells $N \rightarrow \infty$, then we get a quantum advantage with $\tau \rightarrow 0$ in this limit. We observe that the quasiprobability distribution for $q = 1/2$ plays a key role since $g_q''(0)$ is surely real for $q = 1/2$. For $q \neq 1/2$, negativity survives in the limit $N \rightarrow \infty$ if $g_q''(0)$ is complex although τ does not tend to zero as $N \rightarrow \infty$. Of course, we note that the converse of theorem 1 does not hold, since, in principle, we can achieve κ_3/N unbounded for a non-negative probability distribution $\hat{p}_{1/2}(w)$, thus negativity gives just a sufficient condition to achieve $\tau \rightarrow 0$. We show this with the help of a specific example in the next section.

To explain the origin of this quantum advantage, we note that the presence of negativity in $\hat{p}_{1/2}(w)$ implies a nonlocal charging Hamiltonian H_1 . Thus, the quantum advantage in this case is due to the nonlocal structure of H_1 . To understand this, we consider a lattice model with local dimension d and total number of sites N . If H_1 is a r -local Hamiltonian, then $H_1 = \sum_X v_X$, with $v_X = 0$ if $|X| > r$. If $r < \infty$ as $N \rightarrow \infty$, we get

$$|\langle H_1 \tilde{H}_0 H_1 \rangle_\tau| \leq \sum_i \sum_{X|i \in X} \sum_{X'|i \in X'} |\langle v_X \tilde{h}_i v_{X'} \rangle_\tau| \leq cN \quad (36)$$

for some constant c . Then, since κ_3'/N is bounded, κ_3/N is bounded. Similarly, $g_{1/2}^{(n)}(0)$ is bounded for all n . Then, if $\hat{p}_{1/2}(w) < 0$ for some w as $N \rightarrow \infty$, then $g_{1/2}^{(n)}(0)$ is unbounded for some n , which implies that $r \rightarrow \infty$ as $N \rightarrow \infty$, i.e., the charging Hamiltonian H_1 is nonlocal. This charging Hamiltonian generates strong quantum coherence during time evolution; otherwise, the quasiprobability distribution tends to be positive (in this case, it reproduces the two-point measurement scheme probability distribution). Therefore, physical causes of this quantum advantage could be sought in these elements.

V. EXAMPLE

We focus on a local Hilbert space (of a single cell) having dimension $d = 2$. Thus, we consider $h_i = \epsilon_0(\sigma_i^z + 1)/2$, where σ_i^x , σ_i^y , and σ_i^z are the local Pauli matrices of the i th cell. We consider the charging Hamiltonian of Ref. [19]

$$H_1 = \lambda \sum_{j=0}^{k-1} \otimes_{i=1}^r \sigma_{rj+i}^x + \lambda \otimes_{i=1}^s \sigma_{rk+i}^x, \quad (37)$$

where the second term is absent for $s = 0$, with $N = kr + s$, where $k = [N/r]$, $0 \leq s < r$, and $[x]$ is the integer part of x . For $s = 0$, the time-evolved state is

$$|\psi(t)\rangle = e^{-iH_1 t} |0\rangle^{\otimes N} = \otimes_{j=0}^{k-1} |\psi_j(t)\rangle, \quad (38)$$

where

$$|\psi_j(t)\rangle = \sqrt{1-p(t)} \otimes_{i=1}^r |0\rangle_{rj+i} - i\sqrt{p(t)} \otimes_{i=1}^r |1\rangle_{rj+i}, \quad (39)$$

with $p(t) = \sin^2(\lambda t)$. The local states $|0\rangle_i$ and $|1\rangle_i$ are the eigenstates of σ_i^z , with eigenvalues -1 and 1 , respectively. For $s > 0$, the time-evolved state is

$$|\psi(t)\rangle = \otimes_{j=0}^k |\psi_j(t)\rangle, \quad (40)$$

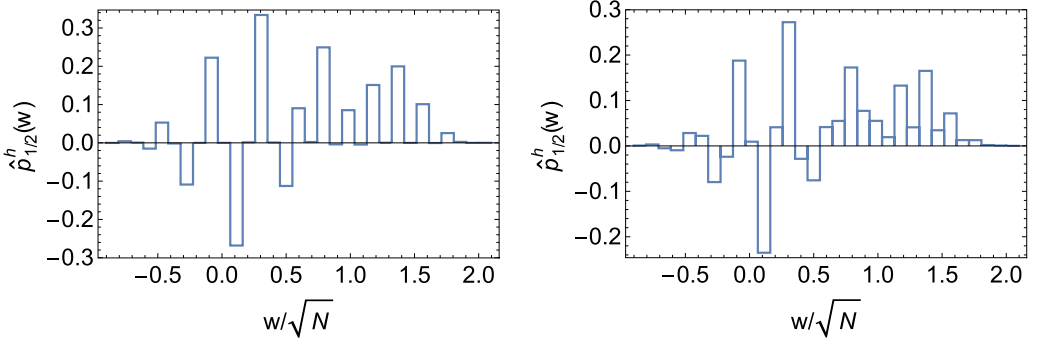


FIG. 1. The histogram $\hat{p}_{1/2}^h(w)$ of the quasiprobability distribution $\hat{p}_{1/2}(w)$. We put $N = 1000$, $r = \lfloor N^{0.75} \rfloor$, $\lambda = r$, and $\epsilon_0 = 1$. We divide the support in a number $M = \lfloor \sqrt{N} \rfloor$ of bins, and the normalization of the plotted quasiprobability histogram is such that the sum of the heights of the bins is one. In the left panel, we put $\alpha = 0$, and in the right panel we put $\alpha = 1$.

where

$$|\psi_k(t)\rangle = \sqrt{1-p(t)} \otimes_{i=1}^s |0\rangle_{rk+i} - i\sqrt{p(t)} \otimes_{i=1}^s |1\rangle_{rk+i}. \quad (41)$$

Thus, the final time is

$$\tau = \frac{\pi}{2\lambda}. \quad (42)$$

For $s = 0$, the maximum and minimum eigenvalues of H_1 are $E_{\max} = \lambda k$ and $E_{\min} = -\lambda k$, then their difference is $\Delta E_1 = 2\lambda k$. In contrast, for $s > 0$ we get $\Delta E_1 = 2\lambda(k+1)$ due to the partial block. Requiring $\Delta E_1 \sim N$ as $N \rightarrow \infty$, we get $\lambda \sim r$, and thus $\tau \sim 1/r$. Then, if $r \rightarrow \infty$ as $N \rightarrow \infty$ we get $\tau \rightarrow 0$ and thus a quantum advantage. We note that $H_1 \in \mathcal{C}$, then Theorem 1 applies. For this model, for $t_1 \in (0, \tau)$ and $t_2 = \tau$, from Eq. (10) we get in the limit $N \rightarrow \infty$

$$X_q(u) \sim e^{iuE_{\max}^0} (c_q(u) - s_q(u)e^{-iur\epsilon_0})^k, \quad (43)$$

where $E_{\max}^0 = N\epsilon_0$, $c_q(u) = \cos(\lambda qu) \cos(\lambda(1-q)u)$, and $s_q(u) = \sin(\lambda qu) \sin(\lambda(1-q)u)$. Thus, $g_q(u)$ reads

$$g_q(u) \sim \frac{1}{r} \ln(c_q(u) - s_q(u)e^{-iur\epsilon_0}), \quad (44)$$

from which we get $g'_q(0) = 0$, $g''_q(0) = -\lambda^2/r \sim r$, and $g'''_q(0) = 6iq(1-q)\epsilon_0\lambda^2 \sim r^2$. Furthermore, $U_I = \otimes_{i=1}^N \sigma_i^x$ and $U_I H_1 U_I = H_I$. Then, condition (1) is satisfied. In order to calculate $\hat{p}_q(w)$, we consider $g_q(u/\sqrt{N})$ as $N \rightarrow \infty$. If $r/\sqrt{N} \rightarrow 0$ as $N \rightarrow \infty$, then $g_q(u/\sqrt{N}) \sim -\lambda^2 u^2/(2rN)$, so that

$$\hat{p}_q(w) \sim \frac{\sqrt{N}}{\sqrt{2\pi}\sigma} e^{-\frac{(\sqrt{N}w - E_{\max}^0)^2}{2\sigma^2}}, \quad (45)$$

with $\sigma^2 = N\lambda^2/r$. Then, $\hat{p}_q(w)$ is a Gaussian, $\hat{p}_q(w) > 0$ although $\tau \rightarrow 0$ if $r \rightarrow \infty$ as $N \rightarrow \infty$. In contrast, if $r/\sqrt{N} \rightarrow \infty$ as $N \rightarrow \infty$, $\hat{p}_{1/2}(w)$ also takes negative values as shown in the left panel of Fig. 1.

Finally, it is interesting to investigate whether the negativity is robust to perturbations. To do this, we consider a perturbation $H_p = \alpha \sum_{i=1}^N \sigma_i^x$, so that the charging Hamiltonian reads $H'_1 = H_1 + H_p$. For $r \rightarrow \infty$ as $N \rightarrow \infty$, at the final time in Eq. (42) the final state is $|\psi(\tau)\rangle \sim |1\rangle^{\otimes N} + O(1/r)$, thus the battery tends to be fully charged as $N \rightarrow \infty$. Concerning the quasiprobability distribution

$p_{1/2}(w)$, we get

$$g_{1/2}(u) \sim \frac{1}{r} \ln (\alpha'_{1/2}(u)c_{1/2}(u) - \beta_{1/2}(u) - (\alpha'_{1/2}(u))^* s_{1/2}(u)e^{-iu\epsilon_0}), \quad (46)$$

where

$$\alpha_{1/2}(u) = \cos^2(\alpha u/2) - e^{-iu\epsilon_0} \sin^2(\alpha u/2), \quad (47)$$

$$\beta_{1/2}(u) = i \sin(\lambda u)(-i \sin(\alpha u)(1 + e^{-iu\epsilon_0})/2)^r. \quad (48)$$

As shown in the right panel of Fig. 1, negativity persists for this commuting perturbation and for the displayed parameters.

VI. CONCLUSIONS

We showed how the quantum advantage is connected to the quasiprobability of the work done in a subinterval of the charging process. If there is some negativity in a quasiprobability ($q = 1/2$) of the work done, then there is a quantum advantage. Thus, the $q = 1/2$ quasiprobability plays a privileged role, while for $q \neq 1/2$ negativity can survive although the duration time does not tend to zero. In particular, for $q = 1/2$, if negativity survives in the limit of a large number of cells, the distribution shows superextensive cumulants, which implies that a quantum advantage is achieved. Basically, under the assumptions used in the proof, the negativity signals nonlocal interactions in the charging Hamiltonian, giving the quantum advantage in the charging of the battery. For these models, this is related with the generation of quantum coherence and entanglement during time evolution. In contrast, from the point of view of the work statistics, the work always admits a representation ($q = 0$) that is a probability distribution (of an observable). We hope that this can help us to better understand the origin of the quantum advantage with a charging time that tends to zero in the limit of a large number of cells, where quantum effects should be very strong. Furthermore, it remains an open problem whether we can check whether the quantum advantage has been achieved by looking at the negativity of a work distribution. In particular, the latter should be measured without destroying the final state of the battery, and a nondemolition diagnostic of the relevant quasiprobability is not supplied here and remains outside the scope of the paper.

DATA AVAILABILITY

There are no publicly available research data or software supporting this manuscript. Requests for further information or data should be sent to the authors.

APPENDIX A: LEGGETT-GARG INEQUALITY

Let us focus on the work w done in the time interval $[t_1, \tau]$, with $t_1 \in (0, \tau)$. We can consider the works w_τ and w_{t_1} done in the time intervals $[0, \tau]$ and $[0, t_1]$. Since $\rho(0)$ is incoherent with respect to the energy basis, these works have the probability distribution $p(w_t) = \sum_{i,k} p_i p_{k|i} \delta(w_t - E_k^t + E_i^0)$, with $t = \tau, t_1$, where $p_i = \langle E_i^0 | \rho(0) | E_i^0 \rangle$ and $p_{k|i} = |\langle E_k^t | U_{t,0} | E_i^0 \rangle|^2$. Thus, if $p_q(w) < 0$ for some w , then there is no joint probability distribution $p(w_\tau, w_{t_1})$, since $w = w_\tau - w_{t_1}$ and thus $p(w_\tau, w_{t_1})$ is such that $\int \delta(w - w_\tau + w_{t_1}) p(w_\tau, w_{t_1}) dw_\tau dw_{t_1} = p_q(w)$. Furthermore, if there exists a joint probability distribution $p(w_\tau, w_{t_1})$, we obtain the inequality for the covariance

$$|\langle w_\tau w_{t_1} \rangle - \langle w_\tau \rangle \langle w_{t_1} \rangle| \leq \sigma_\tau \sigma_{t_1}, \quad (A1)$$

where $\sigma_t^2 = \langle w_t^2 \rangle - \langle w_t \rangle^2$. This inequality can be expressed in terms of the variance $\sigma^2 = \langle w^2 \rangle - \langle w \rangle^2$ as

$$|\sigma_\tau^2 + \sigma_{t_1}^2 - \sigma^2| \leq 2\sigma_\tau \sigma_{t_1}, \quad (A2)$$

which recalls a Leggett-Garg inequality [56,57]. Actually, Leggett-Garg inequalities concern macro-realism and the existence of suitable multitime probability distributions, not merely the displayed covariance bound. In particular, Eq. (A2) does not depend on q and thus on the quasiprobability representation chosen. Therefore, if the inequality in Eq. (A2) is violated, then there is no joint probability $p(w_\tau, w_{t_1})$ and we can get a distribution $p_q(w)$ that is negative for some w . It is easy to check that for the charging process under consideration, the inequality is not violated, since $\sigma_\tau^2 = 0$ and $\sigma_{t_1}^2 = \sigma^2$. However, this does not exclude the fact that $p_q(w)$ is negative for some w . Furthermore, we note that for $q = 0$ the quasiprobability distribution $p_0(w)$ is non-negative, so there is a joint probability distribution $p(w_\tau, w_{t_1})$. This explains why the inequality is satisfied.

APPENDIX B: HISTOGRAM

To determine the histogram $p_q^h(w)$ of the quasiprobability distribution of work $p_q(w)$ from the characteristic function $\chi_q(u)$, we consider the intervals $I_n = [w_n - \Delta w/2, w_n + \Delta w/2]$, where $w_n = n\Delta w$, with n integer. Then, we can determine the histogram by calculating the probability

$$p_n = \int_{I_n} p_q(w) dw = \frac{\Delta w}{2\pi} \int \chi_q(u) \text{sinc}\left(\frac{u\Delta w}{2}\right) e^{-i u w_n} du, \quad (\text{B1})$$

where $\text{sinc}(x) = \sin(x)/x$. Thus,

$$p_q^h(w) = \sum_n p_n \chi_n(w), \quad (\text{B2})$$

where $\chi_n(w) = 1$ for $w \in I_n$ and $\chi_n(w) = 0$ for $w \notin I_n$. To calculate the integral, we can focus on the interval $[-2\pi K/\Delta w, 2\pi K/\Delta w]$, with K large enough. Of course $p_q(w_n) \approx p_n/\Delta w$ for Δw small enough. In the end, we note that the histogram $\hat{p}_q^h(w)$ of the rescaled quasiprobability distribution $\hat{p}_q(w)$ is defined in the same way, and of course has the same form of $p_q^h(w)$ if the support is divided in the same number of bins.

-
- [1] F. Campaioli, F. A. Pollock, and S. Vinjanampathy, Quantum batteries, in *Thermodynamics in the Quantum Regime. Fundamental Theories of Physics*, edited by F. Binder, L. Correa, C. Gogolin, J. Anders, and G. Adesso (Springer, Cham, 2018), Vol. 195.
 - [2] S. Bhattacharjee and A. Dutta, Quantum thermal machines and batteries, *Eur. Phys. J. B* **94**, 239 (2021).
 - [3] F. Campaioli, S. Gherardini, J. Q. Quach, M. Polini, and G. M. Andolina, Colloquium: Quantum batteries, *Rev. Mod. Phys.* **96**, 031001 (2024).
 - [4] R. Alicki and M. Fannes, Entanglement boost for extractable work from ensembles of quantum batteries, *Phys. Rev. E* **87**, 042123 (2013).
 - [5] A. E. Allahverdyan, R. Balian, and T. M. Nieuwenhuizen, Maximal work extraction from finite quantum systems, *Europhys. Lett.* **67**, 565 (2004).
 - [6] G. Francica, J. Goold, F. Plastina, and M. Paternostro, Daemonic ergotropy: Enhanced work extraction from quantum correlations, *npj Quantum Inf.* **3**, 12 (2017).
 - [7] F. Bernards, M. Kleinmann, O. Gühne, and M. Paternostro, Daemonic ergotropy: Generalised measurements and multipartite settings, *Entropy* **21**, 771 (2019).
 - [8] A. Touil, B. Çakmak, and S. Deffner, Ergotropy from quantum and classical correlations, *J. Phys. A: Math. Theor.* **55**, 025301 (2022).
 - [9] C. Cruz, M. F. Anka, M. S. Reis, R. Bachelard, and A. C. Santos, Quantum battery based on quantum discord at room temperature, *Quantum Sci. Technol.* **7**, 025020 (2022).
 - [10] G. Francica, Quantum correlations and ergotropy, *Phys. Rev. E* **105**, L052101 (2022).
 - [11] G. Francica, F. C. Binder, G. Guarneri, M. T. Mitchison, J. Goold, and F. Plastina, Quantum coherence and ergotropy, *Phys. Rev. Lett.* **125**, 180603 (2020).
 - [12] B. Çakmak, Ergotropy from coherences in an open quantum system, *Phys. Rev. E* **102**, 042111 (2020).

- [13] Z. Niu, Y. Wu, Y. Wang, X. Rong, and J. Du, Experimental investigation of coherent ergotropy in a single spin system, *Phys. Rev. Lett.* **133**, 180401 (2024).
- [14] K. V. Hovhannisyanyan, M. Perarnau-Llobet, M. Huber, and A. Acin, Entanglement generation is not necessary for optimal work extraction, *Phys. Rev. Lett.* **111**, 240401 (2013).
- [15] G. Francica, Resource theories with completely passive states and passive states as free states, *Phys. Rev. E* **111**, 014141 (2025).
- [16] F. Binder, S. Vinjanampathy, K. Modi, and J. Goold, Quantacell: Powerful charging of quantum batteries, *New J. Phys.* **17**, 075015 (2015).
- [17] F. Campaioli, F. A. Pollock, F. C. Binder, L. Céleri, J. Goold, S. Vinjanampathy, and K. Modi, Enhancing the charging power of quantum batteries, *Phys. Rev. Lett.* **118**, 150601 (2017).
- [18] J.-Y. Gyhm, D. Šafránek, and D. Rosa, Quantum charging advantage cannot be extensive without global operations, *Phys. Rev. Lett.* **128**, 140501 (2022).
- [19] S. Julia-Farre, T. Salamon, A. Riera, M. N. Bera, and M. Lewenstein, Bounds on the capacity and power of quantum batteries, *Phys. Rev. Res.* **2**, 023113 (2020).
- [20] D. Ferraro, M. Campisi, G. M. Andolina, V. Pellegrini, and M. Polini, High-power collective charging of a solid-state quantum battery, *Phys. Rev. Lett.* **120**, 117702 (2018).
- [21] G. M. Andolina, M. Keck, A. Mari, M. Campisi, V. Giovannetti, and M. Polini, Extractable work, the role of correlations, and asymptotic freedom in quantum batteries, *Phys. Rev. Lett.* **122**, 047702 (2019).
- [22] G. M. Andolina, D. Farina, A. Mari, V. Pellegrini, V. Giovannetti, and M. Polini, Charger-mediated energy transfer in exactly solvable models for quantum batteries, *Phys. Rev. B* **98**, 205423 (2018).
- [23] D. Farina, G. M. Andolina, A. Mari, M. Polini, and V. Giovannetti, Charger-mediated energy transfer for quantum batteries, *Phys. Rev. B* **99**, 035421 (2019).
- [24] Y. Y. Zhang, T. R. Yang, L. Fu, and X. Wang, Powerful harmonic charging in a quantum battery, *Phys. Rev. E* **99**, 052106 (2019).
- [25] G. M. Andolina, M. Keck, A. Mari, V. Giovannetti, and M. Polini, Quantum versus classical many-body batteries, *Phys. Rev. B* **99**, 205437 (2019).
- [26] A. Crescente, M. Carrega, M. Sassetti, and D. Ferraro, Ultrafast charging in a two-photon Dicke quantum battery, *Phys. Rev. B* **102**, 245407 (2020).
- [27] D. Rossini, G. M. Andolina, and M. Polini, Many-body localized quantum batteries, *Phys. Rev. B* **100**, 115142 (2019).
- [28] D. Rossini, G. M. Andolina, D. Rosa, M. Carrega, and M. Polini, Quantum advantage in the charging process of Sachdev-Ye-Kitaev batteries, *Phys. Rev. Lett.* **125**, 236402 (2020).
- [29] D. Rosa, D. Rossini, G. M. Andolina, M. Polini, and M. Carrega, Ultra-stable charging of fast-scrambling SYK quantum batteries, *J. High Energ. Phys.* **11** (2020) 067.
- [30] M. Carrega, J. Kim, and D. Rosa, Unveiling operator growth using spin correlation functions, *Entropy* **23**, 587 (2021).
- [31] J.-Y. Gyhm and U. R. Fischer, Beneficial and detrimental entanglement for quantum battery charging, *AVS Quantum Sci.* **6**, 012001 (2024).
- [32] G. Francica, Quantum advantage in batteries for Sachdev-Ye-Kitaev interactions, *Phys. Rev. A* **110**, 062209 (2024).
- [33] G. Francica, Erratum: Quantum advantage in batteries for Sachdev-Ye-Kitaev interactions [Phys. Rev. A 110, 062209 (2024)], *Phys. Rev. A* **111**, 019901(E) (2025).
- [34] R. Bhatia and C. Davis, A better bound on the variance, *Am. Math. Mon.* **107**, 353 (2000).
- [35] C. Jarzynski, Nonequilibrium equality for free energy differences, *Phys. Rev. Lett.* **78**, 2690 (1997).
- [36] G. E. Crooks, Entropy production fluctuation theorem and the nonequilibrium work relation for free energy differences, *Phys. Rev. E* **60**, 2721 (1999).
- [37] J. Kurchan, A quantum fluctuation theorem, [arXiv:cond-mat/0007360](https://arxiv.org/abs/cond-mat/0007360).
- [38] H. Tasaki, Jarzynski relations for quantum systems and some applications, [arXiv:cond-mat/0009244](https://arxiv.org/abs/cond-mat/0009244).
- [39] P. Talkner, E. Lutz, and P. Hänggi, Fluctuation theorems: Work is not an observable, *Phys. Rev. E* **75**, 050102(R) (2007).
- [40] M. Campisi, P. Hänggi, and P. Talkner, *Colloquium: Quantum fluctuation relations: Foundations and applications*, *Rev. Mod. Phys.* **83**, 771 (2011).

- [41] M. Perarnau-Llobet, E. Bäumer, K. V. Hovhannisyan, M. Huber, and A. Acin, No-go theorem for the characterization of work fluctuations in coherent quantum systems, [Phys. Rev. Lett. **118**, 070601 \(2017\)](#).
- [42] E. Bäumer, M. Lostaglio, M. Perarnau-Llobet, and R. Sampaio, Fluctuating work in coherent quantum systems: Proposals and limitations, in *Thermodynamics in the Quantum Regime. Fundamental Theories of Physics*, edited by F. Binder, L. Correa, C. Gogolin, J. Anders, and G. Adesso (Springer, Cham, 2018), Vol. 195.
- [43] S. Gherardini and G. De Chiara, Quasiprobabilities in quantum thermodynamics and many-body systems, [PRX Quantum **5**, 030201 \(2024\)](#).
- [44] D. R. M. Arvidsson-Shukur, W. F. Braasch Jr., S. De Bièvre, J. Dressel, A. N. Jordan, C. Langrenetz, M. Lostaglio, J. S. Lundeen, and N. Y. Halpern, Properties and applications of the Kirkwood–Dirac distribution, [New J. Phys. **26**, 121201 \(2024\)](#).
- [45] A. E. Allahverdyan, Nonequilibrium quantum fluctuations of work, [Phys. Rev. E **90**, 032137 \(2014\)](#).
- [46] P. Solinas and S. Gasparinetti, Full distribution of work done on a quantum system for arbitrary initial states, [Phys. Rev. E **92**, 042150 \(2015\)](#).
- [47] M. Lostaglio, A. Belenchia, A. Levy, S. Hernández-Gómez, N. Fabbri, and S. Gherardini, Kirkwood–Dirac quasiprobability approach to the statistics of incompatible observables, [Quantum **7**, 1128 \(2023\)](#).
- [48] M. Lostaglio, Quantum fluctuation theorems, contextuality, and work quasiprobabilities, [Phys. Rev. Lett. **120**, 040602 \(2018\)](#).
- [49] G. Francica, Class of quasiprobability distributions of work with initial quantum coherence, [Phys. Rev. E **105**, 014101 \(2022\)](#).
- [50] G. Francica, Most general class of quasiprobability distributions of work, [Phys. Rev. E **106**, 054129 \(2022\)](#).
- [51] G. Francica and L. Dell’Anna, Optimal work extraction from quantum batteries based on the expected utility hypothesis, [Phys. Rev. E **109**, 044119 \(2024\)](#).
- [52] G. Francica and L. Dell’Anna, Role of quantum correlations in daemonic expected utility, [Phys. Rev. A **109**, 052221 \(2024\)](#).
- [53] G. Francica and L. Dell’Anna, Work fluctuation theorems with initial quantum coherence, [Phys. Rev. E **109**, 064138 \(2024\)](#).
- [54] G. Francica and L. Dell’Anna, Quasiprobability distribution of work in the quantum Ising model, [Phys. Rev. E **108**, 014106 \(2023\)](#).
- [55] X. Chen, Z.-C. Gu, and X.-G. Wen, Local unitary transformation, long-range quantum entanglement, wave function renormalization, and topological order, [Phys. Rev. B **82**, 155138 \(2010\)](#).
- [56] A. J. Leggett and A. Garg, Quantum mechanics versus macroscopic realism: Is the flux there when nobody looks? [Phys. Rev. Lett. **54**, 857 \(1985\)](#).
- [57] H. J. D. Miller and J. Anders, Leggett-Garg inequalities for quantum fluctuating work, [Entropy **20**, 200 \(2018\)](#).