

# Fibonacci and fully degenerate conical diffraction cascades: Exploring the Poincaré sphere by path interference

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Tailoring light beams in terms of intensity, phase, and polarization distribution has recently garnered significant attention. This work considers vector beams obtained by the natural conical diffraction phenomenon, occurring when a focused light beam travels along one of the optic axes of a biaxial crystal. If a cascade of several aligned biaxial crystals is used, in its sharpest plane this phenomenon usually gives rise to a concentric ring pattern with spatially varying linear polarization, thus covering the equator of the Poincaré sphere. Here, we propose smart arrangements of conical diffraction cascades based on Fibonacci and degenerate sequences leading to vectorial beams addressing multiple areas of the Poincaré sphere. The present approach is based on the concept of path interference and enables a clear spatial separation between regions exhibiting distinct polarization states (linear, elliptical, circular).

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## I. INTRODUCTION

The shaping and structuring of light beams in terms of intensity, phase, and polarization distributions have attracted significant attention for several decades. Such structured optical fields have enabled a wide range of applications, including micromanipulation, three-dimensional (3D) imaging, and classical and quantum communication [1,2]. Beyond these photonic applications, structured light fields with tailored spatial properties have also emerged as powerful tools for controlling light-matter interactions and engineering quantum states of matter [3,4]. Among the possible light beam structures are vector beams, which have spatially varying polarization across their transverse plane. These peculiar beams can be generated in different ways [5] and one of the most natural methods arises from a linear phenomenon called conical diffraction (CD). This optical effect occurs when a focused beam travels along one of the optic axes of a biaxial crystal (BC), thereby giving rise to a slanted cone with a circular base [6–8]. Even if the input wave is uniformly polarized, CD creates a double circular ring with spatially varying linear polarization, such that any two diametrically opposite points on the rings always have orthogonal linear polarizations. Despite the nearly 200 years that have passed since Hamilton’s prediction [9], this phenomenon has seen a renewed interest in the last two decades, motivated by both fundamental and applied aspects. In particular, CD beams have shown significant capabilities and potential in various modern

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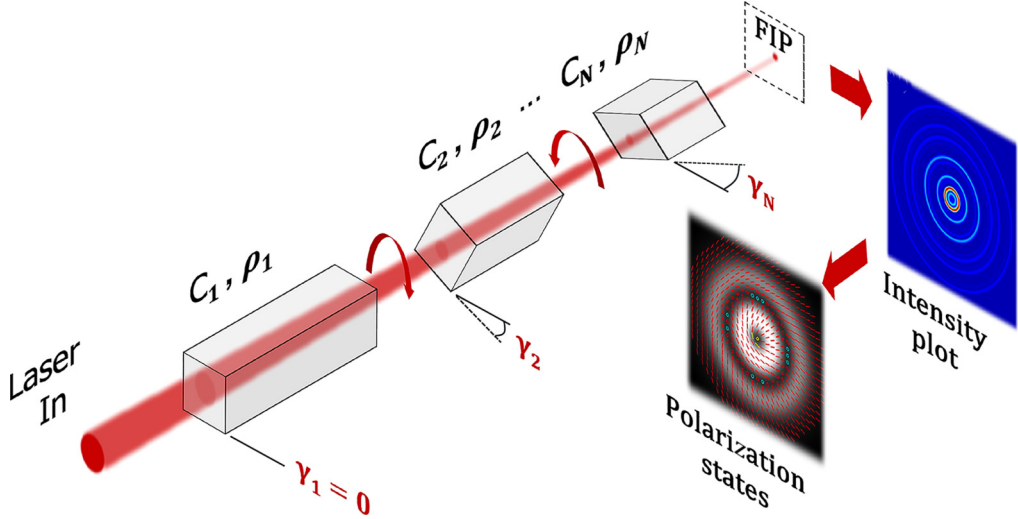


FIG. 1. Schematic of the arrangement for generating conical diffraction beams via a cascaded configuration of  $N$  biaxial crystals  $C_1, \dots, C_N$  possessing CD strengths  $\rho_1, \dots, \rho_N$ . The input beam propagates along the optic axis of each crystal and is focused through all of them to the focal image plane FIP, where the CD ring structure is best visible. The individual BCs can be rotated by an angle  $\gamma_1, \dots, \gamma_N$  around their common direction of the optic axis.

photonics applications, including optical trapping [10,11], free-space optical communication [12], superresolution imaging [13,14], polarimetry [15,16], and polarization 3D light shaping [17]. Recently, within the context of polarization holography, such vector beams have been shown to enable efficient hologram recording regardless of the mutual polarizations of the input object and reference waves. The approach permits a tailoring of the reconstructed or amplified vector waves in terms of intensity and polarization distributions, opening opportunities for their static or dynamic manipulation [18]. In parallel, a notable advancement in the field of CD involves studying cascaded configurations, where two or more crystals are arranged sequentially with their optic axes aligned (Fig. 1) [19–21]. Such a scenario results in an increase in the number of CD rings by a factor of  $2^{N-1}$ , where  $N$  is the number of crystals in the cascade. More complex arrangements involving multiple biaxial crystals with smart intermediary adjustments in wave vector space pave the way for complex shaping of these vector beams leading to CD beams that lack the typical circular symmetry [22,23]. In these peculiar configurations, our group has shown the possibility to generate highly structured noncircular CD patterns composed of individual structures with nonhomogeneous distribution of intensity and polarization around them, as observed in the plane where they are the sharpest (focal image plane, FIP). For an  $N$ -crystal cascade, each structure among the  $2^{N-1}$  ones can be associated with a specific path in the appropriate phase space. Depending on the cascade parameters, these structures may eventually cross each other and thus overlap at given spatial regions on the cross section of the CD beam. While at positions where the structures are distinct the local polarization is generally linear and covers the whole equator of the Poincaré sphere, this is no longer true at positions where the structures (the paths) interfere. It was shown that at the structure crossing and/or overlap positions, mixed (generally elliptical) polarization states can be observed, thus extending the vector wave polarizations outside the equator of the Poincaré sphere [23].

While the above approach requires manipulation in wave vector space between the individual crystals in the cascade, in the present work we discuss how path interference can be exploited even in the absence of such manipulation. Indeed, for a standard cascade, the individual structures are concentric rings that cannot cross each other. However, different rings can be made to spatially overlap upon a smart choice of the cascade parameters. This is achieved when the crystal lengths

follow a Fibonacci-type sequence or, alternatively, when all crystal lengths are identical (fully degenerate cascade). We show that such choices lead to a wider spread of polarization states on specific structures, as observed on the FIP. This spread evolves further upon diffraction. It is also shown that it is possible to spatially isolate, on the same beam, regions exhibiting circular or elliptical polarization from others exhibiting varying linear polarizations. Section II introduces the studied configurations and summarizes the theoretical background on the basis of Berry's theory for CD in cascade [20]. Section III illustrates the effects of path interference in the case of Fibonacci and fully degenerate cascades by means of several examples. Finally, Sec. IV provides a discussion and conclusions.

## II. CASCADE CONFIGURATION AND THEORETICAL BACKGROUND

The configuration analyzed in this work is schematically represented in Fig. 1. It consists of a cascade of  $N$  BCs all aligned along one of their two optic axes. A focused input beam traverses all the BCs and experiences cascaded CD. The plane at which this CD pattern is best resolved is the FIP and corresponds to the focal plane of the input wave if the crystals were isotropic and with a refractive index equal to the one seen by the CD wave [6,24]. We assume here that the input beam is a circularly polarized Gaussian wave and we discuss only the CD pattern at the FIP. At this plane, for sufficiently strong focusing, conventional cascades give a set of well-resolved concentric double rings with spatially varying linear polarizations. Each double ring is separated by a dark ring (Poggendorff ring). For simplicity, in the following we will refer to the double rings simply as "rings."

While remaining aligned with the common direction of the optic axes of all crystals, each BC can freely rotate around this axis. This is associated with an orientation angle  $\gamma_n$  for the  $n$ th crystal (see Fig. 1). If we assumed a unique  $\vec{k}$  vector aligned to the optic axis, the CD cone that would be observed for a single BC spans all Poynting-vector directions associated with this  $\vec{k}$  vector. Importantly, this CD cone is slanted with respect to the optic axis and  $\vec{k}$ , and  $\gamma_n$  indicates in which direction it is slanted. A detailed discussion on the determination of the slanting direction and  $\gamma_n$  can be found in Ref. [25]. Without loss of generality, we can assume  $\gamma_1 = 0$ , so that  $\gamma_n$  is the orientation angle of the  $n$ th BC with respect to the first in the row. The angles  $\gamma_n$  are important parameters of the cascade. They affect the intensity and polarization distributions among the structures composing the CD pattern, as discussed later.

The other crucial parameters for the purpose of the present work are the lengths  $L_n$  of the individual BCs. In this context, it is convenient to introduce the normalized CD strength parameters  $\rho_n \equiv \alpha_n L_n / w$ , which are proportional to these lengths. Here,  $\alpha_n$  is the half-aperture angle of the CD cone for crystal  $n$  and depends on the three main refractive indices of the material. If all BCs are made of the same material, all  $\alpha_n$  reduce to a common value  $\alpha$ . The product  $\alpha_n L_n$  gives the radius of the ring observable if only the  $n^{\text{th}}$  crystal were present. Therefore,  $\rho_n$  corresponds to the normalized radius with respect to the input wave width  $w$ . As described in Refs. [20,26],  $w$  can be taken as the  $1/e$  intensity radius of the incident beam focused by a spherical lens and measured in its focal plane, corresponding to the FIP.

Conical diffraction in cascade has been theoretically modeled by Berry [20] on the basis of a paraxial diffraction theory, where the effect of the BCs on the optical wave was expressed by unitary matrices in Fourier space. These matrices  $U_n$  essentially describe the dephasing upon traversing crystal  $n$  for wave vector components slightly departing from the exact direction of the optic axis. They are a function of both quantities  $\rho_n$  and  $\gamma_n$  discussed above. In the following, we summarize only the principal results of Berry's theory that are relevant to the present work.

It can be shown that the cascade of  $N$  BCs with CD strengths  $\rho_1, \rho_2, \dots, \rho_N$  leads in the FIP to a pattern exhibiting concentric rings with normalized radii:

$$\rho_{\text{tot}} = |\rho_1 \pm \rho_2 \pm \rho_3 \pm \dots \pm \rho_{N-1} \pm \rho_N|. \quad (1)$$

The possible choices of the plus and minus signs in Eq. (1) lead to  $2^{N-1}$  values for the radius  $\rho_{\text{tot}}$ , which correspond to the positions of the  $2^{N-1}$  concentric rings. As pointed out in Ref. [20], each of the above possibilities is identified by a string of + and – and can be referred to as a path. The overall CD pattern is obtained by the coherent superposition of these paths, or, in other words, by superposing the complex optical amplitudes of the CD rings, each being associated with a specific path. In the common case where the CD strengths  $\rho_1, \rho_2, \dots, \rho_N$  are random, the possible values of  $\rho_{\text{tot}}$  generally all differ from each other, meaning that all rings are distinct. However, there are two interesting cases where the same value of  $\rho_{\text{tot}}$  can occur two or more times, so that a few rings can overlap on the same spatial position. As shown by a few examples in the next section, in such situations path interference can lead to interesting effects, specifically on the local polarization distribution.

The first such case concerns Fibonacci cascades. In this case, the values of the CD strengths of the BCs shall be chosen as the elements of a Fibonacci sequence,

$$\rho_{n+1} = \rho_n + \rho_{n-1}. \quad (2)$$

It can be easily shown that for this type of choice the probability of ring overlaps increases strongly if the length of the cascade  $N$  increases. Note that, while Eq. (2) suggests that the BCs should be ordered according to increasing values of  $\rho_n$ , this is not mandatory for our purposes. Indeed, Eq. (1) implies that any permutation among  $\rho_n$  leads to the same set of  $\rho_{\text{tot}}$  values. The only condition is that the numbers  $\rho_1, \rho_2, \dots, \rho_N$  correspond to one of the permutations of the elements of the considered Fibonacci sequence. Different permutations lead to the same number of overlapping rings. Nevertheless, the relative intensity and the local polarization in the observed rings may differ as these also depend on the individual orientations  $\gamma_n$  of the BCs.

The other interesting case is the one of a fully degenerate cascade where  $\rho_1 = \rho_2 = \dots = \rho_N \equiv \rho_0$ . Here, the resulting CD rings are equidistant in the radial direction. If  $N$  is odd there are  $(N + 1)/2$  possible values of  $\rho_{\text{tot}}$  corresponding to  $\rho_{\text{tot}} = (2m + 1)\rho_0$ , with  $m = 0, 1, \dots, (N - 1)/2$ . For  $N$  even, one finds  $N/2 + 1$  values at  $\rho_{\text{tot}} = 2m\rho_0$ , with  $m = 0, 1, \dots, N/2$ , thus including the case where a ring is reduced to a central spot. Clearly, with increasing  $N$  most of the paths will interfere with some others. For instance, for  $N = 7$  we have only 4 distinct ring radii out of the 128 potential ones for the general case.

The amplitude associated with individual paths merits some discussion. The latter depends on the orientation angles  $\gamma_n$  of the individual BCs and, more specifically, on the difference in orientation angles between subsequent crystals, i.e.,

$$\Delta\gamma_n \equiv \gamma_n - \gamma_{n-1}. \quad (3)$$

It can be shown that all paths contribute to the overall CD pattern if all  $\Delta\gamma_n$  differ from 0 or  $\pm\pi$ . These two cases are generally referred to as parallel or antiparallel crystals, respectively. If none of the neighboring BC pairs are either parallel or antiparallel, all the paths are active and, in the general case, all the concentric rings are visible, however, at different levels of intensity. An interesting situation is the one where all BC pairs are crossed so that  $\Delta\gamma_n = \pm\pi/2$  for  $n = 2, \dots, N$ . In this case, the CD phenomenon splits the optical power equally among the  $2^{N-1}$  rings [21]. As a consequence, in the absence of path interference the local intensity on the rings is inversely proportional to their normalized radius  $\rho_{\text{tot}}$  and their circumference.

In contrast, if some of the neighboring BC pairs are either parallel or antiparallel, some of the paths will carry a vanishing amplitude and the corresponding CD rings disappear. This effect can be brought to the extreme if one selects all  $\Delta\gamma_n$  to be either 0 or  $\pm\pi$ . It can be shown that this leads to the selection of an individual path while all others vanish, and thus the survival of only one CD ring in the cascaded pattern. As there are  $2^{N-1}$  possible arrangements fulfilling the above condition, all of the possible paths can be individually selected by a proper choice of the BC arrangement. The consequences in the case of Fibonacci and fully degenerate cascades will be discussed in the next section. Note that this selection feature is also common to the case where the structures composing

the patterns are noncircular as a result of wave vector space manipulation between the BCs, as reported previously in Ref. [23].

We finally briefly illustrate how the calculated patterns in Sec. III are determined by a semianalytical treatment following Ref. [20]. First one determines the coefficients associated with each path, which, as mentioned above, are a function of all the relative BC orientations  $\Delta\gamma_n$ . As the expressions are numerous and lengthy, we do not give them explicitly here. These coefficients are then combined with the numerically evaluated values of the so-called Belsky-Khapalyuk integrals [27,28] defined as

$$B_0(\rho, \zeta; \rho_{\text{tot}}) \equiv \int_0^\infty e^{-\frac{\kappa^2}{2}(1+i\zeta)} \cos(\kappa \rho_{\text{tot}}) J_0(\kappa \rho) \kappa d\kappa \quad (4)$$

and

$$B_1(\rho, \zeta; \rho_{\text{tot}}) \equiv \int_0^\infty e^{-\frac{\kappa^2}{2}(1+i\zeta)} \sin(\kappa \rho_{\text{tot}}) J_1(\kappa \rho) \kappa d\kappa, \quad (5)$$

where  $J_0$  and  $J_1$  are the Bessel functions of the first kind of order 0 and 1, respectively, and  $\rho$  is the normalized radial coordinate in real space,  $\rho \equiv r/w$ . Furthermore,  $\kappa \equiv k_t w$  is the normalized transverse wave vector with respect to the longitudinal direction  $z$  of the optic axis, i.e.,  $k_t \equiv (k_x^2 + k_y^2)^{1/2}$ . The variable  $\zeta \equiv z/(k_0 w^2)$  is a normalized longitudinal coordinate along  $z$  with  $\zeta = 0$  and  $z = 0$  corresponding to the FIP and  $k_0 = 2\pi/\lambda$  being the light wave number for the vacuum wavelength  $\lambda$ . The integrals  $B_0$  and  $B_1$  contain the information on the radial amplitudes of the CD structures associated with each individual path. They are found after azimuthal integration of an initial two-dimensional Fourier integral, expressed in radial coordinates, that contains the product of the transfer matrices  $\mathbf{U}_n$  describing the cascade. By keeping  $\zeta$  fixed, these integrals have to be evaluated numerically as a function of  $\rho$  for all possible values of the parameter  $\rho_{\text{tot}}$  according to Eq. (1), that is, for all the paths. Finally, all the path coefficients and Belsky-Khapalyuk integrals are combined to calculate the four complex elements of a  $2 \times 2$  matrix  $\mathbf{M}(\rho, \varphi, \zeta)$  permitting to obtain the spatial distribution of the complex  $x$  and  $y$  components of the electric displacement vector  $\vec{D}$  from the initial Jones vector of the input wave, which in our case is circularly polarized. This complex  $\vec{D}$  vector contains the whole information on the local intensity ( $\propto |\vec{D}|^2$ ) and on the local polarization of the optical CD wave.

### III. FIBONACCI AND FULLY DEGENERATE CASCADES

#### A. Fibonacci cascades

We start by discussing Fibonacci cascades; examples for series of five BCs are shown in Fig. 2. The results are numerically obtained on the basis of Berry's theory and the procedure described in the previous section. The input wave is supposed to be Gaussian and possesses left-circular polarization.

The first row [Figs. 2(a1)–2(a4)] corresponds to a standard Fibonacci sequence with increasing values of the strength parameters  $\rho_i$  according to Eq. (2), i.e., the ordered sequence  $\{\rho_i\} = (5, 8, 13, 21, 34)$ . Here, subsequent crystals are supposed to be crossed so that  $\Delta\gamma_n = \pm\pi/2$  in Eq. (3). Of the potentially 16 ring radii for a five-BC cascade only 10 survive in this Fibonacci cascade. Indeed, four rings (at  $\rho_{\text{tot}} = 3, 29, 39$ , and 55) correspond to a superposition of two paths, while the ring at  $\rho_{\text{tot}} = 13$  is a threefold superposition. These rings are explicitly identified in Fig. 2(b4). The false color intensity picture [first column, Fig. 2(a1)] shows that the rings have different levels of local intensity, with the two most internal ones  $\rho_{\text{tot}} = 3$  (twofold) and  $\rho_{\text{tot}} = 13$  (threefold), as well as the ring at  $\rho_{\text{tot}} = 29$  (twofold), being the most prominent ones.

The second to fourth columns in Fig. 2 deal with the local polarization distribution. The  $x$  and  $y$  components of the  $\vec{D}$  vector are complex and may be generally expressed as  $D_x = R_x \exp(i\phi_x)$  and  $D_y = R_y \exp(i\phi_y)$ . The second column represents the phase shift  $\Delta\phi \equiv \phi_y - \phi_x$  between the  $y$  and  $x$  components. In this representation, linear polarization corresponds to a phase shift of 0 or  $\pi$

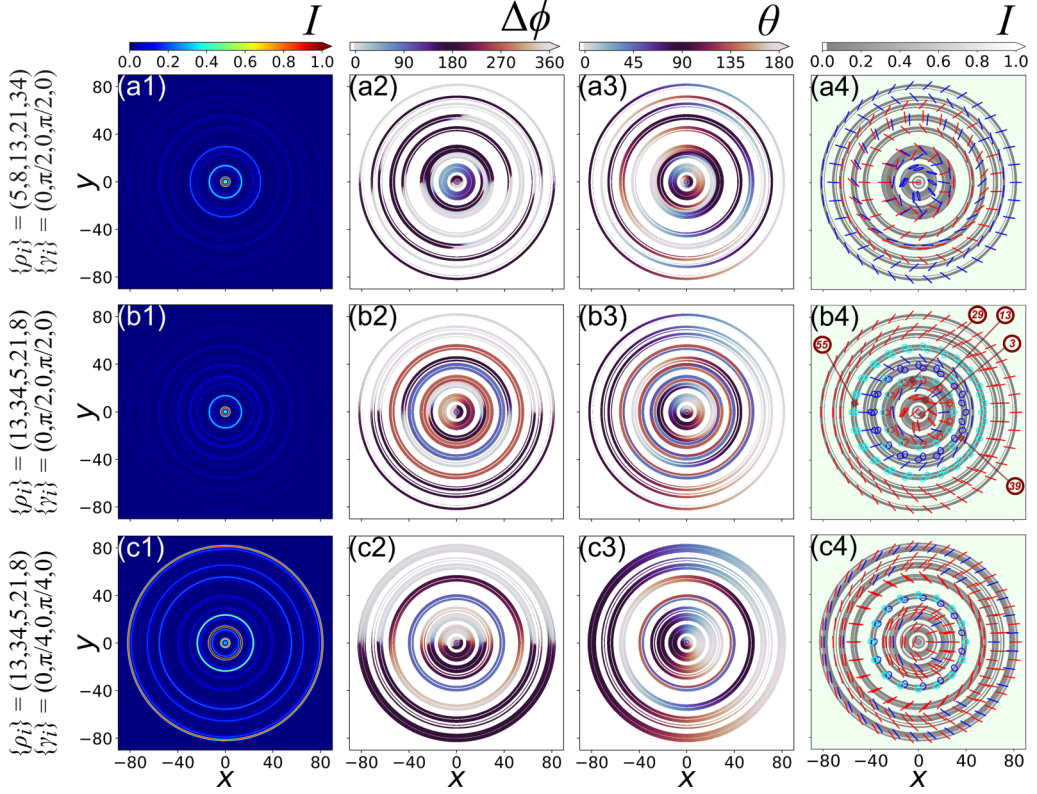


FIG. 2. Conical diffraction structure at the FIP by a Fibonacci cascade of five BCs at positions  $(x, y) = (\rho \cos \varphi, \rho \sin \varphi)$ . The values of CD strength sequence  $\{\rho_i\}$  and the corresponding orientations  $\{\gamma_i\}$  are given on the left of each row. The first row [(a1)–(a4)] is for a standard Fibonacci sequence with increasing values of  $\rho_i$  according to Eq. (2), while the two other rows [(b1)–(c4)] are for a permuted version of the same values. The first two rows are for mutually crossed BCs, for the last row [(c1)–(c4)] the mutual angles are  $45^\circ$ . First column [(a1)–(c1)]: Intensity distribution in the CD structure composed of 10 concentric rings (instead of 16 for general cascades). Second column [(a2)–(c2)]: Phase difference  $\Delta\phi$  between the  $y$  and  $x$  components of the output  $\vec{D}$  field. Third column [(a3)–(c3)]: Angle  $\theta$  for the orientation of the local polarization ellipse. Fourth column [(a4)–(c4)]: Local polarization ellipses on the CD structure. Here, red corresponds to right-handed polarization and blue to left-handed polarization. Cyan circles give regions where the polarization is circular within a small tolerance (see text). The circled numbers in panel (b4) identify the radii  $\rho_{\text{tot}}$  of the five rings having two- or threefold degeneracy due to path interference.

(gray or black color, respectively), while elliptical or circular polarization are associated with the intermediate blueish and brownish colors. Figure 2(a2) shows that for this conventionally ordered Fibonacci sequence only the light in the second ring from inside ( $\rho_{\text{tot}} = 13$ ) departs significantly from a linear polarization. The third column depicts the angle  $\theta \equiv \arctan[(R_y/R_x)\text{sgn}(\cos \Delta\phi)]$ . This is the orientation angle of linear polarization (with  $\theta = 0$  corresponding to horizontal polarization), or, for places where the light is elliptically polarized, the orientation angle of the main axis of the polarization ellipse. As seen in Fig. 2(a3), on each ring the angle  $\theta$  spans all possible directions, with diametrically opposite points associated with orthogonal directions of the polarization ellipse. The latter are explicitly drawn in the representation in the fourth column. As already mentioned, for our standard Fibonacci sequence [Fig. 2(a4)] the polarization is generally linear and the blue and red colors indicate the handedness associated with the (small) residual ellipticity, with blue being associated with left-handed light and red with right-handed light. This handedness is generally

opposite for the two subrings composing a double ring, as seen, for instance, for the case  $\rho_{\text{tot}} = 55$  (fourth ring from the outside), where the polarization directions are explicitly indicated on each subring. As inferred above, Fig. 2(a4) confirms that in the present case only the ring at  $\rho_{\text{tot}} = 13$  exhibits significant ellipticity, which is a consequence of path interference. Please note that, in order to conserve the information on the underlying ring structure, in the second and third columns of Fig. 2 the values of  $\Delta\phi$  and  $\theta$  are plotted only if the corresponding local intensity exceeds a certain threshold, typically in the range 0.1%–0.5% of the overall peak intensity. For the same reasons, the polarization ellipses in the fourth column are overlaid on the ring intensity shown in grayscale.

As mentioned earlier, the crystals whose conical diffraction strengths belong to a Fibonacci sequence do not necessarily need to be ordered with increasing values of  $\rho_i$ . Any permutation of the Fibonacci values leads to the same set of possible ring radii  $\rho_{\text{tot}}$ . The second row in Fig. 2 shows the consequences of one of the possible permutations [BCs ordered according to  $\{\rho_i\} = (13, 34, 5, 21, 8)$ ] with subsequent BCs still in crossed configuration. In Fig. 2(b1), one can recognize the ten expected rings, including the one for a radius  $\rho_{\text{tot}} = 39$ , which was not visible in Fig. 2(a1) due to a perfect destructive interference between the two involved paths. It is evident that the intensity hierarchy among the rings has changed as a result of the different BC ordering. Moreover, this change also leads to a major change in the polarization distribution. Now, in addition to the ring at  $\rho_{\text{tot}} = 13$ , also the rings at  $\rho_{\text{tot}} = 29, 39$ , and  $55$  exhibit strong ellipticity of the light polarization [see Figs. 2(b2) and 2(b4)]. For the rings at  $\rho_{\text{tot}} = 29$  and  $55$ , the light is indeed very close to a circular polarization. Note that the latter is indicated in the diagrams of the fourth column by cyan circles. The criterion for this choice is that the ellipticity angle  $\chi \equiv \arctan(b/a)$  equals  $45^\circ$  (corresponding to exact circular polarization) within a tolerance of  $3^\circ$ ,  $a$  and  $b$  being the lengths of the two main axes of the polarization ellipse.

The intensity and polarization distribution among the rings is influenced not only by the ordering of the BCs, but also by the BC orientation angles  $\gamma_i$ . The third row in Fig. 2 gives the results for the same BC ordering as in the second row, but for smaller relative rotation  $\Delta\gamma_n = \pm\pi/4$  instead of  $\Delta\gamma_n = \pm\pi/2$ , so that the BCs are no longer crossed. Smaller  $\Delta\gamma_n$  tend to increase the power in the outer rings at the expense of the inner ones, as it is evident by the comparison of Fig. 2(c1) with Fig. 2(b1). Now only the rings at  $\rho_{\text{tot}} = 29$  and  $39$  exhibit significant ellipticity of the light polarization, and for  $\rho_{\text{tot}} = 39$ , we have essentially left circular polarization. The threefold ring at  $\rho_{\text{tot}} = 13$ , which exhibited strong ellipticity in the two previous cases, is now associated with essentially linearly polarized light.

The above results clearly show that path interference in Fibonacci cascades leads to an increase of the complexity of CD vector waves. It extends the polarization coverage outside the equator of the Poincaré sphere even at the FIP, where the structure is sharp. This plane is most interesting as it allows to clearly spatially separate regions with different polarization features. However, the most advantageous Fibonacci sequences, with an enhanced number of superposed rings, occur only for cascades of more than five BCs. Such cascades are therefore an interesting solution for generating Poincaré-like beams, although their experimental realization is relatively complex. Nevertheless, in what follows we will see that a fully degenerate cascade configuration can offer increased path interference and polarization diversity with a smaller number of required BCs.

## B. Fully degenerate cascades

As an illustration, Fig. 3 presents fully degenerate cascades composed of two to five BCs, where all BCs share an identical CD strength  $\rho_i = \rho_0 = 10$ . For  $N$  even, the cascade consistently produces a central intensity spot, where the polarization aligns with that of the incident beam. Indeed, as seen in Figs. 3(a2) and 3(c2), the central spot exhibits a relative phase  $\Delta\phi$  of  $90^\circ$ , consistent with left circular polarization. These results are in good agreement with previous theoretical and experimental studies on degenerate cascades of two BCs, which have shown that the central spot maintains the same polarization as the incident beam [29–31].

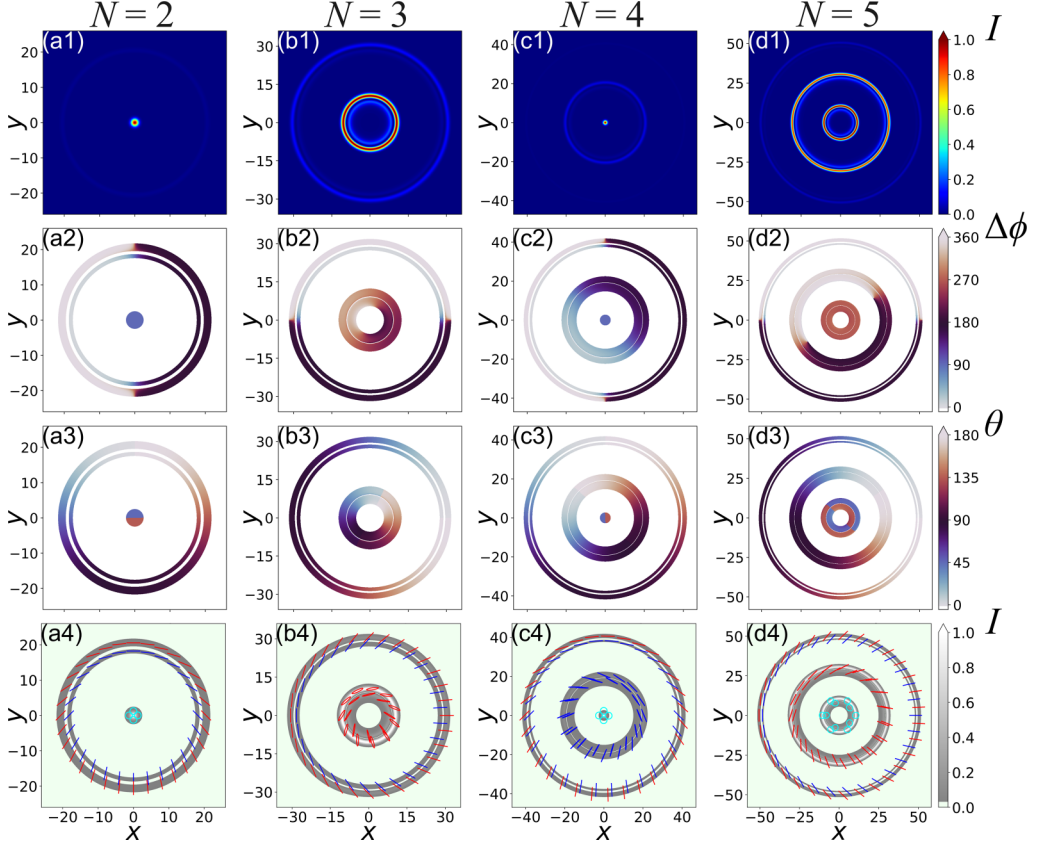


FIG. 3. Fully degenerate cascades with  $\rho_i = \rho_0 = 10$  and mutually crossed BCs  $[\{\gamma_i\} = (0, \pi/2, 0, \pi/2, \dots)]$ . The cascade length  $N$  is given on top of each column. The rows give the distribution of the local intensity  $I$  (first row), and of the relative phase  $\Delta\phi$  (second row), of the main direction of polarization  $\theta$  (third row), as well as the local polarization ellipses (fourth row), as defined in connection to Fig. 2.

The degree of path interference among the rings seen in the first row of Fig. 3 can be easily determined. For  $N = 2$ , no path interference can exist. Furthermore, Eq. (1) implies that all the most external rings in Figs. 3(b1)–3(d1) can be obtained only through a single path (this includes also the hardly visible one at  $\rho_{\text{tot}} = 40$  for  $N = 4$ ). These rings are always associated with linear polarization, as can be verified by the values of  $\Delta\phi$  in the second row and by the explicit representation of the local polarization in the fourth row. For  $N = 3$ , the internal ring at  $\rho_{\text{tot}} = 10$  is threefold, while for  $N = 5$  the ring at  $\rho_{\text{tot}} = 10$  is associated with ten paths and the one at  $\rho_{\text{tot}} = 30$  is fivefold. Finally, for  $N = 4$  the central spot corresponds to the superposition of three paths and the ring at  $\rho_{\text{tot}} = 20$  is a fourfold merging.

Clearly, fully degenerate cascades give simpler CD ring structures with radially equidistant rings facilitating their discrimination. In all but the most external ring, the number of contributing paths is highly increased with respect to the Fibonacci case. However, as already mentioned in Sec. II, not all paths contribute equally to a given ring and the extent of ellipticity depends on the weights associated with the different paths. The example of Fig. 3 shows significant local ellipticity for the internal ring for ( $N = 3$ ) and ( $N = 5$ ). In the former case, the polarization is elliptical and in the latter it is circular [see Figs. 3(b2), 3(b4), 3(d2), and 3(d4)]. To a lesser extent, elliptically polarized light is found also for the intermediate ring for  $N = 4$  [Figs. 3(c2) and 3(c4)]. We have also verified that, as for the Fibonacci case, the intensity and polarization distribution of the ring structure is

modified by a variation of the set of BC orientation angles  $\{\gamma_i\}$ . Note that in the limiting case where all the BCs are either parallel or antiparallel (all  $\Delta\gamma_n$  are either  $= 0$  or  $\pi$ , not shown in Fig. 3) there is only a single surviving path as all the remaining  $2^{N-1} - 1$  ones carry a vanishing contribution. In this case, the single surviving ring is always associated with linear polarizations, similarly to the most external ring in each column of Fig. 3.

Finally, we discuss another intriguing scenario arising when setting  $\rho_0 = 1$ , corresponding to weaker focusing of the input wave. In this case, a single CD pattern with  $\rho_0 = 1$  produces a donut-shaped beam rather than the typical CD double ring, as the Poggendorff ring is absent. Degenerate cascades with  $\rho_0 = 1$  for configurations of three, four, and five BCs are presented in Fig. 4.

For the three-BC cascade, the intensity distribution still forms a donut shape [see Fig. 4(a1)] with a varying relative phase  $\Delta\phi$ . Notably, this variation occurs not only along the azimuthal direction but now also radially to some extent [see Fig. 4(a2)], corresponding to a gradual evolution from elliptical to linear polarization as the radius  $\rho$  increases [see Fig. 4(a4)]. Extending the degenerate cascade with  $\rho_0 = 1$  to four and five BCs results in more complex intensity distributions: a central intensity spot surrounded by a broad ring for four crystals, and a donut shape enclosed by a broad outer ring for five crystals [see Figs. 4(b1) and 4(c1)]. The relative phase  $\Delta\phi$  distribution across the inner donut and outer ring reveals the presence of varying elliptical polarization along the azimuthal direction. Moreover, the elliptical polarization states differ between the inner donut and the outer ring at any given azimuthal angle [see, for instance, Figs. 4(c2) and 4(c4)].

Clearly, as compared to the cases illustrated in Figs. 2 and 3, here low  $\rho_0$  values lead to almost smoothly changing complex vectorial waves with less clear frontiers between regions with different polarization features.

#### IV. DISCUSSION AND CONCLUSION

We have shown that path interference in conical diffraction cascades can lead to a stronger coverage of the Poincaré sphere, while maintaining a well-defined boundary between regions with different local polarization features. Two types of cascades were discussed and shown to enable this polarization diversity.

Fibonacci cascades generally lead to a large number of concentric rings, some of which exhibit path interference. The latter are relatively few if the number of BCs is small and increase gradually with increasing  $N$ . For instance, for  $N = 4$  the CD pattern still exhibits 75% of the maximum number of rings (6 out of 8) and 33% of them are associated with path interference. For the example given in Sec. III ( $N = 5$ ), one observes 62.5% of the maximum number of rings (10 out of 16), half of which exhibit path interference. These numbers evolve further by extending the Fibonacci cascade to a larger number of BCs. Adding a sixth BC with  $\rho_6 = 55$  leads to 17 out of a maximum of 32 rings ( $\approx 53\%$ ) including a central spot associated with  $\rho_{\text{tot}} = 0$ . Of these 17 rings, only 6 are free of path interference, which is found on roughly 65% of the rings. For a cascade with  $N = 7$ , with an additional BC with  $\rho_7 = 89$  one gets 27 rings (out of 64,  $\approx 42\%$ ), of which 20 ( $\approx 74\%$ ) are two-, three-, four-, or fivefold. These numbers indicate that Fibonacci cascades unveil their full potential only for cascades containing at least six or seven BCs, for which the total number of rings is on the order of 20 and some of the ring radii have values close to each other. Indeed, for our choice of starting values of the Fibonacci sequences, for  $N \leq 7$  two rings can get as close as a ratio of 1.029 between their radii, as found for the second and third rings from the outside in a seven-BC Fibonacci cascade.

The case of fully degenerate cascades is somewhat simpler as the number of surviving rings is greatly reduced and their arrangement is radially equidistant. In this case, a high degree of path interference can already be achieved with fewer than five BCs. We have also discussed the case where the conical diffraction strength of the BCs is small, corresponding to a weak focusing of the input Gaussian wave. In this case, one obtains smooth polarization variations across the cross section of the output beam, which is of interest if sharp frontiers between regions exhibiting different kind of polarization are not required.

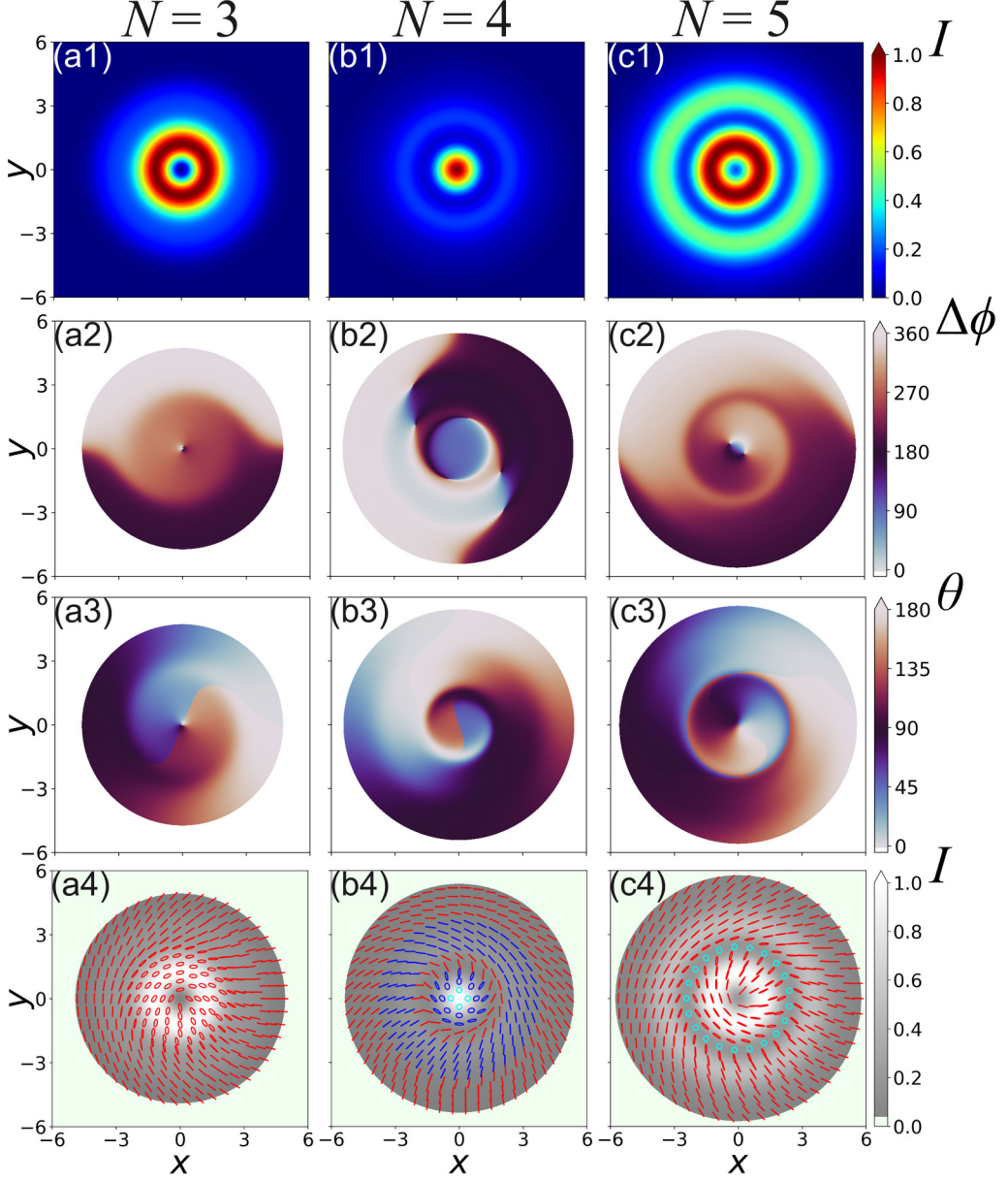


FIG. 4. Fully degenerate cascades with small CD strength parameter ( $\rho_i = \rho_0 = 1$ ) and mutually crossed BCs  $[\{\gamma_i\} = (0, \pi/2, 0, \pi/2, \dots)]$  for cascade lengths of  $N = 3, 4$ , and  $5$ . The rows give the distribution of the local intensity  $I$  (first row), of the relative phase  $\Delta\phi$  (second row), and of the main polarization direction  $\theta$  (third row), as well as the local polarization ellipses (fourth row), as defined in connection to Fig. 2.

For both types of cascades, as seen in the examples in Sec. III, path interference does not always guarantee that the local polarization on the corresponding ring significantly departs from a linear one. The polarization state of an individual ring results from the coherent superposition of all complex field contributions associated with the paths reaching the same radial position. Depending on the cascade parameters, the contributing paths may acquire comparable or different amplitudes. When one contribution dominates, the resulting polarization tends to follow the polarization state

associated with this path, whereas comparable contributions can lead to significant polarization transformations through interference. In particular, significant ellipticity or nearly circular polarization is obtained when the coherent superposition generates orthogonal field components with comparable amplitudes and a relative phase close to  $\pm\pi/2$ . Conversely, the polarization remains linear when the resulting relative phase between the orthogonal components is close to 0 or  $\pi$ . The amplitudes and phases of the individual field contributions from different paths depend on multiple coupled parameters, including the relative orientations of successive BCs (for both cascade types) and the ordering of the BCs (for Fibonacci cascades). Therefore, while the forward calculation of the polarization state is straightforward, designing a cascade that produces a prescribed polarization distribution remains a challenging inverse problem requiring numerical exploration or optimization.

From a practical point of view, there are essentially two possibilities for implementing this kind of BC cascades. The first involves preparing independent BC samples with the correct thicknesses and mounting each one on separate mounts. This implementation allows fine adjustment of the optic axis directions of each sample in the laboratory frame and of their relative orientation angles  $\Delta\gamma_n$ . Although rather bulky, this solution allows one to retain the full versatility of the approach for a given set of sample thicknesses. The second possibility consists of slicing down a thick BC crystal in several plane parallel sheets of the desired thicknesses and gluing them together in a determined order after applying the desired relative rotations by the angles  $\Delta\gamma_n$ . This solution is compact but allows one to obtain only a specific vector wave structure. Here, it is crucial that the relevant optic axis of each sheet is well oriented perpendicular to their surfaces, so that the optic axes remain parallel after the sheets are glued together. Indeed, an excessive precession of the optic axis direction from sheet to sheet would prevent their overall alignment to the input wave propagation direction. For both solutions, the required thickness precision of the individual BC samples or sheets is relatively relaxed. Indeed, the ratio between the radius of a CD ring and its width is typically on the order of  $\rho_{\text{tot}}$ . Therefore, for  $\rho_{\text{tot}} = 100$  a precision of roughly 1% on the thicknesses is sufficient to guarantee an adequate ring overlap; for smaller values of  $\rho_{\text{tot}}$ , even larger errors are allowed.

All the above findings highlight the promising potential of generating Poincaré-like beams, characterized by spatially separated linear, circular, and elliptical polarization states arranged, in our case, in concentric rings. Therefore, the proposed Fibonacci and fully degenerate cascades offer significant opportunities for the control and tailoring of the polarization profiles of CD vector beams, thereby broadening their functionality for a range of applications already demonstrated for vector beams, including optical trapping [32,33], free-space optical communication multiplexing and cryptography [34], single-shot and spatially resolved polarimetric schemes [35], and superresolution microscopy [36], as well as other emerging applications [37,38].

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#### DATA AVAILABILITY

There are no publicly available research data or software supporting this manuscript. Requests for further information or data should be sent to the authors.

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